

Correction to the paper
“On relatively disjoint families of measures, with some
applications to Banach space theory”
this volume pp. 13-36

by

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All references and notation are given in the above-mentioned paper, which appeared in *Studia Mathematica*.

As pointed out to us by I. Singer, our proof of Diendoné's theorem, given preceding Section 2, is incorrect. We restate, then prove the result in question.

THEOREM. *Let (μ_n) be a sequence of bounded finitely additive set functions (complex valued) defined on the discrete set A .*

(a) (Dieudonné [4]) *If $\sup_n |\mu_n(E)| < \infty$ for all $E \subset A$, then*

$$\sup_n \|\mu_n\| < \infty.$$

(b) (Phillips [20]) *If $\lim_n \mu_n(E) = 0$ for all $E \subset A$, then*

$$\lim_n \sum_{j \in A} |\mu_n(\{j\})| = 0.$$

Proof. Let (μ_n) satisfy the hypotheses of (a), put $\mathfrak{F} = \{\mu_n: n = 1, 2, \dots\}$. Let us say that $E \subset A$ is a *bad* set if $\sup_{\mu \in \mathfrak{F}} |\mu(E)| = \infty$. We assume that A is a bad set and argue to a contradiction.

(i) Suppose there exists a bad set E which cannot be written as a disjoint union of two bad sets. Choose $\nu_1 \in \mathfrak{F}$ with $|\nu_1(E)| > 1 + 8 \sup_{\mu \in \mathfrak{F}} |\mu(E)|$. Now choose $F \subset E$ with $|\nu_1(F)| > |\nu_1(E)|/4$. Then $|\nu_1(E \sim F)| \geq |\nu_1(F)| - |\nu_1(E)| \geq |\nu_1(E)|/8$. Now if F is not a bad set, let $E_1 = F$; otherwise let $E_1 = E \sim F$. Thus $|\nu_1(E_1)| \geq |\nu_1(E)|/10$.

Suppose E_1, \dots, E_n and $\nu_1, \dots, \nu_n \in \mathfrak{F}$ have been chosen with $\bigcup_{j=1}^n E_j \subset E$

and $E \sim \bigcup_{j=1}^n E_j$ a bad set. Then $\bigcup_{j=1}^n E_j$ is not a bad set, hence $\sup_{\mu \in \mathfrak{F}} |\mu|(\bigcup_{j=1}^n E_j) = \lambda < \infty$; also $\sup_{\mu \in \mathfrak{F}} |\mu|(E \sim \bigcup_{j=1}^n E_j) = \beta < \infty$. Choose $v_{n+1} \in \mathfrak{F}$ so that $|v_{n+1}|(E) > n + 1 + 10(\lambda + \beta)$. Then $|v_{n+1}|(E \sim \bigcup_{j=1}^n E_j) \geq \frac{9}{10} |v_{n+1}|(E)$. Now choose $F' \subset E \sim \bigcup_{j=1}^n E_j$ with $|v_{n+1}|(F') \geq |v_{n+1}|(E)/5$. Let $E_{n+1} = F'$ if $E \sim \left(\bigcup_{j=1}^n E_j \cup F'\right)$ is a bad set; otherwise let $E_{n+1} = E \sim \left(\bigcup_{j=1}^n E_j \cup F'\right)$. It follows that $|v_{n+1}|(E_{n+1}) \geq |v_{n+1}|(E)/10$.

The sequence of subsets E_1, E_2, \dots of E and functions $v_i \in \mathfrak{F}$ thus constructed are such that the E_i 's are disjoint and $|v_i|(E_i) \geq |v_i|(E)/10 \geq i/10$ for all i .

By Lemma 1.1, we may choose $n_1 < n_2 < \dots$ such that for all i , $|v_{n_i}|(\bigcup_{j \neq i} E_{n_j}) < \frac{1}{2} |v_{n_i}|(E_{n_i})$. (We apply 1.1 to the measures $|v_n|_E / |v_n|(E_{n_i})$ and put $\varepsilon = \frac{1}{2}$; $|v_n|(E) / |v_n|(E_{n_i}) \leq 10$ for all n , of course.) Then putting $G = \bigcup_{j=1}^{\infty} E_{n_j}$, $|v_{n_i}|(G) \geq |v_{n_i}|(E)/20$, whence $|v_{n_i}|(G) \rightarrow \infty$, a contradiction.

(ii) Now assume that every bad set can be written as a disjoint union of two bad sets. Then assuming A is a bad set, there exists an infinite sequence of pairwise-disjoint bad sets, E_1, E_2, \dots

We may then choose for all $i \in \mathbb{N}$ (the positive integers), by induction, $v_i \in \mathfrak{F}$, $N_i \subset \mathbb{N}$, $F_i \subset A$, and $n_i \in \mathbb{N}$ satisfying the relations: $n_i < n_{i+1}$, $N_i \supset N_{i+1}$ with N_i infinite, $n_i \in N_i$, $F_i \subset E_{n_i}$, $|v_i|(E_{n_i}) \geq i + 4 \sum_{j=1}^{i-1} \sup_{\mu \in \mathfrak{F}} |\mu|(F_j)$, $|v_i|(F_i) \geq |v_i|(E_{n_i})/4$, and $|v_i|(\bigcup_{j \in N_{i+1}} E_j) < 1$.

Indeed, let $N_1 = \mathbb{N}$, $n_1 = 1$, and choose v_1 and F_1 appropriately. Having chosen everything up to the i th step, but not N_{i+1} , let $N_i = \bigcup_{j=1}^{\infty} M_j$ with $M_j \cap M_{j'} = \emptyset$ and M_j infinite for all j and $j' \neq j$. Then $\sum_j |v_i|(\bigcup_{r \in M_j} E_r) \leq \|v_i\| < \infty$, hence for some j , $|v_i|(\bigcup_{r \in M_j} E_r) < 1$; put $N_{i+1} = M_j$. Now let n_{i+1} be the first element of N_{i+1} greater than n_i , then choose v_{i+1} and $F_{i+1} \subset E_{n_{i+1}}$ appropriately.

It follows that putting $F = \bigcup_{i=1}^{\infty} F_i$, then for each i ,

$$|v_i|(F) \geq |v_i|(F_i) - \sum_{j=1}^{i-1} |v_i|(F_j) - |v_i|(\bigcup_{j \in N_{i+1}} E_j) \geq \frac{i}{4} - |v_i|(\bigcup_{j \in N_{i+1}} E_j) \geq \frac{i}{4} - 1;$$

whence $|v_i|(F) \rightarrow \infty$, a contradiction. This completes the proof of (a).

To prove (b), we have by (a) and the hypotheses of (b) that $\sup \| \mu_n \| < \infty$. Suppose that the conclusion of (b) were false. Then we could choose a $\delta > 0$, a subsequence (v_n) of the μ_n 's, and a sequence (E_n) of finite disjoint subsets of A such that for all n ,

$$\sum \{ |v_n(\{j\})| : j \in A \} \geq 6\delta,$$

$$\sum \{ |v_n(\{j\})| : j \in \bigcup_{i=1}^{n-1} E_i \} < \delta,$$

$$E_n \subset A \sim \bigcup_{i=1}^{n-1} E_i,$$

and

$$|v_n|(E_n) \geq \frac{1}{5} \sum \{ |v_n(\{j\})| : j \in A \sim \bigcup_{i=1}^{n-1} E_i \}.$$

It would follow that for all n , $|v_n|(E_n) \geq \delta$. Then by Lemma 1.1, there would exist an increasing sequence of indices (n_i) such that for all i ,

$$|v_{n_i}|(\bigcup_{j \neq i} E_{n_j}) < \delta/2.$$

Then putting $E = \bigcup_{j=1}^{\infty} E_{n_j}$,

$$|v_{n_i}|(E) > \delta/2 \quad \text{for all } i,$$

contradicting the hypotheses of (b). Q.E.D.

Page 30 line 14 from the bottom; instead of "3.4" read "3.5".