J Geod (2016) 90:585–587
DOI 10.1007/s00190-016-0907-8

COMMENT

Corrections to "A comparison of the tesseroid, prism and point-mass approaches for mass reductions in gravity field modelling" (Heck and Seitz, 2007) and "Optimized formulas for the gravitational field of a tesseroid" (Grombein et al., 2013)

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Published online: 13 May 2016

Keywords Gravity field modelling · Mass reduction · Tesseroid method · Prism method · Point-mass modelling

1 Summary

A second-order approximation of the tesseroid method has been presented in the paper "A comparison of the tesseroid, prism and point-mass approaches for mass reductions in gravity field modelling" (Heck and Seitz 2007) for the gravitational potential and its first radial derivative. In the paper "Optimized formulas for the gravitational field of a tesseroid" (Grombein et al. 2013) this analytical approach was optimized and extended to all first- and second-order derivatives of the potential.

In both papers the general expression of the Taylor series expansion contains a formal error and needs to be corrected. As will be shown, this correction or rather erratum has no impact on the published and widely used second-order tesseroid formulas.

This comment refers to the article available at doi:10.1007/s00190-013-0636-1 and 10.1007/s00190-006-0094-0.

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2 Erratum to Heck and Seitz (2007)

To compute gravity field functionals like potential, first- or second-order derivatives on the basis of digital terrain or residual terrain models (DTM, RTM), the mass elements have to be discretized. When using point mass, mass line or prism approximation closed formulas can be applied to compute these effects (Nagy et al. 2000, 2002). The drawback of these discretizations is that the masses are re-arranged. If tesseroids are used as mass bodies the discretization follows the natural characteristics of the DTMs. The disadvantage of the tesseroid method is that no analytical solutions of the respective mass integrals exist. Therefore, a Taylor series expansion of the integral kernel can be applied and the terms of the Taylor series are integrated term-wise.

Heck and Seitz (2007) presented an approximation approach for calculating the gravitational potential and its first radial derivative of a spherical tesseroid based on Taylor expansion (see Eqs. (21)–(23) and (32)–(33)). However, in Eqs. (10), (23), and (33) the published denominator $(i + j + k)!$ needs to be corrected to $i!\,j!\,k!$.

In detail, the Taylor expansion of the integral kernel

$$I = \ell^{-1} = \frac{1}{\sqrt{(x' - x)^2 + (y' - y)^2 + (z' - z)^2}}$$

in Eq. (9) (Heck and Seitz 2007) should be changed to

$$I = \sum_{i,j,k} \frac{I_{ijk}}{i!j!k!} \left(x' - x_0\right)^i \left(y' - y_0\right)^j \left(z' - z_0\right)^k .$$

To be consistent with the further derivations the partial derivatives I_{ijk} of the integral kernel I, which are presented in

Eq. (10), should now read

$$I_{ijk} := \left.\frac{\partial^{i+j+k} I(x', y', z')}{\partial x'^i \partial y'^j \partial z'^k}\right|_{\substack{x' = x_0 \\ y' = y_0 \\ z' = z_0}}.$$

The published denominator $(i + j + k)!$ in Eq. (10) is corrected to $i!j!k!$ and is moved to Eq. (9).

Accordingly, the Taylor expansion of the integral kernel of the potential of a tesseroid

$$K = \frac{r'^2 \cos\varphi'}{\ell}$$

in Eq. (22) should be changed to

$$K = \sum_{i,j,k} \frac{K_{ijk}}{i!j!k!} \left(r' - r_0\right)^i \left(\varphi' - \varphi_0\right)^j \left(\lambda' - \lambda_0\right)^k,$$

and the partial derivatives K_{ijk} of the integral kernel K, which are presented in Eq. (23), should now read

$$K_{ijk} := \left.\frac{\partial^{i+j+k} K(r', \varphi', \lambda')}{\partial r'^i \partial \varphi'^j \partial \lambda'^k}\right|_{\substack{r' = r_0 \\ \varphi' = \varphi_0 \\ \lambda' = \lambda_0}}.$$

Analogously, the Taylor expansion of the integral kernel of the attraction of a tesseroid

$$L = \frac{r'^2(r - r'\cos\psi)\cos\varphi'}{\ell^3}$$

in Eq. (32) should be changed to

$$L = \sum_{i,j,k} \frac{L_{ijk}}{i!j!k!} \left(r' - r_0\right)^i \left(\varphi' - \varphi_0\right)^j \left(\lambda' - \lambda_0\right)^k,$$

and the partial derivatives L_{ijk} of the integral kernel L, which are presented in Eq. (33), should now read

$$L_{ijk} := \left.\frac{\partial^{i+j+k} L(r', \varphi', \lambda')}{\partial r'^i \partial \varphi'^j \partial \lambda'^k}\right|_{\substack{r' = r_0 \\ \varphi' = \varphi_0 \\ \lambda' = \lambda_0}}.$$

Note that for even numbers i, j, k with $i + j + k \le 2$ the published and corrected expressions are analytically and numerically identical.

3 Erratum to Grombein et al. (2013)

To increase computational efficiency, Grombein et al. (2013) proposed optimized tesseroid formulas expressed in Cartesian coordinates leading to analytical expressions different from the previous formulas expressed in spherical coordinates, but they are, of course, numerically identical. Akin to Heck and Seitz (2007), the general expression for the Taylor series expansion also occurring in Grombein et al. (2013), Eq. (31), has to be corrected.

In detail, the Taylor expansion in Eq. (31) (Grombein et al. 2013) should read

$$K(P, Q) = \sum_{i,j,k} \frac{K_{ijk}(P, Q_0)}{i!j!k!} \left(r' - r_0\right)^i \left(\varphi' - \varphi_0\right)^j \left(\lambda' - \lambda_0\right)^k,$$

where the published denominator $(i + j + k)!$ in Eq. (31) is corrected to $i!j!k!$.

Also here, for even numbers i, j, k with $i + j + k \le 2$ the published and corrected expressions are analytically and numerically identical.

4 Conclusions and consequences for the tesseroid formulas of second-order

Consequences for Heck and Seitz (2007) The corrections to Eqs. (9), (10), (22), (23), (32), (33) have no further implication on the presented second-order tesseroid formulas that are still correct.

Consequences for Grombein et al. (2013) The correction to Eq. (31) has no further implication on the (second-order) Taylor series approach applied to the optimized tesseroid formulas. Therefore, the presented evaluation rules for the gravitational potential of a tesseroid and its partial derivatives in Eq. (42) are still correct.

The source code as provided on request by the authors B. Heck, T. Grombein and K. Seitz works as intended, even if an earlier version is used.

Various authors (e.g., Wild-Pfeiffer 2008; Tsoulis et al. 2009; Shen and Han 2013) quoted only the second-order tesseroid formulas in their studies, which are correct and work highly efficiently with respect to realistic terrain modelling and computation time. However, if higher order approximations are derived, e.g., fourth-order formulas, one has to consider the correct expression for the general Taylor series expansion in three dimensions.

Acknowledgments The authors Xiao-Le Deng and Wen-Bin Shen would like to acknowledge their support by National 973 Project China (Grant Nos. 2013CB733301 and 2013CB733305), NSFCs (Grant Nos. 41174011, 41429401, 41210006, 41128003, 41021061).

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