# CORRELATION FUNCTIONS AND INVARIANT MEASURES IN CONTINUOUS CONTACT MODEL

YURI KONDRATIEV[1], OLEKSANDR KUTOVIY[2], SERGEY PIROGOV[3]

[1] *Fakultät für Mathematik, Universität Bielefeld,*
*Postfach 10 01 31, 33615 Bielefeld, Germany;*
*Research Center BiBoS, Universität Bielefeld, 33615 Bielefeld, Germany*
*kondrat@mathematik.uni-bielefeld.de*

[2] *Fakultät für Mathematik, Universität Bielefeld,*
*Postfach 10 01 31, 33615 Bielefeld, Germany*
*kutoviy@mathematik.uni-bielefeld.de*

[3] *IITP RAS, Moscow, Russia*
*pirogov@mail.ru*

**Abstract**

We study the continuous version of the contact model. Using an analytic approach we construct the non-equilibrium contact process as a Markov process on configuration space. The construction is based on the analysis of correlation functions evolution. The problem concerning invariant measures as well as asymptotics of correlation functions are also studied.

*Keywords: configuration space; contact model; non-equilibrium Markov process*

*AMS Subject Classification: 60K35, 60J75, 60J80, 82C21, 82C22*

# 1 Introduction

Lattice contact models form an important class of interacting particle systems with rich mathematical properties and many essential applications, see, e.g., [14]. A continuous version of these models was introduced recently in [10]. In the latter paper the existence problem for corresponding spatial Markov process was analyzed in details. This process is a special case of the general birth-and-death processes in the continuum. Namely, we consider configurations, i.e., locally finite subsets $\gamma \subset \mathbb{R}^d$ as values of the process. During the stochastic evolution the points of a configuration independently create new ones distributed in the space according to a dispersion probability density $0 \leq a \in \mathrm{L}^1(\mathbb{R}^d)$ which is an even function. Any existing point has an independent exponentially distributed random life time. The contact process generator is given then on proper functions $F(\gamma)$ by the expression

$$(LF)(\gamma) := \sum_{x\in\gamma}[F(\gamma \setminus x) - F(\gamma)] + \varkappa \int_{\mathbb{R}^d} \sum_{y\in\gamma} a(x-y)[F(\gamma \cup x) - F(\gamma)]dx,$$

where $\varkappa > 0$ is a birth intensity parameter.

The main problem considered in the present paper concerns asymptotics properties of the contact process. First of all, we construct the time evolution of correlation functions for the contact process starting with an initial distribution from a large class of initial states. Corresponding infinite system of evolution equations for correlation functions has a recurrent form and admits a simple analysis. We note, that the intensity parameter has a critical value $\varkappa = 1$. For all other values of this parameter the density of the system tends to $\infty$ or 0 with the time and we cannot expect an appearing of a limiting invariant state.

For the critical value $\varkappa = 1$ and the dimension $d \geq 3$ we prove the existence of a continuous family of invariant measures parameterized by the density values. These invariant measures are described by a simple recurrent relation between their correlation functions and create a concrete class of random point fields which, up to our knowledge, never before was considered in the literature. A specific point of this class is an extremal growth w.r.t. the number of correlation functions. Actually, this growth is a maximal possible one such that the uniqueness of the corresponding measure is still valid. We show that, starting with an admissible initial state, the critical contact process converges to the equilibrium measure uniquely defined by the density of the initial state.

Let us note, that the contact models in the continuum may be used in the epidemiology to model an infection spreading process as well as in the spatial plant ecology where they describe independent growth of a population with a given mortality rate. In such ecological models the case $d = 2$ has a special concrete motivation. As we have mentioned above, invariant measures for our model for $d = 2$ do not exist and the root of this effect is very easy. Namely, in the two-dimensional case correlations between population members are growing in time too fast and the limiting correlation function of second order will diverge to the infinity. To avoid this divergence we may include an additional free

Kawasaki dynamics for points of the configuration (see [5]). This dynamics includes an independent random walk in $\mathbb{R}^d$ for each population member. Then, assuming long tail jumps for the individual random walk process, we can assure the existence of invariant measures for such infinite particle stochastic dynamics. The resulting contact model with Kawasaki dynamics may be used naturally for the study of plankton stochastic dynamics, cf. [19]. Detailed analysis of the discussed model will be given in our forthcoming paper [7].

# 2 Preliminaries

We consider the Euclidian space $\mathbb{R}^d$. By $\mathcal{B}(\mathbb{R}^d)$ we denote the family of all Borel sets in $\mathbb{R}^d$. $\mathcal{B}_b(\mathbb{R}^d)$ denotes the system of all sets in $\mathcal{B}(\mathbb{R}^d)$ which are bounded.

The space of *n-point configuration* is

$$\Gamma_0^{(n)} = \Gamma_{0,\mathbb{R}^d}^{(n)} := \left\{ \eta \subset \mathbb{R}^d \middle| \, |\eta| = n \right\}, \quad n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

where $|A|$ denotes the cardinality of the set $A$.

The space $\Gamma_\Lambda^{(n)} = \Gamma_{0,\Lambda}^{(n)}$ for $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ is defined analogously to the space $\Gamma_0^{(n)}$. As a set $\Gamma_0^{(n)}$ is equivalent to the symmetrization of

$$\widetilde{(\mathbb{R}^d)^n} = \left\{ (x_1, \ldots, x_n) \in (\mathbb{R}^d)^n \,|\, x_k \neq x_l \text{ if } k \neq l \right\},$$

i.e. to the $\widetilde{(\mathbb{R}^d)^n}/S_n$, where $S_n$ is the permutation group of $\{1, \ldots, n\}$. Hence, one can introduce the corresponding topology and Borel $\sigma$-algebra, which we denote by $\mathcal{O}(\Gamma_0^{(n)})$ and $\mathcal{B}(\Gamma_0^{(n)})$, respectively.

The space of *finite configurations*

$$\Gamma_0 := \bigsqcup_{n \in \mathbb{N}_0} \Gamma_0^{(n)}$$

is equipped with the topology $\mathcal{O}(\Gamma_0)$ of disjoint union. Let $\mathcal{B}(\Gamma_0)$ denotes the corresponding Borel $\sigma$-algebra.

A *set* $B \in \mathcal{B}(\Gamma_0)$ is called *bounded* if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $B \subset \bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}$.

We would like to emphasize that due to the structure of $\Gamma_0$, any function on $\Gamma_0$ can be interpreted as a system of symmetrical functions on each component $\Gamma_0^{(n)}$ of $\Gamma_0$.

The *configuration space*

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d \middle| \, |\gamma \cap \Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_b(\mathbb{R}^d) \right\}$$

is equipped with the *vague* topology $\mathcal{O}(\Gamma)$. It is a Polish space (see e.g. [6]). $\mathcal{B}(\Gamma)$ denotes the corresponding Borel $\sigma$-algebra. The *filtration* on $\Gamma$ with a base set $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ is given by

$$\mathcal{B}_\Lambda(\Gamma) := \sigma \left( N_{\Lambda'} \middle| \Lambda' \in \mathcal{B}_b(\mathbb{R}^d), \ \Lambda' \subset \Lambda \right),$$

where $N_\Lambda : \Gamma_0 \to \mathbb{N}_0$ is such that $N_\Lambda(\eta) := |\eta \cap \Lambda|$. For brevity we write $\eta_\Lambda := \eta \cap \Lambda$.

For every $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ the *projection* $p_\Lambda : \Gamma \to \Gamma_\Lambda := \bigsqcup_{n\geq 0} \Gamma_\Lambda^{(n)}$ is defined as

$$p_\Lambda(\gamma) := \gamma_\Lambda$$

One can show that $\Gamma$ is the projective limit of the spaces $\{\Gamma_\Lambda\}_{\Lambda\in\mathcal{B}_b(\mathbb{R}^d)}$ w.r.t. this projections.

In the sequel we will use the following classes of function on $\Gamma_0$:

• $L^0(\Gamma_0)$ - the set of all *measurable functions* on $\Gamma_0$;

• $L^0_{\mathrm{ls}}(\Gamma_0)$ - the set of measurable *functions* with *local support*, i.e. $G \in L^0_{\mathrm{ls}}(\Gamma_0)$ if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ such that $G \restriction_{\Gamma_0\setminus\Gamma_\Lambda} = 0$;

• $L^0_{\mathrm{bs}}(\Gamma_0)$ - the set of measurable *functions* with *bounded support*, i.e. $G \in L^0_{\mathrm{bs}}(\Gamma_0)$ if there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $G \restriction_{\Gamma_0\setminus\bigsqcup_{n=0}^N \Gamma_\Lambda^{(n)}} = 0$;

• $B(\Gamma_0)$ - the set of *bounded* measurable *functions*

• $B_{\mathrm{bs}}(\Gamma_0)$ - the set of *bounded functions* with *bounded support*;

On $\Gamma$ we consider the set of *cylinder functions* $\mathcal{F}L^0(\Gamma)$, i.e. the set of all measurable functions $G \in L^0(\Gamma)$ which are measurable w.r.t. $\mathcal{B}_\Lambda(\Gamma)$ for some $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. These functions are characterized by the following relation:

$$F(\gamma) = F \restriction_{\Gamma_\Lambda} (\gamma_\Lambda).$$

Those cylinder functions which are measurable w.r.t. $\mathcal{B}_\Lambda(\Gamma)$ for fixed $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ we will denote by $\mathcal{F}L^0(\Gamma,\ \mathcal{B}_\Lambda(\Gamma))$.

Next we would like to describe some facts from the harmonic analysis on the configuration space based on [4].

The following mapping between functions on $\Gamma_0$, and functions on $\Gamma$, plays the key role in our further considerations:

$$KG(\gamma) := \sum_{\xi \Subset \gamma} G(\xi), \quad G \in L^0_{\mathrm{ls}}(\Gamma_0) \quad \gamma \in \Gamma,$$

see e.g. [12, 13]. The summation in the latter expression is taken over all finite subconfigurations of $\gamma$, which is denoted by the symbol $\xi \Subset \gamma$.

*K-transform* is linear, positivity preserving, and invertible, with

$$K^{-1}F(\eta) := \sum_{\xi\subset\eta} (-1)^{|\eta\setminus\xi|} F(\xi), \quad F \in \mathcal{F}L^0(\Gamma) \quad \eta \in \Gamma_0. \tag{1}$$

It is easy to see that for any $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and arbitrary $F \in \mathcal{F}L^0(\Gamma,\ \mathcal{B}_\Lambda(\Gamma))$

$$K^{-1}F(\eta) = 1\!\!1_{\Gamma_\Lambda}(\eta) K^{-1}F(\eta), \qquad \forall \eta \in \Gamma_0. \tag{2}$$

The map $K$, as well as the map $K^{-1}$, can be extended to more wide classes of functions. For details and further properties of the map $K$ see, e.g. [4].

One can introduce a *convolution*

$$\begin{array}{rcl} \star : L^0(\Gamma_0)\times L^0(\Gamma_0) & \to & L^0(\Gamma_0) \\ (G_1,G_2) & \mapsto & (G_1\star G_2)(\eta) \end{array} \tag{3}$$

$$:= \sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}^3_\emptyset(\eta)} G_1(\xi_1\cup\xi_2)\, G_2(\xi_2\cup\xi_3),$$

where $\mathcal{P}^3_\emptyset(\eta)$ denotes the set of all partitions $(\xi_1,\xi_2,\xi_3)$ of $\eta$ in 3 parts, i.e., all triples $(\xi_1,\xi_2,\xi_3)$ with $\xi_i\subset\eta$, $\xi_i\cap\xi_j=\emptyset$ if $i\neq j$, and $\xi_1\cup\xi_2\cup\xi_3=\eta$. It has the property that for $G_1,G_2\in L^0_{\mathrm{ls}}(\Gamma_0)$

$$K\,(G_1\star G_2)=KG_1\cdot KG_2.$$

Due to this convolution we can interpret the $K$-transform as the Fourier transform in configuration space analysis, see also [1].

Let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be the set of all probability measures $\mu$ which have *finite local moments* of all orders, i.e.

$$\int_\Gamma |\gamma_\Lambda|^n\mu(d\gamma)<+\infty$$

for all $\Lambda\in\mathcal{B}_b(\mathbb{R}^d)$ and $n\in\mathbb{N}_0$.

A measure $\rho$ on $\Gamma_0$ is called *locally finite* if $\rho(A)<\infty$ for all bounded sets $A$ from $\mathcal{B}(\Gamma_0)$. The set of such measures is denoted by $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$.

A measure $\rho\in\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ is called *positive definite* if

$$\int_{\Gamma_0}(G\star\overline{G})(\eta)\rho(d\eta)\geq 0,\ \ \forall G\in B_{\mathrm{bs}}(\Gamma_0),$$

where $\overline{G}$ is a complex conjugate of $G$.

A measure $\rho$ is called *normalized* if and only if $\rho(\{\emptyset\})=1$.

One can define a transform $K^*:\mathcal{M}^1_{\mathrm{fm}}(\Gamma)\to\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$, which is *dual* to the $K$-transform, i.e., for every $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, $G\in\mathcal{B}_{\mathrm{bs}}(\Gamma_0)$ we have

$$\int_\Gamma KG(\gamma)\mu(d\gamma)=\int_{\Gamma_0} G(\eta)\,(K^*\mu)(d\eta).$$

The measure $\rho_\mu:=K^*\mu$ is called the *correlation measure* of $\mu$. As shown in [4] for $\mu\in\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and any $G\in\mathrm{L}^1(\Gamma_0,\rho_\mu)$ the series

$$KG(\gamma):=\sum_{\eta\Subset\gamma}G(\eta), \tag{4}$$

is $\mu$-a.s. absolutely convergent. Furthermore, $KG\in\mathrm{L}^1(\Gamma,\mu)$ and

$$\int_{\Gamma_0}G(\eta)\,\rho_\mu(d\eta)=\int_\Gamma(KG)(\gamma)\,\mu(d\gamma). \tag{5}$$

Fix a non-atomic and locally finite measure $\sigma$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. For any $n \in \mathbb{N}$ the product measure $\sigma^{\otimes n}$ can be considered by restriction as a measure on $\widetilde{(\mathbb{R}^d)^n}$ and hence on $\Gamma_0^{(n)}$. The measure on $\Gamma_0^{(n)}$ we denote by $\sigma^{(n)}$.

The *Lebesgue-Poisson measure* $\lambda_{z\sigma}$ on $\Gamma_0$ is defined as

$$\lambda_{z\sigma} := \sum_{n=0}^{\infty} \frac{z^n}{n!} \sigma^{(n)}.$$

Here $z > 0$ is the so-called activity parameter. The restriction of $\lambda_{z\sigma}$ to $\Gamma_\Lambda$ will be also denoted by $\lambda_{z\sigma}$. We write $\lambda_z$ instead of $\lambda_{z\sigma}$, if the measure $\sigma$ is considered to be fixed.

The *Poisson measure* $\pi_{z\sigma}$ on $(\Gamma, \mathcal{B}(\Gamma))$ is given as the projective limit of the family of measures $\{\pi_{z\sigma}^\Lambda\}_{\Lambda \in \mathcal{B}_b(\mathbb{R}^d)}$, where $\pi_{z\sigma}^\Lambda$ is the measure on $\Gamma_\Lambda$ defined by $\pi_{z\sigma}^\Lambda := e^{-z\sigma(\Lambda)} \lambda_{z\sigma}$.

A measure $\mu \in \mathcal{M}_{\mathrm{fm}}^1(\Gamma)$ is called *locally absolutely continuous w.r.t.* $\pi_{z\sigma}$ iff $\mu_\Lambda := \mu \circ p_\Lambda^{-1}$ is absolutely continuous with respect to $\pi_{z\sigma}^\Lambda = \pi_{z\sigma} \circ p_\Lambda^{-1}$ for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$. In this case, $\rho_\mu := K^*\mu$ is absolutely continuous w.r.t $\lambda_{z\sigma}$. Let $k_\mu : \Gamma_0 \to \mathbb{R}_+$ be the corresponding Radon-Nikodym derivative, i.e.

$$k_\mu(\eta) := \frac{d\rho_\mu}{d\lambda_{z\sigma}}(\eta), \quad \eta \in \Gamma_0.$$

**Remark 2.1** *The functions*

$$k_\mu^{(n)} : (\mathbb{R}^d)^n \longrightarrow \mathbb{R}_+ \tag{6}$$

$$k_\mu^{(n)}(x_1, \ldots, x_n) := \begin{cases} k_\mu(\{x_1, \ldots, x_n\}), & \text{if } (x_1, \ldots, x_n) \in \widetilde{(\mathbb{R}^d)^n} \\ 0, & \text{otherwise} \end{cases}$$

*are the correlation functions well known in statistical physics, see e.g [17], [18].*

For the technical purposes we also recall the following result:

**Lemma 2.1** *Let $n \in \mathbb{N}$, $n \geq 2$, and $z > 0$ be given. Then*

$$\int_{\Gamma_0} \cdots \int_{\Gamma_0} G(\eta_1 \cup \ldots \cup \eta_n) H(\eta_1, \ldots, \eta_n) d\lambda_{z\sigma}(\eta_1) \ldots d\lambda_{z\sigma}(\eta_n) =$$

$$= \int_{\Gamma_0} G(\eta) \sum_{(\eta_1, \ldots, \eta_n) \in \mathcal{P}_n(\eta)} H(\eta_1, \ldots, \eta_n) d\lambda_{z\sigma}(\eta)$$

*for all measurable functions $G : \Gamma_0 \mapsto \mathbb{R}$ and $H : \Gamma_0 \times \ldots \times \Gamma_0 \mapsto \mathbb{R}$ with respect to which both sides of the equality make sense. Here $\mathcal{P}_n(\eta)$ denotes the set of all ordered partitions of $\eta$ in $n$ parts, which may be empty.*

This lemma is known in the literature as *Minlos lemma* (cf., [9], [15]) and it will be crucial for calculations in many places in the next sections.

# 3 Generators. The symbol of the generator on the space of finite configurations

Let the activity parameter $z$ be equal to 1 and let $0 \le a \in \mathrm{L}^1(\mathbb{R}^d)$ be an arbitrary even function such that

$$\int_{\mathbb{R}^d} a(x)dx = 1.$$

We consider a Markov pre-generator which corresponds to the contact model on the configuration space $\Gamma$, the action of which is given by

$$(LF)(\gamma) := \sum_{x\in\gamma} D_x^- F(\gamma) + \varkappa \int_{\mathbb{R}^d} \sum_{y\in\gamma} a(x-y) D_x^+ F(\gamma) dx, \quad F \in \mathcal{F}L^0(\Gamma),$$

where $D_x^- F(\gamma) = F(\gamma \setminus x) - F(\gamma)$, $D_x^+ F(\gamma) = F(\gamma \cup x) - F(\gamma)$ and $\varkappa > 0$.

**Proposition 3.1** *The image of $L$ under the $K$-transform (or symbol of the operator $L$) on functions $G \in B_{bs}(\Gamma_0)$ has the following form*

$$(\widehat{L}G)(\eta) := (K^{-1}LKG)(\eta) = -|\eta| G(\eta) +$$

$$+\varkappa \int_{\mathbb{R}^d} \sum_{y\in\eta} a(x-y) G((\eta \setminus y) \cup x) dx + \varkappa \int_{\mathbb{R}^d} \sum_{y\in\eta} a(x-y) G(\eta \cup x) dx$$

*Proof.* According to the definition of the operator $\widehat{L}$ we have

$$(\widehat{L}G)(\eta) = I_1(\eta) + I_2(\eta),$$

where

$$I_1(\eta) := K^{-1}\left(\sum_{x\in\cdot} [KG(\cdot \setminus x) - KG(\cdot)]\right)(\eta) =$$

$$= K^{-1}\left(\sum_{x\in\cdot}\left[\sum_{\xi\subset(\cdot\setminus x)} G(\xi) - \sum_{\xi\subset\cdot} G(\xi)\right]\right)(\eta) =$$

$$= K^{-1}\left(\sum_{x\in\cdot}\left[-\sum_{\xi\subset(\cdot\setminus x)} G(\xi\cup x)\right]\right)(\eta) =$$

$$= -\sum_{\zeta\subset\eta} (-1)^{|\eta\setminus\zeta|} \sum_{x\in\zeta} \sum_{\xi\subset\zeta\setminus x} G(\xi \cup x) = -\sum_{\zeta\subset\eta} (-1)^{|\eta\setminus\zeta|} \sum_{x\in\zeta} K\left(G(\cdot\cup x)\right)(\zeta\setminus x).$$

Changing summation in the last expression we get

$$I_1(\eta) = -\sum_{x\in\eta} \sum_{\zeta\in\eta\setminus x} (-1)^{|\eta\setminus(\zeta\cup x)|} K\left(G(\cdot\cup x)\right)(\zeta) =$$

$$= -\sum_{x\in\eta} K^{-1}\left(KG(\cdot\cup x)\right)(\eta\setminus x) = -\sum_{x\in\eta} G(\eta) = -|\eta| G(\eta);$$

Now we compute the second term of $\widehat{L}$

$$I_2(\eta) := K^{-1}\left(\varkappa\int_{\mathbb{R}^d}\sum_{y\in\cdot}a(x-y)\left[KG(\cdot\cup x)-KG(\cdot)\right]dx\right)(\eta)=$$

$$=\varkappa\sum_{\zeta\subset\eta}(-1)^{|\eta\setminus\xi|}\int_{\mathbb{R}^d}\sum_{y\in\zeta}a(x-y)\sum_{\rho\subset\zeta}G(\rho\cup x)dx.$$

Using the fact that

$$\sum_{y\in\zeta}a(x-y)=K\left(a(x-\cdot)1\!\!1_{\{|\cdot|=1\}}(\cdot)\right)(\zeta)$$

we obtain

$$I_2(\eta)=\varkappa\int_{\mathbb{R}^d}K^{-1}\left(K\left(\left(a(x-\cdot)1\!\!1_{\{|\cdot|=1\}}(\cdot)\right)\star G(\cdot\cup x)\right)\right)(\eta)dx=$$

$$=\varkappa\int_{\mathbb{R}^d}\left(\left(a(x-\cdot)1\!\!1_{\{|\cdot|=1\}}(\cdot)\right)\star G(\cdot\cup x)\right)(\eta)dx.$$

By the definition of the convolution, the latter expression can be written as follows

$$\varkappa\int_{\mathbb{R}^d}\sum_{(\xi_1,\xi_2,\xi_3)\in\mathcal{P}^3_\emptyset(\eta)}\left(a(x-\cdot)1\!\!1_{\{|\cdot|=1\}}(\cdot)\right)(\xi_1\cup\xi_2)G(\xi_2\cup\xi_3\cup x)dx.$$

Now, we note that there are only two cases when summands in the last expression are not equal to zero. These cases are $|\xi_1|=1,\ \xi_2=\emptyset$ and $\xi_1=\emptyset,\ |\xi_2|=1$. Therefore,

$$I_2(\eta)=\varkappa\int_{\mathbb{R}^d}\sum_{y\in\eta}a(x-y)G((\eta\setminus y)\cup x)dx+\varkappa\int_{\mathbb{R}^d}\sum_{y\in\eta}a(x-y)G((\eta\setminus y)\cup y\cup x)dx.$$

The latter fact proves the assertion of the proposition. ■

# 4 Construction of the contact process associated with the generator $L$

## 4.1 The adjoint operator to the symbol of $L$

Let a measure $\rho\in\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ be absolutely continuous with respect to the Lebesgue-Poisson measure $\lambda$. By $k(\eta)$, $\eta\in\Gamma_0$ we denote the corresponding density.

**Proposition 4.1** *Assume that*

$$k(\eta)\le C^{|\eta|}|\eta|!,\quad \eta\in\Gamma_0 \tag{7}$$

*for some* $C>0$. *Then,* $\widehat{L}(B_{bs}(\Gamma_0))$ *is a subset of* $\mathrm{L}^1(\Gamma_0,\,\rho)$.

*Proof.* Let $G \in B_{bs}(\Gamma_0)$ be arbitrary and fixed. A direct application of Lemma 2.1 to the calculation of $\mathrm{L}^1$-norm of $\widehat{L}G$ with respect to the measure $\rho$ gives us the necessary result. ■

The operator $\widehat{L}^*$ adjoint to the operator $\widehat{L}$ on the space of correlation functions is defined via the following duality

$$\langle \widehat{L}G,\, k\rangle := \int_{\Gamma_0} \widehat{L}G(\eta)\rho(d\eta) = \int_{\Gamma_0} \widehat{L}G(\eta)k(\eta)\lambda(d\eta) = \langle G,\, \widehat{L}^*k\rangle.$$

In the next proposition we give an explicit representation of the adjoint operator of $\widehat{L}$.

**Proposition 4.2** *The adjoint operator $\widehat{L}^*$ of $\widehat{L}$ on the space of functions which satisfy (7) has the following form:*

$$(\widehat{L}^*k)(\eta) = -|\eta|k(\eta) + \varkappa \sum_{x\in\eta} k(\eta\backslash x) \sum_{y\in\eta\backslash x} a(x-y) + \varkappa \sum_{x\in\eta} \int_{\mathbb{R}^d} a(x-y)k((\eta\backslash x)\cup y)dy.$$

*Proof.* Using the same notation as in Proposition 3.1 we get

$$\int_{\Gamma_0} I_1(\eta)k(\eta)\lambda(d\eta) = -\int_{\Gamma_0} |\eta|G(\eta)k(\eta)\lambda(d\eta) = \int_{\Gamma_0} G(\eta)\left[-|\eta|k(\eta)\right]\lambda(d\eta).$$

For the second part of $\widehat{L}$ we have

$$\int_{\Gamma_0} I_2(\eta)k(\eta)\lambda(d\eta) = J_1 + J_2,$$

where

$$J_1 := \varkappa \int_{\Gamma_0}\int_{\mathbb{R}^d} \sum_{y\in\eta} a(x-y)G((\eta\setminus y)\cup x)dx k(\eta)\lambda(d\eta),$$

and

$$J_2 := \varkappa \int_{\Gamma_0}\int_{\mathbb{R}^d} G(\eta\cup x)\sum_{y\in\eta} a(x-y)k(\eta)dx\lambda(d\eta).$$

Using Lemma 2.1 we obtain

$$J_1 = \varkappa \int_{\Gamma_0}\int_{\mathbb{R}^d} k(\eta\cup y)\left[\int_{\mathbb{R}^d} a(x-y)G(\eta\cup x)dx\right] dy\lambda(d\eta) =$$

$$= \varkappa \int_{\Gamma_0}\int_{\mathbb{R}^d} G(\eta\cup x)\left[\int_{\mathbb{R}^d} a(x-y)k(\eta\cup y)dy\right] dx\lambda(d\eta).$$

Using Lemma 2.1 again for the last expression and for the integral $J_2$, finally we get

$$J_1 = \varkappa \int_{\Gamma_0} G(\eta)\left[\sum_{x\in\eta}\int_{\mathbb{R}^d} a(x-y)k((\eta\setminus x)\cup y)dy\right]\lambda(d\eta).$$

$$J_2 = \varkappa \int_{\Gamma_0} G(\eta) \left[ \sum_{x\in\eta} k(\eta \setminus x) \sum_{y\in\eta\setminus x} a(x-y) \right] \lambda(d\eta).$$

This concludes the proof of the proposition. ■

## 4.2 Time evolution of correlation functions

In this subsection we investigate the evolutional equation associated with the operator $\widehat{L}^*$. It has the following form

$$\frac{\partial k_t}{\partial t}(\eta) = \widehat{L}^* k_t(\eta) = -|\eta| k_t(\eta) + \varkappa \sum_{x\in\eta} k_t(\eta \setminus x) \sum_{y\in\eta\setminus x} a(x-y) +$$

$$+\varkappa \sum_{x\in\eta} \int_{\mathbb{R}^d} a(x-y) k_t((\eta\setminus x)\cup y) dy.$$

Having in mind the relation between functions on the space of finite configurations and collection of symmetrical functions on each component $\Gamma_0^{(n)}$, $n \geq 0$, we rewrite this equation as a system of equations:

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1,\ldots,x_n) = -n\, k_t^{(n)}(x_1,\ldots,x_n) +$$

$$+\varkappa \sum_{i=1}^n k_t^{(n-1)}(x_1,\ldots,\check{x}_i,\ldots,x_n) \sum_{j:\, j\neq i} a(x_i - x_j) +$$

$$+\varkappa \sum_{i=1}^n \int_{\mathbb{R}^d} a(x_i - y) k_t^{(n)}(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n) dy =$$

$$= \widehat{L}_n^* k_t^{(n)}(x_1,\ldots,x_n) + f_t^{(n)}(x_1,\ldots,x_n), \quad n \geq 1,$$

where

$$\widehat{L}_n^* k_t^{(n)}(x_1,\ldots,x_n) := -n\, k_t^{(n)}(x_1,\ldots,x_n) +$$

$$+\varkappa \sum_{i=1}^n \int_{\mathbb{R}^d} a(x_i - y) k_t^{(n)}(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n) dy, \quad n \geq 1$$

and

$$f_t^{(n)}(x_1,\ldots,x_n) := \varkappa \sum_{i=1}^n k_t^{(n-1)}(x_1,\ldots,\check{x}_i,\ldots,x_n) \sum_{j:\, j\neq i} a(x_i - x_j), \quad n \geq 2,$$

$$f_t^{(1)} \equiv 0.$$

Let $n \in \mathbb{N}$ be arbitrary and fixed. We consider the linear Cauchy problem

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1,\ldots,x_n) = \widehat{L}_n^* k_t^{(n)}(x_1,\ldots,x_n) + f_t^{(n)}(x_1,\ldots,x_n), \;\; t \geq 0, \qquad (8)$$

$$k_t^{(n)}(x_1,\ldots,x_n)\Big|_{t=0} := k_0^{(n)}(x_1,\ldots,x_n),$$

in the Banach space $\mathrm{X}_n$, defined as $\mathrm{L}^\infty((\mathbb{R}^d)^n,\,\sigma^{\otimes n})$, where $\sigma$ is the Lebesgue measure on $\mathbb{R}^d$ and $\sigma^{\otimes n}$ is the product measure on $(\mathbb{R}^d)^n$.

**Remark 4.1** *The operator $\widehat{L}_n^*$ in $\mathrm{X}_n$ can be also written in another way*

$$\widehat{L}_n^* k^{(n)}(x_1,\ldots,x_n) = n(\varkappa-1)\,k^{(n)}(x_1,\ldots,x_n) + \sum_{i=1}^n L_a^i k^{(n)}(x_1,\ldots,x_n),$$

*where for each $1\le i\le n$,*

$$L_a^i k^{(n)}(x_1,\ldots,x_n) =$$

$$= \varkappa \int_{\mathbb{R}^d} a(x_i-y)\left[k^{(n)}(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n) - k^{(n)}(x_1,\ldots,x_n)\right] dy$$

*is a generator of a Markov process on $(\mathbb{R}^d)^n$ (see [2]), which describes the jump of the particle placed at the point $(x_1,\ldots,x_i,\ldots,x_n)\in(\mathbb{R}^d)^n$ to the point $(x_1,\ldots,y,\ldots,x_n)\in(\mathbb{R}^d)^n$ with an intensity equal to $a(x_i-y)$.*

**Lemma 4.1** *Let $a\in\mathrm{L}^1(\mathbb{R}^d)$ be a nonnegative even function. Then, for any $n\ge 1$, the operator $\widehat{L}_n^*$ is a bounded linear operator in $\mathrm{X}_n$ as well as in $\mathrm{L}^1((\mathbb{R}^d)^n)$. Moreover, for each $1\le i\le n$, the operator $L_a^i$ is a generator of a contraction semigroup on $\mathrm{X}_n$ and $\mathrm{L}^1((\mathbb{R}^d)^n)$.*

*Proof.* The first part of this theorem is trivial. The second one in the case of the space $\mathrm{X}_n$ follows directly from Remark 4.1 and in the case of $\mathrm{L}^1((\mathbb{R}^d)^n)$ it is a consequence of Beurling-Deny criterion, see e.g. [16]. ■

This Lemma in its turn implies the following result (see e.g. [3]).

**Proposition 4.3** *Let $n\ge 1$ be arbitrary and fixed. The solution to the Cauchy problem (8) in the Banach space $\mathrm{X}_n$ is given by*

$$k_t^{(n)}(x_1,\ldots,x_n) = e^{n(\varkappa-1)t}\left[\bigotimes_{i=1}^n e^{tL_a^i}\right] k_0^{(n)}(x_1,\ldots,x_n) + \varkappa e^{n(\varkappa-1)t}\times \tag{9}$$

$$\times\int_0^t e^{-n(\varkappa-1)s}\left[\bigotimes_{i=1}^n e^{(t-s)L_a^i}\right]\sum_{i=1}^n k_s^{(n-1)}(x_1,\ldots,\check{x}_i,\ldots,x_n)\sum_{j:\,j\ne i} a(x_i-x_j)ds.$$

Next proposition establish a priori estimates for the evolution of correlation functions

**Proposition 4.4** *Let $a \in \mathrm{L}^1(\mathbb{R}^d) \cap \mathrm{L}^\infty(\mathbb{R}^d)$ be an arbitrary nonnegative even function. Suppose that there exists a constant $C > 0$ (independent of $n$) such that for any $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$*

$$k_0^{(n)}(x_1, \ldots, x_n) \le n!\, C^n, \quad for\ \ all\ \ n \ge 0.$$

*Then, for any $t \ge 0$ and almost all (a.a.) $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ w.r.t. the Lebesgue measure*

$$k_t^{(n)}(x_1, \ldots, x_n) \le \varkappa(t)^n (1 + ||a||_{\mathrm{L}^\infty(\mathbb{R}^d)})^n e^{n(\varkappa-1)t} (C + t)^n n!, \tag{10}$$

*where*

$$\varkappa(t) := \max \Big[1,\ \varkappa,\ \varkappa e^{-(\varkappa-1)t}\Big]$$

*holds for all $n \ge 0$.*

*Proof.* The proof uses mathematical induction with respect to $n$. The first induction step (the fulfilment of (10) in the case of $n = 1$) follows from Proposition 4.3. Now assume that for any $t \ge 0$ bound (10) holds for $n - 1$. Using formula (9) and Remark 4.1 we get

$$k_t^{(n)}(x_1, \ldots, x_n) \le e^{n(\varkappa-1)t} C^n n! +$$

$$+\varkappa(n-1)n!(1 + ||a||_{\mathrm{L}^\infty})^n \int_0^t e^{n(\varkappa-1)(t-s)} \varkappa(s)^{n-1} e^{(n-1)(\varkappa-1)s} (C + s)^{n-1} ds \le$$

$$\le e^{n(\varkappa-1)t} C^n n! +$$

$$+\varkappa(n-1)n!(1 + ||a||_{\mathrm{L}^\infty})^n \varkappa(t)^{n-1} e^{n(\varkappa-1)t} \int_0^t e^{-(\varkappa-1)s} (C + s)^{n-1} ds.$$

Using the estimate

$$e^{-(\varkappa-1)s} \le \max\{1,\ e^{-(\varkappa-1)t}\}, \quad \text{for all}\ \ s \in [0,\ t],\ \varkappa > 0$$

we obtain

$$k_t^{(n)}(x_1, \ldots, x_n) \le \varkappa(t)^n (1 + ||a||_{\mathrm{L}^\infty})^n e^{n(\varkappa-1)t} (C + t)^n n!,$$

which concludes the proof of this proposition. ∎

With the help of the previous proposition, we can approximate solutions of the Cauchy problem (8) for $a$ with unbounded support by the solutions of (8) for $a$ with bounded support:

**Corollary 4.1** *Let $0 \le a \in \mathrm{L}^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ be an arbitrary even function such that*

$$\int_{\mathbb{R}^d} a(x)dx = 1 \quad \text{and} \quad a(x) \to 0, \quad |x| \to \infty$$

*and let $k_{t,\,a}^{(n)}$ be a solution to the Cauchy problem (8) in $\mathrm{X}_n$. Suppose, that the conditions of Proposition 4.4 are fulfilled. Then there exists a sequence $\{a_l\}_{l \ge 1} \subset C_0(\mathbb{R}^d)$ such that*

$$k_{t,\,a_l}^{(n)} \to k_{t,\,a}^{(n)} \ \text{ in } \ \mathrm{X}_n \ \text{ as } \ l \to \infty.$$

*Proof.* There exists a sequence $\{a_l\}_{l\geq 1} \subset C_0(\mathbb{R}^d)$ such that

$$a_l \to a, \text{ in } \mathrm{X}_1 \text{ and } \mathrm{L}^1(\mathbb{R}^d) \text{ as } l \to \infty. \tag{11}$$

We will give the rest of the proof of the corollary using mathematical induction. For $n = 1$ the statement is trivial. Now, using induction step $(n-1) \to n$ we estimate the $\mathrm{L}^\infty$-norm of the difference

$$k^{(n)}_{t,\,a_l}(x_1, \dots, x_n) - k^{(n)}_{t,\,a}(x_1, \dots, x_n) = \tag{12}$$

$$= e^{n(\varkappa-1)t}\left(\left[\bigotimes_{i=1}^{n} e^{tL^i_{a_l}}\right] k^{(n)}_0(x_1, \dots, x_n) - \left[\bigotimes_{i=1}^{n} e^{tL^i_{a}}\right] k^{(n)}_0(x_1, \dots, x_n)\right) +$$

$$+\varkappa e^{n(\varkappa-1)t}\times$$

$$\int_0^t e^{n(1-\varkappa)s}\left(\left[\bigotimes_{i=1}^{n} e^{(t-s)L^i_{a_l}}\right]\sum_{i=1}^{n} k^{(n-1)}_{s,\,a_l}(x_1, \dots, \check{x}_i, \dots, x_n)\sum_{j:\,j\neq i} a_l(x_i - x_j) - \right.$$

$$\left. - \left[\bigotimes_{i=1}^{n} e^{(t-s)L^i_{a}}\right]\sum_{i=1}^{n} k^{(n-1)}_{s,\,a}(x_1, \dots, \check{x}_i, \dots, x_n)\sum_{j:\,j\neq i} a(x_i - x_j)\right) ds.$$

Due to Proposition 4.3 the first term on the right hand side of (12) converges to 0 in $\mathrm{X}_n$. Indeed, in our case the strong convergence of semigroups in $\mathrm{L}^\infty$ space with bounded generators follows from the corresponding convergence of generators. But the latter fact is trivial because of the convergence (11).

In order to check the convergence of the second term of (12) to 0 in $\mathrm{X}_n$, let us note that

$$\sum_{i=1}^{n} k^{(n-1)}_{s,\,a_l}(x_1, \dots, \check{x}_i, \dots, x_n)\sum_{j:\,j\neq i} a_l(x_i - x_j)$$

converges to

$$\sum_{i=1}^{n} k^{(n-1)}_{s,\,a}(x_1, \dots, \check{x}_i, \dots, x_n)\sum_{j:\,j\neq i} a(x_i - x_j),$$

in $\mathrm{X}_n$ as $l \to \infty$.

Due to Proposition 4.4

$$\left\|\left[\bigotimes_{i=1}^{n} e^{(t-s)L^i_{a_l}}\right]\sum_{i=1}^{n} k^{(n-1)}_{s,\,a_l}(x_1, \dots, \check{x}_i, \dots, x_n)\sum_{j:\,j\neq i} a_l(x_i - x_j)\right\|_{\mathrm{X}_n}$$

and the same expression with $a_l$ replaced by $a$ are uniformly bounded in $s \in [0,\, t]$. The latter two facts imply the convergence of the second term of (12) to 0 in $\mathrm{X}_n$. ■

Next we solve the following problem: suppose that $(k^{(n)}_0)_{n\geq 0}$ is a system of correlation functions, i.e., there exists a probability measure $\mu_0 \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, locally absolutely continuous with respect to the Poisson measure, whose correlation functions are exactly $(k_0^{(n)})_{n\geq 0}$. We would like to investigate now whether the evolution of $(k_0^{(n)})_{n\geq 0}$ in time preserves the property described above. Namely, whether $(k_t^{(n)})_{n\geq 0}$, at any moment of time $t > 0$, is a system of correlation functions or not. In order to answer this question, one can apply, for example, the result about the reconstruction of probability measures by correlation functions, which has been proposed by A. Lenard in [11] (see also [4]). For the readers' convenience below we give the conditions which have to be checked

- (Lenard positivity) for any $G \in B_{\mathrm{bs}}(\Gamma_0)$ with $KG \geq 0$

$$\int_{\Gamma_0} G(\eta)\rho(d\eta) \geq 0, \tag{13}$$

  where $\rho \in \mathcal{M}(\Gamma_0)$ is a correlation measure which corresponds to the system of correlation functions $(k^{(n)})_{n\geq 0}$ and additionally it is supposed to be locally finite and normalized, i.e $\rho(\{\emptyset\}) = 1$.

**Remark 4.2** *Lenard positivity ensures the existence of $\mu \in \mathcal{M}_{\mathrm{fm}}(\Gamma)$ such that the corresponding correlation measure $\rho_\mu = \rho$.*

- (moment growth) for any bounded set $\Lambda \subset \mathbb{R}^d$ and $j \geq 0$

$$\sum_{n=0}^{\infty} \left(m_{n+j}^{\Lambda}\right)^{-\frac{1}{n}} = \infty,$$

  where

$$m_n^{\Lambda} := (n!)^{-1} \int_{\Lambda} \cdots \int_{\Lambda} k^{(n)}(x_1, \ldots, x_n) dx_1 \ldots dx_n.$$

**Remark 4.3** *The moment growth condition ensures the uniqueness of $\mu \in \mathcal{M}_{\mathrm{fm}}(\Gamma)$ such that the corresponding correlation measure $\rho_\mu = \rho$.*

Further steps will be devoted to the verification of the latter conditions.

**Lemma 4.2** *Let $0 \leq a \in \mathrm{L}^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ be an arbitrary even function such that*

$$\int_{\mathbb{R}^d} a(x)dx = 1 \quad \text{and} \quad a(x) \to 0, \quad |x| \to \infty.$$

*Then, at any moment of time $t > 0$, the function $k_t$ (see (9)) is positive in the sense of (13).*

*Proof.* See **Appendix 1**.

**Remark 4.4** *The moment growth condition for the system of functions* $\left\{k^{(n)}\right\}_{n\geq 1}$ *is fulfilled if there exists a constant* $C > 0$ *(independent of* $n$*) such that*

$$k^{(n)}(x_1, \ldots, x_n) \leq n!\, C^n, \quad for \;\; all \;\; n \geq 0.$$

*Proof.* *The statement of the remark follows from direct calculations.*

The class of all probability measures whose system of correlation functions satisfies the bound given in Remark 4.4 will be denoted by $\mathcal{M}^1_{C,\,fac}(\Gamma)$.

For any system of functions $(k^{(n)})_{n\geq 0}$ we define the generating functional $\mathcal{L}_k : \mathcal{F}_k \to \mathbb{C}$:

$$\mathcal{L}_k(\theta) := \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \theta(x_1) \ldots \theta(x_n) k^{(n)}(x_1, \ldots, x_n) dx_1 \ldots dx_n, \qquad (14)$$

where $\mathcal{F}_k$ is the set of all functions $\theta$ for which (14) exists.

**Remark 4.5** *Let us define for an arbitrary* $\delta > 0$

$$U^1_\delta := \left\{\theta \in \mathrm{L}^1(\mathbb{R}^d) \;|\; ||\theta||_{\mathrm{L}^1(\mathbb{R}^d)} \leq \delta\right\}.$$

*If the system of functions* $(k^{(n)})_{n\geq 0}$ *satisfies the assumption of Remark 4.4, then the functional* $\mathcal{L}_k$ *is holomorphic in* $U^1_\delta$ *for some* $\delta > 0$.

**Remark 4.6** *Assume that the system of functions* $(k^{(n)})_{n\geq 0}$ *is a system of correlation functions for some measure* $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$. *In this case, there exists a connection between generating functional (14) and the measure* $\mu$ *(see e.g. [4]), given by*

$$\mathcal{L}_k(\theta) = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} \theta(x_1) \ldots \theta(x_n) k^{(n)}(x_1, \ldots, x_n) dx_1 \ldots dx_n =$$

$$= \int_\Gamma K\left(\prod_{x\in\gamma} \theta(x)\right) \mu(d\gamma) = \int_\Gamma \prod_{x\in\gamma} (1 + \theta(x))\, \mu(d\gamma), \quad \theta \in \mathcal{F}_k.$$

*The latter functional*

$$\mathcal{L}_\mu(\theta) := \int_\Gamma \prod_{x\in\gamma} (1 + \theta(x))\, \mu(d\gamma), \qquad (15)$$

*is called the Bogoliubov functional of the measure* $\mu$.

Now, we set

$$\mathcal{M}^1_{hol}(\Gamma) := \left\{\mu \in \mathcal{M}^1(\Gamma) \;\middle|\; \exists\, \delta > 0 : \;\; \mathcal{L}_\mu(\theta) \;\text{ is holomorphic in }\; U^1_\delta \right\}.$$

**Remark 4.7** *Remark 4.5 implies that* $\mathcal{M}^1_{C,\,fac}(\Gamma) \subset \mathcal{M}^1_{hol}(\Gamma)$

**Theorem 4.1** *Let $0 \leq a \in \mathrm{L}^1(\mathbb{R}^d) \cap C(\mathbb{R}^d)$ be an even function such that*

$$\int_{\mathbb{R}^d} a(x)dx = 1 \quad \text{and} \quad a(x) \to 0, \quad |x| \to \infty.$$

*Then, for any $\mu \in \mathcal{M}^1_{C,\,fac}(\Gamma)$ there exists a Markov function $X^\mu_t \in \Gamma$ with the initial distribution $\mu$ associated with the generator $L$, such that for any $t > 0$, the corresponding distribution of $X^\mu_t$ is given by $\mu_t \in \mathcal{M}^1_{hol}(\Gamma)$.*

*Proof.* Using the result of A. Lenard about the reconstruction of probability measures by correlation functions (see [11]), Lemma 4.2, Proposition 4.4 and Remark 4.4 we are able to define evolution on $\mathcal{M}^1(\Gamma)$. This gives us the possibility to determine all finite dimensional distributions of $X^\mu_t$. ■

## 4.3 Invariant measures

In this subsection we would like to describe the invariant measures of the contact process on $\Gamma$ constructed in the previous subsection.

In order to explain the expected time asymptotics for the correlation functions of our model let us consider the translation invariant case and the evolution of the first correlation function. Namely, we will assume that the first correlation function does not depend on $x \in \mathbb{R}^d$:

$$k^1_t(x) =: \rho_t, \;\; \text{for all} \;\; t \geq 0.$$

The function $\rho_t$ is called density. In this case, due to the results in the previous subsection, the time evolution of the first correlation function is given by

$$\frac{\partial \rho_t}{\partial t} = (\varkappa - 1)\rho_t,$$

$$\rho_t|_{t=0} = \rho_0.$$

The solution to this equation can be written as follows:

$$\rho_t = \exp\{(\varkappa - 1)t\}\rho_0.$$

We will distinguish the following cases:

1. **Subcritical** ($\varkappa < 1$): $\rho_t \to 0$, as $t$ tends to $\infty$;
2. **Supercritical** ($\varkappa > 1$): $\rho_t \to \infty$, as $t$ tends to $\infty$;
3. **Critical** ($\varkappa = 1$): $\rho_t = \rho_0 = \rho$.

**Remark 4.8** *For the case $\varkappa < 1$, the bound in Proposition 4.4 implies*

$$k^{(n)}_t \to 0 \;\; \text{in} \;\; \mathrm{X}_n \;\; \text{as} \;\; t \to \infty,$$

*for any $n \in \mathbb{N}$.*

Now, coming back to the purpose of this subsection, it becomes clear that invariant measures may exist only in the critical case. Moreover, due to Theorem 4.1, all invariant measures of the contact process can be described in terms of the corresponding system of correlation functions as positive solutions to the following system of equations

$$\widehat{L}_n^* k^{(n)} + f^{(n)} = 0, \quad n \geq 1, \qquad k^{(0)} \equiv 1,$$

where for any $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$

$$f^{(n)}(x_1, \ldots, x_n) := \sum_{i=1}^{n} k^{(n-1)}(x_1, \ldots, \check{x}_i, \ldots, x_n) \sum_{j:\, j \neq i} a(x_i - x_j).$$

This statement can be formulated more precisely:

**Proposition 4.5** *If a measure $\mu \in \mathcal{M}^1(\Gamma)$ is an invariant measure for the contact process $X_t^\mu \in \Gamma$, then the system of corresponding correlation functions of this measure is a solution to the recurrent systems of equation*

$$n\, k^{(n)}(x_1, \ldots, x_n) = \sum_{i=1}^{n} k^{(n-1)}(x_1, \ldots, \check{x}_i, \ldots, x_n) \sum_{j:\, j \neq i} a(x_i - x_j) +$$

$$+ \sum_{i=1}^{n} \int_{\mathbb{R}^d} a(x_i - y) k^{(n)}(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) dy, \quad n \geq 1. \tag{16}$$

Now, we give below the answer to the inverse problem and prove some kind of ergodicity result for our process in the translation invariant case. In this connection, for any $n \in \mathbb{N}$ we will be interested in the time asymptotics of solutions to an auxiliary Cauchy problem

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1, \ldots, x_n) = \widehat{L}_n^* k_t^{(n)}(x_1, \ldots, x_n), \quad t \geq 0, \tag{17}$$

$$k_t^{(n)}(x_1, \ldots, x_n)\Big|_{t=0} := k_0^{(n)}(x_1, \ldots, x_n),$$

in the Banach space $\mathrm{X}_n$ .

**Theorem 4.2** *Let $d \geq 3$ be arbitrary and fixed and let $0 \leq a \in \mathrm{L}^1(\mathbb{R}^d)$ be an arbitrary even continuous function such that*

1. $\int_{\mathbb{R}^d} a(x) dx = 1$,
2. $\int_{\mathbb{R}^d} x_k x_j \, a(x) dx < \infty$, for all $1 \leq k,\, j \leq d$,
3. $\hat{a}(\cdot) := \int_{\mathbb{R}^d} e^{-i(\cdot,\, x)} a(x) dx \in \mathrm{L}^1(\mathbb{R}^d)$.

*Then, for any $\rho \in \mathbb{R}_+$ there exists a unique measure $\mu^\rho \in \mathcal{M}^1(\Gamma)$ such that its system of correlation functions $\left\{k^{(n),\,\rho}\right\}_{n\geq 0}$ is translation invariant, solves equation (16) and satisfies the following estimate*

$$||k^{(n),\,\rho}||_{\mathrm{X}_n} \leq C(\rho)^n (n!)^2, \quad n \geq 1,$$

*for some positive constant $C(\rho)$. Moreover, the first correlation function (density) of $\mu^\rho$ is exactly $\rho \in \mathbb{R}_+$.*

*Let $\mu_t$ be the distribution of $X_t^{\mu_0}$, $\mu_0 \in \mathcal{M}^1_{hol}(\Gamma)$ at time $t \geq 0$ and let $\left\{k_t^{(n)}\right\}_{n\geq 0}$ denotes the system of correlation functions of $\mu_t$. Then, in the critical case ($\varkappa = 1$) the following conditions are fulfilled*

1. $k_t^{(1)} = k_0^{(1)} =: \rho$*;*
2. *for any $n \geq 2$ and any $\varphi \in \mathrm{L}^1((\mathbb{R}^d)^n)$*

$$\left(k_t^{(n)},\, \varphi\right) \to \left(k^{(n),\,\rho},\, \varphi\right) \quad \textit{as} \quad t \to \infty,$$

*where $\left(k_t^{(n)},\, \cdot\right)$ and $\left(k^{(n),\,\rho},\, \cdot\right)$ are notations for the corresponding functionals on $\mathrm{L}^1((\mathbb{R}^d)^n)$.*

*Proof.* We have to show that under the assumptions of Theorem 4.2 the equation (16) has a solution for any initial $k^{(1)} = \rho,\ \rho \in \mathbb{R}_+$ which satisfies the moment growth condition and Lenard positivity (13). For the moment growth condition it is enough to show that the solution has the following property

$$||k^{(n)}||_{\mathrm{X}_n} \leq C^n (n!)^2.$$

We will give the proof using the mathematical induction method. Let us first consider the case $n = 2$. Since we are in the translation invariant case, we have

$$k^{(2)}(x_1, x_2) = k^{(2)}(x_1 - x_2, 0) =: k(x_1 - x_2),$$

where $k$ is even function on $\mathbb{R}^d$. Hence, equation (16) can be rewritten as

$$(a \star k)(x_1 - x_2) - k(x_1 - x_2) = -\rho a(x_1 - x_2), \tag{18}$$

where

$$(a \star k)(x) := \int_{\mathbb{R}^d} a(x - y)k(y)dy.$$

It is clear, that the best method for the investigation of equations with convolutions is the Fourier transform method. Assume equation (18) has a solution $v \in \mathrm{L}^1(\mathbb{R}^d)$. Then, the Fourier transform of $v$ satisfies the following equation

$$\hat{v}(p) = \frac{\rho \hat{a}(p)}{1 - \hat{a}(p)}.$$

If $d \geq 3$ and the conditions of Theorem 4.4 are fulfilled, then $\hat{v}(p)$ has an integrable singularity $\frac{1}{|p|^2}$ at $p = 0$, i.e $\hat{v}(p) \in \mathrm{L}^1(\mathbb{R}^d)$. Therefore,

$$v(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{i(p,x)} \frac{\rho \hat{a}(p)}{1 - \hat{a}(p)} dp \ \in \ \mathrm{L}^\infty(\mathbb{R}^\mathrm{d}). \tag{19}$$

**Remark 4.9** *Let us consider the translation invariant case. Suppose that solution to (18) is a second correlation function. Then, an application of the Fourier transform directly to (18) does not have any a priori sense (in general second correlation function is not integrable). Contrary to the second correlation function, the second Ursell function $u^{(2)}$ in majority of physical applications is integrable in one coordinate. Namely, the function $u$ which is defined by*

$$u(x_1 - x_2) := u^{(2)}(x_1 - x_2, 0) = k(x_1 - x_2) - \rho^2$$

*is integrable on $\mathbb{R}^d$. It is easy to check that the equation for the function $u$ has the same form as (18) for the function $k$ (the set of constants belong to the kernel of the operator $\widehat{L}_2^*$), i.e.*

$$(a \star u)(x_1 - x_2) - u(x_1 - x_2) = -\rho a(x_1 - x_2).$$

Having in mind Remark 4.9 and (19) one can easily check that

$$k^{(2)}(x_1, x_2) = k(x_1 - x_2) = v(x_1 - x_2) + \rho^2 \tag{20}$$

is a solution to (18) in $\mathrm{X}_2$. Moreover,

$$k^{(2)}(x_1, x_2) \leq \rho A + \rho^2 \leq C^2 (2!)^2,$$

where

$$A = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \frac{|\hat{a}(p)|}{1 - \hat{a}(p)} dp,$$

and the constant

$$C \geq \frac{\sqrt{\rho(A + \rho)}}{2}$$

will be chosen later.

The reason why we look for the solution to (18) in $\mathrm{X}_2$ in the form (20) is motivated by the arguments given above and Remark 4.9. We would like to stress that we do not need the existence of a solution to the equation (18) in the space $\mathrm{L}^1(\mathbb{R}^d)$ to define (20) in $\mathrm{X}_2$ rigorously.

Let us consider now equation (16) for $n \geq 3$

$$\widehat{L}_n^* k^{(n)}(x_1, \ldots, x_n) = \tag{21}$$

$$= -\sum_{i=1}^{n} k^{(n-1)}(x_1, \ldots, \check{x}_i, \ldots, x_n) \sum_{j:\, j \neq i} a(x_i - x_j) =: -f^{(n)}(x_1, \ldots, x_n).$$

The following function is a solution to this equation in the Banach space $\mathrm{X}_n$:

$$k^{(n)}(x_1,\ldots,x_n)=\int_0^\infty \left(e^{t\widehat{L}_n^*}f^{(n)}\right)(x_1,\ldots,x_n)dt, \tag{22}$$

provided

$$\int_0^\infty \left(e^{t\widehat{L}_n^*}f^{(n)}\right)(x_1,\ldots,x_n)dt<\infty, \quad \text{for a.a. } (x_1,\ldots,x_n)\in(\mathbb{R}^d)^n$$

and

$$e^{t\widehat{L}_n^*}f^{(n)}\to 0, \qquad t\to\infty.$$

Therefore, in order to clarify the existence of the solution to (21) we should check whether the right hand side of (22) has sense in $\mathrm{X}_n$. Using the mathematical induction step $(n-1)\to n$ and the Markov property of $e^{t\widehat{L}_n^*}$, we get

$$\int_0^\infty \left(e^{t\widehat{L}_n^*}f^{(n)}\right)(x_1,\ldots,x_n)dt\le \tag{23}$$

$$\le\int_0^\infty\left(e^{t\widehat{L}_n^*}C^{n-1}\left((n-1)!\right)^2\sum_{i=1}^n\sum_{j:\,j\neq i}a(\cdot_i-\cdot_j)\right)(x_1,\ldots,x_n)dt=$$

$$=C^{n-1}\left((n-1)!\right)^2\sum_{i=1}^n\sum_{j:\,j\neq i}\int_0^\infty\left(e^{t\left(L_a^i+L_a^j\right)}a(\cdot_i-\cdot_j)\right)(x_i,\,x_j)dt. \tag{24}$$

The contraction property of the semigroup $e^{tL_a^j}$ implies: there exists a subset $N\subset\mathbb{R}^d$, the Lebesgue measure of which is zero, such that for a.a. $x_j\in\mathbb{R}^d$ w.r.t. the Lebesgue measure

$$\int_0^\infty\left(e^{t\left(L_a^i+L_a^j\right)}a(\cdot_i-\cdot_j)\right)(x_i,\,x_j)dt\le$$

$$\le\int_0^\infty\sup_{x_j\in\mathbb{R}^d\setminus N}\left(e^{tL_a^i}a(\cdot_i-x_j)\right)(x_i)dt\le$$

$$\le\frac{1}{(2\pi)^d}\int_0^\infty\sup_{x_j\in\mathbb{R}^d\setminus N}\int_{\mathbb{R}^d}\left|\left(e^{tL_a^i}\widehat{a(\cdot_i-x_j)}\right)(p)\right|dp\,dt=$$

$$=\frac{1}{(2\pi)^d}\int_0^\infty\sup_{x_j\in\mathbb{R}^d\setminus N}\int_{\mathbb{R}^d}e^{t(\hat{a}(p)-1)}\left|\int_{\mathbb{R}^d}e^{-i(p,x)}a(x-x_j)dx\right|dp\,dt=$$

$$=\frac{1}{(2\pi)^d}\int_0^\infty\sup_{x_j\in\mathbb{R}^d\setminus N}\int_{\mathbb{R}^d}e^{t(\hat{a}(p)-1)}\left|e^{-i(p,x_j)}\hat{a}(p)\right|dp\,dt\le$$

$$\le\frac{1}{(2\pi)^d}\int_0^\infty\int_{\mathbb{R}^d}e^{t(\hat{a}(p)-1)}\left|\hat{a}(p)\right|dp\,dt.$$

For any $p \in \mathbb{R}^d \setminus \{0\}$

$$\int_0^\infty e^{t(\hat{a}(p)-1)}dt = \frac{1}{1-\hat{a}(p)}.$$

Moreover, because of (19)

$$\int_{\mathbb{R}^d} \frac{|\hat{a}(p)|}{1-\hat{a}(p)}dp < \infty.$$

Therefore, the Fubini theorem for non-negative functions implies that

$$\int_0^\infty \int_{\mathbb{R}^d} e^{t(\hat{a}(p)-1)} \, |\hat{a}(p)| \, dp \, dt < \infty.$$

Finally, using the result obtained for the case $n = 2$, under the conditions of Theorem 4.2, for almost all $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$ w.r.t. the Lebesgue measure we get

$$\int_0^\infty \left(e^{t\widehat{L}_n^*} f^{(n)}\right)(x_1, \ldots, x_n)dt \leq C^{n-1}A(n!)^2 \leq C^n(n!)^2,$$

where

$$C = \max\left\{A, \; \frac{\sqrt{\rho(A+\rho)}}{2}\right\}.$$

The remaining statement of the first part of the theorem which has to be proved is Lenard positivity for the system of functions $\{k^{(n)}\}_{n\geq 0}$. But it follows directly from the second part of the theorem which describes the time asymptotics of the considered system of correlation functions.

The first statement of the second part of the theorem is trivial. In order to prove the second one, let us consider the following difference

$$k_t^{(n)}(x_1, \ldots, x_n) - k^{(n)}(x_1, \ldots, x_n) = \tag{25}$$
$$= \left[e^{t\widehat{L}_n^*} - 1\!\!1\right] k^{(n)}(x_1, \ldots, x_n) + e^{t\widehat{L}_n^*}\left[k_0^{(n)}(x_1, \ldots, x_n) - k^{(n)}(x_1, \ldots, x_n)\right] +$$
$$+ \int_0^t e^{s\widehat{L}_n^*} f_{t-s}^{(n)}(x_1, \ldots, x_n)ds,$$

where $\{k^{(n)}\}_{n\geq 0}$ is a solution to (16), constructed in the first part of this proof, such that

$$k^{(1)} = k_0^{(1)} = k_t^{(1)} = \rho, \quad t > 0.$$

Since

$$\left[e^{t\widehat{L}_n^*} - 1\!\!1\right] k^{(n)}(x_1, \ldots, x_n) = \int_0^t e^{s\widehat{L}_n^*} \widehat{L}_n^* k^{(n)}(x_1, \ldots, x_n)ds =$$
$$= -\int_0^t e^{s\widehat{L}_n^*} f^{(n)}(x_1, \ldots, x_n),$$

the difference (25) can be rewritten in the form

$$k_t^{(n)}(x_1,\dots,x_n) - k^{(n)}(x_1,\dots,x_n) = \tag{26}$$
$$= e^{t\widehat{L}_n^*}\left[k_0^{(n)}(x_1,\dots,x_n) - k^{(n)}(x_1,\dots,x_n)\right] +$$
$$+\int_0^t e^{s\widehat{L}_n^*}\left[f_{t-s}^{(n)}(x_1,\dots,x_n) - f^{(n)}(x_1,\dots,x_n)\right] ds.$$

Similar to the observations which were proposed above for the investigation of the right hand side of (22) and due to Proposition 4.4, one can easily check that

$$\int_0^\infty e^{s\widehat{L}_n^*} f^{(n)}(x_1,\dots,x_n) ds \in \mathrm{X}_n \tag{27}$$

and

$$\int_0^t e^{s\widehat{L}_n^*} f_{t-s}^{(n)}(x_1,\dots,x_n) ds \in \mathrm{X}_n.$$

As a next step we use the method of mathematical induction. For $n = 1$ the second statement of the second part of the theorem is trivial. Let us assume the inductive step $(n-1) \to n$. Namely, let

$$k_t^{(n-1)} \to k^{(n-1)}, \;\; t\to\infty \;\text{ in }\; \mathrm{X}_{n-1}. \tag{28}$$

This immediately implies the convergence of

$$f_{t-s}^{(n)} \to f^{(n)}, \;\; t\to\infty \;\text{ in }\; \mathrm{X}_n, \tag{29}$$

for any fixed $s\in[0,\infty)$. Therefore, taking into account Proposition 4.4 and (28), one can find a constant $K > 0$ such that for any $t \geq 0$

$$||k_t^{(n-1)}||_{X_{n-1}} \leq K||k^{(n-1)}||_{X_{n-1}}.$$

Now, for an arbitrary $\varepsilon > 0$ there exists $T > 0$ such that for all $t \geq T$

$$\int_T^t e^{s\widehat{L}_n^*}\left[f_{t-s}^{(n)}(x_1,\dots,x_n) - f^{(n)}(x_1,\dots,x_n)\right] ds \leq$$
$$\leq \int_T^t e^{s\widehat{L}_n^*}\left[|f_{t-s}^{(n)}(x_1,\dots,x_n)| + |f^{(n)}(x_1,\dots,x_n)|\right] ds \leq$$
$$\leq 2K\int_T^t ||k^{(n-1)}||_{X_n}\sum_{i=1}^n\sum_{j:\,j\neq i}\left(e^{s\left(L_a^i+L_a^j\right)}a(\cdot_i-\cdot_j)\right)(x_i,\,x_j)ds \leq$$
$$\leq 2K\int_T^\infty ||k^{(n-1)}||_{X_n}\sum_{i=1}^n\sum_{j:\,j\neq i}\left(e^{s\left(L_a^i+L_a^j\right)}a(\cdot_i-\cdot_j)\right)(x_i,\,x_j)ds < \varepsilon.$$

In the latter estimate we have used (23) and the bound for (27).

The convergence (29) and the contraction property of the semigroup $e^{t\widehat{L}_n^*}$ imply

$$\int_0^T e^{s\widehat{L}_n^*}\left[f_{t-s}^{(n)}(x_1,\dots,x_n) - f^{(n)}(x_1,\dots,x_n)\right] ds \to 0, \ \ t\to\infty \ \text{ in } \mathrm{X}_n.$$

Finally,

$$\int_0^t e^{s\widehat{L}_n^*}\left[f_{t-s}^{(n)}(x_1,\dots,x_n) - f^{(n)}(x_1,\dots,x_n)\right] ds \to 0, \ \ t\to\infty \ \text{ in } \mathrm{X}_n.$$

Assuming that

$$e^{t\widehat{L}_n^*}\left[k_0^{(n)}(x_1,\dots,x_n) - k^{(n)}(x_1,\dots,x_n)\right] \to 0, \ \ t\to\infty, \ \text{ in } \mathrm{X}_n. \tag{30}$$

and due to (26), the second statement of the second part of the theorem is now obvious.

Now, let us come back to the assumption (30). This assumption means that the asymptotics, as $t$ tends to $\infty$, of the solution to the Cauchy problem

$$\frac{\partial k_t^{(n)}}{\partial t}(x_1,\dots,x_n) = \widehat{L}_n^* k_t^{(n)}(x_1,\dots,x_n), \ \ t\geq 0, \tag{31}$$

$$k_t^{(n)}(x_1,\dots,x_n)\Big|_{t=0} := k_0^{(n)}(x_1,\dots,x_n) \in \mathrm{X}_n,$$

do not depend on the initial data. The boundness of the operator $\widehat{L}_n^*$ in $\mathrm{X}_n$ implies that the solution $k_t^{(n)} = e^{t\widehat{L}_n^*}k_0^{(n)}$ to the Cauchy problem (31) exists and it is a function from $\mathrm{X}_n$. The latter fact gives us the possibility to look at the solution of (31) in the class of generalized functions $(\mathrm{L}^1((\mathbb{R}^d)^n))' \subset \mathrm{S}'((\mathbb{R}^d)^n)$ (where $\mathrm{S}'((\mathbb{R}^d)^n)$ linear continuous functionals on the class of rapidly decreasing functions on $(\mathbb{R}^d)^n$). The well-definiteness of the Fourier transform for the class $\mathrm{S}'((\mathbb{R}^d)^n)$ gives the possibility to consider the following functional

$$(\widehat{k_t^{(n)}},\varphi) = (k_t^{(n)},\widehat{\varphi}), \quad \varphi\in \mathrm{S}((\mathbb{R}^d)^n)$$

where $\widehat{\varphi}$ is the Fourier transform of $\varphi$. Sometimes we will use notation $\mathcal{F}(\varphi)$ instead of $\widehat{\varphi}$ to avoid complicated notations in large formulas. The functional $\widehat{k_t^{(n)}}$ can be written also in more explicit form

$$(\widehat{k_t^{(n)}},\varphi) = \int_{\mathbb{R}^d}\dots\int_{\mathbb{R}^d} e^{t\widehat{L}_n^*}k_0^{(n)}(x_1,\dots,x_n)\widehat{\varphi}(x_1,\dots,x_n)dx_1\dots dx_n.$$

Since for $\varkappa = 1$

$$\left|\left|\widehat{L}_n^* k\right|\right|_{\mathrm{L}^\infty((\mathbb{R}^d)^n)} \leq n(||a||_{\mathrm{L}^1(\mathbb{R}^d)}+1)||k||_{\mathrm{L}^\infty((\mathbb{R}^d)^n)}$$

we get

$$\left|\sum_{l=0}^{N}\frac{t^l}{l!}\left(\left(\widehat{L}_n^*\right)^l k_0^{(n)}\right)(x_1,\ldots,x_n)\widehat{\varphi}(x_1,\ldots,x_n)\right|\le$$

$$\le\sum_{l=0}^{N}\frac{t^l}{l!}\left(n(||a||_{\mathrm{L}^1(\mathbb{R}^d)}+1)\right)^l ||k_0^{(n)}||_{\mathrm{L}^\infty}\,|\widehat{\varphi}(x_1,\ldots,x_n)|\le$$

$$\le\exp\{tn(||a||_{\mathrm{L}^1(\mathbb{R}^d)}+1)\}||k_0^{(n)}||_{\mathrm{L}^\infty}\,|\widehat{\varphi}(x_1,\ldots,x_n)|\in\mathrm{L}^1((\mathbb{R}^d)^n)$$

for all $N\in\mathbb{N}$. By the Lebesgue dominated convergence theorem

$$(\widehat{k_t^{(n)}},\varphi)=\sum_{l=0}^{\infty}\frac{t^l}{l!}\int_{\mathbb{R}^d}\ldots\int_{\mathbb{R}^d}\left(\left(\widehat{L}_n^*\right)^l k_0^{(n)}\right)(x_1,\ldots,x_n)\widehat{\varphi}(x_1,\ldots,x_n)dx_1\ldots dx_n.$$

Then, using the Newton's binomial formula to write the explicit form for the expression $\left(\left(\widehat{L}_n^*\right)^l k_0^{(n)}\right)(x_1,\ldots,x_n)$ and the Fourier property of the convolution, i.e.

$$\widehat{(a\star k)}(p_1,\ldots,p_n)=\widehat{a}(p_1,\ldots,p_n)\widehat{k}(p_1,\ldots,p_n),\quad\text{for}\quad k\in\mathrm{L}^1((\mathbb{R}^d)^n)$$

one can easily check that

$$(\widehat{k_t^{(n)}},\varphi)=\sum_{l=0}^{\infty}\frac{t^l}{l!}\int_{\mathbb{R}^d}\ldots\int_{\mathbb{R}^d}k_0^{(n)}(x_1,\ldots,x_n)\left(\left(\widehat{L}_n^*\right)^l\widehat{\varphi}\right)(x_1,\ldots,x_n)dx_1\ldots dx_n$$

and

$$\left(\left(\widehat{L}_n^*\right)^l\widehat{\varphi}\right)(x_1,\ldots,x_n)=\mathcal{F}\left(\left(\sum_{i=1}^{n}\widehat{a}(\cdot_i)-n\right)^l\varphi\right)(x_1,\ldots,x_n).$$

Therefore,

$$(\widehat{k_t^{(n)}},\varphi)=\int_{\mathbb{R}^d}\ldots\int_{\mathbb{R}^d}k_0^{(n)}(x_1,\ldots,x_n)\left(e^{t\widehat{L}_n^*}\widehat{\varphi}\right)(x_1,\ldots,x_n)dx_1\ldots dx_n=$$

$$=\int_{\mathbb{R}^d}\ldots\int_{\mathbb{R}^d}k_0^{(n)}(x_1,\ldots,x_n)\mathcal{F}\left(e^{t\left(\sum_{i=1}^{n}\widehat{a}(\cdot_i)-n\right)}\varphi\right)(x_1,\ldots,x_n)dx_1\ldots dx_n.$$

For $\varkappa=1$ the semigroup $e^{t\widehat{L}_n^*}$ is a contraction semigroup in $\mathrm{L}^1((\mathbb{R}^d)^n)$. It follows directly from Lemma 4.1 and Remark 4.1. It is also not difficult to see that

$$\mathcal{F}\left(\exp\left\{t\left(\sum_{i=1}^{n}\widehat{a}(\cdot_i)-n\right)\right\}\varphi\right)(x_1,\ldots,x_n)\to 0,\quad\text{as}\quad t\to\infty$$

pointwisely. The latter two facts imply

$$e^{t\widehat{L}_n^*}\widehat{\varphi}\to 0\quad\text{in}\quad\mathrm{L}^1((\mathbb{R}^d)^n)\quad\text{as}\quad t\to\infty.$$

As result, for any $\varphi \in \mathrm{S}((\mathbb{R}^d)^n)$ we have

$$\left(\widehat{k_t^{(n)}}, \varphi\right) \to 0, \quad \text{as } \ t \to \infty$$

and hence

$$\left(k_t^{(n)}, \varphi\right) \to 0, \quad \text{as } \ t \to \infty.$$

Since $\mathrm{S}((\mathbb{R}^d)^n)$ is dense in $\mathrm{L}^1((\mathbb{R}^d)^n)$ and

$$||k_t^{(n)}||_{(\mathrm{L}^1((\mathbb{R}^d)^n))'} = ||k_t^{(n)}||_{\mathrm{L}^\infty((\mathbb{R}^d)^n)} \le ||k_0^{(n)}||_{\mathrm{L}^\infty((\mathbb{R}^d)^n)}$$

we have: for any $\varphi \in \mathrm{L}^1((\mathbb{R}^d)^n)$

$$\left(k_t^{(n)}, \varphi\right) \to 0, \quad \text{as } \ t \to \infty.$$

This fact concludes the proof of the main theorem. ■

The system of equations (17) corresponds to an infinite number of independent random walks on $\mathbb{R}^d$. Similar to the observations proposed in [5], the investigation of the asymptotics for (17) in the sense of convergence in the norm requires some restrictions on the initial correlation functions. It is more convenient to formulate these restrictions in terms of Ursell functions, which correspond to the correlation functions. Ursell functions are defined as follows

$$u(\eta) := \sum_{i=1}^{|\eta|} \sum_{(\xi_1,\ldots,\xi_i)\in\mathcal{P}_\emptyset^i(\eta)} (-1)^{i-1}(i-1)!\, k(\xi_1)\cdots k(\xi_i), \quad \eta \in \Gamma_0,$$

$$u(\eta) = k(\eta), \ \text{ if } \ |\eta| = 1.$$

The inverse relation is given by

$$k(\eta) := \sum_{i=1}^{|\eta|} \sum_{(\xi_1,\ldots,\xi_i)\in\mathcal{P}_\emptyset^i(\eta)} u(\xi_1)\cdots u(\xi_i), \quad \eta \in \Gamma_0. \tag{32}$$

**Remark 4.10** *Let us denote by $K_{\text{addmissible}}$ the class of translation invariant correlation functions (or the class of corresponding measures), whose Ursell functions satisfy the following assumptions:*

1. *for any $n \ge 2$*
$$\sup_{x\in\mathbb{R}^d} u^{x,\,(n-1)} \in \mathrm{L}^1((\mathbb{R}^d)^{n-1}),$$
*where*
$$u^{x,\,(n-1)}(x_1, ..., x_{n-1}) := u(\{x_1, ..., x_{n-1}, x\});$$
2. *for any $n \ge 2$*
$$\sup_{x\in\mathbb{R}^d} \widehat{u^{x,\,(n-1)}} \in \mathrm{L}^1((\mathbb{R}^d)^{n-1}).$$

*Under the conditions of Theorem 4.2 and for any* $\left\{k_0^{(n)}\right\}_{n\geq 0} \in K_{\text{addmissible}}$

$$k_t^{(n)} \to k^{(n),\,\rho} \quad in \ \mathrm{X}_n \quad as \quad t \to \infty.$$

*Proof.* Below we will show, that for the class of initial correlation functions from $K_{\text{admissible}}$, the assumption (30) is fulfilled. The first important observation, which follows from the definition of the Ursell function, is that evolutional equation for the $n$-th Ursell function $u^{(n)}$, which corresponds to the equation (31), is of the same type. Namely,

$$\frac{\partial u_t^{(n)}}{\partial t}(x_1, \ldots, x_n) = \widehat{L}_n^* u_t^{(n)}(x_1, \ldots, x_n), \;\; t \geq 0, \tag{33}$$

$$u_t^{(n)}(x_1, \ldots, x_n)\Big|_{t=0} := u_0^{(n)}(x_1, \ldots, x_n),$$

Since the operator $\widehat{L}_n^*$ preserves translation invariant functions and the initial function $u_0 \in K_{\text{admissible}}$ is considered to be translation invariant, the evolution of $u_t$ will be also translation invariant.

Let $x \in \mathbb{R}^d$ be arbitrary and fixed. The definition of the class $K_{\text{admissible}}$ and the fact that semigroup $e^{t\widehat{L}_n^*}$ can be presented as the product of semigroups acting in each coordinate of $(x_1, \ldots, x_n) \in (\mathbb{R}^d)^n$, i.e. in the form $\bigotimes_{i=1}^n e^{tL_a^i}$, imply

$$u_t^{x,\,(n-1)} \in \mathrm{L}^1((\mathbb{R}^d)^{n-1}), \quad n \geq 2.$$

Therefore, the Fourier transform of $u_t^{x,\,(n-1)}$ exists and the equation (33) for this function in the Fourier coordinates has the following form

$$\frac{\partial \widehat{u_t^{x,\,(n-1)}}}{\partial t}(p_1, \ldots, p_{n-1}) = \int_{\mathbb{R}^d} a(x-y)\widehat{u_t^{y,\,(n-1)}}(p_1, \ldots, p_{n-1})dy + \tag{34}$$

$$+ \left[\sum_{i=1}^{n-1} \hat{a}(p_i) - n\right] \widehat{u_t^{x,\,(n-1)}}(p_1, \ldots, p_{n-1}).$$

For fixed $(p_1, \ldots, p_{n-1}) \in (\mathbb{R}^d)^{n-1}$, we define

$$\tilde{u}_t(x) := \widehat{u_t^{x,\,(n-1)}}(p_1, \ldots, p_{n-1}).$$

In terms of this function, the equation (34) has the form

$$\frac{\partial \tilde{u}_t}{\partial t}(x) = \int_{\mathbb{R}^d} a(x-y)\tilde{u}_t(y)dy + \left[\sum_{i=1}^{n-1} \hat{a}(p_i) - n\right] \tilde{u}_t(x). \tag{35}$$

Due to the definition of $K_{\text{admissible}}$, the function $\tilde{u}_t(x)$ is bounded. Moreover,

$$\tilde{u}_t(x) := \exp\left\{t\left[\sum_{i=1}^{n-1} \hat{a}(p_i) - (n-1)\right]\right\}\left(e^{t\widehat{L}_1^*}\tilde{u}_0\right)(x)$$

is a solution to (35).

Now,

$$||u_t^{(n)}||_{\mathrm{X}_n} \le \sup_{x\in\mathbb{R}^d} \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} |\widehat{u_t^{x,\,(n-1)}}(p_1,\dots,p_{n-1})| dp_1 \dots dp_{n-1} \le$$

$$\le \int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \exp\left\{ t\left[\sum_{i=1}^{n-1} \hat{a}(p_i) - (n-1)\right]\right\} \sup_{x\in\mathbb{R}^d} |\widehat{u_0^{x,\,(n-1)}}(p_1,\dots,p_{n-1})| dp_1 \dots dp_{n-1}.$$

Since

$$\int_{\mathbb{R}^d} \dots \int_{\mathbb{R}^d} \sup_{x\in\mathbb{R}^d} |\widehat{u_0^{x,\,(n-1)}}(p_1,\dots,p_{n-1})| dp_1 \dots dp_{n-1} < \infty,$$

the following norm

$$||u_t^{(n)}||_{\mathrm{X}_n} \to 0,$$

as $t$ tends to $\infty$. ■

# Appendix 1

According to the definition of the Lenard positivity it is enough to check that

$$\int_{\Gamma_0} G(\eta)\rho_t(d\eta) :=$$

$$= \sum_{n\ge 0} \frac{1}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} G^{(n)}(x_1,\dots,x_n) k_t^{(n)}(x_1,\dots,x_n) dx_1 \dots dx_n \ge 0, \qquad (36)$$

for all $G \in B_{\mathrm{bs}}(\Gamma_0)$, such that $KG \ge 0$. Moreover, due to Corollary 4.1 it is enough to check the latter inequality only in the case of $a \in C_0(\mathbb{R}^d)$.

Let $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be locally absolutely continuous w.r.t. the Poisson measure whose system of correlation functions $\{k^{(n)}\}_{n\ge 0}$ satisfies the assumption of Remark 4.4.

As it was shown in [10], there exists a Markov process $X_t^\gamma$ on the configuration space $\Gamma$ with the corresponding generator $L$ in the case of $a \in C_0(\mathbb{R}^d)$. Next, we consider the following functions on $\Gamma$

$$F^{(n)}(\gamma) = \sum_{\{x_1,\dots,x_n\}\subset\gamma} e^{-\beta|x_1|} \cdots e^{-\beta|x_n|}, \;\; \beta > 0, \;\; n \in \mathbb{N}, \;\; |\gamma| \ge n.$$

Note, that

$$\int_{\Gamma} F^{(n)}(\gamma)\mu(d\gamma) =$$

$$= \frac{1}{n!} \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} e^{-\beta|x_1|} \cdots e^{-\beta|x_n|} k^{(n)}(x_1,\dots,x_n) dx_1 \dots dx_n < \infty, \qquad (37)$$

because of (i) of Remark 4.4. Using the same direct computation as in [10] we obtain

$$LF^{(n)}(\gamma) \le C_1 F^{(n)}(\gamma) + C_2 F^{(n-1)}(\gamma),$$

for some constants $C_1$, $C_2 > 0$. The use of the latter estimate for the function

$$\mathbb{L}^{(N)}(\gamma) := \sum_{n=1}^{N} F^{(n)}(\gamma),$$

gives us the bound

$$L\mathbb{L}^{(N)}(\gamma) \le C\mathbb{L}^{(N)}(\gamma), \quad C > 0.$$

The application of the martingale representation together with the Gronwall inequality implies

$$\mathrm{E}\left[\mathbb{L}^{(N)}(X_t^\gamma)\right] \le \mathbb{L}^{(N)}(\gamma)e^{Ct}. \tag{38}$$

Let $\{\mu_t\}_{t\ge 0}$ be the evolution of $\mu_0$ which corresponds to $X_t^\gamma$ described by the dual Kolmogorov equation

$$\frac{\partial \mu_t}{\partial t} = L^\star \mu_t,$$

$$\mu_t\,|_{t=0} = \mu_0,$$

where $L^\star$ is the adjoint operator to the operator $L$ on $\mathcal{M}^1(\Gamma)$. Then (38) and the bound

$$\mathbb{L}^{(N)}(\gamma) \ge \mathbb{L}^{(N)}(\gamma_\Lambda) \ge \sum_{k=1}^{N} \left(\min_{x\in\Lambda}\{e^{-\beta|x|}\}\right)^k \binom{|\gamma_\Lambda|}{k} \ge$$

$$\ge \begin{cases} \left(1+\min_{x\in\Lambda}\{e^{-\beta|x|}\}\right)^{|\gamma_\Lambda|} - 1, & \text{if } |\gamma_\Lambda| \le N, \\ C_N\left(\min_{x\in\Lambda}\{e^{-\beta|x|}\}\right)^N |\gamma_\Lambda|^N, & \text{otherwise,} \end{cases} \qquad \Lambda \in \mathcal{B}_b(\mathbb{R}^d),$$

with

$$0 < C_N < \frac{1}{N^N}$$

imply

$$\int_\Gamma |\gamma_\Lambda|^N \mu_t(d\gamma) \le \left(\min_{x\in\Lambda}\{e^{-\beta|x|}\}\right)^{-N} C_N^{-1}\left(1 + \int_\Gamma \mathrm{E}\left[\mathbb{L}^{(N)}(X_t^\gamma)\right]\mu_0(d\gamma)\right) \le$$

$$\le \left(\min_{x\in\Lambda}\{e^{-\beta|x|}\}\right)^{-N} C_N^{-1}\left(1 + e^{Ct}\int_\Gamma \mathbb{L}^{(N)}(\gamma)\mu_0(d\gamma)\right) < \infty,$$

where the latter integral is finite because of (37). Therefore, the evolution of states $\{\mu_t\}_{t\ge 0} \subset \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, which means that there exists a Markov evolution of corresponding correlation measures (or corresponding correlation functions) on $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$ associated with the generator $L$. The fulfilment of (36) is now obvious because of the Markov property of the semigroup which corresponds to the evolution of states. ■

**Acknowledgements**. The financial support of DFG through the SFB 701 (Bielefeld University) and German-Ukrainian Project 436 UKR 113/80 and 436 UKR 113/94 is gratefully acknowledged. This work was partially supported by FCT, POCI2010, FEDER.

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