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Correlation Functions for the Heisenberg XXZ-Antiferromagnet

A. G. Izergin[1] and V. E. Korepin[2]

1 Leningrad University, SU-199164 Leningrad, USSR
2 Leningrad Department of V. A. Steklow Mathematical Institute, Fontanka 27, SU-191011 Leningrad, USSR

Abstract. The general method for calculation of correlation functions in integrable quantum models has been given in papers [1, 2]. The correlation function of the third components of local spins for the Heisenberg one-dimensional XXZ-antiferromagnet is calculated in this paper. The answer is a series which gives, in particular, an improved version of the usual perturbative expansion in the anisotropy parameter. The remarkable property of the series obtained is that the long-distance asymptotics of the correlator is given already by the first term. The arguments are given in favour of the convergence of the series.

1. Introduction

The present paper deals with the problem of calculation of correlation functions for the Heisenberg XXZ-model. This model describes an interaction of spins 1/2 located at the sites of one-dimensional lattice, the spin vector at the m^{th} site being

$$\sigma^{(m)} = \{\sigma_j^{(m)}; j = 1, 2, 3; m = 1, 2, \ldots, m\}.$$

Here $\sigma^{(m)}$ are Pauli matrices with usual commutation relations:

$$[\sigma_j^{(m)}, \sigma_k^{(n)}] = 2i\delta_{mn}\varepsilon_{jkl}\sigma_l^{(m)};$$

M denotes the complete number of the sites. The Hamiltonian $\mathscr{H}$ of the model (in the presence of constant magnetic field H directed along the third axis) is written in the following form:

$$\mathscr{H} = -\sum_{m=1}^{M} \{\sigma_1^{(m)}\sigma_1^{(m+1)} + \sigma_2^{(m)}\sigma_2^{(m+1)} + \Delta(\sigma_3^{(m)}\sigma_3^{(m+1)} - 1)\} + H\sum_{m=1}^{M}(1 - \sigma_3^{(m)}). \tag{1.1}$$

The periodical boundary conditions are supposed to be imposed ($\sigma^{(M+1)} \equiv \sigma^{(1)}$). Parameter Δ describes the internal anisotropy of the model. After Bethe [3] constructed eigenfunctions for the isotropic XXX-model ($\Delta = \pm 1$), the ground state and the spectrum of excitations of Hamiltonian (1.1) were found for every Δ [4, 5, 6]. At $\Delta \geqq 1$ the ground state of $\mathscr{H}$ is the ferromagnetic state $|0\rangle$ (all spins up at $H \geqq 0$), and the model describes the one-dimensional ferromagnet. At $\Delta < 1$ the ground eigenstate $|\Omega\rangle$ is of a more complicated structure having zero magnetization ($\langle \Omega|\sigma_3^{(m)}|\Omega\rangle = 0$) at $H = 0$. Thus the model describes the one-dimensional antiferromagnet at $\Delta < 1$. At $\Delta = 0$ one has the XX-model which corresponds to free fermions.

In this paper zero-temperature equal-time correlation function $\langle \sigma_3^{(m)} \sigma_3^{(1)} \rangle$:

$$\langle \sigma_3^{(m)} \sigma_3^{(1)} \rangle \equiv \langle \Omega|\sigma_3^{(m)} \sigma_3^{(1)}|\Omega\rangle / \langle \Omega|\Omega\rangle \tag{1.2}$$

is calculated at the thermodynamical limit ($M \to \infty$). The answer is obtained for the following values of Δ and H:

$$0 \leqq \Delta < 1 ; \quad 0 \leqq H < 2(1 - \Delta) . \tag{1.3}$$

(It should be mentioned that for the ferromagnetic case ($\Delta \geqq 1$) the answer is trivial: $\langle \sigma_3^{(m)} \sigma_3^{(1)} \rangle = \langle \sigma_3^{(1)} \rangle^2 = 1$.) In region (1.3) of parameters ground state $|\Omega\rangle$ is not the ferromagnetic one; that makes the problem rather complicated. To solve it we use the approach to calculation of equal-time correlation functions for integrable models suggested in papers [1, 2]. It is based on the quantum inverse scattering method (QISM) [7] applied to the Heisenberg XXZ-model in paper [8] (correlation functions for the XXZ-ferromagnet ($\Delta \geqq 1$) in the frame of QISM were discussed in [9]).

In papers [1, 2] the correlation function of currents for the nonrelativistic Bose-gas with repulsive delta-function interaction was obtained. This is an example of the completely integrable model with the R-matrix of the XXX-type. The Heisenberg XXZ-model which is discussed in more detail in Sect. 2 is an example of the model with the XXZ R-matrix [8]. The necessary generalization of the approach to these models (including also the sine-Gordon model which will be considered in a separate paper) is made in Sects. 3–5. In the rest of the paper these results are used to calculate correlator (1.2), (1.3) at the thermodynamical limit. The answer is the series which gives, in particular, an improved version of the usual expansion in the anisotropy parameter around the point $\Delta = 0$ (which corresponds to the XX-model, or free fermions). At $\Delta = 0$ the series is reduced to the first term giving at $H = 0$ the known answer [10, 11] for correlator (1.2). It is highly probable that the series obtained is convergent within region (1.3) of parameters. The remarkable property of the series is that the first term already gives the correct asymptotics of correlator (1.2) and (1.3):

$$\langle \sigma_3^{(m)} \sigma_3^{(1)} \rangle - \langle \sigma_3^{(m)} \rangle \langle \sigma_3^{(1)} \rangle \sim 1/m^2 \quad (m \to \infty)$$

(normalized meanvalue $\langle \sigma_3^{(m)} \rangle = \langle \sigma_3^{(1)} \rangle$; for $H = 0$ it is equal to zero). Thus for $H = 0$ the asymptotics obtained is in agreement (for $0 \leqq \Delta < 1$) with the result of papers [12, 13].

In this paper we calculate the zero-temperature correlation function. The approach can be generalized to calculate the correlation function at finite temperature which gives the temperature dependence of the correlation radius. These results will be published later.

2. Heisenberg *XXZ*-Antiferromagnet

Before the generalization of the results of papers [1, 2] to the *XXZ*-case let us state the problem of calculation of equal-time correlation functions for the *XXZ*-antiferromagnet in more detail. It is convenient to go from anisotropy parameter Δ in (1.1) to the "coupling constant" η:

$$\Delta = \cos 2\eta \tag{2.1}$$

so that the region of parameter (1.3) is now

$$0 < 2\eta \leqq \pi/2\,; \qquad 0 \leqq H < 4\sin^2\eta\,. \tag{2.2}$$

The value $2\eta = 0$ corresponds to the isotropic ferromagnet, and $2\eta = \pi/2$ to the XX-model (i.e. to free fermions).

The simplest of the eigenstates of Hamiltonian $\mathscr{H}$ (1.1) [3, 4] is the ferromagnetic (all spins up) state $|0\rangle$:

$$|0\rangle = \bigotimes_{m=1}^{M} |\uparrow\rangle_m \qquad (\sigma_3^{(m)}|\uparrow\rangle_m = |\uparrow\rangle_m)\,; \qquad \mathscr{H}|0\rangle = 0\,. \tag{2.3}$$

Other eigenstates $|\psi_N(\lambda_1, \ldots, \lambda_N)\rangle$ $(N = 1, 2, 3, \ldots)$ can be obtained by putting "particles" with "rapidities" $\lambda_1, \ldots, \lambda_N$ into the ferromagnetic state. These rapidities are complex numbers which should satisfy the following system of equations

$$\begin{aligned} l_0^M(\lambda_j) &= \prod_{\substack{k=1\\k\neq j}}^{N} \exp\{i\Phi(\lambda_j, \lambda_k)\} \quad (j = 1, \ldots, N)\,; \\ l_0(\lambda) &= \sinh(\lambda - i\eta)/\sinh(\lambda + i\eta)\,; \\ \Phi(\lambda, \mu) &= i\ln[\sinh(\lambda - \mu + 2i\eta)/\sinh(\lambda - \mu - 2i\eta)]\,. \end{aligned} \tag{2.4}$$

The momentum and the energy of the state thus obtained are the sums of momenta $p(\lambda_j)$ and energies $h(\lambda_j)$ of individual particles where functions $p(\lambda)$ and $h(\lambda)$ are

$$p(\lambda) = i\ln l_0(\lambda)\,(-2\eta \leqq p \leqq 2\eta \text{ for } \operatorname{Im}\lambda = 0, \pi/2)\,, \tag{2.5}$$

$$\begin{aligned} h(\lambda) &= -2\sin 2\eta(dp/d\lambda) + 2H \\ &= 2\sin^2 2\eta[\sinh(\lambda + i\eta)\sinh(\lambda - i\eta)]^{-1} + 2H\,. \end{aligned} \tag{2.6}$$

One can also define operator $\mathbf{Q}$ of the complete number of these particles as the sum of operators $\mathbf{q}_m$ of the number of particles at the m^{th} site:

$$\mathbf{Q} = \sum_{m=1}^{M} \mathbf{q}_m\,; \qquad \mathbf{q}_m = \tfrac{1}{2}(1 - \sigma_3^{(m)})\,; \qquad [\mathbf{Q}, \mathscr{H}] = 0\,. \tag{2.7}$$

The pseudovacuum is the state without particles ($\mathbf{Q}|0\rangle = 0$); the number of particles at state ψ_N is equal to N. It should be emphasized that the ferromagnetic state $|0\rangle$ (2.3) is not a ground state in case (2.2).

We will consider the *XXZ*-model at the thermodynamical limit ($M \to \infty$). At this limit the values of λ's allowed by system (2.4) include elementary particles with $\operatorname{Im}\lambda = 0$ or $\operatorname{Im}\lambda = \pi/2$, as well as their bound states which were described in detail in paper [6]. The ground state $|\Omega\rangle$ of the Hamiltonian (which will be called the physical vacuum) is obtained in case (2.2) by filling the ferromagnetic state $|0\rangle$ with elementary particles with $\mathrm{Im}\,\lambda=\pi/2$ (only these vacuum particles appear to be of importance for calculation of correlators). We'll write rapidities for vacuum particles in the following form:

$$\lambda=s+i\pi/2\,; \qquad \mathrm{Im}\,s=0\,. \tag{2.8}$$

At $M\to\infty$ the number N of particles in physical vacuum $|\Omega\rangle$ also goes to infinity, the ratio N/M remaining finite. The values of rapidities for vacuum particles fill the Fermi zone $-\Lambda\leqq s\leqq\Lambda$ with the density $\varrho(s)$ [$M\varrho(s)ds$ is equal to the number of particles having rapidities between s and $(s+ds)$]. System (2.4) turns into the linear integral equation for $\varrho(s)$

$$2\pi\varrho(s)-(\mathfrak{K}\varrho)(s)=u(s+i\pi/2)\,, \tag{2.9}$$

where operator $\mathfrak{K}$ acts as follows:

$$(\mathfrak{K}\varrho)(s)=\int\limits_{-\Lambda}^{\Lambda} K(s,t)\varrho(t)dt\,. \tag{2.10}$$

Functions $K(\lambda,\mu)$ and $u(\lambda)$ are

$$\begin{aligned}K(\lambda,\mu)&=K(\lambda-\mu)=\partial\Phi(\lambda,\mu)/\partial\lambda\\ &=\sin 4\eta[\sinh(\lambda-\mu+2i\eta)\sinh(\lambda-\mu-2i\eta)]^{-1}\,;\end{aligned} \tag{2.11}$$

$$\begin{aligned}u(\lambda)&=dp/d\lambda=i\partial\ln l_0(\lambda)/\partial\lambda\\ &=-\sin 2\eta[\sinh(\lambda+i\eta)\sinh(\lambda-i\eta)]^{-1}\,.\end{aligned} \tag{2.12}$$

Notice that at $0<2\eta<\pi/2$ (2.2) functions $K(s,t)$ and $u(s+i\pi/2)$ are positive ($\mathrm{Im}\,s=\mathrm{Im}\,t=0$). "Fermi rapidity" $\Lambda>0$ is uniquely defined by the requirement of the minimization of vacuum energy E_v [14, 6]:

$$E_v/M=\int\limits_{-\Lambda}^{\Lambda} h(s+i\pi/2)\varrho(s)ds\,; \tag{2.13}$$

at a weak magnetic field Λ is given by:

$$\Lambda=[(\pi-2\eta)/\pi]\ln[(8\eta\sin 2\eta)/((\pi-2\eta)H)]\;(H\to 0)\,. \tag{2.14}$$

It should be also noted that at the thermodynamical limit one has

$$0<N/M=\int\limits_{-\Lambda}^{\Lambda}\varrho(s)ds\leqq 1/2\,. \tag{2.15}$$

As $\Lambda=\infty$ at $H=0$, one can easily solve equation for $\varrho(s)$:

$$\begin{aligned}&\varrho(s)=\{2(\pi-2\eta)\cosh[\pi s/(\pi-2\eta)]\}^{-1}\,; \qquad N/M=1/2\\ &(H=0,\Lambda=\infty)\,.\end{aligned} \tag{2.16}$$

So we have described the ground state of $\mathscr{H}$ (1.1) in region (2.2).

Turn now to correlator $\langle\sigma_3^{(m)}\sigma_3^{(1)}\rangle$ (1.2) [due to the translation invariance correlator $\langle\sigma_3^{(m)}\sigma_3^{(n)}\rangle$ depends on $(m-n)$ only and is easily reduced to (1.2)]. It is convenient to express it in terms of correlator $\langle\mathbf{q}_m\mathbf{q}_1\rangle$ of local operators of number of particles $\mathbf{q}_m$, $\mathbf{q}_1$ given by (2.7):

$$\langle\sigma_3^{(m)}\sigma_3^{(1)}\rangle=4\langle\mathbf{q}_m\mathbf{q}_1\rangle-4\langle\mathbf{q}_1\rangle+1\,. \tag{2.17}$$

These quantities are well-known at $m=1,2$ (see, e.g., [15])

$$\langle \mathbf{q}_1^2 \rangle = \langle \mathbf{q}_m^2 \rangle = \langle \mathbf{q}_m \rangle = \int\limits_{-\Delta}^{\Delta} \varrho(s) ds ;$$
$$\langle \sigma_3^{(m)} \rangle = 1 - 2\langle \mathbf{q}_m \rangle ; \qquad \langle \sigma_3^{(m)2} \rangle = 1 \tag{2.18}$$

$$\langle \mathbf{q}_2 \mathbf{q}_1 \rangle = \langle \mathbf{q}_{m+1} \mathbf{q}_m \rangle = -\frac{1}{4} + \langle \mathbf{q}_m \rangle - \frac{1}{4} \frac{\partial}{\partial \Delta} \left(\frac{E_v}{M} \right) ;$$
$$\langle \sigma_3^{(2)} \sigma_3^{(1)} \rangle = \langle \sigma_3^{(m+1)} \sigma_3^{(m)} \rangle = -\frac{\partial}{\partial \Delta} \left(\frac{E_v}{M} \right) \quad (\Delta \equiv \cos 2\eta) . \tag{2.19}$$

Here E_v is vacuum energy (2.13).

To calculate the correlator at $m \geqq 3$ we apply the approach of papers [1, 2]. To do this one introduces operator $\mathbf{Q}_1(m)$ of the number of particles at the first m sites of the lattice:

$$\mathbf{Q}_1(m) = \sum_{j=1}^{m} \mathbf{q}_j . \tag{2.20}$$

This operator is quite analogous to operator $\mathbf{Q}_1$ of papers [1, 2] [see (1.9) in [1]]. Correlator $\langle \mathbf{q}_m \mathbf{q}_1 \rangle$ is expressed in terms of normalized meanvalues of this operator:

$$\langle \mathbf{q}_m \mathbf{q}_1 \rangle = D^{(2)} \langle \mathbf{Q}_1^2(m) \rangle / 2 = (D^{(2)}/2) (\langle \Omega | \mathbf{Q}_1^2(m) | \Omega \rangle / \langle \Omega | \Omega \rangle) \quad (m \geqq 3) , \tag{2.21}$$

where $D^{(2)}$ is the second derivative operator on the lattice and acts on functions $f(m)$ (m-integer) as follows

$$(D^{(2)} f)(m) = f(m) + f(m-2) - 2f(m-1) . \tag{2.22}$$

Turn now to correlator (1.2) which is naturally represented in the following form:

$$\langle \sigma_3^{(m)} \sigma_3^{(1)} \rangle \equiv \langle \sigma_3^{(m)} \rangle \langle \sigma_3^{(1)} \rangle + \langle\!\langle \sigma_3^{(m)} \sigma_3^{(1)} \rangle\!\rangle \tag{2.23}$$

The first term at the right hand side is equal [see (2.18)] to

$$\langle \sigma_3^{(m)} \rangle \langle \sigma_3^{(1)} \rangle = \langle \sigma_3^{(1)} \rangle^2 = \left(1 - 2 \int\limits_{-\Delta}^{\Delta} \varrho(s) ds \right)^2 ; \tag{2.24}$$

it is simply the square of the mean magnetization. The nontrivial part of the correlator can be expressed by means of (2.17), (2.21) as follows:

$$\langle\!\langle \sigma_3^{(m)} \sigma_3^{(1)} \rangle\!\rangle = 2D^{(2)} \langle \mathbf{Q}_1^2(m) \rangle - 4 \left(\int\limits_{-\Delta}^{\Delta} \varrho(s) ds \right)^2 . \tag{2.25}$$

So the problem of calculation of the correlator (1.2) is reduced to calculation of the meanvalue of operator $\mathbf{Q}_1^2(m)$ (2.20). To do this we introduce the two-site generalized model, as is explained in detail in paper [1]. The difference is that now one has to consider the model with the XXZ R-matrix. The necessary generalization is made in Sects. 3–5.

3. The Two-Site Generalized XXZ-Model

The main object in QISM (see, for example, [7]) is the monodromy matrix $T(\lambda)$ of the auxiliary linear problem. In our case it is a 2×2 matrix depending on complex spectral parameter λ:

$$T(\lambda)=\begin{pmatrix} A(\lambda); & B(\lambda) \\ C(\lambda); & D(\lambda)\end{pmatrix}. \tag{3.1}$$

The matrix elements of $T(\lambda)$ do not commute – they are "quantum operators." Their commutation relations are given by

$$R(\lambda,\mu)T(\lambda)\otimes T(\mu)=T(\mu)\otimes T(\lambda)R(\lambda,\mu), \tag{3.2}$$

where $R(\lambda,\mu)$ is the XXZ-model R-matrix:

$$R(\lambda,\mu)=\begin{pmatrix} f(\mu,\lambda) & 0 & 0 & 0 \\ 0 & g(\mu,\lambda) & 1 & 0 \\ 0 & 1 & g(\mu,\lambda) & 0 \\ 0 & 0 & 0 & f(\mu,\lambda)\end{pmatrix}. \tag{3.3}$$

Here functions f and g are

$$f(\lambda,\mu)=\frac{\sinh(\lambda-\mu+2i\eta)}{\sinh(\lambda-\mu)}; \qquad g(\lambda,\mu)=\frac{i\sin 2\eta}{\sinh(\lambda-\mu)}, \tag{3.4}$$

η is a "coupling constant."

For the Heisenberg XXZ-model the monodromy matrix (3.1) is constructed as a matrix product of local L-operators at the sites of the lattice

$$T_H(\lambda)=L_M(\lambda)L_{M-1}(\lambda)\dots L_1(\lambda)$$

[8]. To calculate correlator (1.2), however, it is sufficient to consider monodromy matrices with more simple internal structure. To do this let us introduce the two-site generalized model. It is a model with a monodromy matrix $T(\lambda)$, which is a matrix product of two monodromy matrices:

$$T(\lambda)=T_2(\lambda)T_1(\lambda); \tag{3.5}$$

$$T_i(\lambda)=\begin{pmatrix} A_i(\lambda); & B_i(\lambda) \\ C_i(\lambda); & D_i(\lambda)\end{pmatrix} \quad (i=1,2). \tag{3.6}$$

The matrix $T_1(\lambda)$ can be associated with the first site and $T_2(\lambda)$ with the second site of a lattice with two sites. Matrix elements of $T_i(\lambda)$ are quantum operators which commute at different sites of the lattice. Operators at the same site commute according to the rule (3.2). The monodromy matrix $T_i(\lambda)$ $(i=1,2)$ has the vacuum $|0\rangle_i$ – the state in quantum space with the following properties:

$$\begin{aligned} &C_i(\lambda)|0\rangle_i=0; \qquad A_i(\lambda)|0\rangle_i=a_i(\lambda)|0\rangle_i; \\ &D_i(\lambda)|0\rangle_i=d_i(\lambda)|0\rangle_i; \qquad B_i(\lambda)|0\rangle_i\neq 0. \end{aligned} \tag{3.7}$$

The state $|0\rangle=|0\rangle_2\otimes|0\rangle_1$ is the vacuum for $T(\lambda)$ (3.5), (3.1):

$$\begin{aligned} &C(\lambda)|0\rangle=0; \quad A(\lambda)|0\rangle=a(\lambda)|0\rangle; \quad D(\lambda)|0\rangle=d(\lambda)|0\rangle; \\ &a(\lambda)=a_1(\lambda)a_2(\lambda); \qquad d(\lambda)=d_1(\lambda)d_2(\lambda). \end{aligned} \tag{3.8}$$

Here $a_i(\lambda)$, $d_i(\lambda)$ are c-number functions which are defined by the choice of concrete models. The crucial point is that there exist monodromy matrices $T_i(\lambda)$ for arbitrary functions $a_i(\lambda)$, $d_i(\lambda)$ [16]. It is convenient to use the following notations:

$$l(\lambda)=a_1(\lambda)/d_1(\lambda)\,; \qquad m(\lambda)=a_2(\lambda)/d_2(\lambda)\,; \\ r(\lambda)=a(\lambda)/d(\lambda)=l(\lambda)m(\lambda)\,. \tag{3.9}$$

Different functions $l(\lambda)$ and $m(\lambda)$ correspond to essentially different models. Function $l(\lambda)$ will be the main free functional parameter in the two-site model. It occurs that the dependence of correlation functions on $l(\lambda)$ is rather simple and can be explicitly evaluated. It should be noted that for the Heisenberg XXZ-model functions $r(\lambda)$, $l(\lambda)$, $m(\lambda)$ are periodical functions of λ with the period equal to $i\pi$ [see (3.21)]; further it will be sufficient to consider them as arbitrary periodical functions with this period.

The trace of the monodromy matrix $\tau(\lambda)=A(\lambda)+D(\lambda)$ generates the Hamiltonians of completely integrable systems [see (3.20)]. Eigenfunctions of $\tau(\lambda)$ are of the form

$$|\psi_N(\lambda_1,\ldots,\lambda_N)\rangle=\prod_{j=1}^{N}\mathbb{B}(\lambda_j)|0\rangle\,, \tag{3.10}$$

where

$$\mathbb{B}(\lambda)\equiv B(\lambda)/d(\lambda)\,. \tag{3.11}$$

Here all the λ_j are different [17] and satisfy the system of transcendental equations (s.t.e.)

$$r_j\prod_{\substack{k=1\\k\neq j}}^{N}(f_{jk}/f_{kj})=1 \quad (j=1,\ldots,N)\,; \tag{3.12}$$

$f_{jk}\equiv f(\lambda_j,\lambda_k)$ and $r_j\equiv r(\lambda_j)$. The s.t.e. may be put into the form $\varphi_j=0(\mathrm{mod}\,2\pi)$, where

$$\varphi_j=i\ln r_j+\sum_{\substack{k=1\\k\neq j}}^{N}\Phi_{jk}\,. \tag{3.13}$$

Function $\Phi_{jk}\equiv\Phi(\lambda_j,\lambda_k)=i\ln(f_{jk}/f_{kj})$ here is the same as in (2.4). The corresponding eigenvalue of $\tau(\lambda)$ is

$$\tau(\lambda)|\psi_N(\lambda_1,\ldots,\lambda_N)\rangle=t_N(\lambda;\lambda_1,\ldots,\lambda_N)|\psi_N(\lambda_1,\ldots,\lambda_N)\rangle\,; \\ t_N=a(\lambda)\prod_{j=1}^{N}f(\lambda,\lambda_j)+d(\lambda)\prod_{j=1}^{N}f(\lambda_j,\lambda)\,. \tag{3.14}$$

The dual vacuum $\langle 0|={}_2\langle 0|\otimes{}_1\langle 0|$ satisfies relations

$$\langle 0|B(\lambda)=0\,; \qquad \langle 0|A(\lambda)=a(\lambda)\langle 0|\,; \qquad \langle 0|D(\lambda)=d(\lambda)\langle 0|\,.$$

We put also $\langle 0|0\rangle={}_i\langle 0|0\rangle_i=1$. The dual state

$$\langle\psi_N(\lambda_1,\ldots,\lambda_N)|=\langle 0|\prod_{j=1}^{N}\mathbb{C}(\lambda_j)\,; \qquad \mathbb{C}(\lambda)\equiv C(\lambda)/d(\lambda) \tag{3.15}$$

is an eigenstate of $\tau(\lambda)$; $\langle\psi_N|\tau(\lambda)=t_N\langle\psi_N|$ with the same eigenvalue (3.14) if the s.t.e. (3.12) is valid. The "norm" is equal to [18]:

$$\langle\psi_N(\lambda_1,\ldots,\lambda_N)|\psi_N(\lambda_1,\ldots,\lambda_N)\rangle=(\sin 2\eta)^N\left(\prod_{j\neq k} f_{jk}\right)\det_N(\varphi'), \tag{3.16}$$

where the $N\times N$-matrix φ' is defined as follows:

$$\varphi'_{jk}=\partial\varphi_j/\partial\lambda_k=\delta_{jk}\left(z_j+\sum_{l=1}^{N}K_{jl}\right)-K_{jk}. \tag{3.17}$$

Here $z_j\equiv z(\lambda_j)$; function $z(\lambda)$ is equal to

$$z(\lambda)\equiv i\partial\ln r(\lambda)/\partial\lambda. \tag{3.18}$$

Function $K_{jk}\equiv K(\lambda_j,\lambda_k)=\partial\Phi(\lambda_j,\lambda_k)/\partial\lambda_j$ was given in (2.11). Notice that eigenfunctions corresponding to different sets of λ_j are orthogonal due to different eigenvalues (3.14).

The operators of the number of particles will play an important role. Operators $\mathbf{Q}_i$ of the number of particles at the i^{th} site of the lattice $(i=1,2)$ and $\mathbf{Q}$ of the complete number of particles are defined as follows:

$$\begin{gathered}\mathbf{Q}_i\prod_{k=1}^{n}\mathbb{B}_i(\lambda_k)|0\rangle=n\prod_{k=1}^{n}\mathbb{B}_i(\lambda_k)|0\rangle\quad(i=1,2);\\ \langle 0|\prod_{k=1}^{n}\mathbb{C}_i(\lambda_k)\mathbf{Q}_i=n\langle 0|\prod_{k=1}^{n}\mathbb{C}_i(\lambda_k)\quad(i=1,2);\\ \mathbf{Q}=\mathbf{Q}_1+\mathbf{Q}_2;\qquad \mathbf{Q}|0\rangle=\mathbf{Q}_i|0\rangle=0;\qquad[\mathbf{Q},\tau(\lambda)]=0.\end{gathered} \tag{3.19}$$

Here λ_k are arbitrary and are not supposed to satisfy s.t.e. (3.12).

Let us discuss the connection of the two-site model with the Heisenberg XXZ-model. The monodromy matrix $T_H(\lambda)$ of this model is constructed in a standard way by means of local L-operators. Operators A,B,C,D in (3.1) and (3.6) are functions of spins $\sigma^{(m)}$ and act in the tensor product of local spin spaces. The Hamiltonian $\mathscr{H}$ (1.1) is expressed in terms of $\tau_H(\lambda)=A(\lambda)+D(\lambda)$ by means of the trace identities [8]:

$$\mathscr{H}=2i\sin 2\eta\frac{\partial}{\partial\mu}\ln\tau_H(\mu)|_{\mu=-i\eta}+2M\cos 2\eta+2H\mathbf{Q}, \tag{3.20}$$

so that $\Delta=\cos 2\eta$ is indeed the anisotropy parameter in $\mathscr{H}$. All the formulae of the generalized two-site model are valid also for the Heisenberg XXZ-model provided that one considers $T_1(\lambda)$ in (3.5) as the monodromy matrix from the first site to the m^{th} site of the lattice, and $T_2(\lambda)$ – as the monodromy matrix from the $(m+1)^{\text{th}}$ to the M^{th} site. Functions $a(\lambda)$, $d(\lambda)$, $r(\lambda)$, $l(\lambda)$, and $m(\lambda)$ (3.9) in the Heisenberg model are equal to

$$\begin{gathered}a(\lambda)=a_0^M(\lambda);\qquad d(\lambda)=d_0^M(\lambda);\\ r(\lambda)=l_0^M(\lambda);\qquad l(\lambda)=l_0^m(\lambda);\qquad m(\lambda)=l_0^{M-m}(\lambda).\end{gathered} \tag{3.21}$$

Here $a_0(\lambda)=\sinh(\lambda-i\eta)$, $d_0(\lambda)=\sinh(\lambda+i\eta)$ and function

$$l_0(\lambda)=a_0(\lambda)/d_0(\lambda)$$

is just the same as given in (2.4). Function $z(\lambda)$ (3.18) is expressed in terms of $u(\lambda)$ (2.12):

$$z(\lambda)=i\partial\ln(l_0^M(\lambda))/\partial\lambda=Mu(\lambda)\,. \tag{3.22}$$

Operators $\mathbf{Q}$ and $\mathbf{Q}_1$ (3.19) turn into operators $\mathbf{Q}$ (2.7) and $\mathbf{Q}_1(m)$ (2.20) of the Heisenberg model. The pseudovacuum $|0\rangle$, (3.7) and (3.8) corresponds to the ferromagnetic state (2.3), and the dual pseudovacuum $\langle 0|$ is hermitian conjugated to it: $\langle 0|=|0\rangle^+$. S.t.e. (3.12) goes to system (2.4). State vectors (3.10), (3.15) are thus eigenvectors of $\mathscr{H}$ (1.1). Operators $B(\lambda)$ and $C(\lambda)$ in (3.1) for the Heisenberg model have the involution $C(\lambda)=-B^+(\lambda^*)$; so (3.16) really gives the norm of the wave function [18].

We are interested in correlator (1.2) which is expressed in terms of the vacuum mean value of operator $\mathbf{Q}_1^2$ (2.25). This operator is defined also in the two-site model (3.19) where its matrix elements are easier to calculate due to the arbitrary functional parameter $l(\lambda)$ (3.9). Below we study the mean value $\langle\psi_N|\mathbf{Q}_1^2|\psi_N\rangle$ in the two-site model (Sects. 4 and 5) at first. The proofs of the majority of the results obtained in these sections are similar to those for the XXX-case [1, 2], so we don't give the details of the proofs. The details of the proofs which are different from the XXX-case are given in Appendices 1 and 2.

3. The Mean Value of Operator $\mathbf{Q}_1^2$ and Irreducible Parts

It is convenient to begin with the "generating operator" $\exp\{\alpha\mathbf{Q}_1\}$, α being an arbitrary complex number (operator $\mathbf{Q}_1^2$ is obtained by double differentiation with respect to α at $\alpha=0$). The starting point is the following formula for matrix elements of this operator which is valid in the two-site model, which is proved in [1]:

$$\begin{aligned}\mathbb{M}_N^\alpha &\equiv \langle 0|\prod_{j=1}^{N}\mathbb{C}(\lambda_j^C)\exp\{\alpha\mathbf{Q}_1\}\prod_{k=1}^{N}\mathbb{B}(\lambda_k^B)|0\rangle\\
&=\sum_{\{\lambda^B\}=\{\lambda_{\mathrm{I}}^B\}\cup\{\lambda_{\mathrm{II}}^B\}}\ \sum_{\{\lambda^C\}=\{\lambda_{\mathrm{I}}^C\}\cup\{\lambda_{\mathrm{II}}^C\}}\exp\{\alpha n_1\}\\
&\quad\cdot\langle 0|\prod_{\mathrm{I}}\mathbb{C}_1(\lambda_{\mathrm{I}}^C)\prod_{\mathrm{I}}\mathbb{B}_1(\lambda_{\mathrm{I}}^B)|0\rangle\ \langle 0|\prod_{\mathrm{II}}\mathbb{C}_2(\lambda_{\mathrm{II}}^C)\prod_{\mathrm{II}}\mathbb{B}_2(\lambda_{\mathrm{II}}^B)|0\rangle\\
&\quad\cdot\left(\prod_{\mathrm{I}}m(\lambda_{\mathrm{I}}^B)\right)\left(\prod_{\mathrm{II}}l(\lambda_{\mathrm{II}}^C)\right)\left(\prod_{\mathrm{I,II}}f(\lambda_{\mathrm{I}}^B,\lambda_{\mathrm{II}}^B)\right)\left(\prod_{\mathrm{I,II}}f(\lambda_{\mathrm{II}}^C,\lambda_{\mathrm{I}}^C)\right).\end{aligned} \tag{4.1}$$

Here the sum is taken over all the partitions of the set $\{\lambda_j^B; j=1,\dots,N\}$ into two disjoint subsets $\{\lambda_{\mathrm{I}}^B\}$ and $\{\lambda_{\mathrm{II}}^B\}$ and over similar partitions of the set $\{\lambda^C\}$. These partitions are independent except that $\mathrm{card}\{\lambda_{\mathrm{I}}^B\}=\mathrm{card}\{\lambda_{\mathrm{I}}^C\}=n_1$; $\mathrm{card}\{\lambda_{\mathrm{II}}^B\}=\mathrm{card}\{\lambda_{\mathrm{II}}^C\}=n_2=N-n_1$. Product $\prod_{\mathrm{I}}$ denotes the product over all the $\lambda\in\{\lambda_{\mathrm{I}}\}$, and thus contains n_1 factors. Product $\prod_{\mathrm{I,II}}$ denotes double product over all $\lambda\in\{\lambda_1\}$ and over all $\lambda\in\{\lambda_{\mathrm{II}}\}$ and contains n_1n_2 factors. Notice that values of λ^C,λ^B in (4.1) can be quite arbitrary; we suppose only that $\lambda_j^B\neq\lambda_k^B$ $(j\neq k)$ and $\lambda_j^C\neq\lambda_k^C$ $(j\neq k)$ (see [17]). Operators $\mathbb{B}$ and $\mathbb{C}$ are defined in (3.11) and (3.15).

Formula (4.1) expresses $\mathbb{M}_N^\alpha$ in terms of "scalar products"

$$\langle 0|\prod \mathbb{C}_i(\lambda^C)\prod \mathbb{B}_i(\lambda^B)|0\rangle \quad (i=1,2).$$

These scalar products were investigated in detail in paper [18]. Their properties necessary to prove the following results are given in Appendix 1. The independent variables which scalar products depend on are also discussed in detail there [see (A.7)]. One then concludes that $\mathbb{M}_N^\alpha$ depends on $6N$ independent complex arguments

$$\mathbb{M}_N^\alpha \equiv \mathbb{M}_N^\alpha(\{\lambda^C\}_N, \{\lambda^B\}_N, \{l^C\}_N, \{l^B\}_N, \{m^C\}_N, \{m^B\}_N). \tag{4.2}$$

The variables $l_j^{B,C}$ and $m_j^{B,C}$ introduced here are the values of arbitrary functions $l(\lambda)$ and $m(\lambda)$ (3.9) at points $\lambda_j^{B,C}$ [compare with (A.6)]:

$$l_j^{B,C} \equiv l(\lambda_j^{B,C}); \qquad m_j^{B,C} \equiv m(\lambda_j^{B,C}). \tag{4.3}$$

Due to commutation relations (3.2): $[\mathbb{C}(\lambda), \mathbb{C}(\mu)] = [\mathbb{B}(\lambda), \mathbb{B}(\mu)] = 0$. Hence $\mathbb{M}_N^\alpha$ is a symmetric function with respect to replacement of triples

$$(\lambda_j^C, l_j^C, m_j^C) \leftrightarrow (\lambda_k^C, l_k^C, m_k^C)$$

and with respect to $(\lambda_j^B, l_j^B, m_j^B) \leftrightarrow (\lambda_k^B, l_k^B, m_k^B)$. The main property of $\mathbb{M}_N^\alpha$ is that it has first-order poles at $\lambda_j^C \to \lambda_k^B$, the residue being expressed in terms of $\mathbb{M}_{N-1}^\alpha$. If $\lambda_N^C \to \lambda_N^B \to \lambda_N$ and all the other variables in (4.2) are fixed one has for (4.1) (other possibilities are easily restored from the symmetry):

$$\begin{aligned}
\mathbb{M}_N^\alpha|_{\lambda_N^C \to \lambda_N^B} &= g(\lambda_N^C, \lambda_N^B)\,[l_N^C m_N^B - r_N^B]\,e^\alpha \left(\prod_{j=1}^{N-1} f_{Nj}^C f_{Nj}^B\right) \\
&\cdot \mathbb{M}_{N-1}^\alpha(\{\lambda_j^C\}_{N-1}, \{\lambda_j^B\}_{N-1}, \{\tilde{l}_j^C\}_{N-1}, \{\tilde{l}_j^B\}_{N-1}, \{m_j^C\}_{N-1}, \{m_j^B\}_{N-1}) \\
&+ g(\lambda_N^C, \lambda_N^B)\,[r_N^C - l_N^C m_N^B] \left(\prod_{j=1}^{N-1} f_{Nj}^C f_{Nj}^B\right) \\
&\cdot \mathbb{M}_{N-1}^\alpha(\{\lambda_j^C\}_{N-1}, \{\lambda_j^B\}_{N-1}, \{l_j^C\}_{N-1}, \{l_j^B\}_{N-1}, \{\tilde{m}_j^C\}_{N-1}, \{\tilde{m}_j^B\}_{N-1}).
\end{aligned} \tag{4.4}$$

Here $f_{Nj}^{C,B} \equiv f(\lambda_N, \lambda_j^{C,B})$ and modification of l, m is defined similar to (A.9)

$$\tilde{l}_j^{C,B} = l_j^{C,B}(f_{jN}^{C,B}/f_{Nj}^{C,B}); \qquad \tilde{m}_j^{C,B} = m_j^{C,B}(f_{jN}^{C,B}/f_{Nj}^{C,B}). \tag{4.5}$$

Formula (4.4) is proved using Eqs. (4.1) and (A.8).

To get the mean value of operator $\exp\{\alpha \mathbf{Q}_1\}$ with respect to eigenfunctions (3.10) and (3.15) two steps have to be taken.

(i) The first step is to take the limit $\lambda_j^C \to \lambda_j^B \to \lambda_j$ ($j=1,2,\dots,N$; all λ_j being different). At the physically interesting models, variables l_j, m_j (4.3) are values of smooth functions $l(\lambda)$ and $m(\lambda)$ [see, e.g. (3.21)]. Thus the residue at the pole in (4.4) is equal to zero and the corresponding limit is finite. The dependence of matrix element (4.1) on vacuum eigenvalues at points λ_j is then represented in terms of variables l_j, m_j,

$$l_j \equiv l(\lambda_j); \qquad m_j \equiv m(\lambda_j) \quad (j=1,\dots,N), \tag{4.6}$$

and of variables x_j, y_j,

$$x_j \equiv x(\lambda_j); \qquad y_j \equiv y(\lambda_j) \quad (j=1,\dots,N), \tag{4.7}$$

where we denote:

$$x(\lambda)=i\partial\ln l(\lambda)/\partial\lambda\,;\qquad y(\lambda)=i\partial\ln m(\lambda)/\partial\lambda\,. \tag{4.8}$$

So in this case $\mathbb{M}_N$ depends on $5N$ complex variables

$$\mathbb{M}_N^\alpha|_{\lambda_j^C=\lambda_j^B}\equiv M_N^\alpha(\{\lambda\}_N,\{x\}_N,\{y\}_N,\{l\}_N,\{m\}_N) \tag{4.9}$$

[it is to be compared with (A.10)]. The dependence of M_N^α on each x_j, y_j is linear. As variables z_j (3.18) are equal to $z_j=x_j+y_j$, they are not independent variables.

(ii) To make the second step one has to impose s.t.e. (3.12) on λ_j in (4.9). After this is done one can express variables m_j in terms of other variables due to (3.9). Thus the mean value $\langle\psi_N|\exp\{\alpha\mathbf{Q}_1\}|\psi_N\rangle$ is the function of $4N$ complex variables [cf. with (A.12)]:

$$\begin{aligned}&\langle\psi_N(\{\lambda\}_N)|\exp\{\alpha\mathbf{Q}_1\}|\psi_N(\{\lambda\}_N)\rangle\\&\equiv\mathcal{M}_N^\alpha(\{\lambda\}_N,\{x\}_N,\{y\}_N,\{l\}_N)\\&=M_N^\alpha\left(\{\lambda\}_N,\{x\}_N,\{y\}_N,\{l\}_N,\left\{m_j=l_j^{-1}\prod_{k\neq j}(f_{kj}/f_{jk})\right\}_N\right).\end{aligned} \tag{4.10}$$

The dependence of $\mathcal{M}_N^\alpha$ on each x_j, y_j is linear; using (4.4) one has:

$$\begin{aligned}\partial\mathcal{M}_N^\alpha/\partial x_N=&(\sin2\eta)\exp\{\alpha\}\left(\prod_{j=1}^{N-1}f_{Nj}f_{jN}\right)\\&\cdot\mathcal{M}_{N-1}^\alpha(\{\lambda_j\}_{N-1},\{\tilde{x}_j\}_{N-1},\{y_j\}_{N-1},\{\tilde{l}_j\}_{N-1}),\end{aligned} \tag{4.11}$$

$$\begin{aligned}\partial\mathcal{M}_N^\alpha/\partial y_N=&(\sin2\eta)\left(\prod_{j=1}^{N-1}f_{Nj}f_{jN}\right)\\&\cdot\mathcal{M}_{N-1}^\alpha(\{\lambda_j\}_{N-1},\{x_j\}_{N-1},\{\tilde{y}_j\}_{N-1},\{l_j\}_{N-1}).\end{aligned} \tag{4.12}$$

At the right-hand side here l is modified to $\tilde{l}$ according to the rule (4.5) and the modification of x, y to $\tilde{x}, \tilde{y}$ is made similar to (A.14):

$$\tilde{x}_j=x_j+K_{jN}\,;\qquad\tilde{y}_j=y_j+K_{jN}\quad(j=1,\dots,N-1)\,, \tag{4.13}$$

where $K_{jN}\equiv K(\lambda_j,\lambda_N)$ is defined in (2.11).

Formulae (4.11) and (4.12) are of importance because they give an opportunity to restore meanvalue $\mathcal{M}_N$ (4.10) in terms of values of $\mathcal{M}_n|_{x_j=y_j=0}$ $(j=1,\dots,n\,;\,n\leqq N)$ [2], which is the main result of the analysis of the mean value in the two-site model. To do this one has to introduce irreducible parts. Irreducible part I_N^α of mean value $\langle\psi_N|\exp\{\alpha\mathbf{Q}_1\}|\psi_N\rangle$ (4.10) is defined as follows:

$$I_N^\alpha(\{\lambda\}_N,\{l\}_N)\equiv(\sin2\eta)^{-N}\left(\prod_{j\neq k}f_{jk}\right)^{-1}\mathcal{M}_N^\alpha|_{x_j=y_j=0\,(j=1,\dots,N)}\,, \tag{4.14}$$

the factor before $\mathcal{M}_N^\alpha$ being introduced for convenience. Mean values of powers of operator $\mathbf{Q}_1$ can be obtained by differentiation of the mean value (4.10):

$$\langle\psi_N|\mathbf{Q}_1^n|\psi_N\rangle=\partial^n\langle\psi_N|\exp\{\alpha\mathbf{Q}_1\}|\psi_N\rangle/\partial\alpha^n|_{\alpha=0}\,. \tag{4.15}$$

The irreducible part of $\langle\psi_N|\mathbf{Q}_1^n|\psi_N\rangle$ is $I_N^{(n)}$:

$$I_N^{(n)}(\{\lambda\}_N, \{l\}_N) = \partial^n I_N^\alpha / \partial\alpha^n|_{\alpha=0}. \tag{4.16}$$

It follows from (3.16)–(3.18) that the irreducible parts of the mean value of the unit operator are rather simple:

$$I_N^{(0)} = \delta_{0N}. \tag{4.17}$$

The irreducible parts $I_N^{(1)}$ are equal to zero as will be shown in Appendix 2:

$$I_N^{(1)} \equiv 0 \quad (N \geqq 0). \tag{4.18}$$

The irreducible parts $I_N^{(2)}$ of the N-particle mean values $\langle\psi_N|\mathbf{Q}_1^2|\psi_N\rangle$ are of great interest. Further we denote it simply $I_N^{(2)} \equiv I_N$, suppressing the superscript. Irreducible parts I_N do not vanish for $N \geqq 2$.

It should be mentioned that it is possible to give the definition of irreducible parts I_N directly in the Heisenberg XXZ-model as it was done for nonrelativistic Bose-gas in [1, 2]. At N small it is possible to calculate I_N directly from definitions (4.14) and (4.10) using commutation relations (3.2) (it is easier, however, to use for this purpose form factors, see Appendix 2). One gets after some algebra:

$$I_0 = I_1 = 0; \tag{4.19}$$

$$\begin{aligned} I_2(\{\lambda_1, \lambda_2\}, \{l_1, l_2\}) &= \mathscr{A}_2^1(\lambda_1, \lambda_2)\,[l(\lambda_1) l^{-1}(\lambda_2) - 1] \\ &\quad + \mathscr{A}_2^1(\lambda_2, \lambda_1)\,[l(\lambda_2) l^{-1}(\lambda_1) - 1]; \\ \mathscr{A}_2^1(\lambda_1, \lambda_2) &= -\frac{2}{\sinh^2\lambda_{12}} \frac{\sinh(\lambda_{12} + 2i\eta)}{\sinh(\lambda_{12} - 2i\eta)} \end{aligned} \tag{4.20}$$

(we denote $\lambda_{jk} \equiv \lambda_j - \lambda_k$) and also

$$\begin{aligned} & I_3(\{\lambda_1, \lambda_2, \lambda_3\}, \{l_1, l_2, l_3\}) \\ &\quad = \sum_P \mathscr{A}_3^1(\lambda_{P_1}, \lambda_{P_2}, \lambda_{P_3})\,[l(\lambda_{P_1}) l^{-1}(\lambda_{P_2}) - 1]. \end{aligned} \tag{4.21}$$

The sum here is taken over all permutations P of numbers $1, 2, 3$;

$$\begin{aligned} \mathscr{A}_3^1(\lambda_1, \lambda_2, \lambda_3) &= \frac{4\sin 4\eta}{\sinh^2\lambda_{12}} \frac{\sinh(\lambda_{12} + 2i\eta)}{\sinh(\lambda_{12} - 2i\eta)} \\ &\quad \cdot \frac{1}{\sinh(\lambda_{31} + 2i\eta)\sinh(\lambda_{23} + 2i\eta)} \left(\frac{\sinh\lambda_{32}}{\sinh\lambda_{31}} + \frac{\sinh\lambda_{31}}{\sinh\lambda_{32}}\right). \end{aligned} \tag{4.22}$$

To investigate general properties of I_N it is necessary to use the algebraic structure of QISM (see Appendix 2). As a result, I_N can be represented in the following form:

$$\begin{aligned} I_N(\{\lambda\}_N, \{l\}_N) &= \sum_{\{\lambda\} = \{\lambda_+\} \cup \{\lambda_-\} \cup \{\lambda_0\}}^{0 \leqq n \leqq [N/2]} \prod_{(+)}^{n} l(\lambda_+) \prod_{(-)}^{n} l^{-1}(\lambda_-) \\ &\quad \cdot \mathscr{A}_N^n(\{\lambda_+\}_n, \{\lambda_-\}_n, \{\lambda_0\}_{N-2n}). \end{aligned} \tag{4.23}$$

The sum here is taken over all the partitions of the set $\{\lambda\}_N$ into three disjoint subsets:

$$\operatorname{card}\{\lambda_+\}_n = \operatorname{card}\{\lambda_-\}_n = n\,;$$
$$\operatorname{card}\{\lambda_0\}_{N-2n} = N-2n\,;$$
$$0 \leqq n \leqq [N/2]\,.$$

Coefficients $\mathscr{A}_N^n$ are the Fourier coefficients of the irreducible part I_N. They do not depend on l_j but only on λ_j being a rational functions of $\exp(\lambda)$. Fourier coefficients depend on the R-matrix only and do not depend on the concrete model. All the dependence on concrete models enters through vacuum values $l(\lambda)$ and is written in (4.23) explicitly. The irreducible part has also the following important property [1]:

$$I_N(\{\lambda\}_N, \{l_1 = l_2 = \ldots = l_N = 1\}_N) = 0\,, \tag{4.24}$$

which permits to express coefficient $\mathscr{A}_N^0$ as a linear function of coefficients $\mathscr{A}_N^n$ ($1 \leqq n \leqq [N/2]$) [see, e.g., (4.20) and (4.21)]. Other properties of I_N proved in Appendix 2 are the following ones.

(i) The behavior in coupling constant η:

$$\begin{aligned} &I_N \sim \mathscr{A}_N^n \sim \eta^{N-2} && (\eta \to 0)\,,\\ &I_N \sim \mathscr{A}_N^n \sim \{(\pi/2) - \eta\}^{N-2} && (\eta \to \pi/2)\,,\\ &I_N \sim \mathscr{A}_N^n \sim \{(\pi/2) - 2\eta\}^{N-2} && (\eta \to \pi/4)\,. \end{aligned} \tag{4.25}$$

(ii) $\mathscr{A}_N^n$ are symmetric functions of $\lambda_j^{(+)}$, as well as symmetric functions of $\lambda_j^{(-)}$ and $\lambda_j^{(0)}$ (separately).

(iii) Under the replacement of $\{\lambda_+\} \leftrightarrow \{\lambda_-\}$ one has

$$\begin{aligned} &\mathscr{A}_N^n(\{\lambda_+\}, \{\lambda_-\}, \{\lambda_0\}) = [\mathscr{A}_N^n(\{\lambda_-\}, \{\lambda_+\}, \{\lambda_0\})]^*\,;\\ &\mathscr{A}_N^n(\{\lambda_+\}, \{\lambda_-\}, \{\lambda_0\}) = \Big(\prod_{(+),(-)} [f(\lambda_+, \lambda_-)/f(\lambda_-, \lambda_+)]\Big)\\ &\qquad \cdot \Big(\prod_{(0),(+)} f^{-1}(\lambda_0, \lambda_+)\Big)\Big(\prod_{(0),(-)} f^{-1}(\lambda_-, \lambda_0)\Big)\\ &\qquad \cdot a_N^n(\{\lambda_+\}, \{\lambda_-\}, \{\lambda_0\})\,;\\ &a_N^n(\{\lambda_+\}, \{\lambda_-\}, \{\lambda_0\}) = a_N^n(\{\lambda_-\}, \{\lambda_+\}, \{\lambda_0\})\,. \end{aligned} \tag{4.26}$$

5. The Representation of the Mean Value of Operator $\mathbf{Q}_1^2$ in Terms of Irreducible Parts

This representation is the main result of the analysis of the mean value in the two-site generalized model. Relations (4.11) and (4.12) being established, the proof is the same as in the XXX-case [2] and we only give the result.

Consider the normalized mean value of operator $\mathbf{Q}_1^2$ with respect to eigenfunctions ψ_N (3.10) and (3.15):

$$\begin{aligned} \langle \mathbf{Q}_1^2 \rangle_N &\equiv \langle \psi_N(\{\lambda\}_N) | \mathbf{Q}_1^2 | \psi_N(\{\lambda\}_N) \rangle / \langle \psi_N(\{\lambda\}_N | \psi_N(\{\lambda\}_N) \rangle\\ &= \langle \mathbf{Q}_1^2 \rangle_N(\{\lambda\}_N, \{x\}_N, \{y\}_N, \{l\}_N)\,. \end{aligned} \tag{5.1}$$

We have written the independent variables explicitly [see (3.16) and (4.10)]. The following representation is valid for mean value (5.1):

$$\langle \mathbf{Q}_1^2 \rangle_N = \langle \mathbf{Q}_1^2 \rangle_N^{(0)} + \sum_{n_I=2}^{N} \Gamma_{n_I,N} . \tag{5.2}$$

Define now the right-hand side of this formula.

(i) Quantity $\langle \mathbf{Q}_1^2 \rangle^{(0)}$ corresponds to the contribution of the irreducible part $I_N^{(0)}$ (4.17):

$$\langle \mathbf{Q}_1^2 \rangle_N^{(0)} = \frac{1}{\det_N(\varphi')} \sum_{\{\lambda\}_N = \{\lambda^x\}_{n_x} \cup \{\lambda^y\}_{n_y}} n_x^2 \det_{n_x}(\varphi_x') \det_{n_y}(\varphi_y') . \tag{5.3}$$

Here $\det_N(\varphi')$ is the determinant of the $N \times N$-matrix φ' (3.17). The sum is taken over all the partitions of set $\{\lambda\}_N$ into two disjoint subsets $\{\lambda^x\}_{n_x}$ and $\{\lambda^y\}_{n_y}$, so that $n_x + n_y = N$. Jacobians $\det_{n_x}(\varphi_x')$ and $\det_{n_y}(\varphi_y')$ are the determinants of the $n_x \times n_x$-matrix φ_x' and of the $n_y \times n_y$-matrix φ_y', which are defined similar to (3.17):

$$\begin{gathered}(\varphi_x')_{jk} = \delta_{jk}\left[x(\lambda_j^x) + \sum_{l=1}^{n_x} K(\lambda_j^x, \lambda_l^x) \right] - K(\lambda_j^x, \lambda_k^x) \\ (\lambda_j^x, \lambda_k^x, \lambda_l^x \in \{\lambda^x\}_{n_x}; j, k, l = 1, 2, \ldots, n_x) ;\end{gathered} \tag{5.4}$$

$$\begin{gathered}(\varphi_y')_{jk} = \delta_{jk}\left[y(\lambda_j^y) + \sum_{l=1}^{n_y} K(\lambda_j^y, \lambda_l^y) \right] - K(\lambda_j^y, \lambda_k^y) \\ (\lambda_j^y, \lambda_k^y, \lambda_l^y \in \{\lambda^y\}_{n_y}; j, k, l = 1, 2, \ldots, n_y) .\end{gathered} \tag{5.5}$$

So the first term at the right-hand side of (5.2) depends on variables x_j and y_j only and does not depend on l_j. It should be emphasized that

$$z(\lambda) = x(\lambda) + y(\lambda) , \tag{5.6}$$

so that $\det_N(\varphi')$ depends on the sum z_j of x_j and y_j only.

(ii) Quantities $\Gamma_{n_I,N}$ are defined in a more complicated way:

$$\Gamma_{n_I,N} = [\det_N(\varphi')]^{-1} \sum_{\{\lambda\}_N = \{\lambda^I\}_{n_I} \cup \{\lambda^v\}_{n_v}} \det_{n_v}(\varphi_v') I_{n_I,N}^d(\{\lambda^I\}_{n_I}, \{\lambda^v\}_{n_v}) . \tag{5.7}$$

The sum is taken over all the partitions of set $\{\lambda\}_N$ into two disjoint subsets, namely, the subset $\{\lambda^v\}_{n_v}$ of "vacuum" rapidities and subset $\{\lambda^I\}_{n_I}$; $n_v + n_I = N$. Jacobian $\det_{n_v}(\varphi_v')$ is the determinant of $n_v \times n_v$-matrix φ_v' which is defined similar to (3.17):

$$\begin{gathered}(\varphi_v')_{jk} = \delta_{jk}\left[z(\lambda_j^v) + \sum_{l=1}^{n_v} K(\lambda_j^v, \lambda_l^v) \right] - K(\lambda_j^v, \lambda_k^v) \\ (\lambda_{j,k,l}^v \in \{\lambda^v\}_{h_v}; j, k, l = 1, 2, \ldots, n_v) .\end{gathered} \tag{5.8}$$

Here $z(\lambda)$ is given by (5.6).

"Dressed" irreducible parts I^d at (5.6) are defined as follows:

$$\begin{aligned} I_{n_I,N}^d(\{\lambda^I\}, \{\lambda^v\}) = & \sum_{\{\lambda^I\} = \{\lambda^+\}_n \cup \{\lambda^-\}_n \cup \{\lambda^0\}_{n_0}}^{2n + n_0 = n_I} \mathscr{A}_{n_I}^n(\{\lambda^+\}_n, \{\lambda^-\}_n, \{\lambda^0\}_{n_0}) \\ & \cdot E_{n,n_v}(\{\lambda^+\}, \{\lambda^-\}, \{\lambda^v\}) \left(\prod_{j=1}^{n} l(\lambda_j^+) l^{-1}(\lambda_j^-) \right) . \end{aligned} \tag{5.9}$$

The sum here is taken as in (4.23) (where one has to change $N \to n_I$). So the difference between $I^d_{n_I,N}$ and I_{n_I} (4.23) is in dressing factor E_{n,n_v}, defined as follows:

$$E_{n,n_v}(\{\lambda^+\}_n, \{\lambda^-\}_n, \{\lambda^v\}_{n_v}) = [\det_{n_v}(\varphi'_v)]^{-1}$$
$$\cdot \sum_{\{\lambda^v\}_{n_v} = \{\lambda^x\}_{n_x} \cup \{\lambda^y\}_{n_y}} \det_{n_x}(\varphi'_x) \det_{n_y}(\varphi'_y)$$
$$\cdot \prod_{j=1}^{n_x} \prod_{k=1}^{n} \{f(\lambda_k^+, \lambda_j^x) f^{-1}(\lambda_j^x, \lambda_k^+) f(\lambda_j^x, \lambda_k^-) f^{-1}(\lambda_k^-, \lambda_j^x)\}. \tag{5.10}$$

The sum is over all the partitions of $\{\lambda^v\}_{n_v}$ into $\{\lambda^x\}_{n_x}$ and $\{\lambda_y\}_{n_y}$ $(n_x + n_y = n_v)$. Jacobians $\det_{n_x}(\varphi'_x)$ and $\det_{n_y}(\varphi'_y)$ are defined in (5.4) and (5.5). The dependence on x_j, y_j enters the Jacobians; $\det_{n_x}(\varphi'_x)$ depends on x_j; $\det_{n_y}(\varphi'_y)$ depends on y_j; $\det_N(\varphi')$ and $\det_{n_v}(\varphi'_v)$ depends on $z_j = x_j + y_j$. The dependence on l_j is written explicitly in I^d (5.9). We have defined the right-hand side of (5.2). The behavior of $\Gamma_{K,N}$ in the coupling constant η is the same as the behavior of I_K (4.23) [see (4.25)]:

$$\Gamma_{K,N} \sim \{(\pi/4) - \eta\}^{K-2} \quad (\eta \to \pi/4)(K \geqq 2). \tag{5.11}$$

It should be emphasized that the results of Sects. 3–5 are valid for any integrable model with the XXZ R-matrix which can be solved by means of the algebraic Bethe Ansatz of QISM. In the rest of the paper formula (5.2) is used to calculate correlator (1.2) for the Heisenberg one-dimensional ferromagnet.

6. The Thermodynamical Limit and Dressing Equations

The transition to the Heisenberg XXZ-model will be made in two steps. The first step is taken in this section. We make arbitrary functions $r(\lambda)$ (3.9) and $z(\lambda)$ (3.18) of the two-site model equal to their values (3.21) and (3.22) in the Heisenberg model,

$$r(\lambda) = l_0^M(\lambda); \qquad l_0(\lambda) = \sinh(\lambda - i\eta)/\sinh(\lambda + i\eta); \tag{6.1}$$

$$z(\lambda) = Mu(\lambda); \qquad u(\lambda) = i\partial \ln l_0(\lambda)/\partial\lambda \quad \text{[see (2.12)]}. \tag{6.2}$$

Functions $l(\lambda)$ (3.9) and $x(\lambda)$ (4.8) remain arbitrary. Due to (5.6) function $y(\lambda)$ (4.8) is now expressed in terms of $x(\lambda)$:

$$y(\lambda) = Mu(\lambda) - x(\lambda), \tag{6.3}$$

variables y_j being thus linear functions of variables (4.7):

$$y_j = Mu(\lambda_j) - x_j. \tag{6.4}$$

S.t.e. (3.12) of this model is the same as s.t.e. (2.4) of the Heisenberg model, the thermodynamical limit $(M \to \infty)$ of s.t.e. being also the same.

Turn now to Eq. (5.1) and take for $|\psi_N\rangle$ there the state which corresponds to the physical vacuum $|\Omega\rangle$ in the Heisenberg model. At the thermodynamical limit this state is described by vacuum particle density $\varrho(s)$ (2.9) [vacuum rapidities have property $\operatorname{Im}\lambda = \pi/2$, and we use notation (2.8) putting $\operatorname{Re}\lambda \equiv s$]. Arbitrary function $x(\lambda)$ is supposed to be fixed and finite at $M \to \infty$. Then function $y(\lambda)$ (6.3) goes to infinity at $M \to \infty$.

Consider first the thermodynamical limit of term $\langle \mathbf{Q}_1^2 \rangle_N^{(0)}$ in (5.2) which is given by (5.3). It is convenient to consider quantity E_N [2]:

$$E_N(\alpha, [x(\lambda)]) = [\det_N(\varphi')]^{-1} \cdot \sum_{\{\lambda\}_N = \{\lambda^x\}_{n_x} \cup \{\lambda^y\}_{n_x}} \exp\{\alpha n_x\} \det_{n_x}(\varphi_x') \det_{n_y}(\varphi_y'). \tag{6.5}$$

Here α is supposed to be pure imaginary. To calculate the limit of each term in (6.5) one has to calculate the ratio $\det_{n_y}(\varphi_y')\det_N^{-1}(\varphi')$ in this limit ($M\to\infty$, $N\to\infty$, n_x-fixed, $n_y = N - n_x \to \infty$).

The thermodynamical limit of $\det_N(\varphi')$ giving the norm of wave function (3.16) is rather simple [2, 19]:

$$\det_N(\varphi') = (\det(1 - \mathfrak{K}/2\pi))\left(\prod_{j=1}^{N} [2\pi M \varrho(s_j)]\right). \tag{6.6}$$

Here s_j are real parts of rapidities entering $\det_N(\varphi')$; integral linear operator $\mathfrak{K}$ is defined in (2.10) and (2.11). Using a similar representation for $\det_{n_y}(\varphi_y')$ and properties of operator $\mathfrak{K}$ given in Appendix 3 one calculates the determinant ratio:

$$\lim[\det_{n_y}(\varphi_y')/\det_N(\varphi')] = \exp\left\{-\frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} x\left(s + \frac{i\pi}{2}\right)ds\right\}\prod_{j=1}^{n_x}[\omega_\Lambda(s_j^x)/(2\pi M\varrho(s_j^x))]. \tag{6.7}$$

The weight $\omega_\Lambda(s)$ here is given by

$$\omega_\Lambda(s) = \exp\left\{-(1/2\pi)\int_{-\Lambda}^{\Lambda} K(s,t)dt\right\}. \tag{6.8}$$

For coupling constant η being in region (2.2), one has the following important property:

$$0 < \omega_\Lambda(s) \leqq \omega_\Lambda(\Lambda) < 1; \qquad \omega_\infty = \exp\left\{-\frac{\pi - 4\eta}{\pi}\right\}\left(0 < 2\eta \leqq \frac{\pi}{2}\right). \tag{6.9}$$

The limit of determinant ratio (6.7) having been calculated, one can prove the following statement. The limit $E(\alpha, [x(\lambda)]) = \lim_{N\to\infty} E_N(\alpha, [x(\lambda)])$ does exist and is equal to the sum of the limits of individual terms at the right-hand side of (6.5). Function E satisfies the following properties:

$$2\pi\delta E(\alpha, [x(\lambda)])/\delta x(\mu) = -E(\alpha, [x(\lambda)]) + \exp\{\alpha\}E(\alpha, [x(\lambda) + K(\lambda,\mu)]) \tag{6.10}$$
$$(\operatorname{Im}\lambda = \operatorname{Im}\mu = \pi/2);$$

$$|E(\alpha, [x(\lambda)])| \leqq 1 \quad (\operatorname{Im}\lambda = \pi/2) \tag{6.11}$$

(recall that we consider α to be pure imaginary).

The proof of this statement is made in complete analogy with the proof for Bose-gas given in paper [2].

Linear Eq. (6.10) is solved by Fourier transformation. Putting

$$E\left(\alpha, \left[x\left(s + \frac{i\pi}{2}\right)\right]\right) = \exp\left\{\int_{-\Lambda}^{\Lambda} x\left(t + \frac{i\pi}{2}\right)P(t,\alpha)dt\right\}, \tag{6.12}$$

one has that (6.10) and (6.11) are equivalent to the following integral nonlinear equation and inequality for $P(t,\alpha)$:

$$2\pi P(t,\alpha)=\exp\left\{\alpha+\int_{-\Lambda}^{\Lambda}K(t,s)P(s,\alpha)ds\right\}-1\,;$$
$$\mathrm{Re}\,P(t,\alpha)\leqq 0\,. \qquad (6.13)$$

It is proved in Appendix 4 that for α pure imaginary and $0<2\eta\leqq\pi/2$ (2.2) the solution of system (6.13) does exist and this solution is unique.

The thermodynamical limit $\langle \mathbf{Q}_1^2\rangle^{(0)}$ of $\langle \mathbf{Q}_1^2\rangle_N^{(0)}$ is obtained from $E(\alpha,[x(\lambda)]$ by taking the second derivative in α at $\alpha=0$. The answer is given in (6.24). So the thermodynamical limit of the first term at the right-hand side of (5.2) is calculated.

Other terms $\Gamma_{n_I,N}$ [(5.7) and (5.11)] at this limit ($N\to\infty$, n_I-fixed) are calculated in a similar way. The limiting determinant ratio at (5.7) is obtained similarly to (6.7):

$$\det_{n_v}(\varphi_v')/\det_N(\varphi')\to\prod_{j=1}^{n_I}[2\pi M\varrho(\lambda_j^I)]^{-1}\omega_\Lambda(\lambda_j^I) \qquad (6.14)$$

($N\to\infty$; $n_v=N-n_I\to\infty$; n_I-fixed).

To calculate the limit of dressed irreducible part I^d (5.9) one has to calculate the limiting dressing factor $E_n=\lim E_{n,n_v}$ (5.10). This is to be done essentially in the same way as the calculation of the limit of E_N (6.5). The limiting function E_n does exist and is defined by the following properties:

$$|E_n(\{\lambda^+\}_n,\{\lambda^-\}_n,[x(\lambda)])|\leqq 1(\mathrm{Im}\,\lambda=\pi/2;0<2\eta\leqq\pi/2)\,;$$
$$E_n\equiv 1 \text{ at } x\equiv 0\,. \qquad (6.15)$$

$$2\pi\delta E_n/\delta x(\mu)=-E_n$$
$$+\left(\prod_{j=1}^{n}f(\lambda_j^+,\mu)f^{-1}(\mu,\lambda_j^+)f(\mu,\lambda_j^-)f^{-1}(\lambda_j^-,\mu)\right)E_n([x(\lambda)+K(\lambda,\mu)]) \qquad (6.16)$$
$$(\mathrm{Im}\,\lambda=\mathrm{Im}\,\mu=\pi/2)\,.$$

It is essential that the modulus of the product of the f-factors at the right-hand side is equal to one. Putting

$$E\left(\{\lambda_+\}_n,\{\lambda_-\}_n,\left[x\left(t+\frac{i\pi}{2}\right)\right]\right)$$
$$=\exp\left\{\int_{-\Lambda}^{\Lambda}x\left(t+\frac{i\pi}{2}\right)P_n(t,\{\lambda_+\}_n,\{\lambda_-\}_n)dt\right\}, \qquad (6.17)$$

one obtains that Eq. (6.16) is equivalent to the following nonlinear "dressing" equation for P_n

$$2\pi P_n(t)=\left(\prod_{j=1}^{n}f(s_j^+,t)f^{-1}(t,s_j^+)f(t,s_j^-)f^{-1}(s_j^-,t)\right)\exp\left\{\int_{-\Lambda}^{\Lambda}K(t,s)P_n(s)ds\right\}-1\,. \qquad (6.18)$$

Inequality (6.15) turns into the following inequality for P_n:

$$\mathrm{Re}\,P_n\leqq 0\,. \qquad (6.19)$$

For $0<2\eta\leqq\pi/2$ (2.2) the solution of system (6.18) and (6.19) does exist and is unique. It is proved in Appendix 4, where properties of functions P_n are discussed in detail. Of greatest importance for us are the following properties which can be easily proved using Eqs. (A.38) and (A.39). At t finite and $2\eta\neq\pi/2$, $\mathrm{Re}\,P_n=0$ if and only if $\{\lambda^+\}_n\equiv\{\lambda^-\}_n$ (i.e. sets $\{\lambda^+\}_n$ and $\{\lambda^-\}_n$ are the same):

$$\mathrm{Re}\,P_n(t,\{\lambda\}_n,\{\lambda\}_n)=0\,. \tag{6.20}$$

In any other case one has for $2\eta<\pi/2$:

$$\mathrm{Re}\,P_n(t,\{\lambda^+\}_n,\{\lambda^-\}_n)<0 \quad (t\neq\pm\infty;\{\lambda^+\}_n\neq\{\lambda^-\}_n)\,. \tag{6.21}$$

The behavior at $t\to\pm\infty$ is as follows:

$$P_n(t,\{\lambda^+\}_n,\{\lambda^-\}_n)\to 0 \quad (\text{at } t\to\pm\infty \text{ and } \lambda \text{ fixed})\,. \tag{6.22}$$

Equation (6.14) and (6.17) permit us to calculate the thermodynamical limit $\Gamma_{n_I}=\lim\Gamma_{n_I,N}$ (n_I-fixed) [the answer is given in (6.26)].

Let us give now the answer for the thermodynamical limit of the meanvalue (5.1) and (5.2) for the two-site model considered in this section [see (6.1)–(6.3)]. At this limit state $|\psi_N\rangle$ at (5.1) goes to $|\Omega\rangle$ which corresponds to the physical vacuum in the Heisenberg model. Using results obtained above one has:

$$\langle\mathbf{Q}_1^2\rangle\equiv\frac{\langle\Omega|\mathbf{Q}_1^2|\Omega\rangle}{\langle\Omega|\Omega\rangle}=\langle\mathbf{Q}_1^2\rangle^{(0)}+\sum_{K=2}^{\infty}\Gamma_K\,. \tag{6.23}$$

Here $\langle\mathbf{Q}_1^2\rangle^{(0)}$ is the thermodynamical limit of $\langle\mathbf{Q}_1^2\rangle_N^{(0)}$ in (5.2):

$$\langle\mathbf{Q}_1^2\rangle^{(0)}=\left(\int_{-\Lambda}^{\Lambda}x\left(t+\frac{i\pi}{2}\right)P'(t)dt\right)^2+\int_{-\Lambda}^{\Lambda}x\left(t+\frac{i\pi}{2}\right)P''(t)dt\,, \tag{6.24}$$

where we denote

$$P'(t)=\partial P(t,\alpha)/\partial\alpha|_{\alpha=0}\,;\qquad P''(t)=\partial^2P(t,\alpha)/\partial\alpha^2|_{\alpha=0}\,, \tag{6.25}$$

and function $P(t,\alpha)$ is defined by (6.13).

Quantities Γ_K are contributions of K-particle limiting irreducible parts I_K^d:

$$\Gamma_K=\frac{1}{K!}\int_{-\Lambda}^{\Lambda}\left(\prod_{j=1}^{K}\frac{ds_j\omega_\Lambda(s_j)}{2\pi}\right)I_K^d(\{\lambda\}_K)\quad\left(\lambda_j=s_j+\frac{i\pi}{2};\mathrm{Im}\,s_j=0\right). \tag{6.26}$$

Weight $\omega_\Lambda(s)=\exp\left\{-(1/2\pi)\int_{-\Lambda}^{\Lambda}K(s,t)dt\right\}$ is defined in (6.8). Irreducible parts I_K^d are obtained from $I_{K,N}^d$ (5.9) by changing dressing factors E_{n,n_v} to E_n defined in (6.17) and (6.19):

$$\begin{aligned}I_K^d(\{\lambda\}_K)=&\sum_{\{\lambda\}_K=\{\lambda^+\}_n\cup\{\lambda^-\}_n\cup\{\lambda^0\}_{K-2n}}\mathscr{A}_K^n(\{\lambda^+\}_n,\{\lambda^-\}_n,\{\lambda^0\}_{K-2n})\\&\cdot E_n(\{\lambda^+\}_n,\{\lambda^-\}_n,[x(\lambda)])\left(\prod_{j=1}^{n}l(\lambda_j^+)l^{-1}(\lambda_j^-)\right).\end{aligned} \tag{6.27}$$

Here $\mathscr{A}_K^n$ are Fourier coefficients of the irreducible part [see (4.19)–(4.23)]. It should be emphasized that the behavior of Γ_K in coupling constant η is just the same as of $\Gamma_{K,N}$ [see (5.11)].

7. Correlation Function $\langle\sigma_3^{(m)}\sigma_3^{(1)}\rangle$ for the Heisenberg XXZ-Antiferromagnet

The normalized vacuum mean value of operator $\mathbf{Q}_1^2(m)$ (2.20) is obtained from (6.23)–(6.27) by substituting for arbitrary functions $l(\lambda)$ (3.9) and $x(\lambda)$ (4.8) their values in the Heisenberg model (3.21) [cf. with (6.1) and (6.2)]:

$$l(\lambda)=l_0^m(\lambda)\,; \qquad l_0(\lambda)=\cosh(s-i\eta)/\cosh(s+i\eta)\,, \tag{7.1}$$

$$x(\lambda)=im\partial\ln l_0(\lambda)/\partial\lambda=mu(s+i\pi/2) \quad (\lambda=s+i\pi/2)\,. \tag{7.2}$$

The function $u(s+i\pi/2)>0$ is given by (2.12). It is to be remembered that all the rapidities λ here are the vacuum ones, so we denote $\lambda=s+i\pi/2$; $\operatorname{Im}s=0$ [see (2.8)]. Vacuum rapidities fill the Fermi zone $-\Lambda\leqq s\leqq\Lambda$, "Fermi rapidity" Λ being defined by magnetic field H [see (2.13) and (2.14)]; $\Lambda\to\infty$ at $H\to 0$. Quantities $\langle\mathbf{Q}_1^2\rangle^{(0)}$ and Γ_K in (6.23) are thus functions of the "distance" m and we write for them $\langle\mathbf{Q}_1^2(m)\rangle^{(0)}$ and $\Gamma_K(m)$:

$$\langle\mathbf{Q}_1^2(m)\rangle=\langle\mathbf{Q}_1^2(m)\rangle^{(0)}+\sum_{K=2}^{\infty}\Gamma_K(m)\,. \tag{7.3}$$

Function $P'(t)$ in (6.24) satisfies the following linear integral equation which is readily obtained from (6.13) and (6.25): $2\pi P'(t)-(\mathfrak{K}P')(t)=1$. Using Eq. (2.9) for $\varrho(s)$ one has from (6.24) and (7.2):

$$\langle\mathbf{Q}_1^2(m)\rangle^{(0)}=m^2\left(\int_{-\Lambda}^{\Lambda}\varrho(s)ds\right)^2+m\int_{-\Lambda}^{\Lambda}u(s+i\pi/2)P''(s)ds\,. \tag{7.4}$$

Contributions $\Gamma_K(m)$ of dressed irreducible parts are given by (6.26) with $l(\lambda)$ and $x(\lambda)$ defined in (7.1) and (7.2).

Turn now to correlator (1.2) and (1.3) which is represented in the following form (2.23) and (2.24):

$$\langle\sigma_3^{(m)}\sigma_3^{(1)}\rangle=\left(1-2\int_{-\Lambda}^{\Lambda}\varrho(s)ds\right)^2+\langle\!\langle\sigma_3^{(m)}\sigma_3^{(1)}\rangle\!\rangle\,. \tag{7.5}$$

To obtain the nontrivial part of this correlator one has to apply the derivative $D^{(2)}$ (2.22) to $\langle Q_1^2(m)\rangle$ (7.3) [see (2.25)]. The differentiation of $\langle Q_1^2(m)\rangle^{(0)}$ is easily done due to relations $D^{(2)}m=0$; $D^{(2)}m^2=2$. Thus one gets:

$$\langle\!\langle\sigma_3^{(m)}\sigma_3^{(1)}\rangle\!\rangle=\sum_{K=2}^{\infty}G_K(m)\,, \tag{7.6}$$

where [see (2.25), (6.26), and (7.3)]

$$G_k(m)=2D^{(2)}\Gamma_K(m)=\frac{2}{K!}\int_{-\Lambda}^{\Lambda}\left(\prod_{j=1}^{K}\frac{ds_j\omega_\Lambda(s_j)}{2\pi}\right)D^{(2)}I_K^d(m;\{\lambda_K\}_K)\,. \tag{7.7}$$

Irreducible part $I_K^d(m;\{\lambda\})$ is just $I_K^d(\{\lambda\})$ (6.27), where the substitution (7.1) and (7.2) is made; one thus has

$$\begin{aligned}D^{(2)}I_K^d(m;\{\lambda\}_K)=&\sum_{n>0}\mathscr{A}_K^n(\{\lambda^+\}_n,\{\lambda^-\}_n,\{\lambda^0\}_{K-2n})\\&\cdot\exp\{(m-2)\pi_n(\{s^+\}_n,\{s^-\}_n)\}\left(\prod_{j=1}^{n}l_0^{m-2}(\lambda_j^+)l_0^{-m+2}(\lambda_j^-)\right)\\&\cdot\left[\exp\{\pi_n(\{s^+\}_n,\{s^-\}_n)\}\left(\prod_{j=1}^{n}l_0(\lambda_j^+)l_0^{-1}(\lambda_j^-)\right)-1\right]^2.\end{aligned} \tag{7.8}$$

The sum here is taken as explained after (4.23) and (5.9) but the term with $n=0$ is omitted as it is annihilated by the derivative. Fourier coefficients $\mathscr{A}_K^n$ are defined by (4.23); for $K=2,3$ they are given in (4.20) and (4.22). We introduce also the followith notation:

$$\pi_n(\{s^+\}_n, \{s^-\}_n) = \int_{-\Lambda}^{\Lambda} u(t+i\pi/2)P_n(t, \{\lambda^+\}_n, \{\lambda^-\}_n)dt \qquad (\lambda^\pm = s^\pm + i\pi/2)\,; \tag{7.9}$$

functions P_n here as defined by (6.18) and (6.19). It should be emphasized that functions π_n are the only quantities which are not given in the expression for $G_K(m)$ quite explicitly. They are defined in terms of functions P_n which are solutions of the nonlinear integral equation (6.18). This equation can be solved either perturbatively in $\{(\pi/2)-2\eta\}$ [see (A.41)] or by means of iterations described at Appendix 4 (A.37). The important property of functions π_n (7.9) is $\mathrm{Re}\,\pi_n \leqq 0$, which follows from $u(t+i\pi/2)>0$ (2.12) and $\mathrm{Re}\,P_n \leqq 0$ (6.19). More exactly, using (6.20), (6.21), and (A.41) one has the following statements:

$$\pi_n(\{s^+\}_n, \{s^-\}_n) = 0 \quad \text{at} \quad 2\eta = \pi/2\,; \tag{7.10}$$

$$\pi_n(\{s\}_n, \{s\}_n) = 0 \quad (\{s^+\} = \{s^-\} = \{s\}; 0 < 2\eta \leqq \pi/2)\,; \tag{7.11}$$

$$\mathrm{Re}\,\pi_n(\{s^+\}_n, \{s^-\}_n) \leqq 0\,; \qquad \mathrm{Re}\,\pi_n = O[((\pi/2)-2\eta)^2]\,, \qquad (0 < 2\eta \leqq \pi/2)\,. \tag{7.12}$$

Formulae (7.6)–(7.9) give the final answer for correlator $\langle\!\langle \sigma_3^{(m)} \sigma_3^{(1)} \rangle\!\rangle$. It is represented as a series, the K^{th} term of the series corresponding to making K holes in the physical vacuum. To clarify the structure of series (7.6) we write down the first two terms. Function G_2 is represented as a double integral. Using (4.20), (4.26), and (A.40) one has:

$$G_2(m) = -\frac{1}{\pi^2} \int_{-\Lambda}^{\Lambda} ds_1 ds_2 \omega_\Lambda(s_1)\omega_\Lambda(s_2) \frac{\sinh(s_1-s_2+2i\eta)}{\sinh(s_1-s_2-2i\eta)}$$
$$\cdot \frac{1}{\sinh^2(s_1-s_2)} \exp\{(m-2)\pi_1(s_1,s_2)\} \left(\frac{\cosh(s_1-i\eta)\cosh(s_2+i\eta)}{\cosh(s_1+i\eta)\cosh(s_2-i\eta)}\right)^{m-2}$$
$$\cdot \left[\exp\{\pi_1(s_1,s_2)\} \left(\frac{\cosh(s_1-i\eta)\cosh(s_2+i\eta)}{\cosh(s_1+i\eta)\cosh(s_2-i\eta)}\right) - 1\right]^2, \quad (m \geqq 3)\,. \tag{7.13}$$

Weight $\omega_\Lambda(s)$ being defined in (6.8), the only quantity which is not given in (7.13) explicitly is function π_1 [see (7.9)], which is to be calculated as explained above. Function $\pi_1(s_1, s_2)$ enters also the expression for $G_3(m)$ which is given by the following triple integral:

$$G_3(m) = -\frac{\sin 4\eta}{\pi^3} \int_{-\Lambda}^{\Lambda} \left(\prod_{j=1}^{3} ds_j \omega_\Lambda(s_j)\right) \frac{1}{\sinh(s_3-s_1+2i\eta)\sinh(s_3-s_2-2i\eta)}$$
$$\cdot \left(\frac{\sinh(s_3-s_2)}{\sinh(s_3-s_1)} + \frac{\sinh(s_3-s_1)}{\sinh(s_3-s_2)}\right) \frac{\sinh(s_1-s_2+2i\eta)}{\sinh(s_1-s_2-2i\eta)}$$
$$\cdot \frac{1}{\sinh^2(s_1-s_2)} \exp\{(m-2)\pi_1(s_1,s_2)\} \left(\frac{\cosh(s_1-i\eta)\cosh(s_2+i\eta)}{\cosh(s_1+i\eta)\cosh(s_2-i\eta)}\right)^{m-2}$$
$$\cdot \left[\exp\{\pi_1(s_1,s_2)\} \frac{\cosh(s_1-i\eta)\cosh(s_2+i\eta)}{\cosh(s_1+i\eta)\cosh(s_2-i\eta)} - 1\right]^2, \quad (m \geqq 3)\,. \tag{7.14}$$

Let us discuss now the general structure of the K^{th} term $G_K(m)$. It is represented as the K-multiple integral (7.7). At $s_j \to \infty$ this integral is convergent which makes possible to put $\Lambda = \infty$ (i.e. $H = 0$). The singularities of the integrand at finite real s_j are reduced to first order poles. It follows from the analysis of the limiting process discussed in detail in Sect. 6 that the principal values of corresponding integrals should be taken. With this prescription all the integrals are well defined.

Expression (7.7) for $G_K(m)$ contains the K^{th} power of weight $\omega_\Lambda(s)$ (6.8) possessing property $0 \leqq \omega_\Lambda(s) \leqq \omega_\Lambda(\Lambda) < 1$; $\omega_\infty = \exp\{-(\pi - 4\eta)/\pi\} < 1$ for $0 < 2\eta < \pi/2$, $\Lambda > 0$ (6.9). Therefore the convergence of series (7.6) seems highly probable, though we don't know the complete proof yet. It should be mentioned that region $\pi/2 < 2\eta < \pi$ of the coupling constant also corresponds to an antiferromagnetic case. At these η, however, weight $\omega_\Lambda(s)$ in (7.7) becomes more than 1 and series (7.6) seems to be divergent; to obtain correlator $\langle \sigma_3^{(m)} \sigma_3^{(1)} \rangle$ in this region of the coupling constant one has to make an accurate analytical continuation from region $0 < 2\eta \leqq \pi/2$. The convergence of the series is closely connected with the following phenomenon. S.t.e. (2.4) in the logarithmic form can be obtained from a variational principle [14, 4]. Corresponding action is convex only in our region $0 < 2\eta \leqq \pi/2$, which permits us to prove the existence of the solution of system (2.4).

The behavior of $G_K(m)$ in the coupling constant η is the same as the behavior of Fourier coefficients [see (4.25) and (5.11)]. At $2\eta \to \pi/2$ one has

$$G_K(m) \sim [(\pi/2) - 2\eta]^{K-2} . \tag{7.15}$$

This property shows that the perturbative expansion in the coupling constant $[(\pi/2) - 2\eta]$ is easily obtained from series (7.6). To obtain this expansion up to $[(\pi/2) - 2\eta]^n$, one has to consider n first terms of series (7.6). At $2\eta = \pi/2$ only the term with $K = 2$ survives which gives the correlator for the XX-model. In this case $K(s, t) = 0$ [see (2.11)]. It means that $\varrho(s) = (\pi \cosh 2s)^{-1}$ (2.9), $\omega_\Lambda(s) = 1$ (6.8) and $\pi_1(s_1, s_2) = 0$ (7.10). Integrals in (7.5) and (7.13) can be taken explicitly and one obtains:

$$\langle \sigma_3^{(m)} \sigma_3^{(1)} \rangle = \left(1 - \frac{2q}{\pi}\right)^2 - \frac{4 \sin^2[(m-1)q]}{\pi^2 (m-1)^2} , \tag{7.16}$$

($m \geqq 3$; $2\eta = \pi/2$; $0 \leqq H < 4$). Here q is the Fermi momentum which at $2\eta = \pi/2$ is given as [see (2.5)]

$$q = 2 \operatorname{arctg} \operatorname{th} \Lambda = \operatorname{arctg}(\sqrt{4 - H^2}/H) . \tag{7.17}$$

(Fermi rapidity Λ is simply expressed in terms of magnetic field H for $2\eta = \pi/2 : \cosh 2\Lambda = 2/H$.) The first term at the right-hand side of (7.16) gives the square of magnetization and the second one represents the nontrivial part $\langle\!\langle \sigma_3^{(m)} \sigma_3^{(1)} \rangle\!\rangle$. At $H = 0$ ($\Lambda = \infty$), $q = \pi/2$, and the correlator is especially simple:

$$\langle \sigma_3^{(m)} \sigma_3^{(1)} \rangle = \langle\!\langle \sigma_3^{(m)} \sigma_3^{(1)} \rangle\!\rangle = - \frac{4 \sin^2[(m-1)\pi/2]}{\pi^2 (m-1)^2} , \tag{7.18}$$

($m \geqq 3$; $2\eta = \pi/2$; $H = 0$). This formula reproduces the known answer [10, 11] for the XX-model.

Turn now again to correlator for $0<2\eta\leqq\pi/2$. The most remarkable property of series (7.6) is that the correct asymptotical behavior of the correlator at $m\to\infty$ is given already by the first term. Below we discuss the most obvious results concerning the first two terms of the series.

Consider first $G_2(m)$ (7.13). Integrating by parts in s_1, s_2 and using (7.11) one obtains

$$G_2(m)=-\frac{2a^2}{\pi^2(m-1)^2}\left\{1-\frac{1}{2}(C\exp[(m-1)(-2iq+i\varkappa-r)]+\text{c.c.})\right\},$$
$$(m\gg\pi/2\eta). \tag{7.19}$$

The following notations are used here. (i) q is the Fermi momentum:

$$q=2\,\text{arctg}(\text{tg}\eta\,\text{th}\Lambda). \tag{7.20}$$

(ii) Constant $C=C(\eta,\Lambda)$ does not depend on m and is easily calculated in terms of the function $\pi_1(\Lambda,-\Lambda)$. Of importance for us is the following property:

$$C\to1 \quad \text{at} \quad 2\eta\to\pi/2. \tag{7.21}$$

(iii) Quantity a is given by different formulae in the following two regions of m:

$$\begin{aligned} &a=\omega_\Lambda(\Lambda) \quad (m\gg\sin(\pi-4\eta)\,\text{ch}2\Lambda),\\ &a=\omega_\Lambda(0) \quad (m\ll\sin(\pi-4\eta)\,\text{ch}2\Lambda). \end{aligned} \tag{7.22}$$

Here $\omega_\Lambda(s)=\exp\left\{-(1/2\pi)\int_{-\Lambda}^{\Lambda}K(s,t)dt\right\}$ (6.8). It is to be remembered that in any case $m\gg\pi/2\eta$ [see (7.19)].

(iv) Quantities r and $\varkappa$ are:

$$r=-\text{Re}\,\pi_1(\Lambda,-\Lambda); \qquad \varkappa=\text{Im}\,\pi_1(\Lambda,-\Lambda). \tag{7.23}$$

It follows from (7.10), (7.12) that

$$\begin{aligned} &r=\varkappa=0 \quad \text{at} \quad 2\eta=\pi/2,\\ &r>0; \quad r=O[((\pi/2)-2\eta)^2] \quad \text{at} \quad 0<2\eta<\pi/2. \end{aligned} \tag{7.24}$$

At $H=0$ and $2\eta\to\pi/2$, $r=(\pi-4\eta)^2$. So all the notations used at the right-hand side of (7.19) are explained.

Discuss now asymptotics (7.19) in more detail. There are three different scales M_1, M_2, M_3 in the asymptotics:

$$M_1=\pi/2\eta; \qquad M_2=r^{-1}; \qquad M_3=\sin(\pi-4\eta)\,\text{ch}2\Lambda. \tag{7.25}$$

At $m\gg M_1$ the integrand in (7.13) does oscillate and one thus obtains asymptotics (7.19). The second scale M_2 [see (7.23) and (7.24)] characterizes the region where oscillations in the asymptotics exist. For $m<M_2$ there are oscillations ($\sim\exp[(m-1)(-2iq+i\varkappa)]$); for $m\gg M_2$ they disappear. These two scales are essentially defined by coupling constant η [i.e. by internal anisotropy Δ (2.1) of the antiferromagnet]. The third scale is $M_3\sim\text{ch}2\Lambda$. As Λ is defined by external magnetic field H, M_3 is also essentially defined by the magnetic field. For $m\gg M_3$ and $m\ll M_3$ one has different a in (7.19). Though we are discussing now the first term of series (7.6), the existence of these three scales has grave physical meaning and they have to be present in the exact correlator $\langle\langle\sigma_3^{(m)}\sigma_3^{(1)}\rangle\rangle$ also.

Note that $M_1=1$, $M_2=\infty$, $M_3=0$, and $a=\omega_\Lambda(\Lambda)=\omega_\Lambda(0)=1$ for the XX-case ($2\eta=\pi/2$). Using (7.24) and (7.21) one restores then correlator (7.16)–(7.18) (it is to be mentioned that asymptotic formula (7.19) appears to be exact in this case).

For $2\eta<\pi/2$ terms $G_K(m)$ $(K\geqq 3)$ of series (7.6) also contribute to the asymptotics of the correlator. The contribution of $G_3(m)$ (7.14) can be easily calculated. It is of the same form as (7.19) resulting in corrections to coefficients a and C there, these corrections being especially simple for the nonoscillating terms. Summing up contributions of $G_2(m)$ and $G_3(m)$ one obtains the following results for correlator (7.6).

For nonzero magnetic field $H>0$ the "far" asymptotics and the "near" asymptotics have to be distinguished:

$$\langle\!\langle \sigma_3^{(m)}\sigma_3^{(1)}\rangle\!\rangle \cong G_2(m)+G_3(m) = -\frac{2\omega_\Lambda^2(\Lambda)}{\pi^2(m-1)^2}\left[1+\frac{2}{\pi}\int_{-\Lambda}^{\Lambda} ds\,\omega_\Lambda(s)K(s,\Lambda)\right]$$

$$(0<H<4\sin^2\eta;\, 0<2\eta<\pi/2;\, m\gg\max\{M_1,M_2,M_3\});\quad (7.26)$$

$$\langle\!\langle \sigma_3^{(m)}\sigma_3^{(1)}\rangle\!\rangle \cong G_2(m)+G_3(m) = -\frac{2\omega_\Lambda^2(0)}{\pi^2(m-1)^2}\left[1+\frac{2}{\pi}\int_{-\Lambda}^{\Lambda} ds\,\omega_\Lambda(s)K(s,0)\right]$$

$$(0<H<4\sin^2\eta;\, 0<2\eta<\pi/2;\, M_3\gg m\gg\max\{M_1,M_2\}).\quad (7.27)$$

Weight $\omega_\Lambda(s)$ and function $K(s,t)$ are defined in (6.8) and (2.11). For $M_{1,2,3}$ see (7.25). [It is worth mentioning that for Λ sufficiently large the square bracket in (7.26) turns into $[1-4\omega_\Lambda(\Lambda)\ln\omega_\Lambda(\Lambda)]$, and the square bracket in (7.27) – into $[1-4\omega_\Lambda(0)\ln\omega_\Lambda(0)]$.]

To obtain the asymptotics of the correlator at zero magnetic field one puts $\Lambda\to\infty$. As $M_3\to\infty$ at $\Lambda\to\infty$, one has to use Eq. (7.27). The result is:

$$\langle \sigma_3^{(m)}\sigma_3^{(1)}\rangle = \langle\!\langle \sigma_3^{(m)}\sigma_3^{(1)}\rangle\!\rangle \cong G_2(m)+G_3(m) = -\frac{2\exp\{-2(\pi-4\eta)/\pi\}}{\pi^2(m-1)^2}\left[1+4\frac{(\pi-4\eta)}{\pi}\exp\left\{-\frac{(\pi-4\eta)}{\pi}\right\}\right]$$

$$(H=0, \Lambda=\infty;\, 0<2\eta<\pi/2;\, m\gg\max\{M_1,M_2\}).\quad (7.28)$$

So we have calculated the asymptotics of the first two terms of series (7.6) representing the correlator. It is obvious that other terms $G_K(m)$ $(K\geqq 4)$ do not change behavior $\sim 1/m^2$, changing only the coefficients.

Equations (7.26)–(7.28) are already sufficient to obtain the asymptotics of the correlator up to the first order in $[(\pi/2)-2\eta]$ (i.e. near the XX-model). After elementary calculations one has at nonzero magnetic field $H>0$:

$$\langle\!\langle \sigma_3^{(m)}\sigma_3^{(1)}\rangle\!\rangle = -\frac{2}{\pi^2(m-1)^2}\left[1+\left(\frac{\pi-4\eta}{\pi}\right)\mathrm{th}\,2\Lambda+O[(\pi-4\eta)^2]\right]$$

$$(H>0;\, 0<2\eta<\pi/2;\, m\gg\max\{M_1,M_2,M_3\});\quad (7.29)$$

$$\langle\!\langle \sigma_3^{(m)}\sigma_3^{(1)}\rangle\!\rangle = -\frac{2}{\pi^2(m-1)^2}\left[1+2\left(\frac{\pi-4\eta}{\pi}\right)\mathrm{th}\,\Lambda+O[(\pi-4\eta)^2]\right]$$

$$(H>0;\, 0<2\eta<\pi/2;\, M_3\gg m\gg\max\{M_1,M_2\}).\quad (7.30)$$

For $H=0$ one has

$$\langle\!\langle \sigma_3^{(m)}\sigma_3^{(1)}\rangle\!\rangle = -\frac{2}{\pi^2(m-1)^2}\left[1+2\left(\frac{\pi-4\eta}{\pi}\right)+O[(\pi-4\eta)^2]\right]$$
$$(H=0; 0<2\eta<\pi/2; m\gg \max\{M_1, M_2\}). \tag{7.31}$$

It is to be emphasized that we obtain asymptotics (7.31) at $H=0$ as a limit $\Lambda\to\infty$ of "near" asymptotics (7.30). [The matter is that "far" asymptotics (7.29) is valid if $m\gg M_3=\sin(\pi-4\eta)\,\mathrm{ch}\,2\Lambda$. At $(\pi-4\eta)$ small (but fixed) the corresponding region does not exist at $\Lambda=\infty$.] The first order correction thus obtained is twice as large as the corresponding correction of paper [13] where the correlator at $H=0$ was calculated. If one takes the limit $\Lambda\to\infty$ not in (7.30) but in (7.29) then the coefficient is just the same. Our limiting procedure seems to be more accurate.

8. Conclusion

We have calculated the simplest correlation function $\langle\sigma_3^{(m)}\sigma_3^{(1)}\rangle$ for the one-dimensional Heisenberg antiferromagnet. The method used is quite general, results of Sects. 3–5 of this paper giving a background for calculation of practically any correlation function for integrable models with the R-matrix of the XXZ-type. In particular, we'll calculate the field correlator for the sine-Gordon model in our next paper. The method permits also calculation of the dependence of the correlation radius on the temperature for such models; for the One-Dimensional Bose Gas it was done in [20].

Appendix 1

"Scalar product" $\mathbb{S}_N$ is defined as follows:

$$\mathbb{S}_N = \langle 0|\prod_{j=1}^{N}\mathbb{C}(\lambda_j^C)\prod_{k=1}^{N}\mathbb{B}(\lambda_k^B)|0\rangle. \tag{A.1}$$

Here $\lambda_j^C \neq \lambda_k^C$ $(j\neq k)$; $\lambda_j^B\neq\lambda_k^B$ $(j\neq k)$. The properties of $\mathbb{S}_N$ necessary for us are given below [18]. Scalar products can be in principle calculated by means of (3.2) and (3.7). So one has for example

$$\mathbb{S}_1=\langle 0|\mathbb{C}(\lambda^C)\mathbb{B}(\lambda^B)|0\rangle = g(\lambda^C,\lambda^B)\,[r(\lambda^C)-r(\lambda^B)], \tag{A.2}$$

where g and r are defined in (3.4) and (3.9). For N arbitrary the dependence of $\mathbb{S}_N$ on vacuum eigenvalues $r(\lambda)$ can be explicitly extracted:

$$\mathbb{S}_N=\sum_{\text{part}}\left(\prod_{j=1}^{N} r(\lambda_j^{(pr)})\right)\mathbb{K}_N(\text{part}). \tag{A.3}$$

The sum here is taken over all the partitions of the set $\{\lambda^C\}_N\cup\{\lambda^B\}_N$ into two disjoint subsets $\{\lambda^{(pr)}\}_N$ and $\{\lambda^{(ab)}\}_N$ (subindex N in $\{\lambda\}_N$ means the numbers of elements in this set). Coefficients $\mathbb{K}_N$ do not depend on $r(\lambda)$ and are functions of $2N$ rapidities λ_j^C, λ_k^B, the dependence on each individual λ at all the other λ's fixed being as follows:

$$\mathbb{K}_N=\exp(\lambda)\mathscr{K}(\exp(2\lambda))=\exp(-\lambda)\tilde{\mathscr{K}}(\exp(-2\lambda)). \tag{A.4}$$

Here $\mathscr{K}, \tilde{\mathscr{K}}$ are rational functions of their arguments decreasing at infinity:

$$\begin{aligned} &\mathscr{K}(z)=O(z^{-1}) \quad (|z|\to\infty);\\ &\tilde{\mathscr{K}}(z^{-1})=O(z) \quad (|z|\to 0); \qquad z=\exp(2\lambda). \end{aligned} \tag{A.5}$$

Scalar product $\mathbf{S}_N$ depends on the values of the arbitrary function $r(\lambda)$ at $2N$ points λ^C, λ^B. Due to the arbitrariness of this function these values can be considered as independent variables r_j^C, r_j^B:

$$r_j^C \equiv r(\lambda_j^C); \qquad r_j^B = r(\lambda_j^B). \tag{A.6}$$

So the scalar product is the function of $4N$ complex variables:

$$\mathbf{S}_N = \mathbf{S}_N(\{\lambda^C\}_N, \{\lambda^B\}_N, \{r^C\}_N, \{r^B\}_N), \tag{A.7}$$

which is symmetric with respect to

$$(\lambda_j^C, r_j^C) \leftrightarrow (\lambda_k^C, r_k^C) \quad \text{or} \quad (\lambda_j^B, r_j^B) \to (\lambda_k^B, r_k^B)$$

(separately). The most important property of $\mathbf{S}_N$ is that it has a first-order pole at $\lambda_j^C \to \lambda_k^B$ ($j, k = 1, 2, \ldots, N$), all the other arguments being fixed. At $\lambda_N^C \to \lambda_N^B \to \lambda_N$ one has (the general case is obvious due to the symmetry):

$$\begin{aligned} &\mathbf{S}_N(\{\lambda^C\}_N, \{\lambda^B\}_N, \{r^C\}_N, \{r^B\}_N)|_{\lambda_N^C \to \lambda_N^B}\\ &= g(\lambda_N^C, \lambda_N^B)(r_N^C - r_N^B)\left(\prod_{j=1}^{N-1} f_{Nj}^B f_{Nj}^C\right)\\ &\cdot \mathbf{S}_{N-1}(\{\lambda^C\}_{N-1}, \{\lambda^B\}_{N-1}, \{\tilde{r}^C\}_{N-1}, \{\tilde{r}^B\}_{N-1}). \end{aligned} \tag{A.8}$$

Here $f_{Nj}^{B,C} \equiv f(\lambda_N, \lambda_j^{B,C})$ (3.4) and

$$\tilde{r}(\lambda) = r(\lambda)(f(\lambda, \lambda_N)/f(\lambda_N, \lambda)). \tag{A.9}$$

Function $\mathbf{S}_{N-1}$ in (A.8) does not depend on r_N; $\mathbf{S}_{N-1}$ also depends on λ_N only due to modification (A.9).

It should be emphasized that $\mathbf{S}_N$ has this pole only if function $r(\lambda)$ is not smooth at $\lambda = \lambda_N$. In physical cases $r(\lambda)$ is a smooth function [see, e.g. (3.21)], and the residue at $\lambda = \lambda_N$ is equal to zero. Then the dependence of $\mathbf{S}_N$ on the vacuum eigenvalue at point λ_N is represented naturally in terms of two variables: $r_N = r(\lambda_N)$ and $z_N = i\partial \ln r(\lambda)/\partial\lambda|_{\lambda=\lambda_N}$. The dependence on z_N is linear, the coefficient at z_N being easily obtained from (A.8).

At the limit $\lambda_j^C \to \lambda_j^B \to \lambda_j$ ($j = 1, 2, \ldots, N$) (all λ_j are supposed to be different) the scalar product depends on $3N$ complex variables:

$$\mathbf{S}_N \equiv \mathbf{S}_N(\{\lambda\}_N, \{z\}_N, \{r\}_N)(\{\lambda^C\}_N = \{\lambda^B\}_N \equiv \{\lambda\}_N). \tag{A.10}$$

Here z_j is defined in (3.18). S_N is a linear function of each z_j.

The case where λ_j in (A.10) satisfy s.t.e. (3.12) is of primary importance, because S_N gives the "norm" $\mathscr{N}$ of eigenfunction (3.10):

$$\begin{aligned} &\langle \psi_N(\{\lambda\}_N)|\psi_N(\{\lambda\}_N)\rangle \equiv \mathscr{N}_N(\{\lambda\}_N, \{z\}_N)\\ &= S_N\left(\{\lambda_j\}_N, \{z_j\}_N, \left\{r_j = \prod_{k\neq j}(f(\lambda_k, \lambda_j)/f(\lambda_j, \lambda_k))\right\}_N\right). \end{aligned} \tag{A.11}$$

The norm also is a linear function of each z_j, the coefficient being obtained from (A.8):

$$\partial\mathcal{N}_N/\partial z_N=(\sin 2\eta)\mathcal{N}_{N-1}(\{\lambda\}_{N-1},\{\tilde{z}\}_{N-1})\left(\prod_{k=1}^{N-1} f_{Nk}f_{kN}\right). \tag{A.12}$$

The modification of z_j to $\tilde{z}_j$ is defined as follows:

$$\tilde{z}_j=z_j+K_{jN}\quad (j=1,\dots,N-1), \tag{A.13}$$

where $K_{jN}\equiv K(\lambda_j,\lambda_N)$ (2.11). Equation (A.12) was the basic one in obtaining explicit formula (3.16) [18].

Appendix 2

To investigate properties of irreducible parts (4.14) at N arbitrary it is necessary to introduce form factors. Form factor F_N^α is matrix element (4.1) between different eigenfunctions (3.10) and (3.15),

$$F_N^\alpha\equiv\langle\psi_N(\{\lambda^C\}_N)|\exp\{\alpha\mathbf{Q}_1\}|\psi_N(\{\lambda^B\}_N)\rangle. \tag{A.14}$$

Al λ_j^C,λ_j^B here are different and satisfy s.t.e. (3.12) (each set $\{\lambda^C\},\{\lambda^B\}$ separately). It means that variables m^B in (4.2) can be expressed in terms of λ^B and l^B, and variables m^C – in terms of λ^C and l^C [see (3.9)]. So the form factor depends on $4N$ independent variables:

$$F_N^\alpha=F_N^\alpha(\{\lambda^C\}_N,\{\lambda^B\}_N,\{l^C\}_N,\{l^B\}_N). \tag{A.15}$$

The irreducible part I_N^α (4.14) can be obtained as some limiting value of the form factor [1]:

$$I_N^\alpha(\{\lambda_j\}_N,\{l_j\}_N)=(\sin 2\eta)^{-N}\left(\prod_{j\neq k}^{N} f_{jk}\right)^{-1} \cdot\lim_{\varepsilon\to 0}F_N^\alpha(\{\lambda_j\}_N,\{\lambda_j+\varepsilon\}_N,\{l_j\}_N,\{l_j\}_N). \tag{A.16}$$

The investigation of form factor (A.15) is similar to the one made for the XXX-case [1]. The difference is in analyticity properties in λ's of scalar products [see (A.4) and (A.5)]. Taking this into account one proves the following representation for the form factor:

$$F_N^\alpha(\{\lambda^C\}_N,\{\lambda^B\}_N,\{l^C\}_N,\{l^B\}_N) =\sum_{\text{part}}\left(\prod_{pr} l(\lambda_{pr}^C)\right)\left(\prod_{pr} l^{-1}(\lambda_{pr}^B)\right)\mathbb{R}_N(\text{part}). \tag{A.17}$$

Here the sum is taken over all the partitions of set $\{\lambda^C\}_N$ into two disjoint subsets $\{\lambda_{pr}^C\}_n$ and $\{\lambda_{ab}^C\}_{N-n}$ and over partitions of set $\{\lambda^B\}_N$ into two disjoint subsets $\{\lambda_{pr}^B\}_n$ and $\{\lambda_{ab}^B\}_{N-n}$. These partitions are independent except that

$$\operatorname{card}\{\lambda_{pr}^C\}_n=\operatorname{card}\{\lambda_{pr}^B\}_n=n;\qquad \operatorname{card}\{\lambda_{ab}^C\}_{N-n}=\operatorname{card}\{\lambda_{ab}^B\}_{N-n}=N-n.$$

Product $\prod_{pr} l(\lambda_{pr}^C)$ denotes the product of n factors $l(\lambda_j^C)$; $\lambda_j^C\in\{\lambda_{pr}^C\}_n$. Product $\prod_{pr} l^{-1}(\lambda_{pr}^B)$ denotes the product of n factors $l^{-1}(\lambda_j^B)$; $\lambda_j^B\in\{\lambda_{pr}^B\}_n$. So form factor F_N^α is a linear function of each $l(\lambda_j^C)$ and a linear function of each $l^{-1}(\lambda_j^B)$. Coefficient

$\mathbb{R}_N(\text{part})$ does not depend on l and is a function of λ's only, having the following representation:

$$\mathbb{R}_N(\text{part}) = \sigma_n^\alpha(\{\lambda_{pr}^C\}_n, \{\lambda_{pr}^B\}_n)\sigma_{N-n}^\alpha(\{\lambda_{ab}^B\}_{N-n}, \{\lambda_{ab}^C\}_{N-n}\} \cdot \left\{\prod_{pr}\prod_{ab} f(\lambda_{pr}^C, \lambda_{ab}^C)\right\}\left\{\prod_{pr}\prod_{ab} f(\lambda_{ab}^B, \lambda_{pr}^B)\right\}. \tag{A.18}$$

Product $\prod_{pr}\prod_{ab}$ denotes the independent products over all $\lambda \in \{\lambda_{pr}\}$ and all $\lambda \in \{\lambda_{ab}\}$; this product contains $n(N-n)$ factors. Functions $\sigma_n^\alpha(\{\lambda^C\}_n, \{\lambda^B\}_n)$ $(n=0,1,\ldots)$ are functions of $2n$ rapidities $\lambda^{C,B}$ and are uniquely defined by the following properties.

(1) σ_n^α is a rational function of $\exp(\lambda_j)$; $\lambda_j \in \{\lambda^C\}_n \cup \{\lambda^B\}_n$.

(2) It depends on λ_n^C at other λ's fixed as follows [cf. with (A.4)]:

$$\sigma_n^\alpha = \exp(\lambda_n^C)\tilde{\sigma}_n^\alpha(\exp(2\lambda_n^C)). \tag{A.19}$$

Here $\tilde{\sigma}_n^\alpha$ is a rational function of $\exp(2\lambda_n^C)$ decreasing at $n \geqq 1$ at infinity:

$$\tilde{\sigma}_n^\alpha = O[\exp(-2\lambda_n^C)]\,(\exp(2\lambda_n^C) \to \infty; n \geqq 1). \tag{A.20}$$

(The dependence of σ_n^α on λ_n^B is similar).

(3) Function σ_n^α is a symmetric function of λ_j^C and a symmetric function of λ_j^B.

(4) The only singularities of $\tilde{\sigma}_n^\alpha$ (A.19) are first-order poles at

$$g^{-1}(\lambda_n^C, \lambda_j^B) = 0; \quad j = 1, 2, \ldots, n$$

[g is defined in (3.4)], the residue at $\lambda_n^C = \lambda_n^B = \lambda_n$ being equal to

$$\begin{aligned}\sigma_n^\alpha(\{\lambda^C\}_n, \{\lambda^B\}_n)|_{\lambda_n^C \to \lambda_n^B} &= g(\lambda_n^C, \lambda_n^B)\left\{\exp(\alpha)\prod_{j=1}^{n-1} f_{jn}^C f_{nj}^B - \prod_{j=1}^{n-1} f_{jn}^B f_{nj}^C\right\} \\ &\cdot \sigma_{n-1}^\alpha(\{\lambda^C\}_{n-1}, \{\lambda^B\}_{n-1}).\end{aligned} \tag{A.21}$$

(5) At $n=0,1$:

$$\sigma_0^\alpha \equiv 1; \quad \sigma_1^\alpha(\lambda^C, \lambda^B) = g(\lambda^C, \lambda^B)\{\exp(\alpha) - 1\}. \tag{A.22}$$

Remark also the following important property:

$$\sigma_n^\alpha(\{\lambda^B\}_n, \{\lambda^C\}_n) = \exp(n\alpha)\sigma_n^{-\alpha}(\{\lambda^C\}_n, \{\lambda^B\}_n). \tag{A.23}$$

To investigate irreducible part I_N of the meanvalue of operator $\mathbf{Q}_1^2$ one needs form factors of operators $\mathbf{Q}_1$ and $\mathbf{Q}_1^2$, which can be obtained from (A.14) by differentiation with respect to α at $\alpha=0$. One has then from (A.17) and (A.18):

$$\begin{aligned}F'_N &\equiv \langle\psi_N(\{\lambda^C\}_N|\mathbf{Q}_1|\psi_N(\{\lambda^B\}_N)\rangle \\ &= \left\{\prod_{j=1}^N l(\lambda_j^C)l^{-1}(\lambda_j^B) - 1\right\}\sigma'_N(\{\lambda^C\}_N, \{\lambda^B\}_N),\end{aligned} \tag{A.24}$$

where σ'_N is defined as

$$\sigma'_N(\{\lambda^C\}_N, \{\lambda^B\}_N) \equiv \partial\sigma_N^\alpha(\{\lambda^C\}_N, \{\lambda^B\}_N)/\partial\alpha|_{\alpha=0}. \tag{A.25}$$

Properties of functions σ', which is easily restored from properties (1)–(5) of functions σ^α, define them uniquely and allow us to calculate them. The first three functions are

$$\begin{aligned}&\sigma'_0=0; \quad \sigma'_1(\lambda^C,\lambda^B)=g(\lambda^C,\lambda^B);\\ &\sigma'_2(\{\lambda_1^C,\lambda_2^C\}_2,\{\lambda_1^B,\lambda_2^B\}_2)=2\cos 2\eta g^{-1}(\lambda_1^C+\lambda_2^C,\lambda_1^B+\lambda_2^B)\\ &\cdot g(\lambda_1^C,\lambda_1^B)g(\lambda_1^C,\lambda_2^B)g(\lambda_2^C,\lambda_1^B)g(\lambda_2^C,\lambda_2^B).\end{aligned} \tag{A.26}$$

The following property is obtained from (A.23)

$$\sigma'_N(\{\lambda^C\}_N,\{\lambda^B\}_N)=-\sigma'_N(\{\lambda^B\}_N,\{\lambda^C\}_N). \tag{A.27}$$

One has from (A.25) and (A.21) the following relation

$$\begin{aligned}&\sigma'_N(\{\lambda^C\}_N,\{\lambda^B\}_N)|_{\lambda_N^C\to\lambda_N^B}\\ &=g(\lambda_N^C,\lambda_N^B)\left\{\prod_{j=1}^{N-1} f_{jN}^C f_{Nj}^B-\prod_{j=1}^{N-1} f_{Nj}^C f_{jN}^B\right\}\sigma'_{N-1}(\{\lambda^C\}_{N-1},\{\lambda^B\}_{N-1}),\end{aligned} \tag{A.28}$$

which permits us to establish the behavior of σ' in the coupling constant η:

$$\begin{aligned}&\sigma'_N\sim\eta^{2N-1} \quad (\eta\to 0),\\ &\sigma'_N\sim[(\pi/2)-\eta]^{2N-1} \quad (\eta\to\pi/2),\\ &\sigma'_N\sim[(\pi/4)-\eta]^{N-1} \quad (\eta\to\pi/4).\end{aligned} \tag{A.29}$$

Turn now to the form factor of operator $\mathbf{Q}_1^2$:

$$F''\equiv\langle\psi_N(\{\lambda^C\}_N)|\mathbf{Q}_1^2|\psi_N(\{\lambda^B\}_N)\rangle. \tag{A.30}$$

Using (A.17) and (A.18) one has

$$\begin{aligned}F''_N=&\sigma''_N(\{\lambda^B\}_N,\{\lambda^C\}_N)+\left(\prod_{j=1}^N l(\lambda_j^C)l^{-1}(\lambda_j^B)\right)\sigma''_N(\{\lambda^C\}_N,\{\lambda^B\}_N)\\ &+2\sum_{\text{part}}^{1\leqq n\leqq N-1}\sigma'_n(\{\lambda_{pr}^C\}_n,\{\lambda_{pr}^B\}_n)\sigma'_{N-n}(\{\lambda_{ab}^B\}_{N-n},\{\lambda_{ab}^C\}_{N-n})\\ &\cdot\left\{\prod_{pr} l(\lambda_{pr}^C)l^{-1}(\lambda_{pr}^B)\right\}\left\{\prod_{pr}\prod_{ab} f(\lambda_{pr}^C,\lambda_{ab}^C)\right\}\left\{\prod_{pr}\prod_{ab} f(\lambda_{ab}^B,\lambda_{ab}^C)\right\}.\end{aligned} \tag{A.31}$$

The sum here is taken as is explained after (A.17) but we have written down explicitly the two terms corresponding to the partition $\{\lambda_{pr}^C\}=\emptyset$; $\{\lambda_{pr}^B\}=\emptyset$ and to the partition $\{\lambda_{pr}^C\}=\{\lambda^C\}_N$; $\{\lambda_{pr}^B\}=\{\lambda^B\}_N$. We denote $\operatorname{card}\{\lambda_{pr}^C\}_n=\operatorname{card}\{\lambda_{pr}^B\}=n$ and $\sigma''_n\equiv\partial^2\sigma_n^\alpha/\partial\alpha^2|_{\alpha=0}$.

Let us now consider irreducible parts using Eq. (A.16). One obtains from (A.24) that $I_N^{(1)}=0$, (4.18). For irreducible part $I_N^{(2)}\equiv I_N$ of the meanvalue of operator $\mathbf{Q}_1^2$ one gets from (A.31) representation (4.23). It should be noted that Fourier coefficients $\mathscr{A}_N^n$ $(n\geqq 1)$ in (4.23) are expressed in terms of functions σ' only, function σ'' entering only coefficient $\mathscr{A}_N^0$ which can be expressed as a linear combination of $\mathscr{A}_N^n$ $(n\geqq 1)$ due to (4.24). Using (A.29) one comes to (4.25).

Appendix 3

Some important properties of operator $\mathfrak{K}$ (2.10) and (2.11) are discussed here. At $0<2\eta\leqq\pi/2$ (2.2) and $\lambda=s+i\pi/2$, $\operatorname{Im}s=0$ (2.8) one easily obtains the following estimate for the quadratic form of matrix φ' (3.17):

$$\sum_{j,k} v_j\varphi'_{jk}v_k = M\sum_j v_j^2 u(s_j+i\pi/2) + \sum_{k<l}(v_k-v_l)^2 K(s_k,s_l) > M\sum_j v_j^2 u(s_j+i\pi/2)>0\,. \quad \text{(A.32)}$$

Here v_j is an arbitrary vector with real components; functions $K(s,t)$ and $u(s+i\pi/2)$ defined in (2.11) and (2.12) are positive at $0<2\eta<\pi/2$. Going to the thermodynamical limit $\left[\text{which corresponds to changing } \sum_j \to M\int_{-\Lambda}^{\Lambda} ds\varrho(s)\right]$ one has for arbitrary real function $\psi(s)$:

$$\int_{-\Lambda}^{\Lambda}\psi(s)\,[\delta(s-t)-(1/2\pi)K(s,t)]\psi(t)ds\,dt>(1/2\pi)\int_{-\Lambda}^{\Lambda}\psi^2(s)u(s+i\pi/2)\varrho^{-1}(s)ds\,. \quad \text{(A.33)}$$

Using this formula one gets that eigenvalues of operator $\mathfrak{K}$ satisfy the following inequality:

$$1-\varepsilon>|\mathfrak{K}/2\pi|>0\,, \quad \text{(A.34)}$$

where ε is positive constant (for $H>0$), going to zero as $H\to 0$.

It should be mentioned that the determinant of the linear integral operator $(1-\mathfrak{K}/(2\pi))$ [entering, e.g., (6.6)] is thus a finite positive number at $H>0$; as $H\to 0$ one has due to (2.14):

$$\det(1-\mathfrak{K}/(2\pi))|_{\Lambda\to\infty}=\exp\left\{-\frac{\Lambda}{\pi}\int_{-\infty}^{\infty}dk\left|\ln\left(1-\frac{\tilde{K}(k)}{2\pi}\right)\right|\right\}$$

and

$$\tilde{K}(k)=\int_{-\infty}^{\infty}e^{iks}K(s,0)ds=2\pi\sinh[(\pi-4\eta)k/2]/\sinh(\pi k/2)\,.$$

Appendix 4

Consider the following system consisting of the nonlinear integral equation and the inequality for function $P(t)$:

$$2\pi P(t)=\exp\{\alpha(t)+(\mathfrak{K}P)(t)\}-1 \quad (\operatorname{Re}\alpha(t)=0)\,, \quad \text{(A.35)}$$

$$\operatorname{Re}P(t)\leqq 0\,. \quad \text{(A.36)}$$

Operator $\mathfrak{K}$ is given by (2.10) and (2.11); given function $\alpha(t)$ is supposed to be pure imaginary, system (6.13) is obtained from (A.35) and (A.36) at $\alpha(t)=\alpha=\text{Const}$; for system (6.18) and (6.19) function $\exp\{\alpha(t)\}$ is given by the product of factors f in (6.18). Prove now that for $0<2\eta\leqq\pi/2$ (2.2) a solution of system (A.35), (A.36) does exist and that the solution is unique.

To prove the existence of a solution consider the following sequence of functions $P_k(t)$ $(k=0,1,2,\ldots)$:

$$P_0(t)\equiv 0;\qquad 2\pi P_{k+1}(t)=\exp\{\alpha(t)+(\mathfrak{K}P_k)(t)\}-1. \tag{A.37}$$

Due to the fact that kernel $K(s,t)$ (2.11) of operator $\mathfrak{K}$ is a positive function $[K(s,t)>0$ for $0<2\eta<\pi/2$ and $s,t\neq\pm\infty]$ one easily sees that $\operatorname{Re}P_k(t)\leqq 0$ for any k. Let us prove now that the sequence (A.37) converges which means that the limiting function $P(t)=\lim P_k(t)$ (at $k\to\infty$) exists and is a solution of (A.35) and (A.36). As $\operatorname{Re}\alpha(t)=0$, one has from (A.37):

$$|P_{k+1}(t)-P_k(t)|=|\exp\{(\mathfrak{K}P_k)(t)\}-\exp\{(\mathfrak{K}P_{k-1})(t)\}|/(2\pi).$$

Kernel $K(s,t)$ (2.11) of operator $\mathfrak{K}$ is positive for $0<2\eta<\pi/2$. So

$$\operatorname{Re}\{(\mathfrak{K}P_k)(t)\}\leqq 0$$

for any k. Using inequality $|\exp(z_1)-\exp(z_2)|\leqq|z_1-z_2|$ for $\operatorname{Re}z_1\leqq 0$, $\operatorname{Re}z_2\leqq 0$, one has

$$|P_{k+1}(t)-P_k(t)|\leqq|(\mathfrak{K}(P_k-P_{k-1}))(t)|/(2\pi)\leqq(\mathfrak{K}|P_k-P_{k-1}|)(t)/(2\pi).$$

Due to (A.34) this means that the limit of $P_k(t)$ at $k\to\infty$ exists. Thus the existence of the solution is proved. The uniqueness of the solution can be proved in a similar way. Suppose that two solutions $P^{(1)}(t)$ and $P^{(2)}(t)$ of (A.35) satisfying (A.36) do exist. Then the following inequality must be fulfilled:

$$|P^{(2)}(t)-P^{(1)}|\leqq(\mathfrak{K}|P^{(2)}-P^{(1)}|)(t)/(2\pi).$$

Multiplying both sides of this inequality by $|P^{(2)}(t)-P^{(1)}(t)|$ and integrating them over t, one comes to contradiction with (A.33) if $P^{(2)}(t)\neq P^{(1)}(t)$. So the existence and uniqueness of the solution of system (A.35), (A.36) is proved. It should be mentioned that if $\alpha(t)=\text{Const}$ and $\Lambda=\infty$ one has that $P(t)=\text{Const}$. Hence the solution of (6.13) for $\Lambda=\infty$ is particularly simple.

The most important property of the solution of system (A.35) and (A.36) is that if function $\alpha(t)$ is not equal to zero identically then the real part of function $P(t)$ is strictly less than zero at finite t:

$$\operatorname{Re}P(t)<0\quad(t\neq\pm\infty,\alpha(t)\not\equiv 0). \tag{A.38}$$

To prove it suppose that $\operatorname{Re}P(t_0)=0$ at some $t_0\neq\pm\infty$. Taking the modulus of both sides of (A.36), one has:

$$[1+(2\pi\operatorname{Im}P(t_0))^2]^{1/2}=\exp\left\{\int_{-\Lambda}^{\Lambda}K(t_0,s)\operatorname{Re}P(s)ds\right\}.$$

Due to property $K(t_0,s)>0$ it can be valid if and only if $\operatorname{Re}P(s)\equiv 0$ at any s. One can easily see then that $\operatorname{Im}P(s)\equiv 0$ at any s, and hence $P(s)\equiv 0$ at any s. This can be so only if $\alpha(s)\equiv 0$ at any s. Equation (A.38) is thus proved. The behavior of function $P(t)$ at $t\to\pm\infty$ is defined by the behavior of function $\alpha(t)$. In particular

$$P(t)\to 0 \text{ at } t\to\pm\infty \text{ if } \alpha(t)\to 0 \text{ at } t\to\pm\infty. \tag{A.39}$$

Equations (A.38) and (A.39) lead to Eqs. (6.20)–(6.22).

Now we list important properties of functions $P_n(t,\{\lambda^+\}_n,\{\lambda^-\}_n)$ which are solutions of (6.18) and (6.19):

(1) These functions are symmetric in all λ_j^+ and in all λ_j^- (separately). They also possess the following property:

$$P_n^*(t,\{\lambda^+\}_n,\{\lambda^-\}_n)=P_n(t,\{\lambda^-\}_n,\{\lambda^+\}_n). \tag{A.40}$$

(2) $$P_n(t,\{\lambda^+\}_n,\{\lambda^-\}_n)|_{\lambda_n^+=\lambda_n^-}=P_{n-1}(t,\{\lambda^+\}_{n-1},\{\lambda^-\}_{n-1})$$

(3) $$|P_n(t,\{\lambda^+\}_n,\{\lambda^-\}_n)|\leqq(1/\pi).$$

(4) $P_n\to 0$ at $2\eta\to\pi/2$, the first two terms of the expansion in $\varepsilon=(\pi/2)-2\eta$ being equal to:

$$\begin{aligned}P_n(t)=&-\frac{i\varepsilon}{\pi}\sum_{j=1}^{n}[\tanh(s_j^+-t)-\tanh(s_j^--t)]\\&-\frac{\varepsilon^2}{\pi}\left\{\sum_{j=1}^{n}[\tanh(s_j^+-t)-\text{tahn}(s_j^--t)]\right\}^2\\&+i\varepsilon^2\left\{\sum_{j=1}^{n}\left[\coth(s_j^-+t)-\coth(s_j^++t)\right.\right.\\&\left.\left.+\frac{(s_j^++t)}{\sinh^2(s_j^++t)}-\frac{(s_j^-+t)}{\sinh^2(s_j^-+t)}\right]\right\}+O(\varepsilon^3).\end{aligned} \tag{A.41}$$

(5) At $\eta\to 0$ function $P_n\neq 0$ only if $|t-s_j^\pm|=\eta$ $(j=1,\dots,n)$.

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Note added in proof: After this paper was sent to Commun. Math. Phys. we have learned that the correlation function for the XX-model (free fermions) at nonzero magnetic field $H>0$ was first calculated by B. Sutherland [Sutherland, B.: Correlation functions for two-dimensional ferroelectrics. Phys. Lett. **26**A, 532–533 (1968)].