

Correlation Risk and Optimal Portfolio Choice

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ABSTRACT

In this paper we solve an intertemporal portfolio problem with correlation risk, using a new approach for simultaneously modeling stochastic correlation and volatility. The solutions of the model are in closed form and include an optimal portfolio demand for hedging correlation risk. We calibrate the model and find that the optimal demand to hedge correlation risk is a non negligible fraction of the myopic portfolio, which often dominates the pure volatility hedging demand. The hedging demand for correlation risk is larger in settings with high average correlations and correlation variances. Moreover, it is increasing in the number of assets available for investment as the dimension of uncertainty with regard to the correlation structure becomes proportionally more important.

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THIS PAPER INVESTIGATES OPTIMAL INTERTEMPORAL PORTFOLIO DECISIONS in the presence of correlation risk. We study an incomplete markets economy in which the investment opportunity set is stochastic due to changes in both volatilities and correlations. In such an economy, the investor has a separate hedging demand for correlation risk. Depending on the economic scenario, we show that optimal portfolios can look significantly different from those obtained in a more common economic setting, in which the investment opportunity set is affected only by time-varying expected returns and/or volatilities.

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The practical importance of modeling correlation risk is becoming increasingly evident. To illustrate the portfolio impact of correlation risk, consider the case of a hedge fund with \$1*Bil* of assets trading in the late 1990s. Assume that the fund uses an unconstrained mean-variance portfolio strategy constructed using historical data (one year rolling windows with data sampled at daily frequency). Between April 1996 and early August 1998, the correlation coefficient between the 10-year US Treasury bond yield and the Aaa corporate bond yield ranged between 0.95 and 0.99. On August 17, 1998, Russia defaulted on its sovereign debt. Credit and liquidity spreads increased in all markets and the correlation between US Treasury and Aaa corporate bond yields rapidly declined reaching 0.80 in May 1999 (see Figure 1).

Insert Figure 1 about here.

In early August 1998, the average yields (standard deviation) of the two asset classes were 6.56% (0.65) and 7.24% (0.49) respectively. Based on historical observations and assuming a 5.2% risk free rate, the tangency portfolio would have been long \$3.44*Bil* in corporate bonds and short \$2.44*Bil* in Treasury bonds. The expected portfolio return and standard deviation would have been 10.118% and 0.455, respectively. In May 1999, however, the new tangency portfolio would have been long \$1.6*Bil* in corporate bonds and short \$0.6*Bil* in Treasury bonds. The expected portfolio return has changed from 24.2% to 12.2%. The change in the correlation coefficient alone would have induced a large portfolio reallocation and the hedge fund would have had to liquidate \$1.84*Bil* of corporate bonds.¹ The reallocation effect would have been even larger if the hedge fund had operated subject to a Value-at-Risk (VaR) target, since the optimal leverage would have been even lower after the decrease in correlation. Clearly, such large portfolio reallocations are not ex-ante optimal. If the fund manager had anticipated that the correlation coefficient between the two asset classes was not constant but stochastic, the optimal portfolio would have included a position to hedge unexpected changes in correlations. This paper investigates this issue.

A vast literature has explored the implications of stochastic volatility for portfolio choice. Little is known, however, about the impact of stochastic correlations. In part, this is because a sensible specification of a stochastic correlation process implies tight non-linear restrictions and boundary conditions on the asset return process: Correlations need to be bounded between -1 and $+1$ and the implied covariance matrix needs to be symmetric and positive definite. Therefore, a model with stochastic correlations can easily imply an analytically intractable covariance matrix process. In this paper, we follow a new approach to modeling correlation risk and directly specify the covariance matrix process as a matrix-valued affine diffusion. In this way, the model becomes tractable and the solutions of the intertemporal optimal portfolio problem are easy to interpret economically. The transition probabilities of the diffusion process follow a Wishart distribution and have been studied in Bru (1991) and Gouriéroux, Jasiak, and Sufana (2004). Wishart distributions have also been used in the statistics literature to model Bayesian priors on second moments. The optimal portfolio

¹The reader may find this example evocative of the LTCM debacle. See P. Jorion (1999) for a more detailed discussion.

implications, however, are not known. Since the correlation process is stochastic, we consider an incomplete markets economy in which a constant relative risk aversion agent maximizes utility of terminal wealth. This setting allows us to investigate the effect of the investment horizon on the optimal holdings in risky assets. We solve the model in closed-form and characterize the hedging demand against correlation risk in the optimal portfolio. We then use the model solutions to address a number of questions:

(a) What is the economic importance of correlation risk in optimal portfolio choice? We calibrate the model to historical data on international equity and US bond returns and find that, even for a small number of assets, the hedging demand for correlation risk is larger than the hedging demand for pure volatility risk: Correlation risk has an important impact on optimal portfolio weights. This impact increases when the number of assets grows, since the dimension of uncertainty with regard to the correlation structure becomes proportionally more important.

(b) How sensitive is the correlation hedging demand to the average level and variance of correlations? We find that the correlation hedging demand is larger for high average correlations and for high correlation variances. The economic intuition for the first effect is as follows. If correlations between assets are high and constant, the optimal portfolio will tend to build uneven positions in the assets. It is under these conditions that unexpected changes in the correlation coefficients have the highest potential impact on reducing ex-post utilities. Therefore, the desire to hedge these shocks is higher and the correlation hedging demand tends to be larger. Moreover, the higher is the conditional volatility of the correlation, the larger is the average impact on the utility of the optimal portfolio. Therefore, the correlation hedging demand is larger.

(c) How do both the optimal investment in risky assets and the correlation hedging demand vary with respect to the investment horizon? This question plays an important role in optimal life-cycle decisions as well as for pension fund managers. We find that the absolute correlation hedging demand increases with the investment horizon. If the correlation hedging demand is positive (negative), this feature implies an optimal investment in risky assets that increases (decreases) in the investment horizon.

(d) What is the link between the persistence of correlation shocks and the demand for correlation hedging? The persistence of correlation shocks varies across markets. In highly liquid markets like the Treasury and foreign exchange markets, which are less affected by private information issues, correlation shocks are less persistent. In other markets, frictions – such as asymmetric information and differences in beliefs about future cash-flows – make price deviations from equilibrium more difficult to be arbitrated away. Examples include both developed and emerging equity markets. Consistent with this intuition, we find that the optimal hedging demand against correlation risk increases with the extent of correlation shocks persistence. For example, it is higher for equity portfolios than for portfolios formed out of long-term Treasury bonds and high credit quality US corporate bonds.

Time-varying correlation can become an important source of risk with wide ranging economic implications. Moskowitz (2003), for instance, argues that some pricing anomalies such as momentum

and size effect can be explained by stochastic correlations. Driessen, Maenhout, and Vilkov (2006) document that the implied volatility smile is flatter for individual stock options than for index options and attribute the difference to a priced correlation risk factor. The interest in modeling stochastic correlation has also been spurred by financial innovation. Collateralized Debt Obligations (CDOs) consist of pools of Credit Default Swaps (CDSs) where tranching can create flexible default risk profiles. Since most of these new products involve a portfolio of firms, the time-variation of the correlations is a primary source of pricing and risk management issues. The number of financial instruments whose value depends directly on the correlation process is so large that they form a separate asset class: "correlation derivatives". In this class we find, for example, contracts designed to generate exposure to a foreign financial index (either a foreign interest rate or a stock market index) but with a payoff denominated in the domestic currency. Thus, the pricing, hedging, and risk management of these instruments explicitly depend on the correlation between the foreign index and the exchange rate. Other correlation derivatives include foreign-exchange quanto futures and options such as the Nikkei derivatives traded on the CME. Additional examples are differential swaps, basket options, and rainbow options (e.g. maximum options, minimum options, spread options).

This paper draws upon a large literature on optimal portfolio choice under a stochastic investment opportunity set. One set of papers studies optimal portfolio and consumption problems with a single risky asset and a riskless deposit account.² Kim and Omberg (1996) solve the portfolio problem of an investor optimizing utility of terminal wealth, where the risk-less rate is constant and the risky asset has a mean reverting Sharpe ratio and constant volatility. Wachter (2002) extends this setting to allow for intermediate consumption and derives closed-form solutions in a complete markets setting. Chacko and Viceira (2005) relax both the assumption on the preferences and the volatility. They consider an infinite horizon economy with Epstein-Zin preferences, in which the risky asset volatility follows a mean reverting square-root process. Liu, Longstaff, and Pan (2003) model events affecting market prices and volatility, using the double-jump framework in Duffie, Pan, and Singleton (2000). They show that the optimal policy is similar to that of an investor facing short-selling and borrowing constraints, even if none is imposed. Although their approach allows for a rather general model with stochastic volatility, they focus on a single risky asset economy. We contribute to this literature by investigating an economy with multivariate risk factors, in which the correlation between factors is stochastic and acts as an independent source of risk. Moreover, we investigate the optimal portfolio implications when markets are incomplete. This aspect is especially important when volatilities and correlations are stochastic, as it limits the ability of the portfolio manager to span the state-space using portfolios of marketed assets. In order to derive closed-form solutions, however, we work with CRRA preferences. This assumption is more restrictive than Chacko and Viceira (2005), who consider a more general set of Epstein-Zin preferences.

Portfolio selection problems with multiple risky assets have been considered in a further series of papers. The majority of these, however, assumes that volatilities and correlations are constant.

²Since the available instruments for investment consist of two assets, the resulting portfolio optimization problem is effectively univariate, because the budget constraint allows to eliminate one portfolio weight in the optimization problem.

Examples include Brennan and Xia (2002), who study optimal asset allocations under inflation risk, and Sangvinatsos and Wachter (2005), who investigate the portfolio problem of a long-run investor with both nominal bonds and stocks. A notable exception to the constant volatility assumption is Liu (2007), who shows that the portfolio problem can be characterized by a sequence of differential equations in a model with quadratic returns³ and under additional assumptions. To solve in closed-form a concrete model with a risk-less asset, one risky bond and a stock, he assumes independence between the state variable driving pure term structure risk and the additional risk factor influencing the volatility of the stock return. In that model correlations are stochastic, but are restricted to being functions of stock and bond returns volatilities. Therefore, optimal hedging portfolios do not allow volatility and correlation risk to have separate roles. Our setting avoids deterministic dependencies between volatilities and correlations. Moreover, it can be used to analyze portfolio choice problems of, essentially, arbitrary dimension.

We model the stochastic covariance matrix of returns using a single-regime mean-reverting diffusion process, in which the strength of the mean reversion can generate different degrees of persistence in volatilities, correlations and co-volatilities. To obtain closed-form portfolio solutions, we refrain from introducing also an unpredictable jump component in the joint process for returns and correlations. This approach allows us to study the properties of the optimal hedging demand under a persistent correlation process.⁴ A completely different approach to modeling co-movement in portfolio choice relies on either a Markov switching-regime in correlations or on the introduction of a sequence of unpredictable joint Poisson shocks in asset returns. Ang and Bekaert (2002) consider a dynamic portfolio model with two i.i.d. switching regimes, one of which is characterized by higher correlations and volatilities. Using numerical methods, they find that when the international portfolio manager has access to a risk-free asset, the optimal portfolio is significantly sensitive to asymmetric correlations between the two regimes. Our model is different from theirs because we model an incomplete markets economy in which a single regime features persistent volatility and correlation shocks. Moreover, the analytical solutions of the optimal portfolio allow us to study the contribution of the different hedging demands for volatility and correlation risk to the overall portfolio. Since the solutions hold for an arbitrary number of assets, we can also easily study the behavior of correlation hedging as the number of risky assets increases. In this case, the extent of uncertainty with regard to the correlation structure becomes proportionally more important. Das and Uppal (2004) study systemic risk, modeled as an unpredictable common Poisson shock, in a setting with a constant opportunity set and in the context of international equity diversification. They show that systemic risk reduces the gain from diversification and penalizes the investor from holding levered positions. The structure of systemic risk in their model makes changes in the correlation between assets unpredictable and transitory. In our model, correlation and volatility shocks are persistent.

³I.e., the interest rate, the maximal squared Sharpe ratio, the hedging coefficient vector, and the unspanned covariance matrix are all quadratic functions of a state variables process with quadratic drift and diffusion coefficients.

⁴Persistence of second moments has also proven to be an important dimension to “read” traditional asset pricing puzzles through: See, among others, Barsky and De Long (1990, 1993), Bansal and Yaron (2004) and Parker and Julliard (2005).

Thus, they generate a motive for intertemporal hedging. These features yield substantially different portfolio choice implications.

This article provides theoretical results that may prove useful to interpret the empirical evidence emerging from the literature on hedge fund performance. Increasing evidence shows that, after controlling for market risk, hedge fund alphas are significantly positive and persistent.⁵ For instance, Kosowski, Naik, and Teo (2006) document that the average alpha of 771 long/short hedge funds is 0.51 on a monthly basis, with funds in the top decile showing alphas larger than 1.41. They test and find that the performance is persistent and not due to luck. Clearly, part of this performance can be attributed to managerial ability. However, our results suggest that part of these excess returns may also compensate for the exposure to correlation risk, which is intrinsic to a long/short strategy.

This article also relates to the multivariate GARCH-literature.⁶ Pioneering models in this field, as for instance Bollerslev (1987) and Bollerslev, Engle, and Wooldridge (1988), either restrict the correlation to be constant or do not necessarily imply a positive definite covariance matrix.⁷ Further important contributions include Harvey, Ruiz, and Shephard (1994), who specify a model with correlation dynamics that are driven by the same factors affecting the volatility, and Barndorff-Nielsen and Shephard (2004). An important feature of our model is precisely that correlations can have dynamics not fully correlated with the factors affecting the volatility processes. Many recent multivariate GARCH-models ensure a positive definite covariance matrix that can be estimated by a computationally feasible estimation procedure. Engle (2002) proposes a Dynamic Conditional Correlation (DCC) specification with time-varying correlations and positive definite covariance matrices, which builds upon the estimation of a set of univariate GARCH processes. The DCC-model and those that extend it to include, for instance, volatility and correlation asymmetries, are analytically intractable for dynamic portfolio choice purposes. As in multivariate GARCH-settings, our model incorporates a persistence in volatilities and correlations. However, our model preserves the tractability required to study the resulting optimal portfolio strategies analytically.

The article proceeds as follows: Section I summarizes the empirical properties of the correlation between different assets and the potential implications for portfolio choice. Section II describes the model, the main properties of the implied correlation process, and the solution to the portfolio problem. In Section III, we calibrate the model to historical data and quantify the portfolio impact of correlation risk. Section IV discusses some extensions of the model, which include stochastic interest rates and a time-varying predictability of expected returns, in the spirit of Barberis (2000) and Wachter (2002). Section V concludes. All proofs are in the Appendix.

⁵Papers documenting hedge fund performance include Brown, Goetzmann, and Ibbotson (1999), Fung and Hsieh (2001), Mitchell and Pulvino (2001), Agarwal and Naik (2004), Getmansky, Lo, and Makarov (2004), Busse and Irvine (2005), Kosowski, Naik, and Teo (2006).

⁶See, for instance, Poon and Granger (2003) for a review.

⁷A well known additional issue of these specifications is the "curse of dimensionality". For n assets, one needs to model $\frac{n(n+1)}{2}$ elements of the covariance matrix, which implies that the matrices A and B have $\frac{1}{4}n^2(n+1)^2$ elements.

I. The Time Variation of Correlations

The correlation structure of world markets varies over time. Using data ranging from 1850 to 2005 on 84 international equity markets, Goetzmann, Li, and Ronwenhorst (2005) find average correlations ranging over time from -0.07 to 0.47. Over 1870-1913, correlations were very high. This period is called the "golden age of capitalism" (see Rajan and Zingales, 2001) and was characterized by high average per capita equity market capitalizations and relatively integrated financial markets. "Following this peak" - Goetzmann, Li, and Rouwenhorst (2005) argue - "the only constant is change". The average pairwise correlation among the four major markets (France, Germany, U.S, and U.K.) became -0.073 (1915-1918), 0.228 (1919-1939), 0.046 (1940-1945), 0.110 (1946-1971), and 0.475 (1972-2000).⁸ Not surprisingly, the largest negative correlation has been between the U.S. and Germany during World War II. De Santis, Litterman, Vesval, and Winkelman (2003) confirm these results using daily data and conclude that "both volatilities and correlations vary over time. In addition, volatilities and correlations react with different speed to market news and may follow different trends."⁹

An important strand of the literature has explored in more detail the characteristics of this time-variation. Longin and Solnik (1995) reject the null hypothesis of constant international stock market correlations and find that they increase in periods of high volatility. Ledoit, Santa-Clara, and Wolf (2003) find that the level of correlation depends on the phase of the business cycle. Moreover, Erb, Harvey, and Viskanta (1994) find that international markets tend to be more correlated when countries are simultaneously in a recessionary state. Moskowitz (2003) documents that covariances across portfolio returns are highly correlated with NBER recessions and that average correlations are highly time-varying. Ang and Chen (2002) show that the correlation between US stocks and the aggregate US market is much higher during extreme downside movements than during upside movements. Longin and Solnik (2001) and Barndorff-Nielsen and Shephard (2004) find similar results. Another important strand of the literature has provided direct evidence that market integration and financial liberalization change the correlation of emerging markets' stock returns with a global stock market index (Bekaert and Harvey, 1995, 2000). The implication is that economic policies changing the degree of market integration have structural effects on the comovement of financial markets.

To appreciate the time variation in the correlations of some asset returns, Figure 2 plots the estimated conditional correlations based on Engle's (2002) Dynamic Conditional Correlations (DCC)

⁸They use the Browne and Shapiro (1986) and Neudeck and Wesselman (1990) asymptotic distribution to run two separate tests for the properties of international equity correlations. First, they run an element-by-element χ^2 test of the equality on the vectorized correlation matrix. Second, they test whether the average correlation is constant. They strongly reject both null hypotheses.

⁹To explain differences in international equity correlations, Roll (1992) suggests a Ricardian model based on country industrial specialization. Heston and Rouwenhorst (1994) find, however, that explanations based on fiscal, monetary, legal, cultural, and language differences dominate explanations based on industry effects. A different interpretation is investigated by Ribeiro and Veronesi (2002), who propose a model where excess stock comovements during bad times are obtained endogenously as a reflection of higher uncertainty.

model for US and German equity index daily returns (top panel), as well as for 10 year Treasury and Aaa-corporate bond index daily returns (bottom panel), in the periods 1988–2005 and 1996–2005 respectively.

Insert Figure 2 about here.

In both cases, the estimated parameters in the DCC correlation dynamics are significant at standard significance levels, highlighting some persistence in correlations. In the top panel of Figure 2, the unconditional correlation between US and German index returns is about 0.37. The extent of time variation, however, is large: Estimated correlations range between 0.1 in the late eighties and more than 0.6 towards the end of the sample. Although correlations tend to spike in times of extreme market distress, they are generally quite persistent. In the bottom panel of Figure 2, the average correlation of Treasury bills and Aaa-bond returns is about 0.88. Before 1998, bond correlations were very stable and even higher, most of the time, than their unconditional mean. After the 1998 Russian crisis, correlations virtually collapsed to a lowest value of less than 0.4 and became highly time-varying, to recover to a level close to their unconditional value only towards the end of 2003. Bond correlations also tend to be persistent, with occasional spikes during extreme market distress, as in the most recent international financial crises, but the persistence is lower than the one observed for stocks in our example. Since the results on the persistence of the correlation process might be influenced by the two stage nature of the DCC, we check their robustness by using the rolling windows estimator with correction for serial correlation suggested by De Santis et al. (2003). The results are similar.¹⁰

The higher persistence of correlation shocks between stocks is economically plausible: The higher liquidity of the Treasury bond market and the higher uncertainty of stocks future cash flows is likely to make cross-sectional arbitrage trades between Treasury and corporate bonds more effective, at least outside phases of severe credit crises. Thus, shocks to price differentials in the equity market tend to be more persistent. For this reason, in our model a distinct set of parameters directly controls for the persistence of correlation and volatility shocks. An additional by-product of our specification is to allow for correlation shocks that are not perfectly spanned by shocks in volatilities. This is consistent with the empirical evidence discussed in the literature and highlighted by our two examples above. The potentially large persistence of the correlation processes of some asset classes, along with the imperfect co-dependence of volatilities and correlations, suggest that the implications of correlation risk for portfolio choice can be more important than those of stochastic volatility, which some authors have argued to be small (see Chacko and Viceira, 2005).

¹⁰De Santis et al. (2003) find that the average half-life of the correlation process for the 18 international equity markets with the largest stock market capitalization is about 17 months. Moreover, our DCC setting does not account for potential long-memory patterns in the volatility and correlation processes, which many authors have successfully modeled in several applications, typically using long time series of financial data; see, e.g., Poon and Granger (2003) and Andersen et al. (2005) for a review. It is likely that the impact of correlation risk for optimal portfolio choice under a long-memory dependence will be larger than the one under a short memory.

II. The Model

An investor with Constant Relative Risk Aversion utility over terminal wealth trades three assets, a riskless asset with instantaneous riskless return r , and two risky assets, in a continuous-time frictionless economy on a finite time horizon $[0, T]$. Our analysis extends to opportunity sets consisting of any number of risky assets and correlation factors, without affecting the existence of closed-form solutions and their general structure. We focus on the two-dimensional setting, in order to preserve the key economic intuition about optimal portfolio choice with stochastic correlations from notational complications.

A. The Portfolio Allocation Problem

The investment opportunity set can be stochastic both because of changes in expected returns and changes in conditional variances and covariances. It is well-known that in order to obtain tractable solutions, one needs to impose restrictions on either the functional form of the squared Sharpe-ratio, or the maximal squared Sharpe-ratio in incomplete markets. For instance, in standard settings affine maximal squared Sharpe-ratios imply affine solutions. Given an affine state variable process, two options are available. The first is to assume a constant risk premium and an affine inverse covariance matrix process (see also Chacko and Viceira, 2005); the second is to assume time varying expected returns with an affine covariance matrix process and a constant price of risk (see also Liu, 2001). In this article, we study the portfolio impact of correlation risk in both settings. First, we solve a model in which the maximal squared Sharpe ratio is affine in an affine covariance matrix process. In this setting, we can more easily interpret the model parameters because the relevant state variable is the covariance matrix of returns itself. Second, in Section IV we study the implications of correlation risk in a model with constant risk premia and an affine inverse covariance matrix process. In the first model, squared Sharpe ratios are increasing functions of volatilities, but can be increasing or decreasing in correlation, depending on the sign of the prices of risk. In the second model, squared Sharpe ratios are always decreasing in volatilities and correlation if all assets pay a positive risk premium.

The cum dividend evolution of the price vector $S = (S_1, S_2)'$ of the risky opportunities is described by the bivariate stochastic differential equation:

$$dS(t) = I_S \left[(r\bar{I}_2 + \Lambda(\Sigma, t))dt + \Sigma^{1/2}(t) dW(t) \right] \quad ; \quad I_S = \text{Diag}[S_1, S_2], \quad (1)$$

where $r \in \mathbb{R}_+$, $\bar{I}_2 = (1, 1)'$, W is a standard two-dimensional Brownian motion and $\Sigma^{1/2}$ is the positive square root of the conditional covariance matrix Σ of returns. The available investment opportunity set is stochastic, because of the time varying market price of risk $\Sigma^{-1/2}(t)\Lambda(\Sigma, t)$, which is a function of the stochastic covariance matrix Σ . The constant interest rate assumption is generalized in the model extensions discussed at the end of the paper. The diffusion process for Σ is detailed below. Let $\pi(t) = (\pi_1(t), \pi_2(t))'$ denote the proportion of wealth $X(t)$ invested in the first and the second risky asset. An agent's wealth evolves as:

$$dX(t) = X(t) \left[r + \pi(t)'\Lambda(\Sigma, t) \right] dt + X(t)\pi(t)'\Sigma^{1/2}(t)dW(t). \quad (2)$$

The agent selects the portfolio process π in order to maximize CRRA utility of terminal wealth with RRA coefficient $1 - \gamma$. If $X_0 = X(0)$ denotes the initial wealth, and $\Sigma_0 = \Sigma(0)$ denotes the initial covariance matrix, the investor's optimization problem is:

$$J(X_0, \Sigma_0) = \sup_{\pi} \mathbb{E} \left[\frac{X(T)^\gamma - 1}{\gamma} \right], \quad (3)$$

subject to the dynamic budget constraint (2). This setting allows us to investigate how the optimal portfolio allocation problem varies over the life-cycle of the agent.

To model stochastic covariance matrices in a convenient way, we make use of the continuous-time process introduced by Bru (1991) and studied by Gouriéroux and Sufana (2004) and Gouriéroux, Jasiak, and Sufana (2004). This diffusion process is a matrix-valued extension of the univariate square-root process that gained popularity in the term structure and the stochastic volatility literature; see, e.g., Cox, Ingersoll, and Ross (1985) and Heston (1993). Let Z be a bivariate standard Brownian motion independent of W and define $B(t) = [W(t) \ Z(t)]$ as a 2×2 matrix-valued standard Brownian motion. The diffusion process for Σ is defined as:

$$d\Sigma(t) = [\Omega\Omega' + M\Sigma(t) + \Sigma(t)M'] dt + \Sigma^{1/2}(t)dB(t)Q + Q'dB(t)'\Sigma^{1/2}(t), \quad (4)$$

where Ω , M , Q being 2×2 square matrices (with Ω invertible).

This process satisfies five important properties that make it ideal to model stochastic correlation. First, it implies that if $\Omega\Omega' \geq QQ'$ then Σ is a well defined covariance matrix process. Under this condition, the implied correlation process is well behaved and bounded between -1 and $+1$. Second, if $\Omega\Omega' = kQQ'$ for some $k > 1$ then $\Sigma(t)$ follows a Wishart distribution; see Bru (1991). This distribution has been studied in Bayesian statistics to model priors on multivariate second moments, but it has never been used to study optimal portfolio choice. Third, the process (4) is affine in the sense of Duffie and Kan (1996) and Duffie, Filipovic, and Schachermayer (2003). This feature implies closed-form expressions for all conditional Laplace transforms. Fourth, if $d \ln S_t$ is a vector of returns with Wishart covariance matrix $\Sigma(t)$, then the variance of the return of a portfolio π is a Wishart process. This is generally not the case for settings in which volatilities and correlations are modelled by a multivariate GARCH process because GARCH models are not invariant under linear aggregation. Fifth, process (4) is flexible enough to fit many of the empirical features of financial asset returns, such as leverage and co-leverage, which are found to be important empirical properties in the literature.

The only thing left to specify is the risk premium $\Lambda(\Sigma, t)$ as a function of the state variables. To motivate a choice for the form of $\Lambda(\Sigma, t)$, we notice that under a power utility function, in a Breeden's (1979) consumption-based model, the price of risk of an asset with returns dS/S is equal to $(1 - \gamma)Cov_t[dC/C, dS/S]$, where dC/C is the growth rate of aggregate consumption. If aggregate consumption follows $dC/C = \mu_C dt + a'\Sigma^{1/2}dBb$, where μ_C is the drift of consumption growth and a and b are 2×1 vectors, then the risk premium of the i -th asset in our economy is given by $(1 - \gamma)Cov_t[dC/C, dS^i/S^i] = (1 - \gamma) [a'\Sigma^{1/2}dB(t)b, e_i'\Sigma^{1/2}dB(t)e_1]$, where e_i is the i -th unit vector. Using the property that $Cov_t[dBa, dBb] = a'bIdt$, where I is the 2×2 identity matrix,

it is easy to show that the risk premium is affine in $\Sigma(t)$. This result is more general and holds in any economy whose stochastic discount factor has a diffusion term equal to $a'\Sigma^{1/2}dB(t)b$. Thus, we consider economies in which the vector of market prices of risk is linear in $\Sigma(t)$, that is $\Lambda(\Sigma, t) = \Sigma\lambda$ where $\lambda = (\lambda_1, \lambda_2)' \in \mathbb{R}^2$. The same assumption is made in Heston (1993) and Liu (2001) in the case of a scalar economy.

The first challenge in solving the investment problem (3) subject to the covariance matrix dynamics (4) is that markets are incomplete because of the stochastic covariance matrix. This feature is due to the fact that the two-dimensional Brownian motion Z , appearing in the second column of the matrix of Brownian shocks in the covariance dynamics (4), is independent of the Brownian motion W in the returns dynamics (1). It follows that a multiplicity of equivalent martingale measures exists in our model. To solve the portfolio problem, we consider the dual value function characterization implied by the minimax martingale measure. He and Pearson (1991) prove that this value function can be characterized in terms of the following static problem:¹¹

$$J(X_0, \Sigma_0) = \inf_{\nu} \sup_{\pi} \mathbb{E} \left[\frac{X(T)^\gamma - 1}{\gamma} \right], \quad (5)$$

$$\text{s.t. } \mathbb{E} [\xi_{\nu}(T)X(T)] \leq x, \quad (6)$$

where ν indexes the set of all equivalent martingale measures in the model and ξ_{ν} is in the set of associated state price densities.

In a constant covariance setting with complete markets, the market prices of risk associated with the Brownian innovations W are simply equal to $\Theta = \Sigma^{1/2}\lambda$. When markets are incomplete, He and Pearson (1991) show that each admissible market price of risk can be written as the sum of two orthogonal components, one of which is spanned by the asset returns. Since in our setting there are no frictions, the first component is simply given by $(\lambda'\Sigma^{1/2}, \underline{0})'$ – with $\underline{0}$ a two-dimensional vector of zeros – which prices the shocks to asset returns. The second component is the vector $(\underline{0}', \nu'\Sigma^{1/2})'$, where ν is the two-dimensional vector pricing the shocks that are independent of the asset returns. Let Θ_{ν} be the matrix-valued extension of Θ that prices the matrix of Brownian motions $B = [W, Z]$: $\Theta_{\nu} = \Sigma^{1/2}[\lambda, \nu] = \Sigma^{1/2}(\lambda e_1' + \nu e_2')$. Given Θ_{ν} , the associated martingale measure $\xi_{\nu}(T)$ takes the form:

$$\xi_{\nu}(T) = \exp \left(- \int_0^T \left(r(s) + \frac{1}{2} \text{tr}(\Theta_{\nu}'(s)\Theta_{\nu}(s)) \right) ds - \int_0^T \text{tr}(\Theta_{\nu}(s)'dB(s)) \right),$$

where $\text{tr}(\cdot)$ is the trace operator. In addition, it is well-known that the optimality condition for the optimization over π in problem (5) is $X(T) = (\psi\xi_{\nu}(T))^{\gamma/(\gamma-1)}$, where ψ is the multiplier of the constraint (6). Therefore, problem (5) can be written as:

$$J(X_0, \Sigma_0) = \inf_{\nu} \mathbb{E} \left[\frac{(\psi\xi_{\nu}(T))^{\gamma/(\gamma-1)}}{\gamma} \right] - \frac{1}{\gamma} = X_0^\gamma \inf_{\nu} \frac{1}{\gamma} \mathbb{E} \left[\xi_{\nu}(T)^{\gamma/(\gamma-1)} \right]^{1-\gamma} - \frac{1}{\gamma},$$

and we can focus without loss of generality on the solution of the problem:

$$\widehat{J}(0, \Sigma_0) = \inf_{\nu} \mathbb{E} \left[\xi_{\nu}(T)^{\gamma/(\gamma-1)} \right]. \quad (7)$$

¹¹See also Pliska (1986) and Cox and Huang (1989) for the Markovian complete markets case.

To solve this problem, we solve a corresponding Hamilton-Jacobi-Bellman equation after a change of measure from the physical probability P to a new probability P^γ . P^γ is defined by the following Radon-Nykodim derivative with respect to P :

$$\frac{dP^\gamma}{dP} = \exp \left(-\frac{\gamma}{\gamma-1} \int_0^T \text{tr}(\Theta'_\nu(s)dB(s)) + \frac{\gamma^2}{2(\gamma-1)^2} \int_0^T \text{tr}(\Theta'_\nu(s)\Theta_\nu(s))ds \right).$$

This is the Radon-Nykodim derivative that allows us to remove the stochastic integral in the exponential inside the expectation of optimization problem (7):

$$\widehat{J}(0, \Sigma_0) = \inf_\nu \mathbb{E}^\gamma \left[e^{-\frac{\gamma}{\gamma-1} \int_0^T r(s)ds + \frac{\gamma^2}{2(\gamma-1)^2} \int_0^T \text{tr}(\Theta'_\nu(s)\Theta_\nu(s))ds} \right], \quad (8)$$

where $\mathbb{E}^\gamma[\cdot]$ denotes expectations under the probability P^γ . Note that the expression $\text{tr}(\Theta'_\nu(s)\Theta_\nu(s))$ inside the expectation in equation (8) is affine in $\Sigma(s)$.

Results in Schroder and Skiadas (2003) imply that if the original optimization problem has a solution, the value function of the static problem coincides with the value function of the original problem. The above equality holds for all times, and not just at time 0. Cvitanic and Karatzas (1992) have shown that the solution to the original problem exists under additional restrictions on the utility function, most importantly that the relative risk aversion does not exceed one. Cuoco (1997) proves a more general existence result, imposing minimal restrictions on the utility function.

B. Properties of the Variance-Covariance Process

To appreciate the properties of the covariance process implied by the dynamics of (4), we study in this section the implied correlations' dynamics and the resulting co-dependence features of returns' volatilities and correlations.

B.1. The correlation process

To study the correlation process under a Wishart diffusion (4), we can use Itô's Lemma to compute the correlation dynamics.

Proposition 1 *Let ρ be the correlation diffusion process implied by the covariance matrix dynamics (4). The instantaneous drift and conditional variance of $d\rho(t)$ are given by:*

$$\mathbb{E}_t[d\rho(t)] = [A(t)\rho(t)^2 + B(t)\rho(t) + C(t)] dt, \quad (9)$$

$$\mathbb{E}_t[d\rho(t)^2] = [(1 - \rho^2(t)) (E(t) + G(t)\rho(t))] dt, \quad (10)$$

where coefficients A , B , C , E , G depend exclusively on Σ_{11} , Σ_{22} and the model parameters Ω , M and Q .¹²

¹²The explicit expression for the correlation dynamics is derived in Appendix A.

Even if the covariance matrix process is affine, the correlation dynamics are not affine since the correlation is a nonlinear function of variances and covariances. By construction, the instantaneous returns correlation $\rho = \Sigma_{12}/\sqrt{\Sigma_{11}\Sigma_{22}}$ is bounded in the interval $[-1, 1]$: No explicit additional constraint on the correlation process is needed in order to ensure a well-defined return covariance matrix process. The instantaneous drift and volatility of the correlation process are quadratic and cubic in ρ , respectively. The correlation dynamics are not autonomous: Both the drift and the instantaneous variance have coefficients that depend on the level of the volatilities of the first and second asset return.

The correlation drift and the correlation volatility implied by the covariance matrix process (4) are illustrated in Figure 3, where we make use of parameters calibrated to the estimated DCC dynamics plotted in Figure 2.¹³

Insert Figure 3 about here

The calibrated correlation volatility functions are zero at the boundaries, when $\rho = 1, -1$, and peak in both examples at a correlation level close to zero. The calibrated correlation drifts are nonlinear functions of ρ , which are positive for a broad set of correlation values. Both for stocks and bonds the correlation drift crosses the zero line approximately at the level of the sample unconditional correlation. For correlation values below (above) this level the drift is positive (negative) and induces a mean reversion.¹⁴ The positive unconditional correlations imply an asymmetry in the speed of mean reversion over the support of $\rho(t)$, which is more pronounced in the example with Treasury and corporate bonds. We can study more precisely the mean reversion properties of our (non-linear) correlation process by computing its pull function - see Conley, Hansen, Luttmer, and Scheinkman (1997). The pull function $\wp(x)$ of a nonlinear diffusion process X is the conditional probability that X_t reaches the value $x + \epsilon$ before $x - \epsilon$, if initialized at $X_0 = x$. To first order in ϵ , this probability is given by:

$$\wp(x) = \frac{1}{2} + \frac{\mu_X(x)}{2\sigma_X^2(x)} + o(\epsilon), \quad (11)$$

where μ_X and σ_X are the drift and the volatility function of X . It follows that the ratio $\frac{\mu_X(x)}{2\sigma_X^2(x)}$ is a conditional measure of the speed of mean reversion for nonlinear diffusion processes. We can get more insight into the mean reversion properties of the correlation by studying this ratio for the correlation process implied by the calibrated covariance matrix dynamics. The pull functions of the calibrated correlation processes, shifted by the factor $1/2$ in equation (11), are given in Panels 5 and 6 of Figure 3. They are highly asymmetric around the unconditional mean of the correlation. For bonds, the pull function is very large, in absolute value, when the correlation is slightly above the unconditional mean. Over a broad range of low correlation values, the pull function is much lower in absolute value. Therefore, correlation shocks are on average more persistent below the unconditional

¹³Details on the calibration are provided in Section III.

¹⁴As expected, this implies that for perfect positive and negative correlations the drift is negative and positive, respectively.

mean of the correlation. When we calibrate the model to international equity indices, the average level of the pull function is smaller. This feature reflects the larger persistence of correlation shocks in these markets. The asymmetric shape of the pull function is similar to the bond markets case.

The properties of the correlation process have obvious implications for the shape of the unconditional distribution of the correlation. The stronger the asymmetry of the mean reversion, the larger the asymmetry of the unconditional distribution of the correlation process. Figure 4 presents the unconditional density function of the correlation, which is implied by the parameters calibrated to the DCC dynamics of Figure 2.

Insert Figure 4 about here

As expected, the density of the calibration to Treasury and corporate bonds returns is highly skewed and peaked towards high correlation values. The density of the calibration to equity returns is less asymmetric and has a higher dispersion. This pattern is consistent with the weaker and less asymmetric mean reversion of the correlation in this case.

B.2. Volatility and Correlation Leverage Effects

Due to the linear form of the drift in the dynamics (4), the model generates a linear mean reversion in variances and covariances. The strength of this mean reversion is driven by the matrix M , which is assumed negative semi-definite to ensure the typical mean reverting behavior and stationarity. The matrix Q , instead, controls the co-volatility and the leverage effects between returns, volatilities and correlations. Black's volatility 'leverage' effect, that is the negative correlation between returns and volatility, has often been found to be an empirical feature of stock returns, and it is explicitly modeled by Heston (1993) to reproduce the empirical regularities of option-implied volatility skews.¹⁵ Roll's (1988) correlation 'leverage' effect, that is the negative covariance between returns and average correlation shocks across stocks, is also a feature supported by empirical evidence; see, e.g., Ang and Chen (2002). The mechanism producing the volatility and the correlation leverage effects in our model is standard and is based on the fact that the return dynamics (1) can be instantaneously correlated with the variance-covariance dynamics (4).¹⁶ For instance, we obtain for the first asset, using the properties of Wishart processes:¹⁷

$$\text{corr}_t \left(\frac{dS_1}{S_1}, d\Sigma_{11} \right) = \frac{q_{11}}{\sqrt{q_{11}^2 + q_{12}^2}} \quad , \quad \text{corr}_t \left(\frac{dS_1}{S_1}, d\rho \right) = q_{11} (1 - \rho^2(t)) \sqrt{\frac{\Sigma_{22}(t)}{\Sigma_{11}(t)}}$$

¹⁵Gouriéroux and Sufana (2004) apply a setting with Wishart volatilities and no leverage effects to credit derivatives pricing.

¹⁶This feature is not shared by multivariate GARCH-type models with dynamic correlations (see, e.g., Engle, 2002, Ledoit, Santa Clara, and Wolf, 2003, and Pelletier, 2006), where volatilities and correlations are conditionally uncorrelated with asset returns.

¹⁷The expressions for the second asset are symmetric, with q_{12} replacing q_{11} in the numerator of the first equality and in the second equality, and with Σ_{11}/Σ_{22} replacing Σ_{22}/Σ_{11} in the second equality.

where q_{11} and q_{12} are the components of the first row of the matrix Q . It follows that the parameters in the first row of this matrix control the sign of the co-dependence between returns, volatilities and correlation shocks. When q_{11} and q_{12} are negative, the model implies a volatility-leverage and a correlation-leverage effect.

Figure 5 presents scatter plots of simulated returns in our model, plotted against contemporaneous changes in volatilities and correlations, for the same calibrated parameters of Figure 3.

Insert Figure 5 about here

In the calibration based on US and German stock indexes, both a correlation and a volatility leverage effect emerge, which are highlighted by the negative relationship between returns and correlations (Panels 1 and 3) and between returns and volatilities (Panels 5 and 7). These features are due to the negative calibrated parameters q_{11} and q_{12} in Table II of Section III for this case. In the calibration based on Treasury and corporate bonds, however, returns on the Treasury bonds are characterized by a correlation and a volatility leverage effect (Panels 2 and 6, respectively), but the returns of corporate bonds highlight an opposite pattern (Panels 4 and 8, respectively). This feature is due to the mixed sign of the calibrated parameters q_{11} and q_{12} in Table II for this case.¹⁸

C. The Solution of the Investment Problem

To characterize the portfolio choice implications of process (4), we need to solve a corresponding Hamilton-Jacobi-Bellman equation. Therefore, it is convenient to introduce the infinitesimal generator \mathcal{A} of the process Σ . Since the joint process $(\Sigma_{11}, \Sigma_{22}, \Sigma_{12})$ can be written as a trivariate diffusion process, \mathcal{A} is defined in the standard way, as in Merton (1969), for functions $\phi = \phi(\Sigma)$. Using the particular structure of the dynamics (4) one can additionally show that \mathcal{A} can be written in a very compact and simple matrix form. Precisely, let $\phi = \phi(\Sigma)$ be a smooth function. Then, the generator \mathcal{A} associated with the diffusion process (4) takes the form:

$$\mathcal{A}\phi = \text{tr} \{ (\Omega\Omega' + M\Sigma + \Sigma M') \mathcal{D}\phi + 2\Sigma\mathcal{D}(Q'Q\mathcal{D}\phi) \}, \quad (12)$$

¹⁸The above parametrization of leverage effects is the most parsimonious one that can be used in our model. It is easy to extend it towards more general structures. For instance, one could replace the vector $dW(t)$ of Brownian shocks in the returns dynamics (1) by a simple linear combination of Brownian shocks:

$$\rho_1 dW(t) + \rho_2 dZ(t) + \sqrt{1 - \rho_1^2 - \rho_2^2} d\tilde{Z}(t),$$

where $\tilde{Z}(t)$ is a further two dimensional Brownian motion, independent of $W(t)$ and $Z(t)$, and $\rho_1, \rho_2 \in [-1, 1]$ are two additional correlation parameters. For instance, the co-dependence between volatility and the return of the first asset in this extended setting is:

$$\text{corr}_t \left(\frac{dS_1}{S_1}, d\Sigma_{11} \right) = \frac{\rho_1 q_{11} + \rho_2 q_{12}}{\sqrt{q_{11}^2 + q_{12}^2}}.$$

Therefore, one can add additional degrees of freedom to the model, in order to fit, e.g., a more flexible co-volatility and leverage structure, without losing the closed-form expressions for the optimal portfolios. For the sake of simplicity, we proceed with the parsimonious return specification of equation (1).

where \mathcal{D} is a matrix of differential operators.¹⁹ In this form, it is clear that this operator is affine in Σ , since the argument of the trace is affine in Σ ; see also Bru (1991) and Da Fonseca, Grasselli, and Tebaldi (2005).

We characterize the value function of the static problem (5)–(6) by solving problem (8). To obtain the result, we take advantage of the fact that the process (4) can be shown to follow an affine Wishart process also under the minimax martingale measure, which characterizes the solution of the static incomplete-markets problem. The Bellman equation for problem (8) reads:

$$0 = \frac{\partial \hat{J}}{\partial t} + \inf_{\nu} \left\{ \mathcal{A}^{\nu} \hat{J} + \hat{J} \left[-\frac{\gamma}{\gamma-1} r + \frac{\gamma}{2(\gamma-1)^2} \text{tr}(\Theta'_{\nu} \Theta_{\nu}) \right] \right\},$$

subject to the terminal condition $\hat{J}(T, \Sigma) = 1$. In this equation, the infinitesimal generator \mathcal{A}^{ν} of the covariance matrix dynamics under the equivalent martingale measure indexed by ν is:

$$\mathcal{A}^{\nu} \phi = \mathcal{A} \phi - \frac{\gamma}{\gamma-1} \text{tr} \left\{ (Q'(e_1 \lambda' + e_2 \nu') \Sigma + \Sigma(\lambda e'_1 + \nu e'_2) Q) \mathcal{D} \phi \right\},$$

where e_i is the i -th unit vector in \mathbb{R}^2 ; see the proofs in the Appendix. The affine structure of this generator is preserved and implies that the solution of problem (8) is exponentially affine in Σ , with coefficients obtained as solutions of a system of matrix Riccati differential equations.²⁰ These equations can be solved in closed form.

Proposition 2 *Given the covariance matrix dynamics (4), the value function of problem (3) takes the form:*

$$J(X_0, \Sigma_0) = \frac{X_0^{\gamma} \hat{J}(0, \Sigma_0)^{1-\gamma} - 1}{\gamma},$$

where the function $\hat{J}(t, \Sigma)$ is given by:

$$\hat{J}(t, \Sigma) = \exp(B(t, T) + \text{tr}(A(t, T) \Sigma)), \quad (13)$$

with $B(t, T)$ and the symmetric matrix-valued function $A(t, T)$ solving the system of matrix Riccati differential equations:

$$0 = \frac{dB}{dt} + \text{tr}[A \Omega \Omega'] - \frac{\gamma}{\gamma-1} r, \quad (14)$$

$$0 = \frac{dA}{dt} + \Gamma' A + A \Gamma + 2A \Lambda A + C, \quad (15)$$

under the terminal conditions $B(T, T) = 0$ and $A(T, T) = 0$, where:

$$\Gamma = M - \frac{\gamma}{\gamma-1} Q' \begin{pmatrix} \lambda_1 & \lambda_2 \\ 0 & 0 \end{pmatrix}, \quad \Lambda = Q' \begin{pmatrix} 1 & 0 \\ 0 & 1 - \gamma \end{pmatrix} Q, \quad C = \frac{\gamma}{2(\gamma-1)^2} \begin{pmatrix} \lambda_1^2 & \lambda_1 \lambda_2 \\ \lambda_2 \lambda_1 & \lambda_2^2 \end{pmatrix}.$$

The closed-form solution of the system of matrix Riccati differential equations (14)–(15) is detailed in Appendix A.

¹⁹The matrix differential operator \mathcal{D} is defined by $\mathcal{D} := \left(\frac{\partial}{\partial \Sigma_{ij}} \right)_{1 \leq i, j \leq 2}$.

²⁰See, e.g., Reid (1972) for a review of Riccati differential equations.

REMARK. In the literature on affine term structure models, it is well known that modeling correlated stochastic factors is not straightforward. Duffie and Kan (1996) show that parametric restrictions on the drift matrix of the factor dynamics have to be satisfied for a well-defined affine process to exist. In particular, its out of diagonal elements must have the same sign. This feature restricts the correlation structures that these models can fit (see, e.g, Duffee, 2002). In the Dai and Singleton (2000) classification for affine $A_m(n)$ models, specific restrictions need to be imposed for the model to be solvable: the Gaussian factors are allowed to be correlated, but the correlation between Gaussian and square-root factors must be zero. This issue is well-recognized also in the portfolio choice literature.²¹ An interesting by-product of the results of Proposition 2 is that it provides a simple solution for the portfolio problem (3), without imposing additional restrictions on the dependence structure between the risk factors. \square

One advantage of the exponentially affine form of function \widehat{J} in Proposition 2 is that it allows for a simple description of the partial derivatives of the marginal indirect utilities of wealth with respect to the variance and covariance factors. This property provides us with a simple and easily interpretable solution to the incomplete-markets multivariate portfolio choice problem.

Proposition 3 *Let π be the optimal portfolio obtained under the assumptions of Proposition 2. It then follows,*

$$\pi = \frac{\lambda}{1 - \gamma} + 2 \begin{pmatrix} q_{11}A_{11} + q_{12}A_{12} \\ q_{12}A_{22} + q_{11}A_{12} \end{pmatrix}, \quad (16)$$

where A_{ij} denotes the ij -th component of the matrix A , which characterizes the function $\widehat{J}(t, \Sigma)$ in Proposition 2, and the coefficients q_{ij} are the entries of the matrix Q appearing in the Wishart dynamics (4).

The portfolio policy π is the sum of a myopic demand and a hedging demand. The hedging demands for variance and covariance risk are simple linear functions of the elements of the matrix A . The hedging demands on the first and second assets are, respectively, $2(q_{11}A_{11} + q_{12}A_{12})$ and $2(q_{11}A_{12} + q_{12}A_{22})$. The interpretation is simple and can be linked to the Merton (1969) solution. The matrix A describes how the component $\widehat{J}(t, \Sigma)$ of the indirect utility is affected by the state variables driving the dynamics of Σ . Each element A_{ij} can be rewritten as $A_{ij} = \frac{1}{RRA} \cdot \frac{\varepsilon_{\Sigma_{ij}}}{\Sigma_{ij}}$, where $\varepsilon_{\Sigma_{ij}}$ is the elasticity of the marginal indirect utility $\partial J / \partial X$ with respect to Σ_{ij} , and RRA is the relative risk aversion coefficient.²² Hedging demands stem from both a volatility hedging and a covariance hedging motive. Terms proportional to A_{11} and A_{22} are pure volatility hedging demands deriving from the own volatility risk of assets one and two, respectively. Terms proportional to

²¹Liu (2007) addresses this issue by assuming a triangular factor structure in an affine portfolio problem with two risky assets.

²²Precisely, we have:

$$A_{ij} = -\frac{1}{X \frac{\partial^2 J}{\partial X^2} / \frac{\partial J}{\partial X}} \cdot \frac{\partial J}{\partial \Sigma_{ij} \partial X} / \frac{\partial J}{\partial X} = \frac{1}{1 - \gamma} \cdot \frac{\partial J}{\partial \Sigma_{ij} \partial X} / \frac{\partial J}{\partial X},$$

using the envelope condition.

A_{12} are covariance hedging demands: They are due both to changes in volatility (when correlations are different from zero) and to changes in correlations. The second set of parameters that drive the optimal policy consists of q_{11} and q_{12} , which determine the sign of the co-movement of returns, variances, and covariances.

Despite the simple structure of the hedging policies (16), a rich set of possible hedging demands can arise. For instance, if q_{11} and q_{12} are both negative, then volatility and correlation leverage effects will arise for all assets. If, in addition, A_{11} , A_{22} and A_{12} are negative, then all hedging demands for variance and covariance risk will be positive, and the exposure to all risky assets will be increased by the desire to intertemporally hedge variance and covariance risk. This is the situation that arises for investors with relative risk aversion $1 - \gamma > 1$ when we calibrate our model to international equity returns. However, if the sign of the marginal utility sensitivities, or the sign of the co-movement between returns, variances, and covariances, is mixed, then it is possible to obtain some hedging demand components that are positive and some others that are negative. This is the situation that arises when we calibrate the model to Treasury and corporate bond returns. In this setting, the optimal correlation hedging demand for Treasury bills of an investor with risk aversion $1 - \gamma > 1$ is positive, but that for corporate bonds is negative.

In Proposition 2, the value function is written as a function of current wealth X and the covariance matrix Σ . This structure enables us to easily isolate the hedging demand for variance and covariance risk, but not the demand for correlation risk. The problem is that the hedging demand for covariance risk is caused by both volatility and correlation risk. Therefore, we can split this demand into two further hedging components. A volatility hedging demand against changes in returns covariance due to changes in assets' volatilities, and a correlation hedging demand. This decomposition enables us to quantify the contribution of correlation hedging to the overall hedging demand. Using Proposition 2, it is straightforward to compute these demands, given the fact that $\Sigma_{12} = \rho\sqrt{\Sigma_{11}\Sigma_{22}}$.

Proposition 4 *Let π be the optimal portfolio obtained under the assumptions of Proposition 2. The hedging demand for asset i is the sum of three components π_i^{vol} , $\pi_i^{vol/cov}$, π_i^ρ , which hedge, respectively, pure volatility risk, covariance risk due to volatilities and correlation risk. The explicit expressions for these hedging demands are as follows.*

1. *Pure Volatility hedging:*

$$\pi_1^{vol} = 2q_{11}A_{11}, \quad \pi_2^{vol} = 2q_{12}A_{22}. \quad (17)$$

2. *Covariance hedging due to volatility:*

$$\pi_1^{vol/cov} = 2q_{11}A_{12}\rho\sqrt{\frac{\Sigma_{22}}{\Sigma_{11}}}, \quad \pi_2^{vol/cov} = 2q_{12}A_{12}\rho\sqrt{\frac{\Sigma_{11}}{\Sigma_{22}}}. \quad (18)$$

3. *Correlation hedging:*

$$\pi_1^\rho = 2A_{12} \left(q_{12} - q_{11}\rho\sqrt{\frac{\Sigma_{22}}{\Sigma_{11}}} \right), \quad \pi_2^\rho = 2A_{12} \left(q_{11} - q_{12}\rho\sqrt{\frac{\Sigma_{11}}{\Sigma_{22}}} \right). \quad (19)$$

The pure volatility hedging demands for assets one and two in equation (17) are proportional to A_{11} and A_{22} , respectively. Their signs also depend on the correlation between volatility shocks and returns, via the coefficients q_{11} and q_{12} . When A_{11} and A_{22} are negative, positive (negative) hedging demands against volatility risk arise if and only if returns and volatility shocks are negatively (positively) correlated. In equation (18), covariance hedging demands due to volatility are proportional to the correlation level and to the sensitivity A_{12} of the marginal utility of wealth to changes in covariances. This is intuitive. Higher correlations imply a higher impact of a change in volatility on covariances, as well as a larger risk of an adverse covariance movement. Depending on the portfolio setting, the sign of A_{12} can be either positive or negative. In settings of low average correlations, in which portfolio diversification is the main objective, an increase in the correlation between assets implies an increase in the maximal squared Sharpe ratio of the optimally invested portfolio: Current correlations and the future risk-return tradeoff are positively related. Investors with risk aversion $1 - \gamma > 1$, which are characterized by a utility function bounded from above, tend to select portfolios that avoid large losses at the end of the investment horizon. If volatilities and correlations covary negatively (positively) with returns, these investors will have a positive (negative) covariance hedging demand due to volatility, because current losses in the portfolio strategy are hedged (exacerbated) intertemporally by the better (worse) future risk-return tradeoff.²³ This situation arises when we calibrate the model to international equity returns. However, for settings of large positive correlations, in which optimal portfolios can even imply spread positions between assets, the opposite might happen. In these cases, a higher correlation decreases the maximal square Sharpe ratio and implies a negative relation between current correlations and the future risk-return tradeoff. This situation occurs when we calibrate the model to Treasury and corporate bond returns. The correlation hedging demand in equation (19) is also proportional to the sensitivity A_{12} of the marginal utility of wealth to changes in covariances. This is intuitive. The larger A_{12} , the larger the sensitivity of the marginal utility of wealth to correlation shocks, and the stronger the correlation hedging motive. Depending on the sign of q_{11} and q_{12} , correlation hedging can be increasing or decreasing in the correlation level and the ratio of the volatilities of the two assets. For example, π_1^ρ is increasing in ρ and $\sqrt{\Sigma_{22}/\Sigma_{11}}$ when q_{11} is negative. The relative importance of π_i^ρ and $\pi_i^{vol/col}$ for the two assets depends on the relative size of q_{12} , q_{11} , ρ and $\sqrt{\Sigma_{11}/\Sigma_{22}}$.

III. Hedging Correlation Risk

In order to quantify correlation hedging in realistic portfolio choice settings, we calibrate our model to the data. We analyze two portfolio choice scenarios. In the first scenario, we study an international equity portfolio manager and consider a portfolio of US and German stock indices. We investigate how correlation risk affects the desire to optimally diversify international equity risk. The second scenario explores the case of a market-neutral hedge fund that uses spread trades to maximize returns. The investor tries to build a near-arbitrage portfolio using two risky assets that are most of the time almost perfectly positively correlated. In this case, the investor tries to optimally hedge

²³The opposite happens for investors with risk aversion $1 - \gamma < 1$, which have a utility function bounded from below by zero.

the risk of a leveraged position in one asset, using a corresponding short position in another asset. For this setting, we consider 10 year Treasury and Aaa corporate bonds. In both cases, we obtain the time series of the conditional covariance matrix for our calibrations by estimating Engle's (2002) multivariate DCC model.

Scenario I: International equity diversification. For equities, we use daily S&P100 and DAX index data, from January 1988 to December 2005. Panel A of Table I presents the estimated unconditional moments of returns, volatilities and correlations for equities.

Insert Table I about here.

During this period, the unconditional mean of US and German stock returns is about 13% and 11%, respectively. The higher unconditional volatility of German returns arises together with a higher volatility of volatility. The unconditional correlation between the two stock indices is about 37% and the unconditional volatility of correlation is about 15%. These features generate an obvious incentive for diversification.

Scenario II: Market neutral spread trading. For bonds, we use daily data of 10 years Treasury and Aaa corporate bond return indices supplied by Lehman Brothers, from April 1996 to December 2005.²⁴ The average return on Aaa bonds in Panel B of Table I is about 1.84%, slightly higher than the 1.65% mean return of Treasury bills. The unconditional correlation is about 88%, which is more than double the correlation between the S&P100 and the DAX index returns. The volatility of bonds correlation is about 9.5%, which is approximately two-thirds the volatility of stock returns' correlation. The unconditional volatility and the volatility of volatility of Treasury bills are slightly higher than those of corporate bonds. These features generate an obvious incentive for exploiting near-arbitrage opportunities between the Treasury and the corporate bond markets.

The optimal portfolio strategies in the two scenarios are substantially different. In the first scenario, investors exploit the low average correlation to diversify risk between the US and the German equity markets. To this end, they build two long positions in the corresponding assets. In this case, correlation hedging tends to hedge unanticipated changes in the correlation structure, which might reduce the benefits of international diversification. In the second scenario, investors exploit the large average correlation and the low correlation volatility to develop a near arbitrage strategy between the Treasury and the corporate bond markets. To this end, they take a long position in corporate bonds, financed by a short position in Treasury bonds. Therefore, correlation hedging tends to hedge unanticipated changes in correlations that might reduce the effectiveness of this near-arbitrage strategy.

²⁴The same exercise applied to a sample of monthly bond returns over a longer sample starting in January 1988 yielded similar results.

A. The Size of Correlation Hedging

We first calibrate the coefficient vector λ to match average risk premia, given estimates for the covariance matrix of returns. We then calibrate the covariance process (4) to the second moments of volatilities and correlations. Using calibrated parameters, we can then decompose the total hedging demand for asset $i = 1, 2$ into a pure volatility hedging component π_i^{vol} , a covariance hedging component for volatility $\pi_i^{vol/cov}$, and a correlation hedging demand π_i^ρ , as described in Proposition 4. Table II presents the calibrated parameters M and Q for the covariance matrix dynamics in equation (4) under the two portfolio scenarios.

Insert Table II about here.

The negative calibrated parameters q_{11}, q_{12} in the first row of matrix Q in the international equity diversification scenario imply a volatility and a correlation leverage effect for all returns. In the market neutral spread trading scenario, q_{11} is negative, but q_{12} is positive. Calibrated Treasury bond returns exhibit a leverage effect in volatilities and correlations, but calibrated corporate bond returns co-move positively with correlations and their volatility. In particular, this setting implies that when correlations decrease the spread between corporate bonds and Treasury bill returns increases, consistent with the "flight to quality effect" from corporate to Treasury bonds. The negative coefficients in matrix M for stocks and bonds reflect the mean reversion in variances and covariances. The weaker calibrated mean reversion for stocks is due to the higher volatility of the volatility and the correlation. Therefore, shocks in stock correlations are on average more persistent than for bonds.

To compute the size of correlation hedging, covariance hedging due to volatility, and pure volatility hedging, we initialize $\Sigma(t)$ at its unconditional value. We then compute the optimal hedging demand components $\pi_i^{vol}, \pi_i^{vol/cov}, \pi_i^\rho, i = 1, 2$, as defined in Proposition 4, when the average correlation and the correlation volatility deviate from their sample value. We study the case in which the investment horizon is five years and the relative risk aversion parameter is $1 - \gamma = 3$. Calibrated hedging components are expressed as a percentage of the corresponding absolute myopic Merton portfolio.

Scenario I: International equity diversification. In this first calibration, we move the average correlation over a grid in the interval $[0.25, 0.5]$ and hold fixed the remaining moments of returns. Consistent with the literature on univariate portfolio selection with stochastic volatility, pure volatility hedging is a small fraction of the myopic portfolio: On average its absolute size is less than 2.5% of the myopic portfolio, as illustrated in Panels 2 and 4 (triangle points) of Figure 6.

Insert Figure 6 about here.

The overall hedging demand for US and German stocks is dominated by correlation hedging: The hedging demand for correlation risk is about 7% of the myopic portfolio at the sample average correlation and it increases up to 9% for average correlations around 48% (see Panels 1 and 3). The absolute size of the correlation hedging is increasing in the average correlation. This result is

consistent with the intuition: When the average correlation is high, the available risky assets are less able to span the risk due to unexpected shocks in the returns covariance matrix. Moreover, negative shocks in the level of the conditional correlation tend to be more persistent, because of the asymmetric form of the correlation mean reversion. The covariance hedging component due to volatility in Panels 2 and 4 (circle points) is on average about 3% of the myopic portfolio at the sample average correlation. It is typically larger than the volatility hedging demand: For the investment in the US equity index it is more than double the corresponding volatility hedging demand. In the international equity diversification scenario, the calibrated parameters imply a leverage effect for volatilities and correlations of all assets. In addition, the marginal utility of wealth sensitivities of an investor with risk aversion $1 - \gamma > 1$ to the variance and covariance factors are negative. Therefore, all hedging demand components are positive and cumulate in the same direction. The cumulated hedging demand is on average 11% of the myopic demand at the sample average correlation, and about 15% of the myopic portfolio for average correlations of around 48%.

In Figure 7, we present the comparative statics of the hedging demands with respect to the correlation volatility.

Insert Figure 7 about here.

We consider values for the correlation volatility over a grid in the interval $[0.05, 0.24]$, and hold fixed the other moments. The absolute size of correlation hedging is increasing in the correlation volatility: For correlation volatilities of about 24%, correlation hedging in US stocks can be as large as 11.5% of the myopic portfolio (see Panel 1). For correlation volatilities near zero, correlation hedging tends to vanish. This is intuitive: A higher correlation volatility implies a higher risk that the myopic portfolio will be ex-post sub-optimal. The pattern of the pure volatility hedging demand in Panels 2 and 4 (dotted points) is flatter than that of the hedging demand for correlation risk. Moreover, the maximal volatility hedging component is less than 2% and 3.5% of the myopic portfolio, for US and German stocks respectively.

Scenario II: Market neutral spread trading. To obtain comparative statics results for market neutral spread trades, we consider values for the average correlation of bond returns over a grid in the interval $[0.75, 0.98]$. Pure volatility hedging is a small fraction of the myopic portfolio: Its absolute size in Panels 6 and 8 of Figure 6 (triangle points) is less than approximately 2% of the myopic portfolio at the sample correlation. The hedging components against covariance risk caused by volatility changes in the same panels (circle points) is on average 3.5% of the myopic portfolio. The overall hedging demand in bonds is mostly due to correlation hedging, as illustrated by Panels 5 and 7 of Figure 6: correlation hedging is about 7% of Merton myopic portfolio for correlations of around 0.87 and about 8.5%, on average, for a correlation of 0.98. Correlation hedging is increasing in the average correlation level. The basic intuition is similar to the one of the previous scenario. However, the calibrated parameters imply a correlation and volatility leverage effect only for Treasury bonds, together with an opposite pattern for corporate bonds. In addition, the marginal utility sensitivities to volatility are negative for both asset classes, but the sensitivity to correlations and covariances

is positive. It follows that the market neutral spread trading scenario implies a positive (negative) correlation hedging demand for Treasury Bills (corporate bonds). In a similar vein, the hedging demand for covariance risk due to volatility is negative (positive) for Treasury bills (corporate bonds).

To explore the effect of the correlation volatility, we consider values in the interval $[0.01, 0.16]$, and hold the remaining moments fixed. Volatility hedging is a relatively flat function with respect to the correlation volatility, both for Treasury Bills and corporate bonds, and is on average 2% of the myopic portfolio (see Figure 7, Panels 6 and 8, triangle points). The hedging component for covariance risk due to volatility is larger and is about 3.5% of the myopic demand (Panels 6 and 8, dotted points). The absolute size of the correlation hedging demand can be as large as 9% of the myopic portfolio, when the correlation volatility is about 16% (see Panels 5 and 7). As expected, we also find that it is an increasing function of the correlation volatility and that it tends to vanish at zero, when correlation risk also vanishes. The basic intuition is similar to that for the equity scenario.

A.1. Time horizon

An important question addressed by the optimal portfolio choice literature is how the optimal allocation in risky assets varies with respect to the investment horizon. This question is key for investment professionals working in the pension fund industry and for individuals deciding the composition of their retirement accounts. Brennan, Schwartz, and Lagnado (1997), Barberis (2000), Kim and Omberg (1996), and Wachter (2002) address this issue in the context of time-varying expected returns. When volatilities are constant, they find that the optimal investment in risky assets increases in the investment horizon. When volatilities are constant, Kim and Omberg (1996) show that for the investor with utility over terminal wealth and for $1 - \gamma > 1$ the optimal allocation increases in the investment horizon, as long as the risk premium is positive. Wachter (2002) extends this result to the case of utility over intertemporal consumption under the assumption of no uncertainty about the correlation structure of asset returns. Depending on the characteristics of this uncertainty, however, it is reasonable to expect that the optimal demand for hedging correlation risk could mitigate, or strengthen, the speculative components. Our model offers a simple theoretical framework to investigate the role of uncertainty on the nature of this relationship. In both the international equity diversification and the market neutral spread trading scenarios, we find that for realistic parameter calibrations the total optimal allocation to risky assets of an investor with risk aversion $1 - \gamma > 1$ increases with the investment horizon. Figure 8 illustrates these effects.

Insert Figure 8 about here.

All (absolute) hedging demand components are increasing in the investment horizon, both for equities and for bonds. At short investment horizons of, e.g., 3 months, all hedging demands are small (less than 1%) and similar. The correlation hedging demand increases faster than the other hedging components. For instance, in the case of US index returns and at a ten year horizon, the difference between volatility and correlation hedging (less than 2% and about 9%, respectively) is

quite striking. These results show that in some cases the results can change substantially when correlation risk is taken into account.

A.2. Higher-Dimensional Portfolio Choice Settings

The previous examples considered portfolio settings with two risky assets. As the investment horizon increases, we can expect two basic effects to influence the structure of correlation and volatility hedging. First, the sensitivity of the marginal utility of wealth to the single correlation and volatility factors will tend to be smaller, because optimal portfolios' weights will tend, on average, to be smaller. Second, the number of correlation factors increases quadratically with the dimension of the investment universe, but the number of volatility factors increases linearly. Therefore, we can expect the role of correlation hedging to be even more important in portfolio choice problems with many risky assets. To investigate this issue quantitatively, we calibrate the model to an international equity diversification scenario with three risky assets and add the FTSE100 index to the investment opportunity set consisting of the S&P100 and the DAX indices. It is straightforward to modify the proofs of Proposition 2 and 3 to cover the general n -dimensional case. With these results, we compute the optimal portfolios for the model with three risky assets, after having calibrated a 3×3 Wishart covariance process to the time series of US, UK, and German index returns. Table III summarizes the calibrated moments of returns in this setting (Panel A) and the calibrated hedging components for correlation, volatility and covariance risk due to volatility (Panel B).

Insert Table III about here.

Correlation hedging for US and UK stocks is, on average, about 9.1% of the myopic portfolio, larger than the correlation hedging demand for US stocks in the setting with two assets, which was about 7.2%. The correlation hedging demand for German stocks is 6.95%, which is slightly higher than that with two risky assets. This evidence further supports the important role played by correlation hedging: The total average hedging demand is about 14% of the myopic portfolio, while the average pure volatility hedging demand is only 2.4%.

IV. Robustness and Extensions

A. Risk Aversion and Interest Rate Level

The finding that correlation hedging is the most important hedging demand component, when variances and covariances are stochastic, does not hinge on the choice of the relative risk aversion parameter and the level of the interest rate in our calibrations. In Figure 9, we plot for the international diversification scenario both the hedging portfolio weights and the hedging portfolio fractions of the myopic demand as functions of the relative risk aversion.

Insert Figure 9 about here.

Absolute hedging portfolio weights for stocks and bonds peak at a relative risk aversion of about two, which is smaller than the one used in our calibrations. Hedging portfolios as percentages of the myopic demand are monotonically increasing in the relative risk aversion. In both cases, correlation hedging dominates the other hedging demands for all risk aversions considered. The dominating role of correlation hedging is also confirmed by Figure 10, where we plot the single hedging components, in the international diversification and the market neutral spread trading scenarios, for different average interest rate levels from 1% to 5%.

Insert Figure 10 about here.

B. Utility over consumption

The previous results focus on the case of an agent maximizing utility of terminal wealth. It is interesting to discuss how these results could change in the case of an investor maximizing utility over intertemporal consumption. When markets are complete and volatility is constant, Wachter (2002) shows analytically that the optimal portfolio weights of a CRRA investor maximizing intertemporal utility of consumption are equal to a weighted average of the portfolio weights solving a sequence of individual terminal wealth problems. In this context, wealth can be interpreted as a bond that pays consumption as its coupon. The value of wealth is equal to the sum of the "zeros" that pay the optimal consumption at each date. For instance, if an agent is concerned not just about his retirement account but also about the purchase of a house, the optimal portfolio is a weighted average of two separate portfolios: a "retirement portfolio" and a "house portfolio". When markets are incomplete, the previous analytical result in general does not hold. However, under the minimax martingale measure, markets are completed by the addition of a set of securities that span the state-space and receive zero portfolio weights in equilibrium (He and Pearson, 1991). Under this specific martingale measure, the main results of the previous section still hold. Clearly, since the duration of lifetime consumption is lower than the investment horizon of the agent maximizing utility over terminal wealth, the absolute demand for risky assets would be proportionally lower.

C. Additional Dynamic Settings

The previous sections studied correlation risk for a portfolio choice problem in which expected excess returns are linearly related to the covariance of returns (Cox, Ingersoll, and Ross, 1985, Heston, 1993 and Liu, 2001), interest rates are constant, and the covariance process, although stochastic, is autonomous. We investigate the implications of these three assumptions with regards to the properties of the optimal portfolio policies. We consider an economy in which: First, the risk premium is constant. Second, interest rates are stochastic and are a function of the same state variables that are driving asset prices. Third, additional risk factors, like the interest rate or an aggregate market indicator, can affect the dynamic properties of the covariance matrix $\Sigma(t)$ (i.e. its speed of mean reversion and volatility).²⁵ The economy that we consider implies that the square

²⁵In this way, local asymmetries in the covariance matrix dynamics can be introduced in the model. To model asymmetric correlations across regimes, Ang and Bekaert (2002) use an i.i.d. switching regimes setting, in which one

Sharpe ratios are a decreasing function of the returns covariance matrix, and therefore it allows for a direct comparison with Chacko and Viceira (2005), who study a similar, but univariate, economy with constant interest rates. Under these assumptions, it is known that to obtain closed-form solutions it is convenient to model directly the information matrix, i.e. Σ^{-1} , as opposed to Σ .

To introduce additional risk factors, we model Σ^{-1} as a projection of a larger rank matrix Y . For instance, if the new risk factor is the stochastic interest rate, then the 2×2 matrix Σ^{-1} can be obtained from a 3×3 Wishart process $Y(t)$:

$$\Sigma^{-1} = SY S',$$

where the 2×3 matrix S is such that $SS' = id_{2 \times 2}$.²⁶ Given the larger rank of $Y(t)$, the stochastic risk-less rate $r(t)$ can be defined directly from $Y(t)$:

$$r(t) = r_0 + tr(Y(t)D), \quad (20)$$

where $r_0 > 0$ and D is a 3×3 matrix. Notice that the non-negativity of $r(t)$ is ensured simply by assuming that the symmetric matrix D is positive definite.

The process Y satisfies the Wishart dynamics:

$$dY(t) = [\Omega\Omega' + MY(t) + Y(t)M']dt + Y^{1/2}(t)dBQ + Q'dB'Y^{1/2}(t), \quad (21)$$

where the matrices Ω , M and Q are now of dimension 3×3 and where $B = [W, Z_1, Z_2]$, with Z_1 and Z_2 being each a three-dimensional Brownian motion, independent of the first three-dimensional Brownian motion W . It then follows that $\Sigma = SY^{-1}S'$, and it is natural to define $\Sigma^{1/2}$ as the 2×3 matrix $SY^{-1/2}$.²⁷ The return process is given by:

$$dS(t) = I_S \left[\begin{pmatrix} r(t) + \mu_1^e \\ r(t) + \mu_2^e \end{pmatrix} dt + \Sigma^{1/2}(t)dW(t) \right], \quad (22)$$

where the excess return vector $\mu^e = (\mu_1^e, \mu_2^e)' \in \mathbb{R}^2$ is constant and $r(t)$ is given by equation (20).

This setting is effectively a six-factor model in which some interest rate risk factors might be linked to the covariance matrix of stock returns, depending on the specific form of the matrix D in equation (20). It is straightforward to verify that the square Sharpe ratio in this model is affine in Y . Therefore, we can mimick the steps of the proof of Propositions 2, 3, and 4 and solve in closed-form the portfolio choice problem (3) in this extended dynamic setting as well.

of the regimes is characterized by higher correlations and volatilities.

²⁶ A possible choice for S is a 2×3 selection matrix, like e.g.:

$$S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

In this case, $SS' = id_{2 \times 2}$ and $SY S'$ is the 2×2 upper diagonal sub-block of Y .

²⁷ A proof of this statement is presented in Appendix A.

Proposition 5 *The solution of the portfolio problem (3) for the returns dynamics (21)-(22) and under a stochastic interest rate (20) is:*

$$J(X_0, Y_0) = \frac{X_0^\gamma \widehat{J}(0, Y_0)^{1-\gamma} - 1}{\gamma},$$

where

$$\widehat{J}(t, Y) = \exp(B(t, T) + \text{tr}(A(t, T)Y)),$$

with $B(t, T)$ and the symmetric matrix-valued function $A(t, T) := A_1(t, T) + A_2(t, T)$ solving in closed-form the following system of matrix Riccati differential equations:

$$-\frac{dB}{dt} = -\frac{\gamma}{\gamma-1}r_0 + \text{tr}(A\Omega\Omega'), \quad (23)$$

$$-\frac{dA_i}{dt} = \Gamma'_i A_i + A'_i \Gamma_i + 2A'_i \Lambda_i A_i + C_i, \quad (24)$$

where $i = 1, 2$, subject to $A_1(T, T) = A_2(T, T) = 0$. In these equations, the coefficients Γ_1, Γ_2 are given by:

$$\Gamma_1 = MS'S - \frac{\gamma}{\gamma-1}Q'e_1\mu^{e'}S \quad , \quad \Gamma_2 = M(id_{3 \times 3} - S'S).$$

The coefficients Λ_1, Λ_2 are given by:

$$\Lambda_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\gamma & 0 \\ 0 & 0 & 1-\gamma \end{pmatrix} \quad , \quad \Lambda_2 = \begin{pmatrix} 1-\gamma & 0 & 0 \\ 0 & 1-\gamma & 0 \\ 0 & 0 & 1-\gamma \end{pmatrix}$$

and the coefficients C_1, C_2 are:

$$C_1 = \frac{\gamma}{2(\gamma-1)^2}S'\mu^e\mu^{e'}S - \frac{\gamma}{\gamma-1}S'SDS'S \quad , \quad C_2 = -\frac{\gamma}{\gamma-1}(id_{3 \times 3} - S'S)D(id_{3 \times 3} - S'S).$$

Finally, the optimal policy for this portfolio problem reads:

$$\pi = \frac{1}{1-\gamma}\Sigma^{-1} \begin{pmatrix} \mu_1^e \\ \mu_2^e \end{pmatrix} + 2\Sigma^{-1}S \begin{pmatrix} q_{11}A_{11} + q_{12}A_{12} + q_{13}A_{13} \\ q_{11}A_{21} + q_{12}A_{22} + q_{13}A_{23} \\ q_{11}A_{31} + q_{12}A_{32} + q_{13}A_{33} \end{pmatrix}. \quad (25)$$

The optimal policy (25) consists of a myopic and an intertemporal hedging portfolio, which are both proportional to the stochastic inverse covariance matrix. As noted by Chacko and Viceira (2005), in the univariate setting the relative size of the hedging and myopic demands is independent of the current level of volatility. This property also holds in the multivariate case, in the sense that both policies are proportional to the inverse covariance matrix Σ^{-1} . We calibrate the Wishart process Y to the specification used to model the covariance matrix Σ for the special case $D = 0$ (constant interest rate) and under the restriction of an autonomous covariance matrix dynamics. We then compare the results with the findings in the previous sections. This setting is the exact multivariate version of the univariate model studied in Chacko and Viceira (2005). We find again that correlation

hedging is the major determinant of the hedging demand: It amounts, on average, to approximately 60% of the total hedging demand for a Relative Risk Aversion parameter of three and an investment horizon of five years. As expected, the sign of the hedging demands in this setting is different since the squared Sharpe ratio implied by the model is decreasing in the covariance matrix. The myopic portfolio is time varying, via the variation of the inverse covariance matrix Σ^{-1} . This variation is also partly reflected in the time variation of hedging demands. Figure 11 presents a summary of the calibrated myopic portfolio weights (left panels) and hedging demands (right panels) in the international equity diversification and the market neutral spread trading scenarios, when expected excess returns are assumed constant.

Insert Figure 11 about here.

D. Multivariate Portfolio Choice and Predictability

The previous models can be extended to study an important issue in multivariate portfolio choice, such as the joint impact of correlation risk and predictability on the optimal hedging demand. A large literature has studied the consequences of predictability for portfolio choice with a single risky asset. Campbell and Viceira (1999), Kim and Omberg (1996), and Wachter (2002) propose tractable portfolio choice settings that can be handled analytically. The first paper uses approximate solutions to characterize the portfolio of an infinitely-lived investor. The second one derives closed-form solutions for an incomplete-market setting with power utility over terminal wealth. The last one, obtains closed-form optimal portfolios for a complete markets setting with intermediate consumption. Important evidence on predictability has also been studied by Barberis (2000), who shows the portfolio choice impact of predictability under estimation risk, and Xia (2001), who documents the effect of learning about predictability.

Using our previous results, it is easy to model predictability in our framework. Let Y be a matrix containing all the priced factors and let the risk premium on each asset depend on a time-varying predicting factor and on asset returns' volatilities, and correlations:

$$\Lambda(t, Y) = \begin{pmatrix} \lambda_1 \Sigma_{11} + \lambda_2 \Sigma_{12} + Y_{13} \\ \lambda_1 \Sigma_{21} + \lambda_2 \Sigma_{22} + Y_{23} \end{pmatrix}. \quad (26)$$

Given the 3×3 Wishart process Y in equation (21), let $\Sigma = SY S'$, where the matrix S is such that $S' S = id_{2 \times 2}$, and define $\Sigma^{1/2} = SY^{1/2}$.²⁸ To complete the specification, the risk premium is modeled as $\Lambda(t, Y) = SY(t)\lambda$, with $\lambda = (\lambda_1, \lambda_2, 1)' \in \mathbb{R}^3$. The resulting returns dynamics are given by:

$$dS(t) = I_S \left[(r\bar{I}_2 + SY(t)\lambda) dt + \Sigma^{1/2}(t)dW(t) \right], \quad (27)$$

with the risk premium satisfying (26). In this specification, expected excess returns are driven by the variance-covariance risk factors and the additional predictability factors Y_{13} and Y_{23} , which can

²⁸In the most simple case, for instance, S can be chosen to be a selection matrix of the form $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$.

be correlated with the covariance matrix dynamics. Clearly, if $\lambda_1 = \lambda_2 = 0$ the predictability factors are the only variables affecting risk premia, as in Kim and Omberg (1996) and Wachter (2002), who assume a predictability factor following an Ornstein-Uhlenbeck process and a constant volatility of returns. In all other cases, returns variances and covariances will also affect risk premia. It is easy to verify that the squared Sharpe ratio in this and similar models is equal to $\lambda'Y(t)\lambda$. Therefore, the same solution method used in Proposition 2 and 3 leads to closed-form solutions for the optimal portfolio.

V. Discussion and Conclusions

In this article we have analyzed the importance of correlation risk for portfolio decisions, and have proposed a new simple approach to embed stochastic correlation in optimal portfolio choice. The flexibility of the model allows us to study correlation hedging in settings that can account for several stylized features of asset returns, including volatility/correlation persistence and leverage effects. We show that correlation hedging contributes to the overall portfolio in a qualitatively and quantitatively significant way. Examples of calibrated economies show that correlation hedging can be a non-negligible fraction of Merton's myopic portfolio, while volatility hedging is typically small. The correlation hedging demand is larger for settings with extreme average correlations and higher correlation variances, and is increasing in the number of assets available for investment, as the dimension of uncertainty with regard to the correlation structure becomes proportionally more important.

The proposed approach to modeling time-varying correlations, as well as the analytical optimal portfolio results, can prove useful in investigating a number of further economic questions. For instance, an important strand of the empirical asset pricing literature has investigated the characteristics of hedge funds performance.²⁹ Kosowski, Naik, and Teo (2006) document that, after controlling for market risk, hedge fund alphas are significantly positive and persistent. Hedge funds in the top decile of the return distribution have alphas well in excess of 1% per month. They test and find that the performance is persistent and not due to luck. A proposed interpretation of this evidence is that these funds have superior managerial ability. However, about 34% of the hedge funds classify as long/short funds and 7% as fixed income arbitrage funds. Our results suggest that part of these excess returns may also compensate for the exposure to correlation risk, which is key in long/short strategies. Since these alphas are typically obtained without explicitly controlling for exposure to correlation risk, it would be a very interesting empirical question to investigate the extent of the link between exposure to correlation risk and hedge funds alphas.

Correlation risk plays also a direct role in the pricing, hedging, and risk management of correlation derivatives, such as quantos. In these financial instruments, the underlying asset is denominated in one currency, while the instrument itself is settled in another currency at some fixed exchange rate. Such products are attractive for portfolio managers or hedge funds who wish to have exposure

²⁹See, for instance, Brown, Goetzmann and Ibbotson (1999), Fung and Hsieh (2001), Mitchell and Pulvino (2001), Agarwal and Naik (2004), Getmansky, Lo, and Makarov (2004), Busse and Irvine (2005), Kosowski, Naik, and Teo (2006).

to a foreign asset, without carrying the corresponding exchange rate risk. Well-known examples include differential swaps (also known as quantity-adjusted swaps, guaranteed exchange rate swaps, Libor differential swaps), quanto options, quanto equity swaps, and quanto futures (such as the Nikkei Future traded on the CME). In these cases, the pricing, hedging, and risk management of these instruments depend directly on the correlation between the risk factors (see Reiner, 1992 and Dravid, Richardson, and Sun, 1993). For these reasons, these instruments are also referred to by practitioners as "correlation products". In differential swaps, for instance, the dealer commits to paying a floating rate on a fixed US dollar notional amount, rather than on a fixed amount in the foreign currency, as with a typical cross-currency swap. This commitment exposes the dealer to changes in the correlation between the Libor and the exchange rates. Since static hedging strategies are generally not feasible, the dealer must manage the residual correlation risk by using optimal portfolio techniques to model correlation risk.

Correlation risk also plays an important role in the trading of multi-asset (rainbow) options. A multi-asset option is a derivative security whose payoff depends on the future values of several, possibly correlated, underlying assets. Typical examples are basket, spread, outside barrier options and options on the minimum/maximum of several assets. Since equity correlation risk cannot be hedged as precisely as volatility risk, the current market practice of equity derivatives desks is simply to monitor their correlation exposure and try to avoid risk peaks in certain correlation pairs by managing the product flow via dynamic price margins, and by using index options to hedge the "average" correlation risk within the index basket. However, specific correlation risk in individual stock pairs remains. The results of this article provide an alternative strategy for managing correlation risk.

Finally, correlation risk is a key issue in the credit derivative market as the likelihood of a default of one credit may affect the likelihood of default of another (default correlation). Examples of correlation-based products also include instruments written on baskets of credits, such as CDOs and first-to-default (FTD) swaps. Investors in FTD structures sell protection on a reference portfolio of names and assume exposure to the first default that takes place within the pre-defined basket of credits. Upon occurrence of a default, the protection seller needs to identify the protection buyer for the loss on the defaulted credit, calculated on the basis of the full face value of the FTD basket. Clearly, the ex-ante value of these instruments is highly sensitive to the correlations structure in the basket.

References

- Agarwal, V., Naik, N. Y., 2004, Risk and portfolio decisions involving hedge funds, *Review of Financial Studies* 17, 63-98.
- Andersen, T. G., Bollerslev, T., Christoffersen, P. F., and F. Diebold, 2005, Volatility Forecasting, *Working Paper*.
- Ang, A. and G. Bekaert, 2002, International Asset Allocation with Regime Shifts, *Review of Financial Studies* 15, 1137-1187.
- Ang, A. and J. Chen, 2002, Asymmetric Correlations of Equity Portfolios, *Journal of Financial Economics* 63, 443-494.

- Bansal, R. and A. Yaron, 2004, Risks for the Long-run: A Potential Resolution of Asset Pricing Puzzles, *Journal of Finance* 59, 1481-1509.
- Barberis, N., 2000, Investing for the Long-Run when Returns are Predictable, *Journal of Finance* 55, 225-264.
- Barndorff-Nielsen, O. E. and N. Shephard, 2004, Econometric Analysis of Realized Covariation: High Frequency Covariance, Regression and Correlation in Financial Economics, *Econometrica* 72, 885-925.
- Barsky, R. B. and J. B. De Long, 1990, Bull and Bear Markets in the Twentieth Century, *Journal of Economic History* 50, 1-17.
- Barsky, R. B. and J. B. DeLong, 1993, Why Does the Stock Market Fluctuate ?, *Quarterly Journal of Economics* 108, 291-312.
- Bekaert, G. and C. R. Harvey, 1995, Time-varying World Market Integration, *Journal of Finance* 50, 403-444.
- Bekaert, G. and C. R. Harvey, 2000, Foreign Speculators and Emerging Equity Markets, *Journal of Finance* 55, 565-613.
- Bollerslev, T., 1987, A Conditionally Heteroskedastic Time Series Model for Speculative Prices and Rates of Return, *Review of Economics and Statistics* 69, 542-547.
- Bollerslev, T., Chou, R., and K. Kroner, 1992, ARCH Modelling in Finance: A Review of the Theory and Empirical Evidence, *Journal of Econometrics* 52, 5-59.
- Bollerslev, T., Engle, R., and J. Woolridge, 1988, A Capital Asset Pricing Model with Time-Varying Covariances, *Journal of Political Economy* 96, 116-131.
- Brown, S., Goetzmann, W., and R. Ibbotson, 1999, Offshore Hedge Funds: Survival and Performance, 1989-95, *Journal of Business* 72, 91-117.
- Bru, M. F., 1991, Wishart Processes, *Journal of Theoretical and Applied Probability* 4, 725-743.
- Brennan, M. J. and Y. Xia, 2001, Assessing Asset Pricing Anomalies, *Review of Financial Studies* 14, 905-942.
- Brennan, M. J., Schwartz, E.S., and R. Lagnado, 1997, Strategic Asset Allocation, *Journal of Economic Dynamics and Control* 21, 1377-1403.
- Browne, M.W and A. Shapiro, 1986, The Asymptotic Covariance Matrix of Sample Correlation Coefficients under General Conditions, *Linear Algebra and Its Applications* 82, 169-176.
- Busse, J. and P. Irvine, 2006, Bayesian Alphas and Mutual Fund Persistence, *The Journal of Finance* 61, 2251-2288.
- Campbell, J. and L. Viceira, 1999, Consumption and Portfolio Decisions when Expected Returns Are Time Varying, *The Quarterly Journal of Economics* 114, 433-495.
- Chacko, G. and L. Viceira, 2005, Dynamic Consumption and Portfolio Choice with Stochastic Volatility in Incomplete Markets, *Review of Financial Studies* 18, 1369-1402.
- Conley, T. G., Hansen, L. P., Luttmer, G. J., and J. A. Scheinkman, 1997, Short-Term Interest Rates as Subordinated Diffusions, *Review of Financial Studies* 10, 3, 525-577.
- Cox, J. C., Ingersoll, J. E., and S. Ross, 1985, An Intertemporal General Equilibrium Model of Asset Prices, *Econometrica* 53, 2, 363-384.
- Cox, J. C. and Chi-Fu Huang, 1989, Optimal Consumption and Portfolio Policies when Asset Prices Follow a Diffusion, *J. Econ. Theory* 49, 33-83.

- Cuoco, D., 1997, Optimal Consumption and Equilibrium Prices with Portfolio Constraints and Stochastic Income, *Journal of Economic Theory* 72, 33–73.
- Cvitanic, J. and I. Karatzas, 1992, Convex Duality in Constrained Portfolio Optimization, *Annals of Applied Probability* 2, 767-818.
- Da Fonseca, J., Grasselli, M., and C. Tebaldi, 2005, Wishart Multi-Dimensional Stochastic Volatility, *Working Paper*, University of Verona.
- Dai, Q. and K. Singleton, 2000, Specification Analysis of Affine Term Structure Models, *Journal of Finance* 50, 1943-1978.
- Das, S. R. and R. Uppal, 2004, Systemic Risk and International Portfolio Choice, *Journal of Finance* 59, 2809-2834.
- De Santis, G., Litterman, B., Vesval A., and K. Winkelmann, 2003, Covariance Matrix Estimation, *Modern Investment Management*, Bob Litterman and the Quantitative Resources Group, Goldman Sachs Asset Management, John Wiley & Sons, New Jersey.
- Dravid, A., Richardson, M., and T. S. Sun, 1993, Pricing Foreign Index Contingent Claims: An Application to Nikkei Index Warrants, *Journal of Derivatives* 1, 1, 33-51.
- Driessen, J., Maenhout, P., and G. Vilkov, 2006, Option-Implied Correlations and the Price of Correlation Risk, *Working paper*, University of Amsterdam.
- Duffee, G. R., 2002, Term Premia and Interest Rates Forecasts in Affine Models, *Journal of Finance* 52, 405-443.
- Duffie, D., Filipovic, D., and W. Schachermayer, 2003, Affine Processes and Applications in Finance, *Annals of Applied Probability* 13, 984-1053.
- Duffie, D. and R. Kan, 1996, A Yield-Factor Model of Interest Rates, *Mathematical Finance* 4, 379-406.
- Duffie D., Pan J., and K. Singleton, 2000, Transform Analysis and Asset Pricing for Affine Jump Diffusions, *Econometrica* 68, 6, 1343-1376.
- Engle, R., 2002, Dynamic Conditional Correlation - a Simple Class of Multivariate GARCH Models, *Journal of Business and Economic Statistics* 20, 339-350.
- Engle, R. F., and K. F. Kroner, 1995, Multivariate Simultaneous Generalized ARCH, *Econometric Theory* 11, 1, 122-50.
- Erb, C., Harvey, C. and T. Viskanta, 1994, Forecasting International Equity Correlations, *Financial Analysts Journal* November/December, 32-45.
- Fung, W. and D. Hsieh, 2001, The Risk in Hedge Fund Strategies: Theory and Evidence from Trend Followers, *Review of Financial Studies* 14, 313-341.
- Getmansky, M., Lo, A., and I. Makarov, 2004, An Econometric Model of Serial Correlation and Illiquidity of Hedge Fund Returns, *Journal of Financial Economics* 74, 529-610.
- Goetzmann, W. N., Li, L., and K. G. Rouwenhorst, 2005, Long-Term Global Market Correlations, *Journal of Business*, 78, 1-38.
- Gouriéroux, C. and R. Sufana, 2004, Derivative Pricing with Multivariate Stochastic Volatility: Application to Credit Risk, *Working paper*, University of Toronto.
- Gouriéroux, C., Jasiak, J., and R. Sufana, 2004, The Wishart Autoregressive Process of Multivariate Stochastic Volatility, *Journal of Econometrics*, forthcoming.

- Harvey, A., Ruiz, E., and N. Shephard, 1994, Multivariate Stochastic Variance Models, *Review of Economic Studies* 61, 2, 247-64.
- He, H. and N. D. Pearson, 1991, Consumption and Portfolio Policies with Incomplete Markets and Short Sales Constraints: The Infinite Dimensional Case, *Journal of Economic Theory* 54, 259-304.
- Heston, S. L., 1993, A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options, *Review of Financial Studies* 6, 327-343.
- Heston, S. L. and L. G. Rouwenhorst, 1994, Does Industrial Structure Explain the Benefits of International Diversification? *Journal of Financial Economics* 36, 3-27.
- Jorion, P., 1999, Risk Management Lessons from Long-Term Capital Management, *Working paper*.
- Kim, T. S. and E. Omberg, 1996, Dynamic Nonmyopic Portfolio Behavior, *Review of Financial Studies* 9, 141-161.
- Kosowski, R., Naik, N., and M. Teo, 2007, Does Money Grow on Hedge Funds? A Bayesian and Bootstrap Analysis, *Journal of Financial Economics*, forthcoming.
- Ledoit, O., Santa-Clara, P., and M. Wolf, 2003, Flexible Multivariate GARCH Modeling with an Application to International Stock Markets, *Review of Economics and Statistics* 85, 735-747.
- Liu, J., 2001, Dynamic Portfolio Choice and Risk Aversion, *Working paper*, Rady School of Management.
- Liu, J., Longstaff F., and J. Pan, 2003, Dynamic Asset Allocation with Event Risk, *Journal of Finance* 58, 231-259.
- Liu, J., 2007, Portfolio Selection in Stochastic Environments, *Review of Financial Studies*, 20, 1-39.
- Longin, F. and B. Solnik, 1995, Is the Correlation in International Equity Returns Constant: 1960-1990? *Journal of International Money and Finance* 14, 1, 3-26.
- Longin, F. and B. Solnik, 2001, Extreme Correlation of International Equity Markets, *Journal of Finance* 56, 649-675.
- Merton, R. C., 1969, Lifetime Portfolio Selection: The Continuous-Time Case, *Review of Economics and Statistics* 51, 247-257.
- Mitchell, M. and T. Pulvino, 2001, Characteristics of Risk in Risk Arbitrage, *Journal of Finance* 56, 2135-2175.
- Moskowitz, T., 2003, An Analysis of Covariance Risk and Pricing Anomalies, *Review of Financial Studies* 16, 417-457.
- Munk, C. and C. Sorensen, 2004, Optimal Consumption and Investment Strategies with Stochastic Interest Rates, *Journal of Banking and Finance* 28, 1987-2013.
- Neudecker, H. and A. M. Wesselman, 1990, The Asymptotic Covariance of Sample Correlation Matrix, *Linear Algebra and Its Applications* 127, 589-599.
- Parker, J. A. and C. Julliard, 2005, Consumption Risk and the Cross-section of Expected Returns, *Journal of Political Economy*, 113, 185-222.
- Pelletier, D., 2006, Regime Switching for Dynamic Correlations, *Journal of Econometrics* 127, 445-473.
- Pliska, S., 1986, A Stochastic Calculus Model of Continuous Trading: Optimal Portfolios, *Mathematics of Operations Research* 11, 371-382.
- Poon, S. H. and C. Granger, 2003, Forecasting Volatility in Financial Markets: A Review, *Journal of Economic Literature* 41, 478-539.

- Rajan, R. and L. Zingales, 2001, The Great Reversals: The Politics of Financial Developments in the 20th Century, NBER Working Paper No. W8178, National Bureau of Economic Research, Cambridge, MA.
- Reid, W. T., 1972, *Riccati Differential Equations*, Academic Press, New York.
- Reiner, E., 1992, Quanto Mechanics, *Risk* 5, 10, 59-63.
- Ribeiro, R. and P. Veronesi, 2002, The Excess Comovement of International Stock Markets in Bad Times: A Rational Expectations Equilibrium Model, *Working paper*, University of Chicago
- Roll, R., 1988, The International Crash of October, 1987, *Financial Analysts Journal* September-October, 19-35.
- Roll, R., 1992, Industrial Structure and the Comparative Behavior of International Stock Market Indices, *Journal of Finance* 47, 3-41.
- Sangvinatsos, A. and J. Watcher, 2005, Does the Failure of the Expectations Hypothesis Matter for Long-Term Investors? *Journal of Finance* 60, 179-230.
- Schroder, M. and C. Skiadas, 2003, Optimal Lifetime Consumption-Portfolio Strategies under Trading Constraints and Generalized Recursive Preferences, *Stochastic Processes and Their Applications* 108, 155-202.
- Wachter, J., 2002, Portfolio and Consumption Decisions Under Mean-Reverting Returns: An Exact Solution for Complete Markets, *Journal of Financial and Quantitative Analysis* 37, 63-91.
- Xia, Y., 2001, Learning about Predictability: The Effect of Parameter Uncertainty on Dynamic Asset Allocation, *Journal of Finance* 56, 205-246.

Appendix A: Proofs of Theorems

Proof of Proposition 1: The dynamics of the correlation process implied by the Wishart covariance matrix diffusion (4) is computed using Itô's Lemma. Let

$$\rho(t) = \frac{\Sigma_{12}(t)}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} \quad (\text{A1})$$

be the instantaneous correlation between the returns of the first and the second risky assets and denote by σ_{ij} and q_{ij} the ij -th component of the volatility matrix $\Sigma^{1/2}$ and the matrix Q in equation (4), respectively. Applying Itô's Lemma to (A1) and using the dynamics for Σ_{11} , Σ_{22} and Σ_{12} , implied by (4), it follows:

$$\begin{aligned} d\rho = & \left[-\frac{\rho}{2} \left(\frac{q_{11}^2 + q_{21}^2}{\Sigma_{11}} + \frac{q_{12}^2 + q_{22}^2}{\Sigma_{22}} \right) + (\rho^2 - 2) \frac{q_{11}q_{12} + q_{21}q_{22}}{\sqrt{\Sigma_{11}\Sigma_{22}}} + (1 - \rho^2) \frac{m_{21}\Sigma_{11} + m_{12}\Sigma_{22}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \right] dt \\ & - \left[\frac{\rho}{\Sigma_{11}\Sigma_{22}} (\Sigma_{22}\sigma_{11}q_{11} + \Sigma_{11}\sigma_{12}q_{12}) - \frac{\sigma_{12}q_{11} + \sigma_{11}q_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \right] dW_1 \\ & - \left[\frac{\rho}{\Sigma_{11}\Sigma_{22}} (\Sigma_{11}\sigma_{22}q_{12} + \Sigma_{22}\sigma_{21}q_{11}) - \frac{\sigma_{22}q_{11} + \sigma_{21}q_{12}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \right] dW_2 \\ & - \left[\frac{\rho}{\Sigma_{11}\Sigma_{22}} (\Sigma_{22}\sigma_{11}q_{21} + \Sigma_{11}\sigma_{12}q_{22}) - \frac{\sigma_{11}q_{22} + \sigma_{12}q_{21}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \right] dZ_1 \\ & - \left[\frac{\rho}{\Sigma_{11}\Sigma_{22}} (\Sigma_{11}\sigma_{22}q_{22} + \Sigma_{22}\sigma_{21}q_{21}) - \frac{\sigma_{21}q_{22} + \sigma_{22}q_{21}}{\sqrt{\Sigma_{11}\Sigma_{22}}} \right] dZ_2 \end{aligned} \quad (\text{A2})$$

Therefore, the instantaneous drift of the correlation process is a quadratic polynomial with state dependent coefficients:

$$\mathbb{E} [d\rho(t) | \mathcal{F}_t] = [A(t)\rho(t)^2 + B(t)\rho(t) + C(t)] dt, \quad (\text{A3})$$

where coefficients $A(t)$, $B(t)$ and $C(t)$ are given by:

$$A(t) = \frac{q_{11}q_{12} + q_{21}q_{22}}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} - m_{21}\sqrt{\frac{\Sigma_{11}(t)}{\Sigma_{22}(t)}} - m_{12}\sqrt{\frac{\Sigma_{22}(t)}{\Sigma_{11}(t)}}, \quad (\text{A4})$$

$$B(t) = -\frac{1}{2} \left(\frac{q_{11}^2 + q_{21}^2}{\Sigma_{11}(t)} + \frac{q_{12}^2 + q_{22}^2}{\Sigma_{22}(t)} \right), \quad (\text{A5})$$

$$C(t) = -2\frac{q_{11}q_{12} + q_{21}q_{22}}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} + m_{21}\sqrt{\frac{\Sigma_{11}(t)}{\Sigma_{22}(t)}} + m_{12}\sqrt{\frac{\Sigma_{22}(t)}{\Sigma_{11}(t)}}. \quad (\text{A6})$$

The instantaneous conditional variance of the correlation process is easily obtained from equation (A2) using the independence of Brownian motions Z and W , and it is a third order polynomial with state dependent coefficients:

$$\mathbb{E} [d\rho(t)^2 | \mathcal{F}_t] = \left[(1 - \rho^2(t)) \left(\frac{q_{11}^2 + q_{21}^2}{\Sigma_{11}(t)} + \frac{q_{12}^2 + q_{22}^2}{\Sigma_{22}(t)} + 2\rho(t) \frac{q_{11}q_{12} + q_{21}q_{22}}{\sqrt{\Sigma_{11}(t)\Sigma_{22}(t)}} \right) \right] dt.$$

This concludes the proof. □

Proof of Proposition 2: Since markets are incomplete, we follow He and Pearson (1991) and represent any market price of risk as the sum of two orthogonal components, one of which is spanned by the asset returns. Since in our setting there are no frictions, the first component is $\begin{bmatrix} \Sigma^{1/2}\lambda \\ \underline{0} \end{bmatrix}$, where λ is the vector that prices the shocks to asset returns and $\underline{0}$ is a two-dimensional vector of zeros. The second component is $\begin{bmatrix} \underline{0} \\ \Sigma^{1/2}\nu \end{bmatrix}$, where ν is the vector pricing the shocks that are independent of the asset returns. Let Θ_ν be the matrix-valued extension of Θ that prices the matrix of Brownian motions $B = [W, Z]$:

$$\Theta_\nu = \Sigma^{1/2} \begin{bmatrix} \lambda & \nu \end{bmatrix} = \Sigma^{1/2}(\lambda e'_1 + \nu e'_2), \quad (\text{A7})$$

where $e_1 = (1, 0)'$ and $e_2 = (0, 1)'$. Given Θ_ν , the associated martingale measure implies a process ξ_ν of stochastic discount factors, defined for $t \in [0, T]$ by:

$$\xi_\nu(t) = e^{-rt - \text{tr}\left(\int_0^t \Theta'_\nu(s) dB(s) + \frac{1}{2} \int_0^t \Theta'_\nu(s) \Theta_\nu(s) ds\right)}. \quad (\text{A8})$$

Our dynamic portfolio choice problem admits an equivalent static representation by means of the following dual problem, as shown by He and Pearson (1991):

$$J(x, \Sigma_0) = \inf_\nu \sup_\pi \mathbb{E} \left[\frac{X(T)^\gamma - 1}{\gamma} \right], \quad (\text{A9})$$

$$\text{s.t. } \mathbb{E}[\xi_\nu(T)X(T)] \leq x, \quad (\text{A10})$$

where $X(0) = x$. In what follows, we focus on the solution of problem (A9)-(A10). The optimality conditions for the innermost maximization is:

$$X(T) = (\psi \xi_\nu(T))^{\frac{1}{\gamma-1}}, \quad (\text{A11})$$

where the Lagrange multiplier for the static budget constraint is

$$\psi = x^{\gamma-1} \mathbb{E} \left[\xi_\nu(T)^{\frac{\gamma}{\gamma-1}} \right]^{1-\gamma}.$$

It then follows:

$$J(x, \Sigma_0) = x^\gamma \inf_\nu \frac{1}{\gamma} \mathbb{E} \left[\xi_\nu(T)^{\frac{\gamma}{\gamma-1}} \right]^{1-\gamma} - \frac{1}{\gamma}. \quad (\text{A12})$$

Using (A8) and (A12), one can notice that the solution requires the computation of the expected value of the exponential of a stochastic integral. A simple change of measure reduces the problem to the calculation of the expectation of the exponential of a deterministic integral. Let P^γ be the probability measure defined by the following Radon-Nykodim derivative with respect to the physical measure P :

$$\frac{dP^\gamma}{dP} = e^{-\text{tr}\left(\frac{\gamma}{\gamma-1} \int_0^T \Theta'_\nu(s) dB(s) + \frac{1}{2} \frac{\gamma^2}{(\gamma-1)^2} \int_0^T \Theta'_\nu(s) \Theta_\nu(s) ds\right)}. \quad (\text{A13})$$

We denote expectations under P^γ by $\mathbb{E}^\gamma[\cdot]$. Then, the minimizer of (A12) is the solution of the following problem:³⁰

$$\begin{aligned} \hat{J}(0, \Sigma_0) &= \inf_\nu \mathbb{E} \left[\xi_\nu(T)^{\frac{\gamma}{\gamma-1}} \right] \\ &= \inf_\nu \mathbb{E}^\gamma \left[e^{-\frac{\gamma}{\gamma-1} rT + \frac{\gamma}{2(\gamma-1)^2} \text{tr}\left(\int_0^T \Theta'_\nu(s) \Theta_\nu(s) ds\right)} \right] \\ &= \inf_\nu \mathbb{E}^\gamma \left[e^{-\frac{\gamma}{\gamma-1} rT + \frac{\gamma}{2(\gamma-1)^2} \text{tr}\left(\int_0^T \Sigma(s) ds (\lambda \lambda' + \nu \nu')\right)} \right]. \end{aligned} \quad (\text{A14})$$

Notice that the expression in the exponential of the expectation in (A14) is affine in Σ . By Girsanov Theorem, under the measure P^γ the stochastic process B^γ , defined as

$$B^\gamma(t) = B(t) + \frac{\gamma}{\gamma-1} \int_0^t \Theta_\nu(s) ds$$

is a 2×2 matrix of standard Brownian motions. Therefore, the process (4) is an affine process also under the new probability measure P^γ :

$$\begin{aligned} d\Sigma(t) &= \left[\Omega \Omega' + \left(M - \frac{\gamma}{\gamma-1} Q'(e_1 \lambda' + e_2 \nu') \right) \Sigma(t) + \Sigma(t) \left(M - \frac{\gamma}{\gamma-1} Q'(e_1 \lambda' + e_2 \nu') \right)' \right] dt \\ &\quad + \Sigma^{1/2}(t) dB^\gamma(t) Q + Q' dB(t)^{\gamma'} \Sigma^{1/2}(t). \end{aligned} \quad (\text{A15})$$

³⁰Strictly speaking, this holds for $\gamma \in (0, 1)$. For $\gamma < 0$, minimizations are replaced by maximizations and all formulas follow with the same type of arguments.

Using Feynman Kac formula, it is known that if the optimal ν and \hat{J} solve the probabilistic problem (A14), then they must also be a solution of the following Hamilton Jacobi Bellman (HJB) equation:

$$0 = \frac{\partial \hat{J}}{\partial t} + \inf_{\nu} \left\{ \mathcal{A}\hat{J} + \hat{J} \left[-\frac{\gamma}{\gamma-1}r + \frac{\gamma}{2(\gamma-1)^2} \text{tr}(\Sigma(\lambda\lambda' + \nu\nu')) \right] \right\}, \quad (\text{A16})$$

subject to the terminal condition $\hat{J}(T, \Sigma) = 1$, where \mathcal{A} is the infinitesimal generator of the matrix-valued diffusion (A15), which is given by:

$$\begin{aligned} \mathcal{A} = & \text{tr} \left(\left(\Omega\Omega' + \left(M - \frac{\gamma}{\gamma-1} Q'(e_1\lambda' + e_2\nu') \right) \Sigma + \Sigma \left(M - \frac{\gamma}{\gamma-1} Q'(e_1\lambda' + e_2\nu') \right)' \right) \mathcal{D} \right) \\ & + \text{tr}(2\Sigma\mathcal{D}Q'Q\mathcal{D}), \end{aligned} \quad (\text{A17})$$

where

$$\mathcal{D} := \begin{pmatrix} \frac{\partial}{\partial \Sigma_{11}} & \frac{\partial}{\partial \Sigma_{12}} \\ \frac{\partial}{\partial \Sigma_{21}} & \frac{\partial}{\partial \Sigma_{22}} \end{pmatrix}. \quad (\text{A18})$$

The generator is affine in Σ . The optimality condition for the optimal control ν , implied by HJB equation (A16), is:

$$\frac{1}{\gamma-1} \Sigma\nu = \frac{\partial}{\partial \nu} \text{tr} \left((Q'(\nu e_2)'\Sigma + \Sigma\nu e_2'Q) \frac{\mathcal{D}\hat{J}}{\hat{J}} \right) = \frac{\partial}{\partial \nu} \text{tr} \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} Q'(\nu e_2)'\Sigma + \Sigma\nu e_2'Q \frac{\mathcal{D}\hat{J}}{\hat{J}} \right).$$

Applying rules for the derivative of trace operators, the right hand side can be written as $\Sigma \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q'e_2$. It follows that

$$\nu = (\gamma-1) \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q'e_2.$$

We now compute the generator (A17) associated with this solution. To this end, note that

$$\lambda e_1' + \nu e_2' = \lambda e_1' + (\gamma-1) \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q'e_2 e_2'.$$

Substituting the expression for ν obtained in equation (A17), we obtain the generator

$$\begin{aligned} \mathcal{A} = & \text{tr} \left(\left(\Omega\Omega' + \left(M - \frac{\gamma}{\gamma-1} Q'e_1\lambda' \right) \Sigma + \Sigma \left(M - \frac{\gamma}{\gamma-1} Q'e_1\lambda' \right)' \right) \mathcal{D} + 2\Sigma\mathcal{D}Q'Q\mathcal{D} \right) \\ & - \gamma \text{tr} \left(\left(Q'e_2 e_2' Q \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) \Sigma + \Sigma \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q'e_2 e_2' Q \right) \mathcal{D} \right) \\ = & \text{tr} \left(\left(\Omega\Omega' + \left(M - \frac{\gamma}{\gamma-1} Q'e_1\lambda' \right) \Sigma + \Sigma \left(M - \frac{\gamma}{\gamma-1} Q'e_1\lambda' \right)' \right) \mathcal{D} + 2\Sigma\mathcal{D}Q'Q\mathcal{D} \right) \\ & - \gamma \hat{J} \text{tr} \left(\Sigma \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q'e_2 e_2' Q \left(\frac{\mathcal{D}}{\hat{J}} + \frac{\mathcal{D}'}{\hat{J}} \right)' \right). \end{aligned}$$

Substitution of the last expression for \mathcal{A} into the HJB equation (A16) yields the following partial differential equation for \hat{J} :

$$\begin{aligned} -\frac{\partial \hat{J}}{\partial t} = & \text{tr} \left(\left(\Omega\Omega' + \left(M - \frac{\gamma}{\gamma-1} Q'e_1\lambda' \right) \Sigma + \Sigma \left(M - \frac{\gamma}{\gamma-1} Q'e_1\lambda' \right)' \right) \mathcal{D} + 2\Sigma\mathcal{D}Q'Q\mathcal{D} \right) \hat{J} \\ & + \frac{\gamma}{\gamma-1} \hat{J} \left(-r + \frac{\text{tr}(\Sigma\lambda\lambda')}{2(\gamma-1)} \right) - \frac{\gamma}{2} \hat{J} \text{tr} \left(\Sigma \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q'e_2 e_2' Q \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right)' \right), \end{aligned}$$

subject to the boundary condition $\hat{J}(\Sigma, T) = 1$. The affine structure of this problem suggests an exponentially affine functional form for its solution:

$$\hat{J}(t, \Sigma) = \exp(B(t, T) + \text{tr}(A(t, T)\Sigma)),$$

for some state independent coefficients $B(t, T)$ and $A(t, T)$. After inserting this functional form into the differential equation for \hat{J} , the guess can be easily verified. The coefficients B and A are the solutions of the following system of Riccati equations:

$$\begin{aligned} -\frac{dB}{dt} = & \text{tr}(A\Omega\Omega') - \frac{\gamma}{\gamma-1}r, \\ -\text{tr} \left(\frac{dA}{dt} \Sigma \right) = & \text{tr} \left(\Gamma' A \Sigma + A \Gamma \Sigma + 2A Q' Q A \Sigma - \frac{\gamma}{2} (A' + A) Q' e_2 e_2' Q (A' + A) \Sigma + \frac{\gamma}{2(\gamma-1)^2} \lambda \lambda' \Sigma \right), \end{aligned}$$

with terminal conditions $B(T, T) = 0$ and $A(T, T) = 0$, where

$$\Gamma = M - \frac{\gamma}{\gamma - 1} Q' e_1 \lambda'.$$

For a symmetric matrix function A , the second differential equation implies the following matrix Riccati equation:

$$0 = \frac{dA}{dt} + \Gamma' A + A \Gamma + 2A Q' (id_{2 \times 2} - \gamma e_2 e_2') Q A + \frac{\gamma}{2(\gamma - 1)^2} \lambda \lambda'. \quad (\text{A19})$$

Defining

$$\Lambda = Q' (id_{2 \times 2} - \gamma e_2 e_2') Q = Q' \begin{pmatrix} 1 & 0 \\ 0 & 1 - \gamma \end{pmatrix} Q, \quad C = \frac{\gamma}{2(\gamma - 1)^2} \lambda \lambda', \quad (\text{A20})$$

the system of matrix Riccati equations in the statement of Proposition 2 is obtained. These differential equations are completely integrable, so that closed-form expressions for \hat{J} (and hence for J) can be computed.

Given a solution for A , function B is obtained by simple integration:

$$B(t, T) = \frac{\gamma}{\gamma - 1} r(T - t) - \text{tr} \left(\int_t^T A(s, T) \Omega \Omega' ds \right).$$

The solution for A is:

$$A(t, T) = \tilde{A} + e^{\Theta'(T-t)} \left[-\tilde{A}^{-1} + 2 \int_t^T e^{\Theta(T-s)} \Lambda e^{\Theta'(T-s)} ds \right]^{-1} e^{\Theta(T-t)}, \quad (\text{A21})$$

where $\Theta = \Gamma + 2\Lambda\tilde{A}$, e^\cdot is the matrix exponential function and matrix \tilde{A} is the solution of the quadratic system

$$\Gamma' \tilde{A} + \tilde{A} \Gamma + 2\tilde{A} \Lambda \tilde{A} + C = 0. \quad (\text{A22})$$

To prove that (A21) is indeed the solution for A , we apply the variation of constants method. Let $\tau = T - t$ denote time to maturity. With this variable, the Riccati differential equation for A is:

$$\frac{dA}{d\tau} = \Gamma' A(\tau) + A(\tau) \Gamma + 2A(\tau) \Lambda A(\tau) + C, \quad (\text{A23})$$

subject to initial condition $A(0) = 0$. Let A^* be the solution of the following quadratic matrix equation

$$\Gamma' A^* + A^* \Gamma + 2A^* \Lambda A^* + C = 0,$$

and define $A(\tau) := K(\tau) + A^*$. After inserting this guess for A into equation (A23), we see that $K(\tau)$ must solve the matrix differential equation

$$\frac{dK}{d\tau} = \Gamma^{*'} K(\tau) + K(\tau) \Gamma^* + 2K(\tau) \Lambda K(\tau), \quad (\text{A24})$$

subject to initial condition $K(0) = -A^*$, where $\Gamma^* := \Gamma + 2\Lambda A^*$. Define

$$K(\tau) = \exp(\Gamma^* \tau)' H(\tau) \exp(\Gamma^* \tau),$$

for some matrix valued function $H(\tau)$. It then follows,

$$\begin{aligned} \frac{dK}{d\tau} &= \Gamma^{*'} \exp(\Gamma^* \tau)' H(\tau) \exp(\Gamma^* \tau) + \exp(\Gamma^* \tau)' H(\tau) \exp(\Gamma^* \tau) \Gamma^* + \exp(\Gamma^* \tau)' \frac{dH(\tau)}{d\tau} \exp(\Gamma^* \tau) \\ &= \Gamma^{*'} K(\tau) + K(\tau) \Gamma^* + \exp(\Gamma^* \tau)' \frac{dH(\tau)}{d\tau} \exp(\Gamma^* \tau). \end{aligned}$$

By comparing the last expression with (A24), the function $H(\tau)$ must satisfy the following matrix differential equation:

$$\frac{dH}{d\tau} = 2H(\tau) \exp(\Gamma^* \tau) \Lambda \exp(\Gamma^* \tau)' H(\tau),$$

subject to initial condition $H(0) = -A^*$. After integrating this equation, using the definition of $K(\tau)$ and recalling that $A(\tau) = K(\tau) + A^*$, we obtain (A22). This concludes the proof. \square

Proof of Proposition 3: In order to recover the optimal portfolio policy we have, from the proof of Proposition 2:

$$X^*(t) =: \frac{1}{\xi_{\nu^*}(t)} \mathbb{E}[\xi_{\nu^*}(T) X^*(T) | \mathcal{F}_t] = \psi^{\frac{1}{\gamma-1}} \xi_{\nu^*}(t)^{\frac{1}{\gamma-1}} \widehat{J}(t, \Sigma(t)). \quad (\text{A25})$$

For the Wishart dynamics (4), Itô's lemma applied to both sides of (A25) gives, for every state Σ :

$$X^*(t) \text{tr} \left(\begin{bmatrix} \pi_1 & \pi_2 \\ 0 & 0 \end{bmatrix} \Sigma^{1/2} dB \right) = X^*(t) \text{tr} \left(\frac{1}{1-\gamma} \Theta'_{\nu^*} dB + \frac{\mathcal{D}\widehat{J}}{\widehat{J}} \left(\Sigma^{1/2} dBQ + Q' dB' \Sigma^{1/2} \right) \right). \quad (\text{A26})$$

This implies

$$\Sigma^{1/2} \begin{bmatrix} \pi_1 & 0 \\ \pi_2 & 0 \end{bmatrix} = \frac{1}{1-\gamma} \Theta_{\nu^*} + 2\Sigma^{1/2} A Q'.$$

We conclude that portfolio weight $\pi = (\pi_1, \pi_2)'$ is

$$\pi = \frac{\lambda}{1-\gamma} + 2A Q' e_1 = \frac{1}{1-\gamma} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} + 2 \begin{pmatrix} q_{11}A_{11} + q_{12}A_{12} \\ q_{12}A_{22} + q_{11}A_{12} \end{pmatrix}. \quad (\text{A27})$$

This concludes the proof of the proposition. □

Proof of Proposition 4: To obtain the optimal hedging demand in terms of the state variables Σ_{11} , Σ_{22} , and ρ , write $\Sigma_{12} = \rho\sqrt{\Sigma_{11}\Sigma_{22}}$ and note that:

$$\frac{-X}{\frac{\partial^2 J}{\partial^2 X}} \frac{\partial^2 J}{\partial \rho \partial X} = \frac{-X}{\frac{\partial^2 J}{\partial^2 X}} \frac{\partial^2 J}{\partial \Sigma_{12} \partial X} \frac{\partial \Sigma_{12}}{\partial \rho} = A_{12} \sqrt{\Sigma_{11}\Sigma_{22}}.$$

This is the closed form expression for the wealth-scaled ratio of marginal utilities with respect to ρ and X . For volatilities, the same argument gives, for $i, j = 1, 2$, where $i \neq j$:

$$\frac{-X}{\frac{\partial^2 J}{\partial^2 X}} \frac{\partial^2 J}{\partial \Sigma_{ii} \partial X} = A_{ii} + \frac{-X}{\frac{\partial^2 J}{\partial^2 X}} \frac{\partial^2 J}{\partial \Sigma_{12} \partial X} \frac{\partial \Sigma_{12}}{\partial \Sigma_{ii}} = A_{ii} + A_{12} \frac{\rho}{2} \sqrt{\frac{\Sigma_{jj}}{\Sigma_{ii}}}. \quad (\text{A28})$$

This is the closed form expression for the wealth-scaled ratio of marginal utilities with respect to Σ_{ii} and X , when ρ is treated as an explicit state variable, in addition to Σ_{11} and Σ_{22} . The first term on the right hand side of equation (A28) is the one that corresponds to the direct effect of Σ_{ii} on the value function. The second term is the one that corresponds to the indirect effect of Σ_{ii} , via the feedback of Σ_{ii} on Σ_{12} . To compute the corresponding hedging demand it is then enough to use Merton (1969)'s results and to calculate the projection coefficients of $d\Sigma_{11}$, $d\Sigma_{22}$ and $d\rho$ on the space spanned by dS_1/S_1 and dS_2/S_2 , using the available dynamics. After collecting terms proportional to $A_{12}\sqrt{\Sigma_{11}\Sigma_{22}}$, A_{11} , A_{22} , $A_{12}\frac{\rho}{2}\sqrt{\Sigma_{11}/\Sigma_{22}}$ and $A_{12}\frac{\rho}{2}\sqrt{\Sigma_{22}/\Sigma_{11}}$, respectively, the desired decomposition follows. This concludes the proof of the proposition. □

Proof of Proposition 5: We first prove a useful technical result on the form of the inverse covariance matrix $\Sigma^{-1} = (SY S')^{-1}$ when $SS' = id_{2 \times 2}$.

Lemma 1 *Let $SS' = id_{2 \times 2}$. It then follows:*

$$(SY S')^{-1} = SY^{-1} S'.$$

Proof of Lemma 1: Since $SY S'$ is symmetric, we have:

$$SY S' = Q\Lambda Q' \quad , \quad (SY S')^{-1} = Q\Lambda^{-1} Q', \quad (\text{A29})$$

where Q is a 2×2 matrix of eigenvectors of $SY S'$ and Λ a diagonal 2×2 matrix of eigenvalues. Similarly,

$$Y = \bar{Q}\bar{\Lambda}\bar{Q}' \quad , \quad Y^{-1} = \bar{Q}\bar{\Lambda}^{-1}\bar{Q}',$$

where \bar{Q} is a 3×3 matrix of eigenvectors of Y and $\bar{\Lambda}$ a diagonal matrix of eigenvalues. We first show that the eigenvectors of $SY S'$ are all vectors q_i such that $S'q_i$ is an eigenvector of Y . Indeed, let $\bar{q}_i = S'q_i$ be an eigenvector of Y . It then follows,

$$SY S'q_i = SY\bar{q}_i = \lambda_i S\bar{q}_i = \lambda_i q_i,$$

where λ_i is an eigenvalue of both $SY S$ and Y . In particular, the non-zero elements of Λ are a subset of the nonzero elements of $\bar{\Lambda}$. We also have, for all eigenvectors q_i of $SY S$:

$$S\bar{q}_i = SS'q_i = q_i.$$

Since S has rank 2, one eigenvector \bar{q}_i of $SY S'$ must be such that $S\bar{q}_i = 0$. Without loss of generality, let this eigenvector be \bar{q}_3 . We then have:

$$S\bar{Q} = [\quad Q \quad 0_{2 \times 1}]$$

and

$$SY^{-1} S' = S\bar{Q}\bar{\Lambda}^{-1}\bar{Q}' S' = [\quad Q \quad 0_{2 \times 1}] \bar{\Lambda}^{-1} \begin{pmatrix} Q' \\ 0_{1 \times 2} \end{pmatrix} = Q\Lambda^{-1} Q',$$

because the non zero elements in Λ are a subset of those in $\bar{\Lambda}$. From (A29), we conclude:

$$(SY S')^{-1} = SY^{-1} S',$$

as desired. This concludes the proof of the Lemma.

We now proceed with the proof of Proposition 5. Let $\nu = [\nu_1, \nu_2, \nu_3]$, where $\nu_1, \nu_2, \nu_3 \in \mathbb{R}^3$. It turns out, that the value function can be written in the form:

$$J(x, Y_0) = x^\gamma \inf_{\nu} \frac{1}{\gamma} \mathbb{E} \left[\xi_\nu(T)^{\frac{\gamma}{\gamma-1}} \right]^{1-\gamma} - \frac{1}{\gamma} = \frac{x^\gamma \widehat{J}(0, Y_0)^{1-\gamma} - 1}{\gamma},$$

subject to the constraint $S\nu_1 = 0$, where

$$\begin{aligned} \mathbb{E} \left[\xi_\nu(T)^{\frac{\gamma}{\gamma-1}} \right] &= \mathbb{E}^\gamma \left[e^{-\frac{\gamma}{\gamma-1} \int_0^T r(s) ds + \frac{\gamma}{2(\gamma-1)^2} \text{tr}(\int_0^T \Sigma(s)^{-1} ds \mu^e \mu^{e'} + \int_0^T Y(s) ds (\nu_1 \nu_1' + \nu_2 \nu_2' + \nu_3 \nu_3'))} \right] \\ &= \mathbb{E}^\gamma \left[e^{-\frac{\gamma}{\gamma-1} (r_0 + \text{tr}(\int_0^T Y(s) ds D)) + \frac{\gamma}{2(\gamma-1)^2} \text{tr}(\int_0^T Y(s) ds (S' \mu^e \mu^{e'} S + \nu_1 \nu_1' + \nu_2 \nu_2' + \nu_3 \nu_3'))} \right], \end{aligned} \quad (\text{A30})$$

for a probability measure P^γ defined by the density:

$$\frac{dP^\gamma}{dP} = e^{-\text{tr} \left(\frac{\gamma}{\gamma-1} \int_0^T \Theta'_\nu(s) dB + \frac{1}{2} \frac{\gamma^2}{(\gamma-1)^2} \int_0^T \Theta'_\nu(s) \Theta_\nu(s) ds \right)}.$$

The 3×3 matrix Θ_ν of market prices of risk is defined by:

$$\Theta_\nu = [\quad \Sigma^{1/2'} \Sigma^{-1} \mu^e \quad 0_{3 \times 1} \quad 0_{3 \times 1}] + [\quad \tilde{\nu}_1 \quad \tilde{\nu}_2 \quad \tilde{\nu}_3],$$

where $\tilde{\nu}_1, \tilde{\nu}_2, \tilde{\nu}_3 \in \mathbb{R}^3$ and $\Sigma^{1/2} \tilde{\nu}_1 = 0$. For convenience, we parameterize Θ_ν as:

$$\Theta_\nu = Y^{-1/2} [\quad S' SY S' \mu^e \quad 0_{3 \times 1} \quad 0_{3 \times 1}] + Y^{1/2} [\quad \nu_1 \quad \nu_2 \quad \nu_3],$$

where $\nu_i = Y^{-1/2}\tilde{\nu}_i$ for $i = 1, 2, 3$. In this parametrization, the constraint $\Sigma^{1/2}\tilde{\nu}_1 = 0$ reads $S\nu_1 = 0$, and the objective function (A30) follows. Similarly to the previous portfolio choice settings, the following decomposition of Θ_ν applies:

$$\Theta_\nu = Y^{-1/2}S'SY S'\mu^e e'_1 + Y^{1/2}\nu,$$

where e_i is the i -th unit vector in \mathbb{R}^3 . The dynamics of Y under the probability P^γ are:

$$\begin{aligned} dY &= \left[\Omega\Omega' + \left(M - \frac{\gamma}{1-\gamma}Q'\nu' \right) Y - \frac{\gamma}{1-\gamma}Q'e_1\mu^{e'}SY S'S \right. \\ &\quad \left. + Y \left(M - \frac{\gamma}{1-\gamma}Q'\nu' \right)' - \frac{\gamma}{1-\gamma}S'SY S'\mu^e e'_1 Q \right] dt + Y^{1/2}dB^\gamma Q + Q'dB^{\gamma'}Y^{1/2}. \end{aligned}$$

These dynamics are affine in Y . It follows, that the infinitesimal generator of the process Y is:

$$\begin{aligned} \mathcal{A} &= tr \left(\left(\Omega\Omega' + \left(M - \frac{\gamma}{1-\gamma}Q'\nu' \right) Y - \frac{\gamma}{1-\gamma}Q'e_1\mu^{e'}SY S'S \right) \mathcal{D} \right) \\ &\quad + tr \left(\left(Y \left(M - \frac{\gamma}{1-\gamma}Q'\nu' \right)' - \frac{\gamma}{1-\gamma}S'SY S'\mu^e e'_1 Q \right) \mathcal{D} \right) + tr (2Y\mathcal{D}Q'Q\mathcal{D}). \end{aligned} \quad (\text{A31})$$

The HJB equation for \hat{J} can be then written as:

$$0 = \frac{\partial \hat{J}}{\partial t} - \frac{\gamma}{\gamma-1}\hat{J}(r_0 + tr(YD)) + \inf_\nu \left\{ \mathcal{A}\hat{J} + \frac{\gamma}{2(\gamma-1)^2}\hat{J}tr(Y(S'\mu^e\mu^{e'}S + \nu_1\nu'_1 + \nu_2\nu'_2 + \nu_3\nu'_3)) \right\}, \quad (\text{A32})$$

subject to $S\nu_1 = 0$ and the terminal condition $\hat{J}(T, Y) = 1$. For $i = 2, 3$, the optimal control ν_i is given by:

$$\nu_i = (\gamma-1) \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q'e_i,$$

The optimal control ν_1 satisfies the optimality condition

$$\frac{1}{\gamma-1}Y\nu_1 = Y \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q'e_1 + S'\lambda,$$

where $\lambda \in \mathbb{R}^2$ is the vector of Lagrange multipliers of the constraint $S\nu_1 = 0$. This gives:

$$\nu_1 = (\gamma-1) \left(\left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q'e_1 + Y^{-1}S'\lambda \right).$$

Using the constraint

$$0 = S\nu_1 = (\gamma-1) \left(S \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q'e_1 + SY^{-1}S'\lambda \right),$$

it follows:

$$\lambda = -(\gamma-1)SY S'S \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q'e_1.$$

Therefore, the optimal control ν_1 reads explicitly:

$$\nu_1 = (\gamma-1) (id_{3 \times 3} - Y^{-1}S'SY S'S) \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q'e_1.$$

It then follows:

$$\begin{aligned} \nu &= \nu_1 e'_1 + \nu_2 e'_2 + \nu_3 e'_3 \\ &= (\gamma-1) \left[\left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q' - Y^{-1}S'SY S'S \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q'e_1 e'_1 \right]. \end{aligned}$$

After inserting the expression for the optimal ν in equation (A31), we obtain the generator:

$$\begin{aligned} \mathcal{A} &= tr \left(\left(\Omega\Omega' + MY - \frac{\gamma}{1-\gamma}Q'e_1\mu^{e'}SY S'S + YM' - \frac{\gamma}{1-\gamma}S'SY S'\mu^e e'_1 Q \right) \mathcal{D} + 2Y\mathcal{D}Q'Q\mathcal{D} \right) \\ &\quad - \gamma tr \left(\left(Q'Q \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Y - Q'e_1 e'_1 Q \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) S'SY S'S \right) \mathcal{D} \right) \\ &\quad - \gamma tr \left(\left(Y \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q'Q - S'SY S'S \left(\frac{\mathcal{D}\hat{J}}{\hat{J}} + \frac{\mathcal{D}\hat{J}'}{\hat{J}} \right) Q'e_1 e'_1 Q \right) \mathcal{D} \right). \end{aligned}$$

Using the properties of the trace operator, we obtain the more compact expression:

$$\begin{aligned} \mathcal{A} &= \text{tr} \left(\left(\Omega\Omega' + MY - \frac{\gamma}{1-\gamma} Q'e_1\mu^{e'}SY S'S + YM' - \frac{\gamma}{1-\gamma} S'SY S'\mu^e e_1'Q \right) \mathcal{D} + 2Y\mathcal{D}Q'Q\mathcal{D} \right) \\ &\quad - \gamma \widehat{J} \text{tr} \left(\left(Y \left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}} \right) Q'Q - S'SY S'S \left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}} \right) Q'e_1e_1'Q \right) \left(\frac{\mathcal{D}}{\widehat{J}} + \frac{\mathcal{D}'}{\widehat{J}} \right) \right). \end{aligned}$$

Substitution of the last expression for \mathcal{A} in the HJB equation (A32), using the form of the optimal control ν and the identity $SS' = id_{2 \times 2}$, yields the following partial differential equation for \widehat{J} :

$$\begin{aligned} -\frac{\partial \widehat{J}}{\partial t} &= \text{tr} \left(\left(\Omega\Omega' + MY - \frac{\gamma}{1-\gamma} Q'e_1\mu^{e'}SY S'S + YM' - \frac{\gamma}{1-\gamma} S'SY S'\mu^e e_1'Q \right) \mathcal{D} + 2Y\mathcal{D}Q'Q\mathcal{D} \right) \widehat{J} \\ &\quad - \frac{\gamma}{2} \widehat{J} \text{tr} \left(\left(Y \left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}} \right) Q'Q - S'SY S'S \left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}} \right) Q'e_1e_1'Q \right) \left(\frac{\mathcal{D}\widehat{J}}{\widehat{J}} + \frac{\mathcal{D}\widehat{J}'}{\widehat{J}} \right) \right) \\ &\quad - \frac{\gamma}{\gamma-1} \widehat{J}(r_0 + \text{tr}(YD)) + \frac{\gamma}{2(\gamma-1)^2} \widehat{J} \text{tr}(Y S'\mu^e \mu^{e'} S), \end{aligned}$$

subject to the terminal condition $\widehat{J}(T, Y) = 1$. The affine structure of this problem suggests an affine functional form for \widehat{J} :

$$\widehat{J}(t, Y) = \exp(B(t, T) + \text{tr}(A(t, T)Y)),$$

where the functions $B(t, T)$ and $A(t, T)$ are state-independent and satisfy the boundary condition $B(T, T) = 0$ and $A(T, T) = 0$. It follows that B and A satisfy the following system of differential equations:

$$\begin{aligned} -\frac{dB}{dt} &= -\frac{\gamma}{\gamma-1} r_0 + \text{tr}(A\Omega\Omega') \\ -\text{tr} \left(\frac{dA}{dt} Y \right) &= \text{tr} \left(\left(M'A + AM - \frac{\gamma}{\gamma-1} (S'SAQ'e_1\mu^{e'}S + S'\mu^e e_1'QAS'S) \right) Y \right) \\ &\quad + \text{tr} \left(\left(2((1-\gamma)AQ'QA + \gamma S'SAQ'e_1e_1'QAS'S) + \frac{\gamma}{2(\gamma-1)^2} S'\mu^e \mu^{e'} S - \frac{\gamma}{\gamma-1} D \right) Y \right) \end{aligned}$$

with terminal condition $B(T, T) = 0$ and $A(T, T) = 0$. In order to solve the equation for A , it is useful to apply the following decomposition:

$$\begin{aligned} \frac{dA}{dt} Y &= \frac{dA}{dt} S'SY + \frac{dA}{dt} (id_{3 \times 3} - S'S)Y \\ &= \frac{dA}{dt} S'S(S'SY) + \frac{dA}{dt} (id_{3 \times 3} - S'S)(id_{3 \times 3} - S'S)Y. \end{aligned}$$

We can use this decomposition to obtain two systems of matrix Riccati differential equations for $A_1 := AS'S$ and $A_2 := A(id_{3 \times 3} - S'S)$. The system of differential equations for A_1 is:

$$\begin{aligned} -\frac{dA_1}{dt} &= S'SM'A_1 - \frac{\gamma}{\gamma-1} S'\mu^e e_1'QA_1 + A_1MS'S - \frac{\gamma}{\gamma-1} A_1Q'e_1\mu^{e'}S \\ &\quad + 2(1-\gamma)A_1Q'QA_1 + \gamma A_1Q'e_1e_1'QA_1 + \frac{\gamma}{2(\gamma-1)^2} S'\mu^e \mu^{e'} S - \frac{\gamma}{\gamma-1} S'SDS'S \\ &= \left(S'SM' - \frac{\gamma}{\gamma-1} S'\mu^e e_1'Q \right) A_1 + A_1 \left(MS'S - \frac{\gamma}{\gamma-1} Q'e_1\mu^{e'}S \right) \\ &\quad + 2A_1Q' \left((1-\gamma)id_{3 \times 3} + \gamma e_1e_1' \right) QA_1 + \frac{\gamma}{2(\gamma-1)^2} S'\mu^e \mu^{e'} S - \frac{\gamma}{\gamma-1} S'SDS'S, \end{aligned}$$

subject to the terminal condition $A_1(T, T) = 0$. The system of matrix Riccati differential equations for A_2 is:

$$\begin{aligned} -\frac{dA_2}{dt} &= (id_{3 \times 3} - S'S)M'A_2 + A_2M(id_{3 \times 3} - S'S) + 2(1-\gamma)A_2Q'QA_2 \\ &\quad - \frac{\gamma}{\gamma-1} (id_{3 \times 3} - S'S)D(id_{3 \times 3} - S'S), \end{aligned}$$

subject to the terminal condition $A_2(T, T) = 0$. The unique solution for A is obtained as $A = A_1 + A_2$.

In order to recover the optimal portfolio policy, we follow similar arguments as in the proof of Proposition 3 and use the relation:

$$X^*(t) =: \frac{1}{\xi_{\nu^*}(t)} \mathbb{E}[\xi_{\nu^*}(T) X^*(T) | \mathcal{F}_t] = \psi^{\frac{1}{\gamma-1}} \xi_{\nu^*}(t)^{\frac{1}{\gamma-1}} \widehat{J}(t, Y(t)). \quad (\text{A33})$$

Itô's lemma applied to both sides of (A33) gives, for every state Σ :

$$X^*(t) \operatorname{tr} \left(\begin{bmatrix} \pi_1 & \pi_2 \\ 0 & 0 \end{bmatrix} \Sigma^{1/2} dB \right) = X^*(t) \operatorname{tr} \left(\frac{1}{1-\gamma} \Theta_{\nu^*}' dB + \frac{\mathcal{D}\hat{J}}{\hat{J}} \left(Y^{1/2} dBQ + Q' dB' Y^{1/2} \right) \right).$$

This implies (recall that $\Sigma^{1/2} = SY^{-1/2}$):

$$Y^{-1/2} S' \begin{bmatrix} \pi_1 & 0 \\ \pi_2 & 0 \end{bmatrix} = \frac{1}{1-\gamma} \Theta_{\nu^*}' + 2Y^{1/2} A Q'.$$

We conclude that the portfolio weight $\pi = (\pi_1, \pi_2)'$ is

$$\begin{aligned} \pi &= \frac{1}{1-\gamma} \Sigma^{-1} \mu^e + 2\Sigma^{-1} S A Q' e_1 \\ &= \frac{1}{1-\gamma} \Sigma^{-1} \begin{pmatrix} \mu_1^e \\ \mu_2^e \end{pmatrix} + 2\Sigma^{-1} S \begin{pmatrix} q_{11}A_{11} + q_{12}A_{12} + q_{13}A_{13} \\ q_{11}A_{21} + q_{12}A_{22} + q_{13}A_{23} \\ q_{11}A_{31} + q_{12}A_{32} + q_{13}A_{33} \end{pmatrix}. \end{aligned}$$

This concludes the proof of Proposition 5. □

Table I
Unconditional Moments of Asset Returns

Panel A: Unconditional mean, volatility and volatility of volatility of the S&P100 and the DAX index daily returns (first two rows) and of Treasury bond and Aaa Lehmann Brothers corporate bond index daily returns (last two rows), for the periods January 1988 to December 2005 and April 1996 to December 2005, respectively. Unconditional volatilities of volatilities are estimated as the unconditional volatility of conditional volatilities estimated by Engle's (2002) Dynamic Conditional Correlations model. *Panel B:* Unconditional mean and volatility of correlations for the S&P100 and DAX index daily returns, for the periods January 1988 to December 2005 and April 1996 to December 2005, respectively. Unconditional volatilities of correlations are estimated as the unconditional volatility of conditional correlations estimated by Engle's (2002) Dynamic Conditional Correlations model.

Panel A

	Mean of Returns	Volatility of Returns	Volatility of volatility
US	0.1350	0.1597	0.0574
Germany	0.1106	0.2013	0.0856
Treasuries	0.0165	0.0906	0.0107
Aaa	0.0184	0.0857	0.0100

Panel B

	Mean of Correlation	Volatility of Correlation
US-Germany	0.3672	0.1469
Treasuries-Aaa	0.8841	0.0953

Table II
Calibrated Parameters of the Wishart covariance matrix process

Calibrated matrices M and Q for returns dynamics (1) under Wishart covariance matrix process (4) for $\Omega\Omega' = kQQ'$ and $k = 10$. Parameters have been calibrated to the estimated unconditional moments of returns and volatilities in Table I for US/German equity index returns and Treasury bond/Aaa corporate bond index returns.

	M		Q	
US-Germany	-0.2816	0.0147	-0.0009	-0.0030
	0.0502	-0.1548	0.0028	0.0022
Treasuries-Aaa	-0.4563	0.0273	-0.0007	0.0020
	0.0659	-0.3951	0.0024	0.0019

Table III
The 3-assets case.

Panel A: Unconditional mean, volatility, volatility of volatility, unconditional mean and volatility of correlations for the S&P100, the DAX and the FTSE100 index daily returns (first two rows) . The sample period is from January 1988 to December 2005. Unconditional volatilities of volatilities, correlations and volatility of correlations are estimated by Engle's (2002) Dynamic Conditional Correlations model. *Panel B:* Hedging portfolio policies for the 3-assets extension of the model with returns dynamics (1) under Wishart covariance matrix process (4) for $\Omega\Omega' = kQQ'$ and $k = 10$. Hedging policies are reported as fractions of the corresponding myopic components. Parameters have been calibrated to the estimated unconditional moments of returns and volatilities in Panel A for US/German/English equity index returns.

Panel A			
	Mean of Returns	Volatility of Returns	Volatility of volatility
US	0.1350	0.1597	0.0574
Germany	0.1106	0.2013	0.0856
UK	0.1123	0.1376	0.0520
	Mean of Correlation	Volatility of Correlation	
US-Germany	0.3672	0.1469	
US-UK	0.3577	0.1427	
UK-Germany	0.5578	0.1677	
Panel B			
	Correlation Hedging	Volatility Hedging: Volatility Component	Volatility Hedging: Covariance Component
S&P100	0.0923	0.0185	0.0354
DAX30	0.0695	0.0281	0.0318
FTSE100	0.0901	0.0242	0.0298

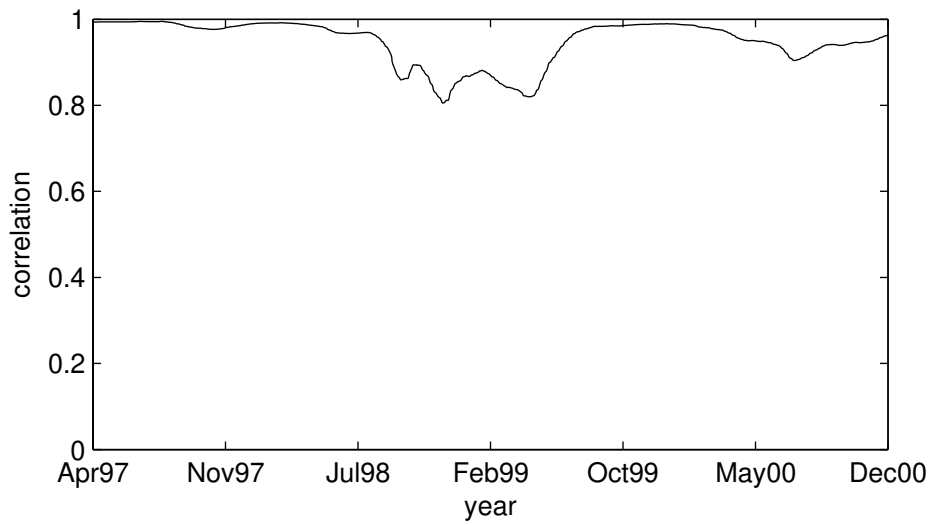


Figure 1. Rolling correlations of Treasury bonds and Aaa corporate bonds. The Figure presents correlations for Treasury and Aaa Corporate bond index daily returns over the period April 1997 to December 2000. Dynamic correlations are estimated by a simple empirical correlation estimator based on a rolling window of 260 days, updated daily.

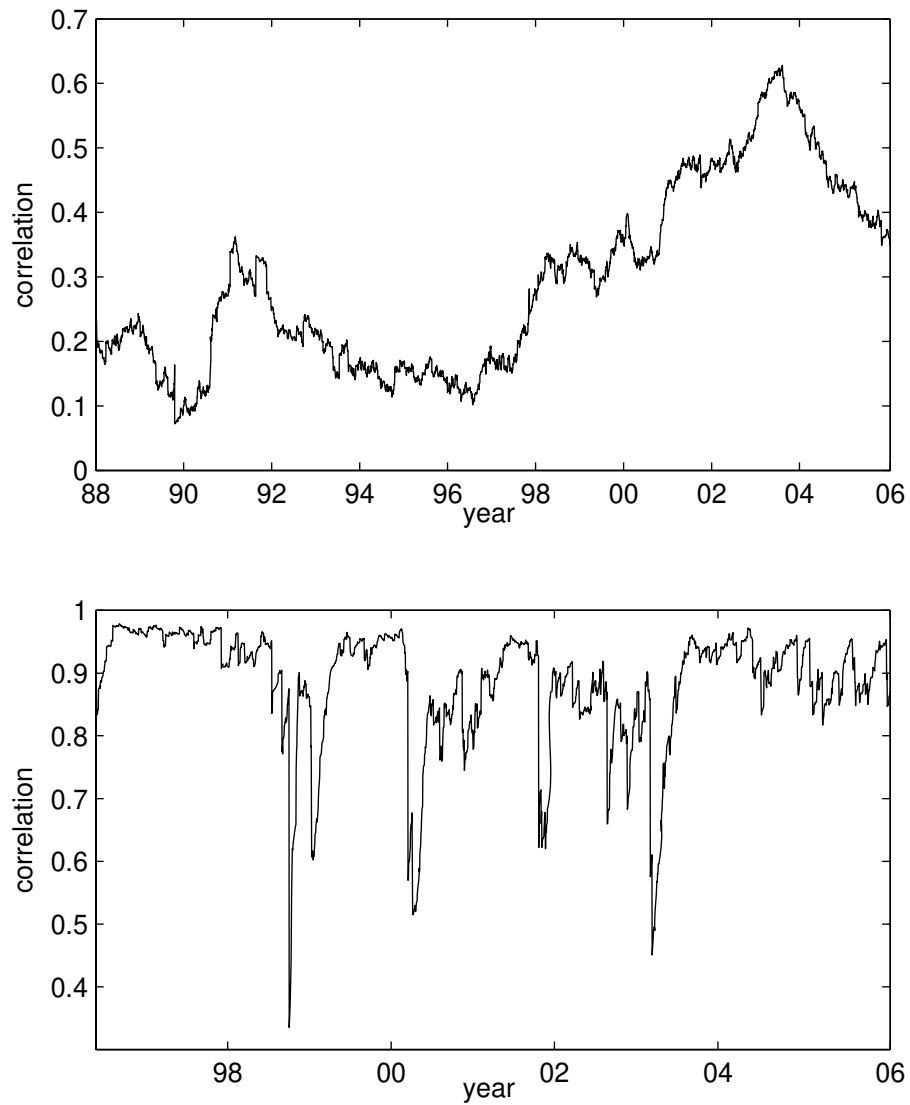


Figure 2. DCC conditional correlations. The upper panel presents estimated conditional correlations from Engle's (2002) DCC model for US and German daily equity index returns, over the period January 1988 to December 2005. The bottom panel presents estimated conditional correlations from Engle's (2002) DCC model for daily Treasury and Aaa corporate bond index returns, over the period April 1996 to December 2005.

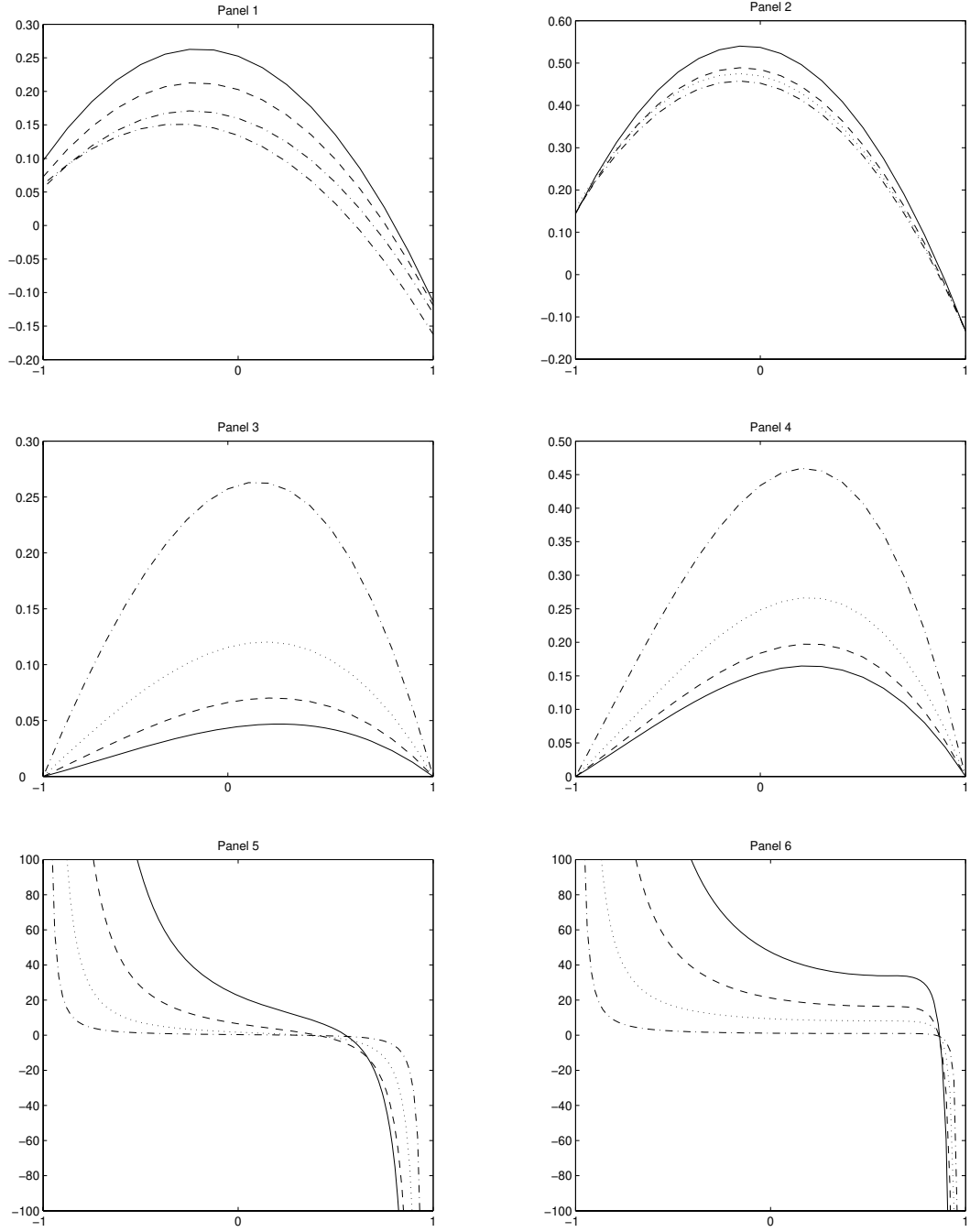


Figure 3. Drift and instantaneous volatility of correlation dynamics. This figure presents the instantaneous drift (Panels 1 and 2), volatility (Panels 3 and 4), and pool function (Panels 5 and 6) of the correlation process implied by the diffusion process (4), as a function of the correlation level ρ and for different values of the volatility ratio $\sqrt{\Sigma_{11}/\Sigma_{22}}$ (0.5, continuous line, 0.75 dashed line, 1, dotted line, 1.25 dashed and dotted line). In Panels on the left column, model parameters have been calibrated to US and German daily equity index returns, over the period January 1988 to December 2005, as described in Section 4, and Σ_{22} has been fixed at the estimated level of German returns unconditional variance. In Panels on the right column, model parameters have been calibrated to US Treasury and Aaa Corporate daily bond index returns, over the period April 1996 to December 2005, as described in Section 4, and Σ_{22} has been fixed at the estimated level of Aaa bond returns unconditional variance.

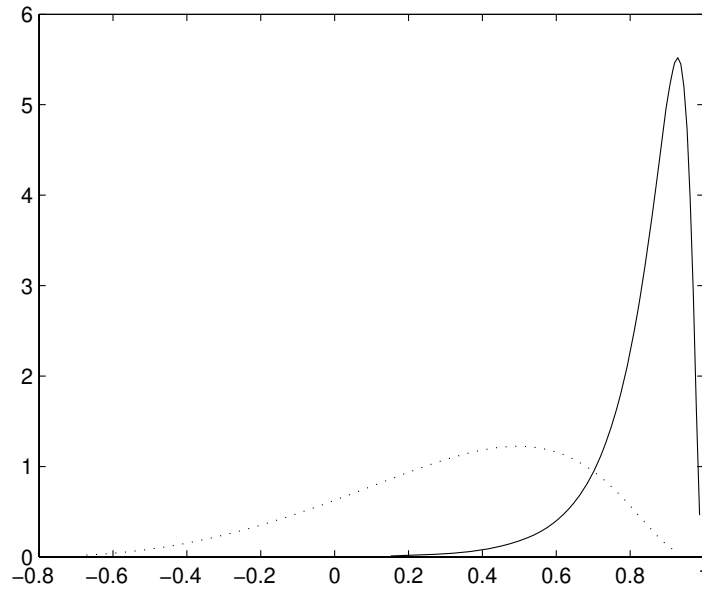


Figure 4. Properties of correlation dynamics. This picture presents estimates of the unconditional probability density function of return correlation implied by the Wishart dynamics (4). The dotted line draws the density of correlation between the S&P100 and the DAX30 index, as arising from the calibration of our model to the unconditional moments of US and German daily equity index returns over the period January 1988 to December 2005. The continuous line draws the density of correlation between Treasury and Aaa corporate bonds, as arising from the calibration of our model to the unconditional moments of Treasury bond and Aaa monthly bond index returns, over the period April 1996 to December 2005. These densities are estimated using a kernel density estimator, with Gaussian kernel, on a sample of 5000 simulated realizations of the correlation.

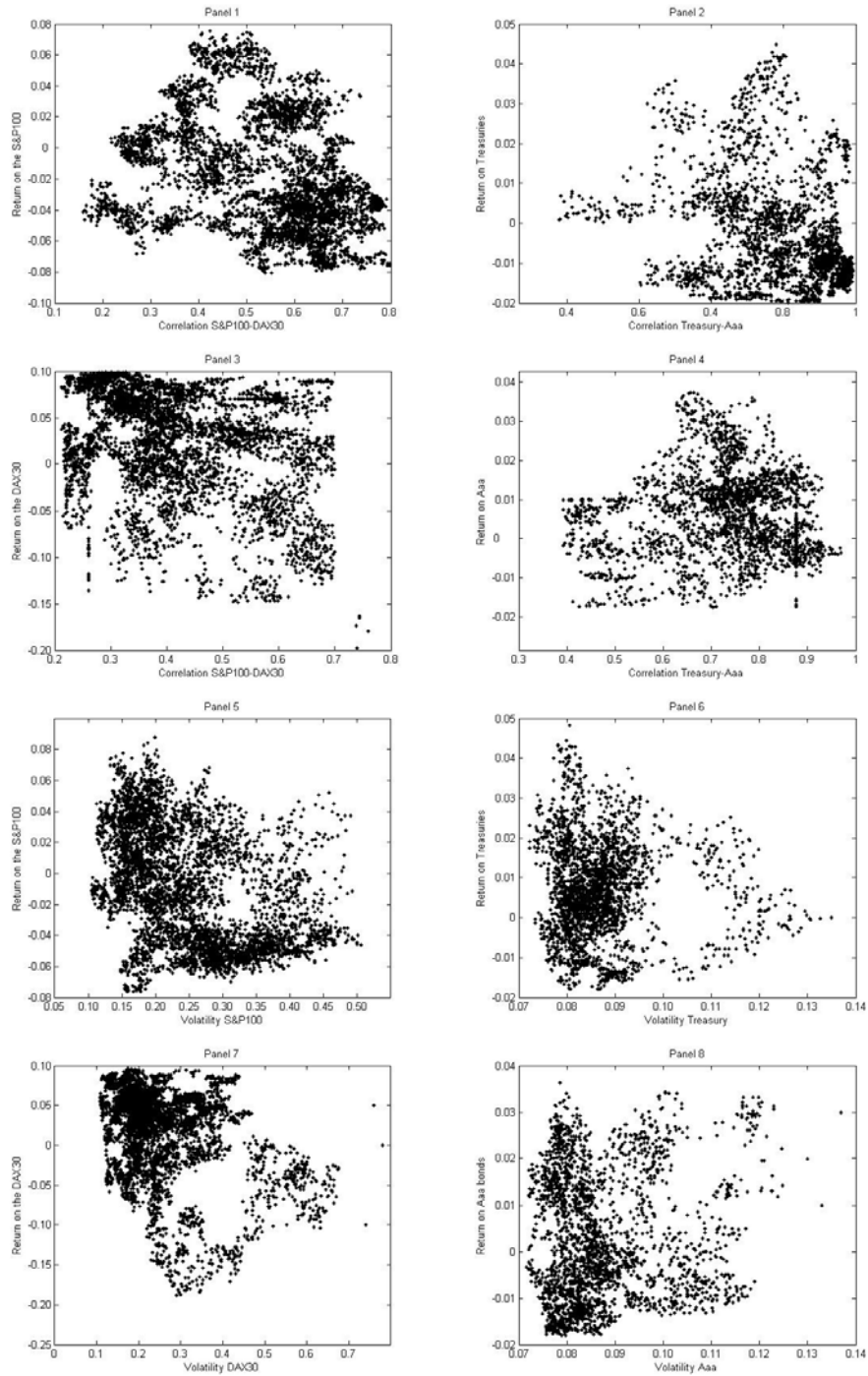


Figure 5. Volatility and correlation-leverage. This figure presents scatter plots of simulated returns, returns volatilities and correlations of *i*) US and German equity indexes (Panels on the left column), and *ii*) Treasury and Aaa Corporate bonds indexes (Panels on the right column). Parameters of the Wishart dynamics (4) have been calibrated to the unconditional moments of time series of these indexes. Panels 1-4 present scatter plots of returns correlations plotted against returns of US and German equity indexes (Panels 1 and 3, respectively), and Treasury and Aaa corporate bonds (Panels 2 and 4, respectively). Panels 5-8 depict scatter plots of returns volatilities plotted against returns of US and German equity indexes (Panels 5 and 7, respectively), and Treasury and Aaa corporate bonds (Panels 6 and 8, respectively).

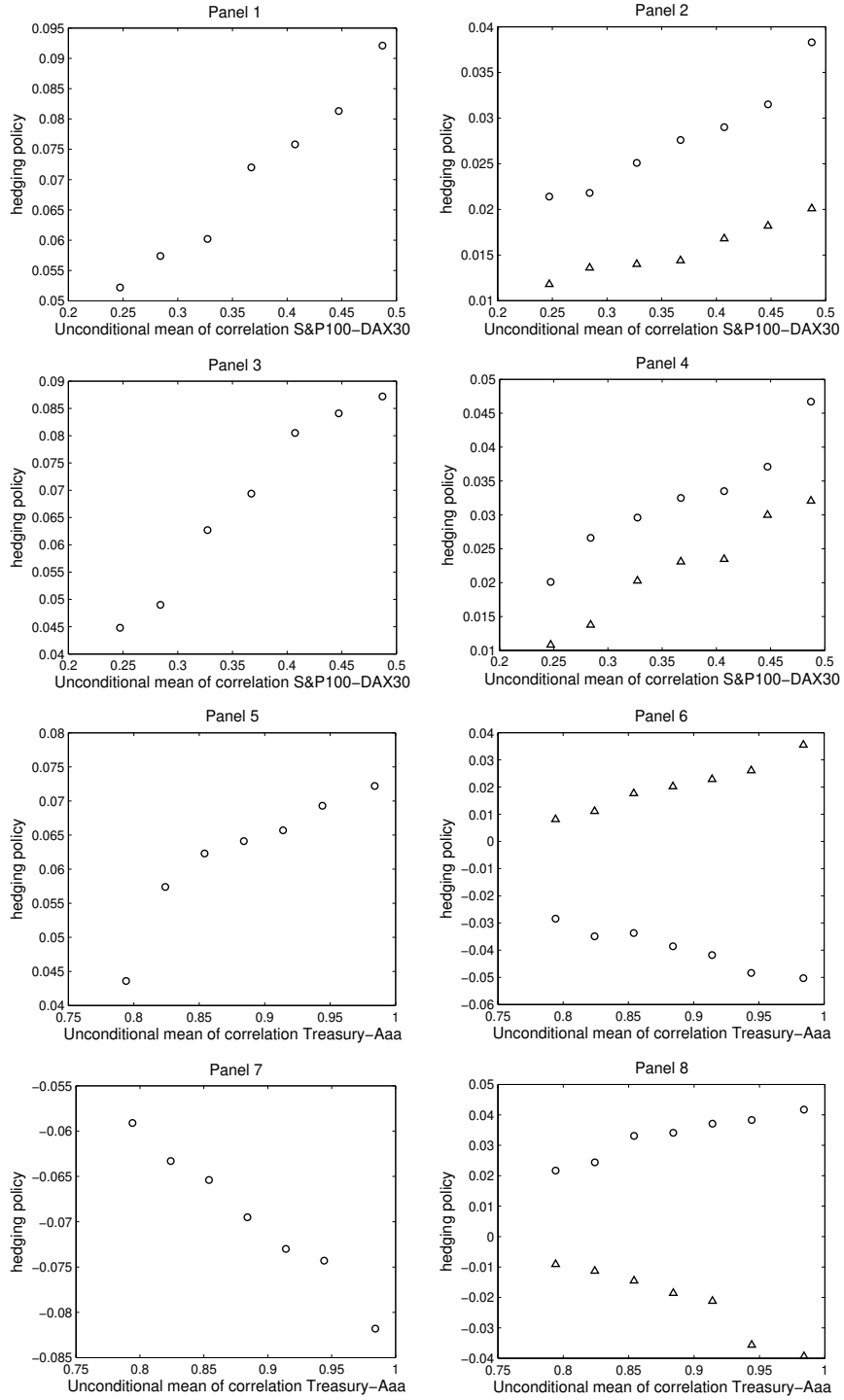


Figure 6. First calibration. Unconditional mean, standard deviation of returns volatilities and unconditional volatility of correlation are fixed at the DCC estimates. The correlation mean varies around the unconditional correlation estimate. Left panels plot correlation hedging as a fraction of the absolute myopic demands. Right panels plot volatility hedging as a fraction of the absolute myopic component (pure volatility hedging, triangle points, and hedging due to volatility influence on covariance risk, circle points). In Panels 1-4, the opportunity set is composed of the US (Panels 1-2) and German (Panels 3-4) equity indices, whereas in Panels 5-8 it is composed of US Treasury bond (Panels 5-6) and US Aaa Corporate bond (Panels 7-8) indices. The investment horizon is 5 years. The relative risk aversion coefficient is $1 - \gamma = 3$.

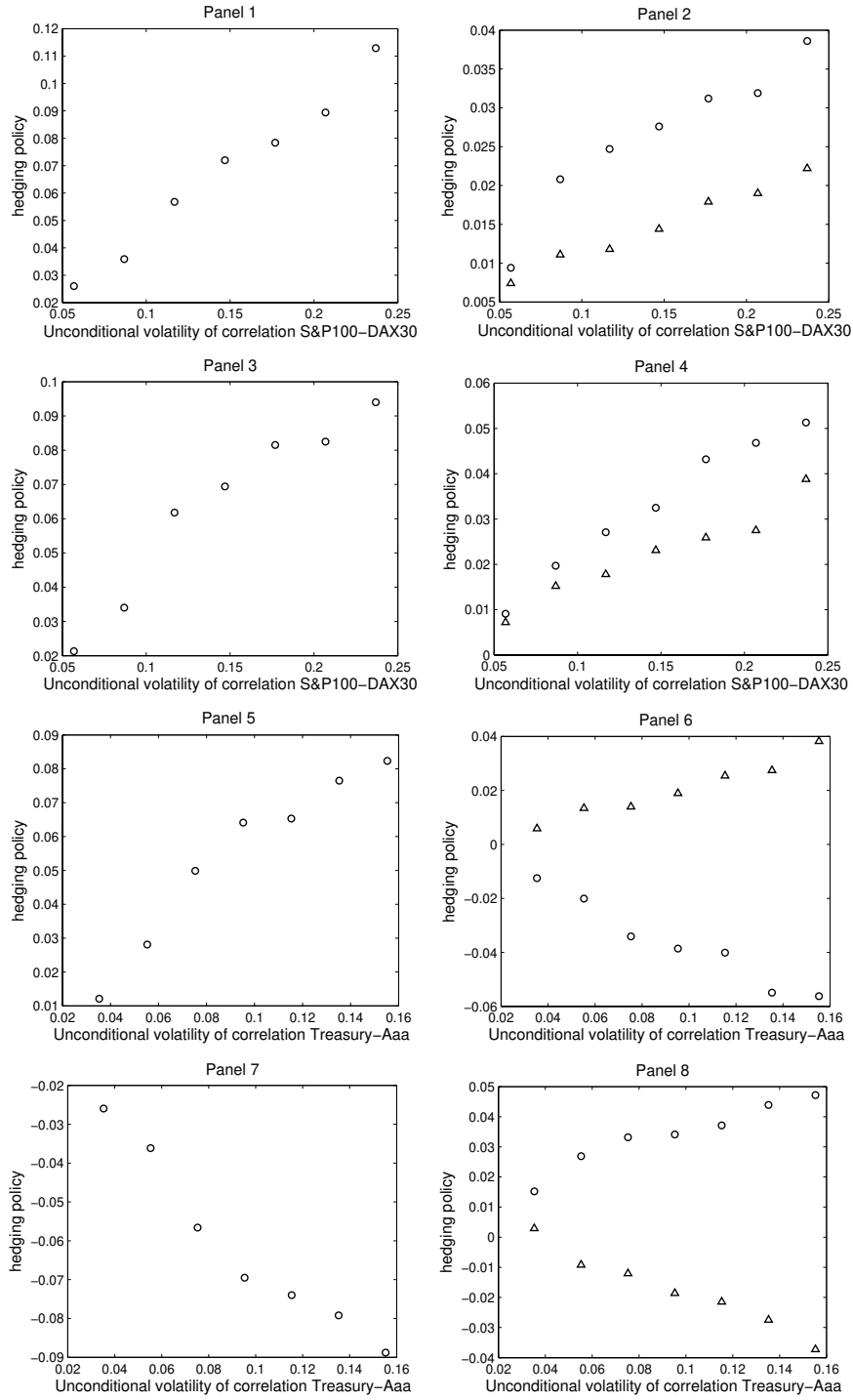


Figure 7. Second calibration. Unconditional mean, standard deviation of returns volatilities and unconditional mean of correlation are fixed at the DCC estimates. The correlation volatility varies around the unconditional correlation volatility implied by the DCC estimate. Left panels plot correlation hedging as a fraction of absolute myopic demands. Right panels plot volatility hedging as a fraction of the absolute myopic components (pure volatility hedging, triangle points, and hedging due to volatility influence on covariance risk, circle points). In Panels 1-4, the investment opportunity set is composed of US (Panels 1-2) and German (Panels 3-4) equity indices, whereas in Panels 5-8 it is composed of US Treasury bond (Panels 5-6) and US Aaa Corporate bond (Panels 7-8) indices. The investment time-horizon is 5 years. The relative risk aversion coefficient is $1 - \gamma = 3$.

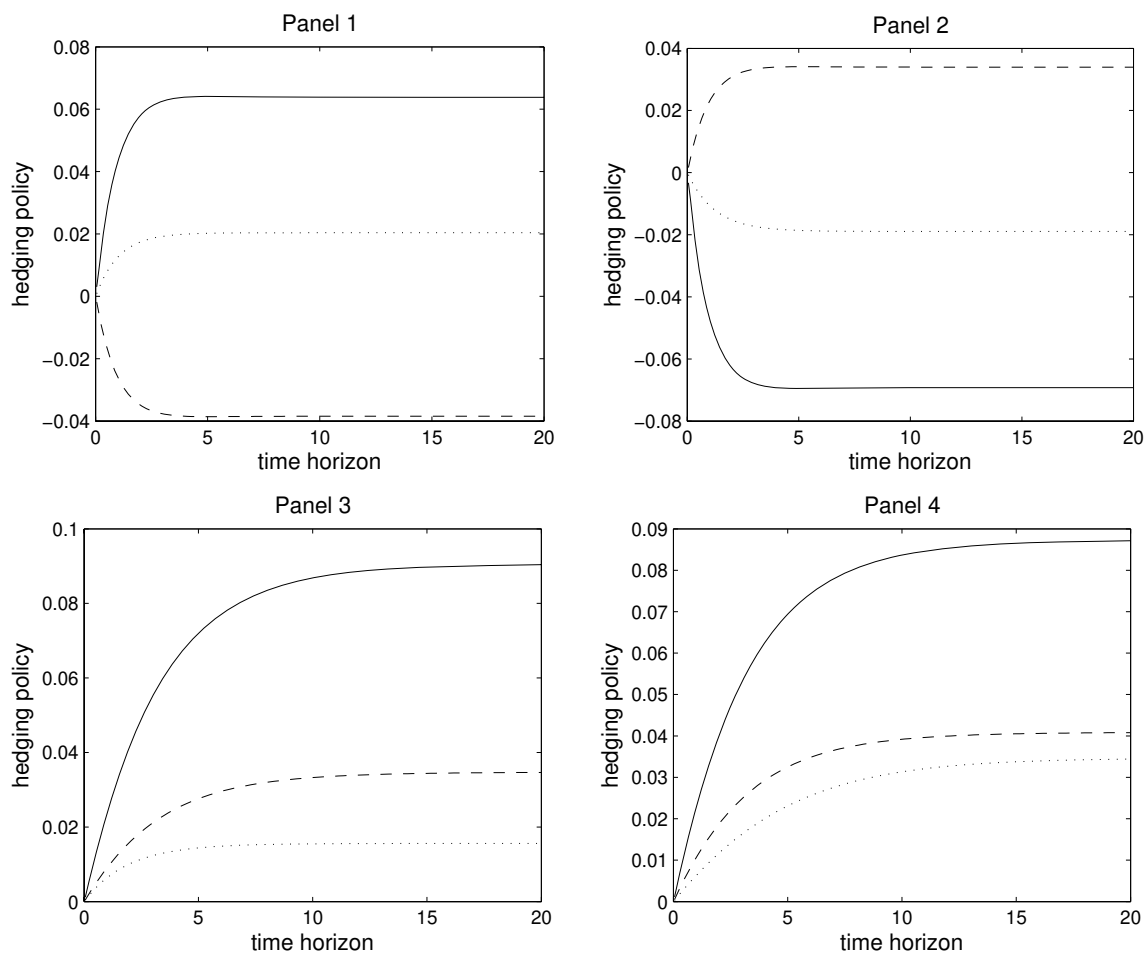


Figure 8. Horizon effect. Portfolio allocations due to correlation hedging (solid lines) and volatility hedging (pure volatility hedging, dotted lines, and hedging volatility influence on covariance risk, dashed lines). Policies are plotted as fractions of the corresponding absolute myopic allocations, for US Treasury bonds (Panel 1) and US Aaa corporate bonds (Panel 2), and for the S&P100 (Panel 3) and the DAX30 index (Panel 4). Time horizons are up to 20 years. The relative risk aversion coefficient used is $1 - \gamma = 3$.

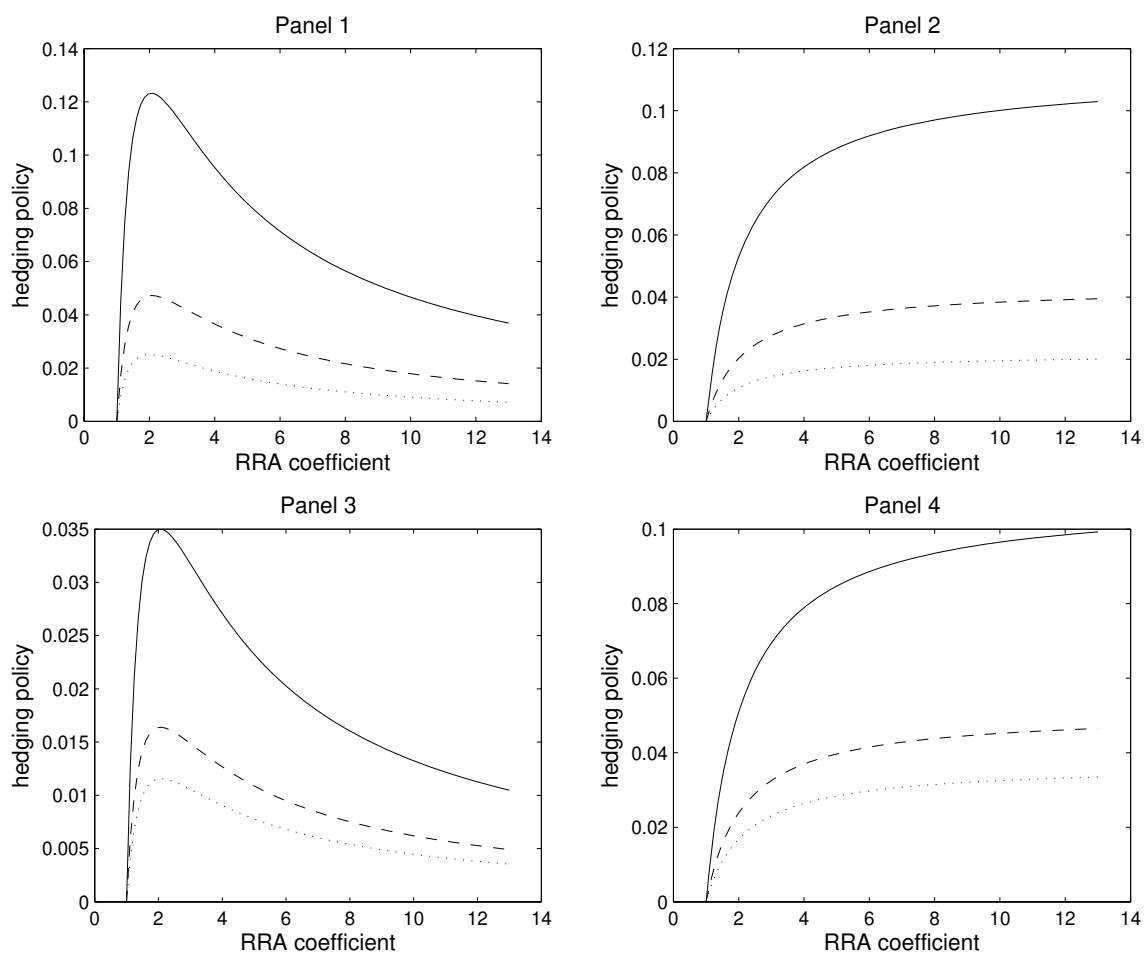


Figure 9. Risk aversion dependence. Portfolio allocation due to correlation hedging (solid lines) and volatility hedging (pure volatility hedging, dotted lines, and volatility influence on covariance risk, dashed lines). Policies for the S&P100 index are in Panels 1 and 2, and policies for the DAX30 index are in Panels 3 and 4. Panels 1 and 3 plot hedging portfolio weights, whereas Panels 2 and 4 plot hedging portfolio weights scaled by the corresponding absolute myopic allocations. The Relative Risk Aversion coefficient ranges from 1 to 13. The time horizon is 5 years.

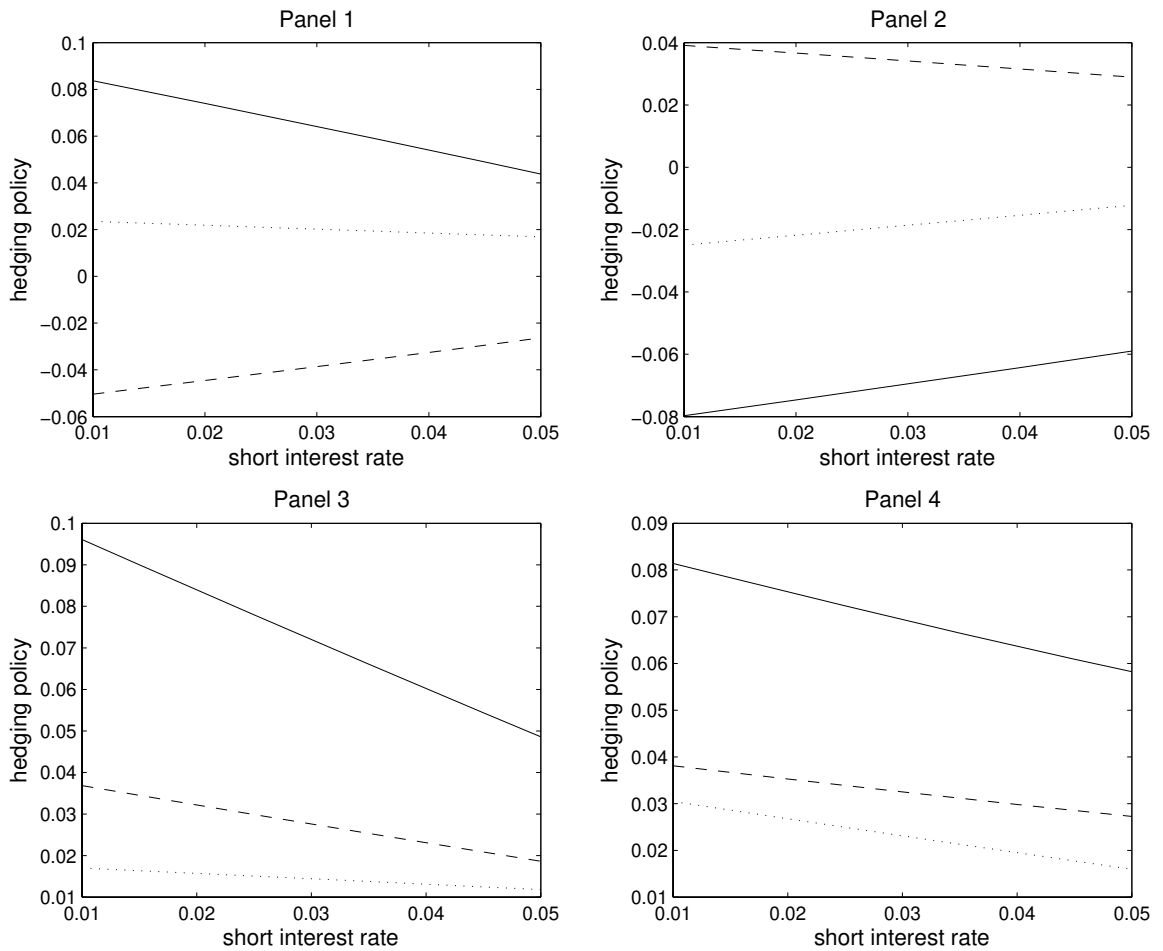


Figure 10. Dependence on calibrated λ . Portfolio allocation due to correlation hedging (solid lines) and volatility hedging (pure volatility hedging, dotted lines, and volatility influence on covariance risk, dashed lines) as the calibrated parameter λ changes for increasing interest rate. Policies for US Treasury bonds and Aaa Corporate bonds are in Panels 1 and 2, respectively. Policies for the S&P100 index and the DAX30 index are in Panels 3 and 4. The Relative Risk Aversion coefficient is 3 and the time horizon is 5 years.

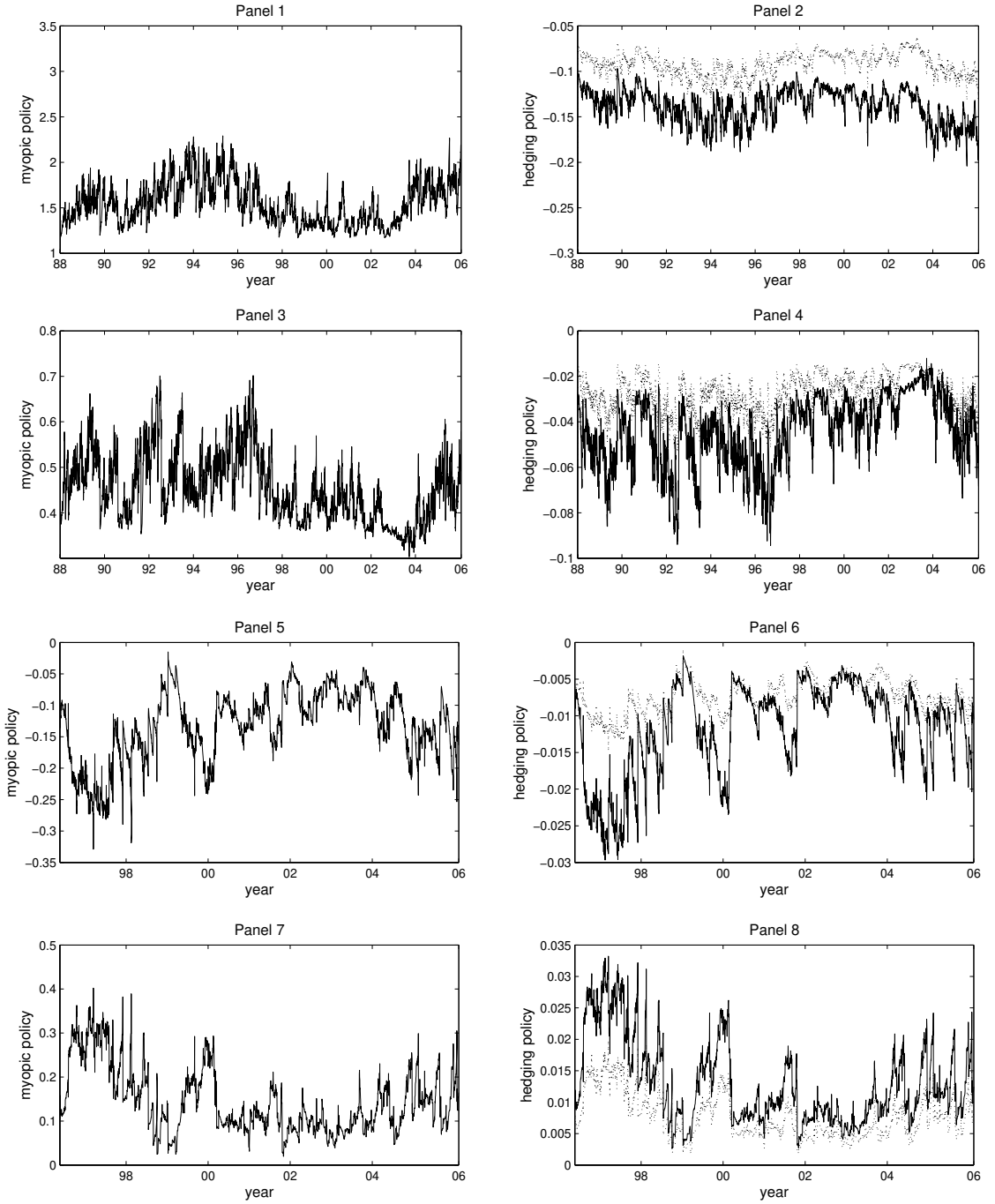


Figure 11. Conditional calibration for the model with constant expected returns. Conditional portfolio policy for the model in Proposition 5, with Y a 3×3 -dimensional Wishart process, $\Sigma^{-1} = SYS'$ and $D = 0$. Policies are plotted for the US (Panels 1 and 2) and the German (Panels 3 and 4) equity indices, and for US Treasury bonds (Panels 5 and 6) and US Aaa corporate bonds (Panels 7 and 8). Panels on the left report myopic policies. Panels on the right report total hedging demands (solid lines) and the component of hedging demands due to correlation hedging (dotted lines). The relative risk aversion coefficient used is $1 - \gamma = 3$ and the time horizon is 5 years.