CORRELATIONS OF THE RIEMANN ZETA FUNCTION

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ABSTRACT. Assuming the Riemann hypothesis, we investigate the shifted moments of the zeta function

$$M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T) = \int_T^{2T} \prod_{k=1}^m |\zeta(\tfrac{1}{2} + i(t+\alpha_k))|^{2\beta_k} dt$$

introduced by Chandee [4], where $\boldsymbol{\alpha} = \boldsymbol{\alpha}(T) = (\alpha_1, \dots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1 \dots, \beta_m)$ satisfy $|\alpha_k| \le T/2$ and $\beta_k \ge 0$. We shall prove that

$$M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T) \ll_{\boldsymbol{\beta}} T(\log T)^{\beta_1^2 + \cdots + \beta_m^2} \prod_{1 \le j < k \le m} |\zeta(1 + i(\alpha_j - \alpha_k) + 1/\log T)|^{2\beta_j\beta_k}.$$

This improves upon the previous best known bounds due to Chandee [4] and Ng, Shen, and Wong [21], particularly when the differences $|\alpha_j - \alpha_k|$ are unbounded as $T \to \infty$. The key insight is to combine work of Heap, Radziwiłł, and Soundararajan [13] and work of the author [7] with the work of Harper [12] on the moments of the zeta function.

1. INTRODUCTION

This paper is concerned with the shifted moments

$$M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T) = \int_T^{2T} \prod_{k=1}^m |\zeta(\tfrac{1}{2} + i(t+\alpha_k))|^{2\beta_k} dt, \tag{1}$$

where $\boldsymbol{\alpha} = \boldsymbol{\alpha}(T) = (\alpha_1, \dots, \alpha_m)$ and $\boldsymbol{\beta} = (\beta_1 \dots, \beta_m)$ satisfy $|\alpha_k| \le T/2$ and $\beta_k \ge 0$. These were first studied in general by Chandee [4], who gave lower bounds assuming the β_k are integers, $\alpha_k = O(\log\log T)$, and $|\alpha_j - \alpha_k| = O(1)$. Chandee also gave upper bounds assuming RH when $|\alpha_j - \alpha_k| = O(1)$ and $\alpha_k = O(\log T)$ which are sharp up to a $(\log T)^{\varepsilon}$ loss. Subsequently Ng, Shen, and Wong [21] removed the $(\log T)^{\varepsilon}$ loss in the special case where $\boldsymbol{\beta} = (\beta, \beta)$ by using the work of Harper [12] on the moments of the zeta, and they also gave bounds in the larger regime $|\alpha_1 + \alpha_2| \le T^{0.6}$. More precisely, in this range they proved

$$M_{(\alpha_1,\alpha_2),(\beta,\beta)}(T) \ll T(\log T)^{2\beta^2} F(\alpha_1, \alpha_2, T)^{2\beta^2}$$

where

$$F(\alpha_1, \alpha_2, T) = \begin{cases} \min\left(|\alpha_1 - \alpha_2|^{-1}, \log T\right) & |\alpha_1 - \alpha_2| \le 1/100 \\ \log(2 + |\alpha_1 - \alpha_2|) & |\alpha_1 - \alpha_2| > 1/100 \end{cases}.$$

Some special cases of the shifted moments $M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T)$ and related objects have been studied unconditionally. For example the integral

$$\int_T^{2T} \zeta(\tfrac{1}{2} + i(t+\alpha_1))\zeta(\tfrac{1}{2} - i(t+\alpha_2))dt$$

akin to $M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T)$ with $\boldsymbol{\beta} = (\frac{1}{2}, \frac{1}{2})$ is fairly well understood. Here the current state of the art comes from Atkinson's formula for the mean square of zeta [1] and Bettin's work on the second moment of zeta with shifts of size $T^{2-\varepsilon}$ [2]. The current state of the art for $M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T)$ with $\boldsymbol{\beta} = (1, 1)$ is due to

Motohashi's explicit formula for the fourth moment of zeta [19, 20] and Kovaleva's work showing Motohashi's formula holds for shifts of size $T^{6/5-\varepsilon}$ [17]. Finally in the case where $\boldsymbol{\beta} = (\beta, \beta)$, sharp upper bounds for $\beta \leq 1$ and lower bounds for all $\beta \geq 0$ were obtained by the author [7]. The goal of this paper is, assuming RH, to extend the work of Ng, Shen, and Wong [21] to arbitrary $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ and to give stronger bounds in the regime where the $|\alpha_j - \alpha_k|$ are unbounded.

Theorem 1.1. *If $\beta_k \geq 0$ and $|\alpha_k| \leq T/2$ for $k = 1, \dots, m$, then*

$$M_{\boldsymbol{\alpha},\boldsymbol{\beta}}(T) \ll T(\log T)^{\beta_1^2+\cdots+\beta_m^2} \prod_{1 \leq j < k \leq m} |\zeta(1 + i(\alpha_j - \alpha_k) + 1/\log T)|^{2\beta_j\beta_k}.$$

Remark. Throughout this paper we will assume T is large in terms of $\boldsymbol{\beta}$, and whenever we use the Vinogradov notation $\ll$ we allow the implicit constant to depend on $\boldsymbol{\beta}$. When we use big-Oh notation, however, then the implicit constants will be absolute unless there is a subscript $O_{\boldsymbol{\beta}}$ to indicate $\boldsymbol{\beta}$ dependence.

Note that our result agrees with Chandee's conjecture for the size of $M_{(\alpha_1,\alpha_2),(\beta,\beta)}(T)$ in the regime $|\alpha_1 - \alpha_2| = O(1)$, and without this restriction our bound is the same order of magnitude predicted by the famous recipe of Conrey, Farmer, Keating, Rubinstein, and Snaith [6]. At heart, Theorem 1.1 is a statement about how $\zeta(\frac{1}{2}+it)$ and $\zeta(\frac{1}{2}+i(t+\alpha))$ are correlated for $t \in [T, 2T]$ and $|\alpha| \leq T/2$. More precisely, it predicts that $\zeta(\frac{1}{2} + it)$ and $\zeta(\frac{1}{2} + i(t + \alpha))$ are perfectly correlated on average when $|\alpha| \leq 1/\log T$, and decorrelate like $|\zeta(1 + i\alpha)|$ for $|\alpha| > 1/\log T$. When $\alpha \leq 1$, the Laurent expansion for zeta shows that we obtain the same correlations predicted from random matrix theory. For larger α, the correlation is of order 1 on average, which can be seen by estimating moments of zeta to the right of the 1-line. There are, however, long range correlations coming from the primes; more precisely, from the extreme values of zeta on the one line. This is to be expected, for the Keating Snaith philosophy only predicts that random matrix theory is a good model for $\zeta(\frac{1}{2} + it)$ in short intervals.

The starting point of the proof is to use the method of Soundararajan [24] and Harper [12] to bound $\log|\zeta(\frac{1}{2} + it)|$ by a short Dirichlet polynomial. Instead of following the argument of Harper, however, we treat the exponential of this short Dirichlet polynomial in a way similar to the work of Heap, Radziwiłł, and Soundararajan [13]. Using this method, the integrals that arise can be evaluated by simply using the mean value theorem for Dirichlet polynomials. These mean values are much easier to evaluate uniformly in the shifts α_k than the integrals of products of shifted cosines that appear when using Harper's method [12] as Ng, Shen, and Wong do in [21]. This difference allows us to obtain upper bounds for general shifts $\boldsymbol{\alpha}$ and exponents $\boldsymbol{\beta}$. The final ingredient is a more precise estimate of the following sum

$$\sum_{p \leq X} \frac{\cos(\delta \log p)}{p}$$

coming from the theory of pretentious multiplicative functions, see Lemma 2.6. This idea appeared in the author's previous work on studying the second moment of moments of zeta in short intervals [7]. This more precise estimate is what allows us to improve the bound of Ng, Shen, and Wong [21] in the regime where $|\alpha_j - \alpha_k|$ is unbounded.

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2. Preliminary tools and notation

We will start by using the following lemma, which is due to Soundararajan [24] and Harper [12].

Lemma 2.1. *Assume RH, let $t \in [T, 2T]$, and $|\alpha| \le T/2$. Then for $2 \le X \le T^2$*

$$\log|\zeta(\tfrac{1}{2} + i(t+\alpha))| \le \mathrm{Re} \sum_{p \le X} \frac{1}{p^{1/2+1/\log X + i(t+\alpha)}} \frac{\log X/p}{\log X}$$
$$+ \sum_{p \le \min(\sqrt{X}, \log T)} \frac{1}{2p^{1+2i(t+\alpha)}} + \frac{\log T}{\log X} + O(1).$$

Throughout it will be useful to break the set of primes into certain intervals. Set

$$\beta_* := \sum_{k \le m} \max(1, \beta_k).$$

We choose a sequence with $T_0 = 2$ and

$$T_j = T^{e^{j-1}/(\log_2 T)^2}$$

for $j > 0$ where we use $\log_j$ to denote the j-fold iterated logarithm. We will choose L to be the largest integer such that $T_L \le T^{1/200\beta_*^2}$. Given any $1 \le j \le L$ define

$$\mathcal{P}_{j,X}(s) = \sum_{p \in (T_{j-1}, T_j]} \frac{1}{p^s} \frac{\log X/p}{\log X}.$$

More generally, given an interval $I_j \subseteq (T_{j-1}, T_j]$ we will set

$$\mathcal{P}_{I_j,X}(s) = \sum_{p \in I_j} \frac{1}{p^s} \frac{\log X/p}{\log X}.$$

If $\mathcal{P}_{I_j,X}(s)$ is not too large, then we will be able to efficiently approximate $\exp(\beta \mathcal{P}_{I_j,X}(s))$ with its Taylor series. Indeed, if we choose cutoff parameters $K_j = (\log_2 T)^{3/2} e^{-j/2}$ for $j \ge 1$ and set

$$\mathcal{N}_{I_j,X}(s;\beta) := \sum_{n \le 20\beta_* K_j} \frac{\beta^n \mathcal{P}_{I_j,X}(s)^n}{n!}$$

where $\beta \le \beta_*$, then we have the following analog of lemma 1 of [13]:

Lemma 2.2. *If $\beta \le \beta_*$, $I_j \subset (T_{j-1}, T_j]$, and $|\mathcal{P}_{I_j,X}(s)| \le 2K_j$ for some $1 \le j \le L$ then*

$$\exp(2\beta \mathrm{Re}\, \mathcal{P}_{I_j,X}(s)) \le (1 + e^{-10K_j\beta_*})^{-1} |\mathcal{N}_{I_j,X}(s;\beta)|^2.$$

Proof. Since $|\mathcal{P}_{I_j,X}(s)| \le 2K_j$, series expansion gives

$$|\exp(\beta \mathcal{P}_{I_j,X}(s))| - e^{-20\beta_* K_j} \le |\mathcal{N}_{I_j,X}(s;\beta)|.$$

By assumption $\exp(-2K_j\beta_*) \le |\exp(\beta \mathcal{P}_{I_j,X}(s))| \le \exp(2K_j\beta_*)$, so the claim readily follows. □

We will first bound the shifted moment of zeta when all of the shifts $t+\alpha_k$ lie in the "good" set

$$\mathcal{G} := \{t \in [T/2, 5T/2] : |P_{j,T_L}(1+1/\log T_L + it)| \le K_j \text{ for all } 1 \le j \le L\}. \tag{2}$$

In this case we may use Lemma 2.1 with $X = T_L$ in tandem with Lemma 2.2 to reduce the problem to computing the mean value of certain Dirichlet polynomial. We will accomplish this with the following mean value theorem of Montgomery and Vaughan (see for example theorem 9.1 of [10]).

Lemma 2.3. *Given any complex numbers a_n*

$$\int_T^{2T} \left|\sum_{n\le N} \frac{a_n}{n^{it}}\right|^2 dt = (T+O(N)) \sum_{n\le N} |a_n|^2.$$

We will also make use of the property that Dirichlet polynomials supported on distinct sets of primes are approximately independent in the mean square sense. The precise formulation we will use is the following splitting lemma which appears in equation (16) of [14]

Lemma 2.4. *Suppose for $1 \le j \le L$ we have j disjoint intervals I_j and Dirichlet polynomials $A_j(s) = \sum_n a_j(n) n^{-s}$ such that $a_j(n)$ vanishes unless n is composed of primes in I_j. Then if $\prod_{j\le L} A_j(s)$ is a Dirichlet polynomial of length N*

$$\int_T^{2T} \prod_{j\le L} |A_j(\tfrac12+it)|^2 dt = (T+O(N)) \prod_{j\le L} \left(\frac{1}{T}\int_T^{2T} |A_j(\tfrac12+it)|^2 dt\right)$$

The following variant due to Soundararajan [24, lemma 3] will also be useful for handling moments of Dirichlet polynomials supported on primes.

Lemma 2.5. *Let k be a natural number and suppose $N^k \le T/\log T$. Then given any complex numbers a_p*

$$\int_T^{2T} \left|\sum_{p\le N} \frac{a_p}{p^{it}}\right|^{2k} dt \ll Tk! \left(\sum_{p\le N} |a_p|^2\right)^k.$$

During the main mean value calculation, we will need to bound a certain product over primes. This product will be controlled with the following lemma, which is a special case of lemma 3.2 of [16]:

Lemma 2.6. *Given $\delta \in \mathbb{R}$ and $X \ge 2$*

$$\sum_{p\le X} \frac{\cos(\delta \log p)}{p} = \log|\zeta(1+1/\log X + i\delta)| + O(1).$$

To handle the shifted moment of zeta when some of the shifts $t+\alpha_k$ lie in the "bad" set $[T/2, 5T/2] \setminus \mathcal{G}$, we take advantage of the incremental structure present. For each $1 \le j \le L$, define

$$\mathcal{B}_j := \{t \in [T/2, 5T/2] : |\mathcal{P}_{r,T_s}(\tfrac12 + 1/\log T_s + it)| \le K_j \text{ for all } 1 \le r < j \text{ and } r \le s \le L$$
$$\text{but } |\mathcal{P}_{j,T_s}(\tfrac12 + 1/\log T_s + it)| > K_j \text{ for some } j \le s \le L\}.$$

Notice that

$$[T/2, 5T/2] \setminus \mathcal{G} = \bigsqcup_{j\le L} \mathcal{B}_j.$$

On the bad sets $\mathcal{B}_j$ the series expansion $\mathcal{N}_{j,T_s}$ of $\exp(P_{j,T_s})$ is a poor approximation, so we are forced to estimate $\log\zeta$ using only the primes up to T_{j-1}. While the resulting Dirichlet polynomial is too short to obtain sharp bounds, we can hedge this loss by multiplying by a suitably large even power of $|P_{j,T_s}|/K_j$, which is larger than 1 on $\mathcal{B}_j$. If we then extend the range of integration to all of $[T, 2T]$, we can still win as the event $|\mathcal{P}_{j,T_s}(1+1/\log T_s + it)| > K_j$ is quite rare. For example, we will make use of the following bound.

Lemma 2.7. *If meas*(S) *denotes the measure of a set* $S \subset [T/2, 5T/2]$, *then*

$$\textit{meas}(\mathcal{B}_1) \ll Te^{-(\log_2 T)^2/5}.$$

Proof. Taking the union bound over all $1 \le s \le L$, we may bound the measure of $\mathcal{B}_1$ by

$$\sum_{s\le L} \frac{1}{K_1^{2k}} \int_{T/2}^{5T/2} |\mathcal{P}_{1,T_s}(\tfrac{1}{2}+1/\log T_s + it)|^{2k} dt$$

for some positive integer k. Trivially bounding $\log(T_s/p)/\log T_s$ by 1, Lemma 2.3 shows this quantity has order at most

$$(T+T_1^k)Lk!\left(\frac{e}{(\log_2 T)^2}\right)^k.$$

We may now conclude by taking $k = \lceil(\log_2 T)^2/e\rceil$ and using Stirling's approximation, noting that $L \le \log_3 T$. ☐

Finally we will need to control the contribution of the prime squares. To this end denote for each $1 \le l \le \log_2 T$ the Dirichlet polynomials

$$Q_l(s) := \sum_{e^l < p \le e^{l+1}} \frac{1}{2p^{2s}}$$

and the sets

$$\mathcal{C}_l = \{t \in [T/2, 5T/2] : |Q_l(\tfrac{1}{2}+it)| > J_l \text{ and } |Q_r(\tfrac{1}{2}+it)| \le J_r \text{ for all } l < r \le \log_2 T\},$$

where we take $J_l := e^{-l/10}$ and follow the convention that $t \in \mathcal{C}_0$ if t does not lie in $\mathcal{C}_l$ for any $1 \le l \le \log_2 T$. If $t \in \mathcal{C}_0$ then the contribution of the prime squares is negligible. On the other hand the integrals over the $\mathcal{C}_l$ are handled in a similar fashion to the integrals over the $\mathcal{B}_j$: first by multiplying by a large even power of $|Q_l(\frac{1}{2}+it)|/J_l$ then by extending the range of integration and using the mean value theorem for Dirichlet polynomials. As with the $\mathcal{B}_j$, the events $\mathcal{C}_l$ are quite rare:

Lemma 2.8. *For* $1 \le l \le \log_2 T$

$$\textit{meas}(\mathcal{C}_l) \ll Te^{-le^{3l/4}}.$$

Proof. The same reasoning used in Lemma 2.7 shows that the measure of $\mathcal{C}_l$ is bounded by

$$\frac{1}{J_l^{2k}} \int_{T/2}^{5T/2} |Q_l(\tfrac{1}{2}+it)|^{2k} dt.$$

Applying Lemma 2.5 with $k = \lceil le^{4l/5}\rceil$ to this integral readily gives the claim. ☐

The proof of Theorem 1.1 is based on the following partition of $[T, 2T]$: Given a subset A of $[m] := \{1, \ldots, m\}$ define

$$\mathcal{G}_A := \{t \in [T, 2T] : t + \alpha_k \in \mathcal{G} \text{ if and only if } k \in A\}.$$

Then we can decompose $[T, 2T]$ into the disjoint union

$$[T, 2T] = \bigsqcup_{A \subseteq [m]} \mathcal{G}_A. \tag{3}$$

In section 3, we will handle the integral over the set $\mathcal{G}_{[m]}$ where all of the shifts $t + \alpha_k$ are good. In section 4, we will handle the cases where some of the shifts $t + \alpha_k$ are bad. We will have to further partition the sets $\mathcal{G}_A$ with $A \subsetneq [m]$ according to which of the sets $\mathcal{B}_j$ the bad shifts $t + \alpha_k$ lie in. There are really no additional new ideas in this section, however, and the difficulty here mostly comes from the need for rather messy notation.

3. Moments over good shifts

We will further partition $\mathcal{G}_{[m]}$ using the subsets

$$\mathcal{C}_\ell = \{t \in [T, 2T] : t + \alpha_k \in \mathcal{C}_{\ell(k)}\}$$

where $\ell : [m] \to \{0, 1, \ldots, \lfloor \log_2 T \rfloor\}$ is an arbitrary function. First we will show that we may restrict our attention to functions ℓ such that $|\ell|_\infty \le 2 \log_3 T$, where $|\ell|_\infty$ denotes the maximum of ℓ.

Proposition 3.1. *For all* ℓ

$$\int_{\mathcal{C}_\ell} \prod_{k=1}^m |\zeta(\tfrac{1}{2} + i(t + \alpha_k))|^{2\beta_k} dt \ll T \exp\left(-\frac{|\ell|_\infty}{2} \exp\left(3|\ell|_\infty/4\right)\right) (\log T)^{2\beta_*^2}.$$

Therefore

$$\sum_{\substack{\ell \\ |\ell|_\infty > 2\log_3 T}} \int_{\mathcal{C}_\ell} \prod_{k=1}^m |\zeta(\tfrac{1}{2} + i(t + \alpha_k))|^{2\beta_k} dt \ll T \exp(-(\log_2 T)^{3/2} (\log_3 T)/10).$$

Remark. Since the shifts satisfy $|\alpha_j - \alpha_k| \le T$, the error term in this proposition (and in later calculations) is of smaller order then the upper bound in Theorem 1.1 by the standard estimate $|\zeta(1 + 1/\log T + it)| \gg \zeta(2 + 2/\log T)/\zeta(1 + 1/\log T) \gg 1/\log T$.

Proof. First note that $\mathcal{C}_\ell$ is contained in a translate of $\mathcal{C}_{|\ell|_\infty}$. Therefore by Lemma 2.8, Harper's bound (say) for the moments of zeta [12], and Hölder's inequality we find that

$$\int_{\mathcal{C}_\ell} \prod_{k=1}^m |\zeta(\tfrac{1}{2} + i(t + \alpha_k))|^{2\beta_k} dt \ll \left(\text{meas}(\mathcal{C}_{|\ell|_\infty})\right)^{1/2} \left(\int_T^{2T} \prod_{k=1}^m |\zeta(\tfrac{1}{2} + i(t + \alpha_k))|^{4\beta_k} dt\right)^{1/2}$$
$$\ll T \exp\left(-\frac{|\ell|_\infty}{2} \exp\left(3|\ell|_\infty/4\right)\right) (\log T)^{2\beta_*^2}.$$

To bound the sum over ℓ, first note that the number of ℓ with $|\ell|_\infty = l$ at most $(l + 1)^m$. Then the second claim follows from the geometric series formula, say, after using the crude bound $\exp(3|\ell|_\infty/4) \ge (\log_2 T)^{3/2}$. □

We shall therefore restrict our attention to ℓ with $|\ell|_\infty \le \log_3 T$ going forward. Now for each ℓ denote by $I_{j,\ell}$ the interval $(T_{j-1}, T_j] \cap (2^{|\ell|_\infty+1}, \infty)$. Our assumption on ℓ implies the interval $I_{j,\ell}$ only differs from $(T_{j-1}, T_j]$ when $j = 1$; however, it will be notationally simpler to use $I_{j,\ell}$ for all j rather than to treat the first interval separately. Notice also that for $t \in \mathcal{C}_l$

$$\left| \mathrm{Re} \sum_{p \le e^{l+1}} \frac{1}{p^{1/2+1/\log T_L + it}} \frac{\log(X/p)}{\log X} + \mathrm{Re} \sum_{p \le \log_2 T} \frac{1}{2p^{1+2it}} \right| = O(e^{l/2}). \tag{4}$$

Therefore for fixed ℓ we may use Lemma 2.1 with $X = T_L$ to find

$$\int_{\mathcal{G}_{[m]} \cap \mathcal{C}_\ell} \prod_{k=1}^{m} |\zeta(\tfrac{1}{2} + i(t + \alpha_k))|^{2\beta_k} dt$$

$$\ll \int_{\mathcal{G}_{[m]} \cap \mathcal{C}_\ell} \prod_{k=1}^{m} \exp\left(2\beta_k \mathrm{Re} \left(\sum_{j=1}^{L} \mathcal{P}_{j,T_L}(\tfrac{1}{2} + 1/\log T_L + i(t+\alpha_k)) + \sum_{l \le \log_2 T} Q_l(\tfrac{1}{2} + i(t + \alpha_k)) \right) \right) dt$$

$$\ll e^{O(\beta_* e^{|\ell|_\infty/2})} \int_{\mathcal{G}_{[m]} \cap \mathcal{C}_\ell} \prod_{k=1}^{m} \prod_{j=1}^{L} \exp\left(2\beta_k \mathrm{Re}\, \mathcal{P}_{I_{j,\ell},T_L}(\tfrac{1}{2} + 1/\log T_L + i(t+\alpha_k))\right) dt$$

Next recall $K_1 = (\log_2 T)^{3/2} e^{-1/2}$, so by (4) and the assumption that $|\ell|_\infty \le 2\log_3 T$ it follows that

$$|\mathcal{P}_{I_{1,\ell},T_L}(\tfrac{1}{2} + 1/\log T_L + i(t+\alpha_k))| \le 2K_1$$

for all $k \le m$ and $t \in \mathcal{G}_{[m]} \cap \mathcal{C}_\ell$. Furthermore when $1 < j \le L$

$$|\mathcal{P}_{I_{j,\ell},T_L}(\tfrac{1}{2} + 1/\log T_L + i(t+\alpha_k))| \le K_j$$

for all $k \le m$ and $t \in \mathcal{G}_{[m]} \cap \mathcal{C}_\ell$ because $I_{j,\ell} = (T_{j-1}, T_j]$ for $j > 1$. Therefore Lemma 2.2 allows us to bound the integral over $\mathcal{G}_{[m]} \cap \mathcal{C}_\ell$ by

$$\ll e^{O(\beta_* e^{|\ell|_\infty/2})} \int_{\mathcal{G}_{[m]} \cap \mathcal{C}_\ell} \prod_{k=1}^{m} \prod_{j=1}^{L} (1 + e^{-10K_j\beta_*})^{-1} |\mathcal{N}_{I_{j,\ell},T_L}(\tfrac{1}{2} + 1/\log T_L + i(t+\alpha_k); \beta_k)|^2 dt$$

$$\ll e^{O(\beta_* e^{|\ell|_\infty/2})} \int_{T}^{2T} \left| e^{|\ell|_\infty/10} Q_{|\ell|_\infty}(\tfrac{1}{2} + it + i\alpha_{k_*}) \right|^{2\lceil |\ell|_\infty e^{4|\ell|_\infty/5} \rceil}$$

$$\times \prod_{j=1}^{L} \prod_{k=1}^{m} |\mathcal{N}_{I_{j,\ell},T_L,\alpha_k}(1 + 1/\log T_L + it; \beta_k)|^2 dt, \tag{5}$$

where k_* is arbitrarily chosen so that $\ell(k_*) = |\ell|_\infty$ and

$$\mathcal{N}_{I_{j,\ell},X,\alpha}(s;\beta) := \mathcal{N}_{I_{j,\ell},X}(s + i\alpha; \beta).$$

Here we have used that $|e^{|\ell|_\infty/10} Q_{|\ell|_\infty}(\tfrac{1}{2} + it + i\alpha_{k_*})| \ge 1$ on $\mathcal{C}_\ell$, that $\prod_{j=1}^{L}(1 + e^{-10K_j\beta_*})^{-1} < \infty$, and nonnegativity to extend the integral to the entire interval $[T, 2T]$.

We are now in a setting where me may use the mean value theorem for Dirichlet polynomials. First note that $Q_l(s)^{\lceil le^{4l/5} \rceil}$ has length at most $\exp(5(\log_3 T)^2 (\log_2 T)^{8/5}) = T^{o(1)}$ for $l \le 2\log_3 T$ and that $\prod_{k=1}^{m} \mathcal{N}_{I_{j,\ell},T_L,\alpha_k}(s;\beta_k)$ has length at most $T_j^{20\beta_*^2 K_j}$. Therefore $\prod_{j \le L} \prod_{k=1}^{m} \mathcal{N}_{I_{j,\ell},T_L,\alpha_k}(s;\beta_k)$

has length at most $T^{1/2}$ as $T_L \le T^{1/200\beta_*^2}$, so the integrand in (5) is a short Dirichlet polynomial. Thus by Lemma 2.4 it suffices to compute the mean values

$$\int_T^{2T} \left|e^{|\ell|_\infty/10} Q_{|\ell|_\infty}(\tfrac{1}{2}+it+i\alpha_{k_*})\right|^{2\lceil |\ell|_\infty e^{4|\ell|_\infty/5}\rceil} dt$$

and

$$\int_T^{2T} \prod_{k=1}^m |\mathcal{N}_{I_{j,\ell},T_L,\alpha_k}(1+1/\log T_L+it;\beta_k)|^2 dt$$

for each $1 \le j \le L$. The estimate for the moment of $Q_{|\ell|_\infty}$ is straightforward.

Proposition 3.2. *For fixed* ℓ

$$\int_T^{2T} \left|e^{|\ell|_\infty/10} Q_{|\ell|_\infty}(\tfrac{1}{2}+it+i\alpha_{k_*})\right|^{2\lceil |\ell|_\infty e^{4|\ell|_\infty/5}\rceil} dt \ll Te^{-|\ell|_\infty e^{|\ell|_\infty/2}}.$$

Proof. By Lemma 2.5

$$\int_T^{2T} |Q_{|\ell|_\infty}(\tfrac{1}{2}+it+i\alpha_{k_*})|^{2\lceil |\ell|_\infty e^{4|\ell|_\infty/5}\rceil} dt$$

$$\ll T\lceil |\ell|_\infty e^{4|\ell|_\infty/5}\rceil! \left(\sum_{e^{|\ell|_\infty}<p\le e^{|\ell|_\infty+1}} \frac{1}{2p^2}\right)^{\lceil |\ell|_\infty e^{4|\ell|_\infty/5}\rceil}.$$

Because

$$\sum_{e^{|\ell|_\infty}<p\le e^{|\ell|_\infty+1}} \frac{1}{2p^2} \ll \frac{1}{|\ell|_\infty e^{|\ell|_\infty}}$$

the claim follows by a routine application of Stirling's approximation. □

To estimate the mean values of the $\mathcal{N}_{I_{j,\ell},X,\alpha}(s;\beta)$ we must analyze their coefficients. Denote $a_X(p) := \log(X/p)/\log X$. We may then write

$$\mathcal{N}_{I_{j,\ell},X,\alpha}(s;\beta) = \sum_{\substack{p|n \Rightarrow p \in I_{j,\ell} \\ \Omega(n) \le 20\beta_* K_j}} \frac{\beta^{\Omega(n)} g_X(n) n^{-i\alpha}}{n^s}$$

where $\Omega(n)$ is the number of prime factors of n counting multiplicity and

$$g_X(n) := \prod_{p^r \| n} \frac{a_X(p)^r}{r!}.$$

Note that $g_X(n)$ is a multiplicative function and $a_X(p) \in [0,1]$. We do not need to completely understand all of the coefficients of the $\mathcal{N}_{I_{j,\ell},X,\alpha}(s;\beta)$; the following lemma will suffice.

Lemma 3.3. *Write*

$$\prod_{k=1}^m \mathcal{N}_{I_{j,\ell},X,\alpha_k}(s;\beta_k) = \sum_{n\ge 1} \frac{b_{I_{j,\ell},X,\boldsymbol{\alpha},\boldsymbol{\beta}}(n)}{n^s}.$$

Then $b_{I_{j,\ell},X,\boldsymbol{\alpha},\boldsymbol{\beta}}$ *is supported on integers composed of primes in* $I_{j,\ell}$*, and for* $p \in I_{j,\ell}$

$$b_{I_{j,\ell},X,\boldsymbol{\alpha},\boldsymbol{\beta}}(p) = a_X(p) \sum_{k=1}^m \beta_k p^{-i\alpha_k}.$$

In general we have the bound

$$|b_{I_{j,\ell},X,\boldsymbol{\alpha},\boldsymbol{\beta}}(p^r)| \le \frac{\beta_*^r k^r}{r!}.$$

Proof. The first two properties are immediate by expanding the coefficients $b_{I_{j,\ell},X,\boldsymbol{\alpha},\boldsymbol{\beta}}(n)$ using the Dirichlet convolution. To deduce the upper bound, first note that the p^r coefficient of each $\mathcal{N}_{I_{j,\ell},X,\alpha}(s;\beta)$ is bounded by $\beta^r/r!$. Whence

$$|b_{I_{j,\ell},X,\boldsymbol{\alpha},\boldsymbol{\beta}}(p^r)| \le \sum_{r_1+\cdots+r_k=r} \prod_{l=1}^{k} \frac{\beta_l^{r_l}}{r_l!} \le \frac{\beta_*^r}{r!} \sum_{r_1+\cdots+r_k=r} \binom{r}{r_1,\ldots,r_k} = \frac{\beta_*^r k^r}{r!}.$$

□

We can now compute

Proposition 3.4. *For* $1 \le j \le s \le L$ *and*

$$\int_T^{2T} \prod_{k=1}^{m} |\mathcal{N}_{I_{j,\ell},T_s,\alpha_k}(1+1/\log T_s + it; \beta_k)|^2 dt \le (T+O(T^{1/2})) \prod_{p\in I_{j,\ell}} \left(1 + \frac{|b_{I_{j,\ell},X,\boldsymbol{\alpha},\boldsymbol{\beta}}(p)|^2}{p} + O_{\boldsymbol{\beta}}\left(\frac{1}{p^2}\right)\right).$$

Proof. By Lemma 2.3 and multiplicativity the mean value of interest equals

$$(T+O(T^{1/2})) \sum_{\substack{p|n \Rightarrow p\in I_{j,\ell} \\ \Omega(n)\le 20\beta_* K_j}} \frac{|b_{I_{j,\ell},X,\boldsymbol{\alpha},\boldsymbol{\beta}}(n)|^2}{n^{1+1/\log T_s}} \le (T+O(T^{1/2})) \prod_{p\in I_{j,\ell}} \left(1 + \sum_{r\ge 1} \frac{|b_{I_{j,\ell},X,\boldsymbol{\alpha},\boldsymbol{\beta}}(p^r)|^2}{p^r}\right),$$

where in the final step we disposed of the condition on $\Omega(n)$ using nonnegativity. Now just notice the contribution of terms with $r \ge 2$ is $\ll 1/p^2$ by the upper bound in Lemma 3.3. □

Now we can use Lemma 2.4 to estimate (5). If we combine Propositions 3.1, 3.2, and 3.4 and sum over ℓ with $|\ell|_\infty \le 2\log_3 T$ we conclude

Proposition 3.5. *On the good set we have the bound*

$$\int_{\mathcal{G}_{[m]}} \prod_{k\le m} |\zeta(\tfrac{1}{2} + i(t+\alpha_k))|^{2\beta_k} dt \ll T(\log T)^{\beta_1^2+\cdots+\beta_m^2} \prod_{1\le j<k\le m} |\zeta(1+i(\alpha_j-\alpha_k)+1/\log T)|^{2\beta_j\beta_k}.$$

Proof. Recalling that at most $(l+1)^k$ of the functions ℓ have $|\ell|_\infty = l$, we see the integral of interest is of order at most

$$T \sum_{0\le l\le 2\log_3 T} (l+1)^k e^{-le^{l/2}} \prod_{2^{l+1}<p\le T_L} \left(1 + \frac{1}{p}\sum_{j,k\le m} \frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}} + O_{\boldsymbol{\beta}}\left(\frac{1}{p^2}\right)\right)$$

$$\ll T \sum_{0\le l\le 2\log_3 T} (l+1)^k e^{-le^{l/2}} \times (l+1)^{\beta_*^2} \prod_{p\le T_L} \left(1 + \frac{1}{p}\sum_{j,k\le m} \frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}} + O_{\boldsymbol{\beta}}\left(\frac{1}{p^2}\right)\right).$$

This is

$$\ll T \prod_{p\le T_L} \left(1 + \frac{\beta_1^2+\cdots+\beta_m^2}{p}\right) \prod_{1\le j<k\le L} \prod_{p\le T_L} \left(1 + \frac{2\beta_j\beta_k\cos((\alpha_j-\alpha_k)\log p)}{p}\right).$$

The first product is visibly $\ll (\log T_L)^{\beta_1^2+\cdots+\beta_m^2}$, and we can bound the remaining products with Lemma 2.6. The claim now follows since $\log T_L \asymp \log T$. □

4. MOMENTS OVER BAD SHIFTS

We now consider the integral over $\mathcal{G}_A$ where A is a proper subset of $[m]$. We will partition $\mathcal{G}_A$ using maps $f : [m] \setminus A \to [L]$ such that $f(\alpha_k) \in \mathcal{B}_{f(k)}$ for $k \notin A$. If we write

$$\mathcal{G}_{A,f} = \{t \in \mathcal{G}_A : f(\alpha_k) \in \mathcal{B}_{f(k)} \text{ for } k \notin A\},$$

then $\mathcal{G}_A$ is the disjoint union of the $\mathcal{G}_{A,f}$ over all $f : [m] \setminus A \to [L]$. Therefore the task at hand is bounding shifted moments over the $\mathcal{G}_{A,f}$. Without loss of generality we will write $A = [m] \setminus [a]$. That is we assume that the first a shifts are bad for some $a \in [m]$, so f is a map from $[a]$ to $[L]$. For such a set A and a map f, we will write $\mathcal{G}_{a,f}$ for $\mathcal{G}_{A,f}$. We may also assume that f is an increasing function, and by Proposition 3.1 we restrict the range of integration to $\mathcal{G}[A] \cap \mathcal{C}_\ell$ for some $\ell : [m] \to \{0, \ldots, \lfloor 2\log_3 T \rfloor\}$. Since f is increasing, we can only work with the primes up to $T_{f(1)-1}$. We first handle the case with $f(1) = 1$.

Proposition 4.1. *If $f(1) = 1$, then*

$$\int_{\mathcal{G}_{a,f}} \prod_{k=1}^{m} |\zeta(\tfrac{1}{2} + i(t + \alpha_k))|^{2\beta_k} dt \ll T e^{-(\log_2 T)^2/20}.$$

Therefore

$$\sum_{\substack{f:[a]\to[L] \\ f(1)=1}} \int_{\mathcal{G}_{a,f}} \prod_{k=1}^{m} |\zeta(\tfrac{1}{2} + i(t + \alpha_k))|^{2\beta_k} dt \ll T e^{-(\log_2 T)^2/30}.$$

Proof. Because $\mathcal{G}_{a,f}$ is contained in a translate of $\mathcal{B}_1$ the first bound is a consequence of Lemma 2.7 and the Cauchy-Schwarz inequality. To bound the sum over f, just notice that there are at most L^a such maps, and $L \leq \log_3 T$. $\square$

Now for fixed a, f, and ℓ with $f(1) > 1$ we may use Lemma 2.1 with $X = T_{f(1)-1}$ in the same manner as the previous section to find

$$\int_{\mathcal{G}_{a,f}\cap\mathcal{C}_\ell} \prod_{k=1}^m |\zeta(\tfrac{1}{2}+i(t+\alpha_k))|^{2\beta_k} dt$$
$$\ll \int_{\mathcal{G}_{[m]}\cap\mathcal{C}_\ell} \prod_{k=1}^m \exp\Bigg(2\beta_k \mathrm{Re}\Bigg(\sum_{j<f(1)} \mathcal{P}_{j,T_{f(1)-1}}(\tfrac{1}{2}+1/\log T_{f(1)-1}+i(t+\alpha_k))$$
$$+\sum_{l\le \log_2 T} Q_l(\tfrac{1}{2}+i(t+\alpha_k)) + (\log_2 T)^2 e^{2-f(1)}\Bigg)\Bigg)dt$$
$$\ll e^{O(\beta_* e^{|\ell|_\infty/2})+e^{2-f(1)}\beta_*(\log_2 T)^2}$$
$$\times \int_{\mathcal{G}_{[m]}\cap\mathcal{C}_\ell} \prod_{k=1}^m \prod_{j<f(1)} \exp\left(2\beta_k \mathrm{Re}\,\mathcal{P}_{I_{j,\ell},T_{f(1)-1}}(\tfrac{1}{2}+1/\log T_{f(1)-1}+i(t+\alpha_k))\right) dt$$
$$\ll e^{O(\beta_* e^{|\ell|_\infty/2})+e^{2-f(1)}\beta_*(\log_2 T)^2}$$
$$\times \sum_{s\le L}\int_T^{2T} \prod_{k=1}^m \prod_{j<f(1)} \exp\left(2\beta_k \mathrm{Re}\,\mathcal{P}_{I_{j,\ell},T_{f(1)-1}}(\tfrac{1}{2}+1/\log T_{f(1)-1}+i(t+\alpha_k))\right)$$
$$\times |\mathcal{P}_{f(1)}(\tfrac{1}{2}+1/\log T_s+i(t+\alpha_1))/K_{f(1)}|^{2\lceil K_{f(1)}^2/2\rceil}$$
$$\times \left|e^{|\ell|_\infty/10} Q_{|\ell|_\infty}(\tfrac{1}{2}+it+i\alpha_{k_*})\right|^{2\lceil |\ell|_\infty e^{4|\ell|_\infty/5}\rceil} dt$$

where $\ell(k_*) = |\ell|_\infty$ as in (5), only now we have also used the definition of the bad set $\mathcal{B}_{f(1)}$ and used a union bound over all $s \le L$. By Lemma 2.4, all that remains is to control the moments of $\mathcal{P}_{f(1)}$ slightly off of the half line.

Proposition 4.2. *For each $s \le L$*

$$\int_T^{2T} |\mathcal{P}_{f(1)}(\tfrac{1}{2}+1/\log T_s+i(t+\alpha_1))/K_{f(1)}|^{2\lceil K_{f(1)}^2/2\rceil} dt \ll Te^{-(\log_2 T)^3 e^{-f(1)}/3}.$$

Proof. Trivially bounding $p^{-i\alpha_1}$ and $\log(X_s/p)/\log(X_s)$ by 1, Lemma 2.5 gives a bound of

$$TK_{f(1)}^{-2k} k! \left(\sum_{p\in(T_{j-1},T_j]} \frac{1}{p}\right)^k$$

where $k = \lceil K_{f(1)}^2/2\rceil$. The quantity in parentheses is asymptotic to 1, so is at most 2 for large T, say. The conclusion is now routine. □

We now have all the necessary tools to bound the shifted moment (1) over bad sets.

Proposition 4.3. *On the bad sets*

$$\sum_{a,\ell,f}\int_{\mathcal{G}_{a,f}\cap\mathcal{C}_\ell} \prod_{k=1}^m |\zeta(\tfrac{1}{2}+i(t+\alpha_k))|^{2\beta_k} dt \ll T(\log T)^{\beta_1^2+\cdots+\beta_m^2} \prod_{1\le j<k\le m} |\zeta(1+i(\alpha_j-\alpha_k)+1/\log T)|^{2\beta_j\beta_k},$$

where the summation is taken over $1 \le a \le m$, functions $\ell : [m] \to \{0, 1, \ldots, \lfloor 2\log_3 T\rfloor\}$ and increasing functions $f : [a] \to [L]$.

Proof. First recall for each $l \leq 2\log_3 T$ there are at most $(l+1)^m$ functions ℓ with $|\ell|_\infty = l$, and given $b \in [L]$, there are at most L^a functions f with $f(1) = b$. By Proposition 4.1 we may also assume $b > 1$. Using Lemma 2.4 along with Propositions 3.2, 3.4, and 4.2, we may bound the sum with $b \geq 2$ by

$$T \sum_{\substack{a\leq m \\ 2\leq b\leq L \\ 0\leq l\leq 2\log_3 T}} L^a (l+1)^m \exp\left(O\left(\beta_* e^{l/2}\right) + e^{2-b}\beta_*(\log_2 T)^2 - l e^{l/2} - e^{-b}(\log_2 T)^3/3\right)$$

$$\times \prod_{2^{l+1} < p \leq T_{b-1}} \left(1 + \sum_{1\leq j,k\leq m} \frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}} + O_{\boldsymbol{\beta}}\left(\frac{1}{p^2}\right)\right)$$

$$\ll T\sum_{b\leq L} L^m \exp\left(e^{2-b}\beta_*(\log_2 T)^2 - e^{-b}(\log_2 T)^3/3\right) \prod_{p\leq T_{b-1}} \left(1 + \sum_{1\leq j,k\leq m} \frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}}\right).$$

Next notice that

$$\prod_{p\in(T_{b-1},T_L]} \left(1 + \sum_{1\leq j,k\leq m} \frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}}\right) \ll (\log T)^{\beta_*^2}.$$

Therefore the desired sum is

$$\ll T(\log T)^{\beta_*^2} \sum_{b\leq L} L^m \exp\left(-c_{\boldsymbol{\beta}} e^{-b}(\log_2 T)^3\right) \prod_{p\leq T_L} \left(1 + \sum_{1\leq j,k\leq m} \frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}}\right).$$

for some $c_{\boldsymbol{\beta}} > 0$. Since $L \leq \log_3 T$, this quantity is

$$\ll T(\log T)^{\beta_*^2} L^{m+1} \exp\left(-c_{\boldsymbol{\beta}}(\log_2 T)^2\right) \prod_{p\leq T_L} \left(1 + \sum_{1\leq j,k\leq m} \frac{\beta_j\beta_k}{p^{i(\alpha_j-\alpha_k)}}\right).$$

This is admissible as $L^{m+1}(\log T)^{\beta_*^2} e^{-c_{\boldsymbol{\beta}}(\log_2 T)^2}$ is $\ll_A (\log T)^{-A}$ for all $A > 0$. $\square$

In view of (3), this completes the proof of Theorem 1.1.

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