Correspondence and canonicity in non-classical logic

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INSTITUTE FOR LOGIC LANGUAGE AND COMPUTATION

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To my Parents

Contents

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Amsterdam Sumit Sourabh
May, 2015.

Chapter 1

Introduction

The focus and main contribution of this thesis is the understanding of the mechanisms underlying correspondence and canonicity, and their application to the development of a uniform correspondence and canonicity theory for a wide family of non-classical logics which includes but is not limited to regular distributive modal logics, and bi-intuitionistic modal mu-calculus. Uniformity is the critical feature of our results, and it is made possible by our methodology, which crucially relies on algebraic and order-topological notions and tools. In the present introduction, we give a brief overview of correspondence and canonicity theory in the basic setting of the normal modal logic **K**, followed by a discussion which motivates the results in the thesis in the light of previous relevant results. We assume the reader to be familiar with the basic semantics of modal logic, including its algebraic semantics.

Modal logics are an important class of logics other than classical logic. In their modern form they were introduced in the 1930s, as enriched formal languages in which one can express and reason about *modes of truth*, e.g., the *possible*, *necessary*, *usual* or *past* truth of propositions [119]. Syntactically, the language of modal logic is an expansion of classical propositional logic with new connectives, so as to have formulas such as $\Box\varphi$ or $\Diamond\varphi$, the intended meaning of which respectively is 'φ is necessary/obligatory/always true in the past...' and 'φ is possible/permitted/sometimes true in the past...'. Modal logics are widely applied in fields as diverse as program verification in theoretical computer science [91], natural language semantics in formal philosophy [18], multi-agent systems in AI [71], foundations of arithmetic [4], game theory and economics [131], and categorization theory in social science [132].We refer to [30, 16, 41] as standard textbooks on modal logic, and to [80] for an excellent overview of the historical development of the mathematical theory of modal logic.

Relational semantics. The key to the success of modal logic is the peculiar but natural way it is interpreted in relational structures, paired with the ubiquity of these structures in science and philosophy. A *relational structure*, also called a *Kripke frame* in honour of Saul Kripke who introduced this interpretation [108], consists of a non-empty set (the domain) and a collection of relations (possibly of different arity). The basic version of this notion is given by structures (W, R) with non-empty domain W and one binary relation R. When regarded as interpretations for modal formulas, relational structures typically represent an environment or system which is subject to *variation* in some respects. The domain W should be thought of as the collection of all the alternative states of affairs allowed by the system. Elements in W can then stand for, e.g. the stages of a process evolving through time, parallel realities, conceivable alternatives over the same scenario due to insufficient knowledge, misperception, etc. The relation R should be thought of as specifying, e.g., the allowed transitions in the evolution of the system, the number of alternatives over a situation which are still entertained by an agent, etc. The contingent aspects of the system or environment under consideration are encoded by a *valuation function* V, assigning subsets of the domain to propositional variables, effectively telling us which pieces of atomic information hold at which states in the system. A relational structure with a valuation, or *Kripke model*, (W, R, V) can obviously be seen as a relational structure $(W, R, \{V(p) \mid p \in \mathsf{Prop}\})$ with $V(p)$ being a unary predicate for each atomic proposition p. Thus both relational structures and Kripke models can serve as models for classical first- and second-order logic.

The theory of relational semantics for modal logic was further developed by Kripke in [109], where he introduced an explicit accessibility relation, and gave a semantic interpretation of normal modal logics **S4** and **S5** using this relation. In [111] and [110], Kripke extended the relational semantics to non-normal modal logics and first-order modal logics, respectively. Another important contribution is due to Lemmon and Scott [118], who introduced *canonical models*. A classic monograph from this period is *An Essay in Classical Modal Logic* by Segerberg [139].

It is important to emphasize that the perspective offered on relational structures by modal logic differs from that of classical languages in certain important respects: firstly, modal logic is intrinsically *local*. A modal formula is interpreted *at a given point* in a structure or Kripke model, and its truth value is unaffected by what happens in the model further than a certain number of relational steps away from this point. Secondly, modal semantics differentiates between *absolute* and *contingent* information. In a Kripke model, the accessibility relations represent the invariant "structure of possibility", while the propositional valuation represents the truth value of pieces of contingent information across these possibilities. Thirdly, modal logic operates on at least two different levels: on *models*, where only one particular distribution of contingent information is taken into account, and on *Kripke frames* (relational structures) where implicitly all contingencies are universally quantified.

Correspondence and Canonicity. *Correspondence theory* arises as the subfield of the model theory of modal logic aimed at answering the question of how precisely these languages — modal, first-order, second-order, and possibly others — interact and overlap in their shared semantic environment. Modal logic inhabits a sphere somewhere between first and second-order logic. Over models, modal languages constitute well-behaved fragments of first-order languages. An important result is the characterization of modal logic as the bisimulation-invariant fragment of first-order logic by van Benthem [13]. On the other hand, a modal formula is valid on a frame if it is true under all possible valuations. Validity thus involves quantification over all subsets of the domain — clearly a second-order notion — making modal languages expressive fragments of universal monadic second-order logic when interpreted over relational structures. However, many of these second-order conditions are in fact equivalent to simple first-order ones. Further natural questions thus arise: which properties, as expressed in a given classical logic, are expressible by means of modal formulas? Which modal formulas express elementary (i.e. first-order definable) properties of relational structures? Modal correspondence theory over frames — which is the focus of this thesis — seeks answers to these questions.

Being able to compare modal logic to classical logic brings several benefits to both logics: fragments of first-order logic inherit the good computational behaviour of modal logic, and conversely the theory of first-order logic (e.g. compactness and Löwenheim-Skolem theorems) and many existing automated proof tools may be made applicable to modal languages. Correspondence theory, broadly understood, has become a standard modus operandi in such fields as program verification, where different logical languages are interpreted over the same structures so that e.g. decidability of one logic can be bounded by the decidability of the other. Besides these technical aspects, there are also additional benefits: indeed, correspondence theory, on the frame level, acts as a meta-semantic tool which makes it possible to understand the meaning of a modal axiom in terms of the first-order condition expressed by its correspondent. For example, the modal axiom $\Diamond p \to p$ can be understood as the 'reflexivity' axiom, and $\Diamond\Diamond p \to \Diamond p$ as the 'transitivity' axiom. This understanding has significantly improved the intuitive appeal of modal logic, and hence has broadened its applicability outside its traditional target areas, and is now also being adopted e.g. in the social sciences.

The line of research in correspondence theory which concerns the present thesis is Sahlqvist correspondence theory. The best known result in this area was achieved by Sahlqvist [136], and involves the syntactic characterization of a class of formulas in the basic modal language (the so-called *Sahlqvist formulas*) which are guaranteed to express elementary conditions which are effectively computable from the given modal formula. Prior to Sahlqvist's result, other examples of correspondence for less general classes of formulas existed in the literature. The earliest examples are due to Jónsson and Tarski [98, 99], who proved correspondence results for particular cases of reflexivity, symmetry and transitivity on a complete Boolean algebra with operators using algebraic methods. *Modal reduction principles* (MRPs) are modal formulas of the form $\mathbf{M}p \to \mathbf{N}p$, where $\mathbf{M}, \mathbf{N}$ are (possibly empty) sequences of modal operators. They were introduced by Fitch [67] who showed that MRPs of the form[1] $\Diamond^i p \to \mathbf{N}p$ and $\mathbf{M}p \to \Box^i p$ have the corresponding first-order properties. This work was extended by van Benthem [9], independently of Sahlqvist, to show that MRPs of the form $\Diamond^i\Box^j p \to \mathbf{N}p$ are elementary. In [10], van Benthem provided a complete classification of the MRPs that define first-order properties on frames. Goldblatt [83, 84] showed that an axiom schema devised by Lemmon and Scott[2] [118] corresponds to an elementary frame condition. For a more extensive survey, we refer to the Handbook chapter on *Correspondence theory* by van Benthem [13].

The effective computation of the elementary condition corresponding to a Sahlqvist formula takes the form of an algorithm, known as the Sahlqvist-van Benthem algorithm. The focus on the algorithmic approach to correspondence has gained momentum in recent years. Notable examples are the algorithms SCAN [72], DLS [147] and SQEMA [48]. SCAN is a resolution-based algorithm and can successfully compute the first-order frame correspondent of every Sahlqvist formula. The same is true of DLS, which, in contrast, is based on the Ackermann's lemma [2]. The algorithm SQEMA was designed specifically for modal logic, and computes first-order frame correspondents for all *inductive formulas*. Like Sahlqvist formulas, inductive formulas [89] are syntactically defined, and form a class which properly extends the class of Sahlqvist formulas.

Sahlqvist canonicity provides "the other half" of Sahlqvist theorem. It states that every Sahlqvist formula is canonical, and hence axiomatizes a modal logic which is complete with respect to the elementary class of frames defined by its first-order correspondent. The original proof of canonicity of Sahlqvist formulas is due to Sahlqvist [136], and uses canonical frames. Correspondence and canonicity together establish that Sahlqvist logics are semantically complete with respect to first-order definable classes of relational structures. We will expand more on the algebraic approach to canonicity later in this chapter. Before moving on to that, we present a brief overview of algebraic methods in modal logic.

Algebraic methods in modal logic. Before the introduction of relational semantics for modal logic, algebras were the main semantic tool for modal logic (cf. e.g. the work of Lemmon [116, 117]). The origins of algebraic methods in logic can be traced back to the pioneering work of Boole in 1840s [33], who introduced 'laws of thought' as an algebraic system. This led to the axiomatization of *Boolean algebras*, which provide *algebraic semantics* for classical propositional logic. In his celebrated work [142], Stone proved a representation theorem and

[1] $\Diamond^0$ is the empty sequence, $\Diamond^1 = \Diamond, \Diamond^{i+1} = \Diamond\Diamond^i$. $\Box^i$ is defined similarly.

[2] Given a pair $n = (n_1, \ldots, n_k)$ and $m = (m_1, \ldots, m_k)$ of k-tuples of natural numbers, the Lemmon-Scott axiom α^m_n is the formula $\Diamond^{m_1}\Box^{n_1}p_1 \wedge \ldots \wedge \Diamond^{m_k}\Box^{n_k}p_k \to \alpha(p_1, \ldots, p_k)$, where $\alpha(p_1, \ldots, p_k)$ is a positive modal formula.

duality for Boolean algebras. Stone's result inspired McKinsey and Tarski to prove a representation theorem for closure algebras (i.e., **S4**-algebras) [123]. Closure algebras naturally generalize to Boolean algebra with operators (BAOs), or modal algebras, which provide algebraic semantics for normal modal logics. In their seminal work [98, 99], Jónsson and Tarski proved a representation theorem for BAOs. Interestingly enough, their main motivation was not the application of this theory to modal logic, but to other families of BAOs, namely relation algebras [148] and cylindric algebras [114].

Until 1970s, existing results in algebraic logic such as Jónsson and Tarski's were not explicitly applied to modal logic. However, the appearance of frame incompleteness results shifted the attention of modal logicians to algebraic methods. The first frame incompleteness result was due to Thomason [149], who constructed an incomplete temporal logic. Other examples of frame incomplete modal logics appeared in papers by Fine [65] and van Benthem [11]. An important contribution of the algebraic approach to modal logic is the work of Blok [31, 32], who investigated the degree of incompleteness of modal logics by studying lattices of varieties of BAOs.

The unification of the algebraic and the relational approaches to modal logic began with the work of Thomason [150] who developed a duality between the categories of frames with p-morphisms and complete BAOs with homomorphisms preserving arbitrary meets. This was followed by the influential work of Goldblatt [83, 84], who proved a full duality between the categories of BAOs with homomorphisms preserving finite meets and descriptive general frames with p-morphisms. In [79], Goldblatt generalized his duality result to BAOs with an arbitrary similarity type. Sambin and Vaccaro [137] proved a topological version of the duality for BAOs using modal spaces.

The use of algebraic methods in modal logic also extends to Sahlqvist theory. Jónsson and Tarski [98] study canonicity in the context of *canonical extensions* as preservation of term identities under the construction of canonical extension. In particular, the canonicity for simple Sahlqvist formulas (cf. Page 24 for definition) follows from their preservation result. Moreover, as mentioned earlier, they also prove correspondence results for particular examples of reflexivity, symmetry and transitivity using algebraic methods in [98]. In [135], de Rijke and Venema define Sahlqvist identities in BAOs with an arbitrary similarity type, and show that they are elementary and canonical.

Three approaches to algebraic Sahlqvist theory. Below, we expand on three prominent algebraic and order-topological approaches to Sahlqvist theory in the literature, which are particularly relevant to this thesis, and can be respectively traced back to Sambin and Vaccaro [138], Jónsson [97], and Ghilardi and

$$
\begin{array}{ccc}
\mathcal{G} \Vdash \varphi & & \mathcal{G}^\sharp \Vdash \varphi \\
\Updownarrow & & \Updownarrow \\
\mathcal{G} \models \mathrm{FO}(\varphi) & \Leftrightarrow & \mathcal{G}^\sharp \models \mathrm{FO}(\varphi).
\end{array}
$$

Figure 1.1: Canonicity-via-correspondence

Meloni [77][3].

Using order-topological methods, Sambin and Vaccaro [138] gave a simpler proof of Sahlqvist canonicity than the original one in [136]. Their strategy, commonly referred to as *canonicity-via-correspondence*, makes use of the existence of a first-order correspondent for any given Sahlqvist formula *in the first-order language of the underlying frame*. The first crucial observation is that the truth of a first-order sentence in a Kripke frame, seen as a relational model, is independent of the assignment being admissible or arbitrary, as illustrated by the horizontal bi-implication in Figure 1.1. Secondly, it is also important to note that the syntactic shape of the Sahlqvist formulas guarantees that the vertical bi-implication on the right hand side of the diagram ($\mathcal{G}^\sharp \Vdash \varphi \Leftrightarrow \mathcal{G}^\sharp \models \mathrm{FO}(\varphi)$) holds. These two main observations ensure that the canonicity result follows. An important step in Sambin and Vaccaro's proof is a celebrated result of Esakia's (the Esakia lemma) [64], which crucially makes use of the compactness of the underlying topological space. Goranko and Vakarelov [89] extended Sambin and Vaccaro's canonicity-via-correspondence strategy to prove the canonicity of inductive formulas.

The second approach to Sahlqvist canonicity was introduced by Jónsson [97] extending techniques in [98], and heavily relying on the theory of canonical extensions [98, 74, 73]. The specific feature of this methodology is that it does not make use of the fact that Sahlqvist formulas have a first-order correspondent. The proof strategy is illustrated in Figure 1.2, and can be briefly summarized as follows. A modal inequality $\varphi \leq \psi$ is canonical if the following implication holds for every modal algebra[4] $\mathbb{A}$

$$\mathbb{A} \models \varphi \leq \psi \qquad \Rightarrow \qquad \mathbb{A}^\delta \models \varphi \leq \psi$$

where $\mathbb{A}^\delta$ denotes the canonical extension of $\mathbb{A}$. The implication above can be equivalently restated as follows:

$$\varphi^{\mathbb{A}} \leq \psi^{\mathbb{A}} \qquad \Rightarrow \qquad \varphi^{\mathbb{A}^\delta} \leq \psi^{\mathbb{A}^\delta},$$

[3]By definition, proving that a given modal formula is *canonical* means proving that its validity transfers from any descriptive general frame to its underlying Kripke frame.

[4]A *modal algebra* is a Boolean algebra with a unary operation which preserves finite and empty join.

$$
\begin{array}{c}
\mathbb{A} \models \varphi \leq \psi \\
\Updownarrow \\
\varphi^{\mathbb{A}} \leq \psi^{\mathbb{A}} \\
\Downarrow \\
\varphi^{\mathbb{A}^\delta} \leq (\varphi^{\mathbb{A}})^\sigma \leq (\psi^{\mathbb{A}})^\sigma \leq \psi^{\mathbb{A}^\delta} \\
\sigma\text{-expanding} \qquad \sigma\text{-contracting} \\
\Downarrow \\
\varphi^{\mathbb{A}^\delta} \leq \psi^{\mathbb{A}^\delta} \\
\Updownarrow \\
\mathbb{A}^\delta \models \varphi \leq \psi
\end{array}
$$

Figure 1.2: Jónsson-style algebraic canonicity

where $\varphi^{\mathbb{A}}$ and $\varphi^{\mathbb{A}^\delta}$ (resp. $\psi^{\mathbb{A}}$ and $\psi^{\mathbb{A}^\delta}$) denote the term maps induced by φ (resp. ψ) on $\mathbb{A}$ and $\mathbb{A}^\delta$, respectively, and the inequalities are interpreted pointwise. Assuming $\varphi^{\mathbb{A}} \leq \psi^{\mathbb{A}}$, in order to prove $\varphi^{\mathbb{A}^\delta} \leq \psi^{\mathbb{A}^\delta}$, Jónsson's strategy is based on proving the following chain of inequalities holds:

$$\varphi^{\mathbb{A}^\delta} \leq (\varphi^{\mathbb{A}})^\sigma \leq (\psi^{\mathbb{A}})^\sigma \leq \psi^{\mathbb{A}^\delta},$$

where $(\varphi^{\mathbb{A}})^\sigma$ and $(\psi^{\mathbb{A}})^\sigma$ respectively denote the *σ-extensions* of the term maps $\varphi^{\mathbb{A}}$ and $\psi^{\mathbb{A}}$. The first and third inequalities are obtained as a consequence of the order-theoretic properties of the term maps $\varphi^{\mathbb{A}}$ and $\psi^{\mathbb{A}}$, which are in their turn guaranteed by the syntactic shape of φ and ψ.

Following the methodology introduced in [97], Gehrke, Nagahashi and Venema [75] proved the canonicity of Sahlqvist inequalities in distributive modal logic. Also, in [59], Dunn, Gehrke and Palmigiano proved completeness via canonicity for a finite collection of inequalities axiomatizing fragments of certain well-known substructural logics.

The third approach to canonicity, independent of Jónsson's, is due to Ghilardi and Meloni [77], and provides a constructive proof of canonicity in the setting of intermediate and intuitionistic modal logics. This environment is constructive in the sense that it is based on filter- and ideal-completions in place of canonical extensions, dually arising from set-based structures. Unlike canonical extensions, these completions are not required to "have enough points", and hence they provide a mathematical environment supporting an intuitionistic metatheory. This approach has been later generalized by Suzuki for the canonicity of substructural and lattice-based logics [144, 146] and posets [145]. We summarize the three approaches to algebraic canonicity, along with the two main themes investigated in this thesis, in Figure 1.3.

Similarities and differences: towards unification. Jónsson's proof strategy is similar to Ghilardi and Meloni's in the use of algebras instead of set-based models, and is similar to Sambin and Vaccaro's in its use of "non-constructive" algebras endowed with topological structures as a tool for canonicity. Moreover, both Jónsson and Ghilardi and Meloni treat canonicity independently of correspondence. However, in contrast to Ghilardi and Meloni and similarly to Sambin and Vaccaro, Jónsson's environment is point-set, in the sense that it uses points of the original algebra to obtain its canonical extension. In addition, the proof of canonicity of Sahlqvist formulas/inequalities following Jónsson's strategy does not straightforwardly generalize to inductive formulas/inequalities. This contrasts with the canonicity-via-correspondence approach, which smoothly transfers from the Sahlqvist to the inductive case.

In recent work [159, 129], we give a proof of algebraic canonicity of inductive formulas/inequalities using a combination of Jónsson's and Sambin and Vaccaro's strategies. It would be interesting to extend Ghilardi and Meloni's constructive proof for canonicity to inductive formulas — possibly, using techniques from Sambin and Vaccaro's strategy.

Subsequent results. Other results in correspondence and canonicity theory have been contributed by many different authors, and concern polyadic modal logic [85, 88], graded modal logic [153], hybrid logic [87, 37, 43] monotone modal logic [93], positive modal logic [39], relevant modal logic [140], substructural logics [113, 144], many-valued modal logic [104], modal mu-calculus [17, 27] and coalgebraic modal logic [120, 55].

Unified correspondence theory. The contributions of the first part of the present thesis belong to *unified correspondence theory*, a very recent line of research which provides a uniform methodological setting accounting for extending the existing Sahlqvist-type results to the logics mentioned above[5]. Unified correspondence takes the discrete duality between *perfect algebras* and the relational semantics of a given logic as its starting point, and regards the phenomenon of correspondence as a by-product of this duality. The two main tools introduced by unified correspondence are: a uniform identification of the Sahlqvist, Inductive and Recursive class for every logic, which is based on the order-theoretic properties of the interpretation of the logical connectives, and the algorithm/calculus for correspondence ALBA[6].

[5]The coalgebraic setting is not yet accounted for by the unified correspondence theory.
[6]ALBA is the acronym of Ackermann Lemma Based Algorithm.

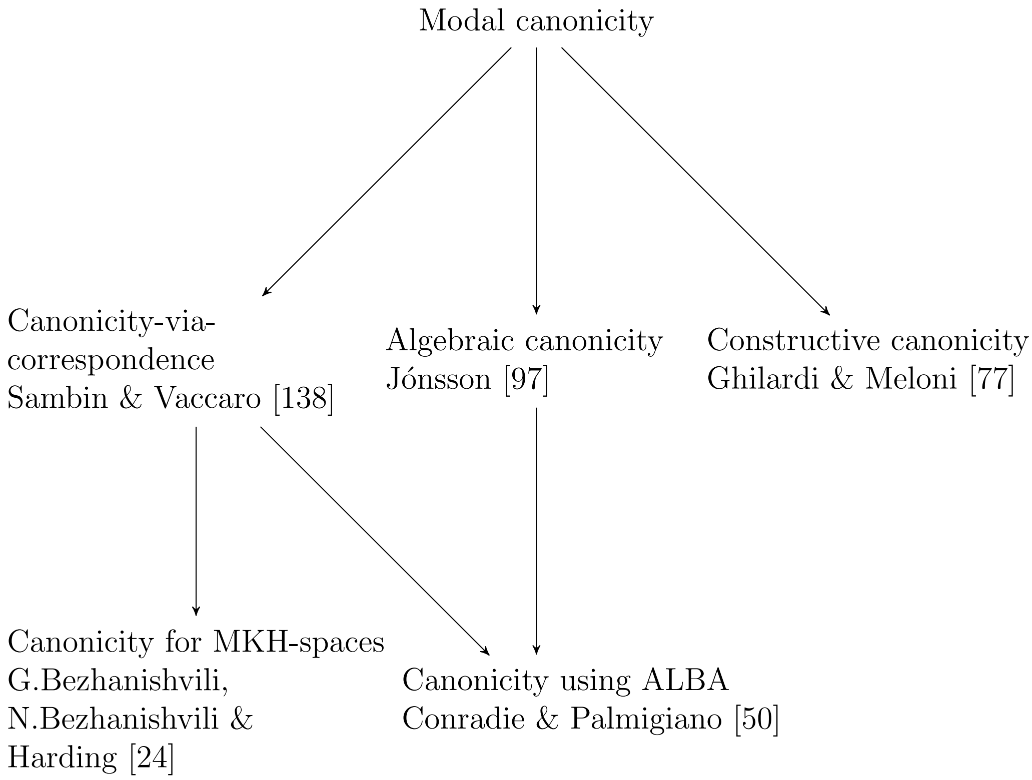

Figure 1.3: Canonicity strategies

A fundamental precursor of ALBA is the SQEMA algorithm [48] for classical normal modal logic. Indeed, two fundamental features that ALBA inherits from SQEMA are its being based on the Ackermann's lemma, and its taking modal formulas as input. Like ALBA, SQEMA is a calculus of rewrite rules, and succeeds in computing the first-order frame correspondent of all inductive formulas in classical (i.e. Boolean) normal modal logic. In [48], it is shown that all SQEMA-reducible formulas are canonical by means of a canonicity-via-correspondence argument. The soundness of the SQEMA rewrite rules is intended to be checked on relational structures. In contrast, the soundness of the ALBA rules is checked on perfect algebras, a setting which naturally supports the interpretation of an expanded language containing the syntactic adjoints and residuals of all connectives.

The first paper in the unified correspondence line proper is [50], in which the ALBA algorithm has been introduced for the language of Distributive Modal Logic (DML), a modal logic framework with the logic of distributive lattices as its propositional base. ALBA is sound on perfect DML-algebras, which are the dual counterparts of DML-frames, and uniformly succeeds on the class of inductive

DML-inequalities (i.e. the counterparts of the inductive formulas of [89]), proven to be canonical again by means of a canonicity-via-correspondence argument. Methodologically, as hinted at early on, the dualities and adjunctions between the relational and the algebraic semantics of DML make it possible to distil the order-theoretic and algebraic significance of the SQEMA reduction steps from the model-theoretic setting, and hence to recast them into an algebraic setting more general than the Boolean one. This can be extended even further, namely to general (i.e. not necessarily distributive) lattices [51]. These results make it clear that correspondence results can be formulated and proved abstracting away from specific logical signatures, and only in terms of the order-theoretic properties of the algebraic interpretations of logical connectives. Hence, these results set the stage for the research program referred to as unified correspondence theory.

A specific feature of the unified correspondence approach is that correspondence results can be adapted to possibly different semantic environments for one and the same given logic. This is effected via enforcing a neat separation between: (a) the algorithmic reduction, which is effected via the manipulation of inequalities and quasi-inequalities of a certain quantified propositional language which expands the primitive given one, and (b) the translation of the fragment of this expanded language free of atomic propositions into the correspondence language associated with the given semantic environment. Another crucial feature of this approach is that the syntactic identification of the Sahlqvist/Inductive class of formulas/inequalities is a *positive* classification, since it is given in terms of the order-theoretic properties of the interpretation of the logical connectives. This contrasts with previous approaches in the literature to generalizing the syntactic identification of such classes, which were either extensional, and hence limited to a specific setting, or were given in terms of a *negative* classification (forbidden configurations of connectives). Both these approaches proved not amenable to be naturally extended to different logical settings.

The success and outreach of this theory manifest itself also in the fact that the algorithm ALBA can be modularly adapted to various settings. In this thesis we extend the algorithm ALBA to regular modal logic and bi-intuitionistic modal mu-calculus in Chapters 4 and 5, respectively. Moreover, in Chapter 6, we introduce an enhanced version of ALBA to prove canonicity in the presence of additional axioms (i.e. *relativized canonicity*) for a class of inequalities in distributive lattice expansions. The setting of regular modal logic, in which the modal operators are non-normal, provides an appropriate mathematical environment to investigate the scope of the theory of unified correspondence. With the development described in Chapter 4, the state-of-the-art in correspondence theory transfers to non-normal (regular) modal logic on a non-modal base naturally interpreted on bounded distributive lattices. The setting of fixed-point operators over an intuitionistic base is mathematically quite rich and powerful. The unified correspondence approach described in Chapter 5 allows us to develop correspondence theory for fixed-point operators in a uniform and systematic way by studying the preservation of order-theoretic properties from certain terms $G(x)$ to $\mu x.G(x)$ (resp. $\nu x.G(x)$).

In addition to the results presented in this thesis, the unified correspondence framework covers a range of other logics — including substructural logics [51], hybrid logic [54], and monotone modal logic [128]. In [44], ALBA developed for mu-calculus is used to prove canonicity for two classes of mu-inequalities. Other applications include the identification of syntactic shapes of axioms which can be translated into well-performing structural rules of a display calculus [90]. Recent work [159, 129] extends the Jónsson-style proof of canonicity [97, 75] to the inductive formulas in DML. As mentioned early on, the techniques in [97, 75] do not trivially generalize to the inductive formulas. This proof makes use of ALBA in a novel way which, interestingly, does not involve correspondence, thus bringing together the two algebraic approaches to Sahlqvist theory.

Duality and canonicity for compact Hausdorff spaces. The second part of the thesis focusses on order-topological methods for correspondence and canonicity. *Dualities* in mathematics refer to back-and-forth mappings between classes of mathematical objects, thereby inducing a translation of their properties. An important example is the celebrated Stone duality [142] between Boolean algebras and Stone spaces[7]. Other notable examples include Priestley duality [133, 134] between distributive lattices and Priestley spaces[8] and Esakia duality [64] between Heyting algebras and Esakia spaces[9]. In logic, dualities play an important role as they establish a connection between the syntactic or algebraic, and semantic or spatial perspectives to logic.

In case of Stone spaces, while compactness and Hausdorffness are quite standard topological properties, having a clopen basis is a very specific property. This reasoning led de Vries [57] to establishing a new duality between what we now call *de Vries algebras*, that is, complete Boolean algebras with a binary relation $\prec$, and compact Hausdorff spaces (cf. Figure 1.4). A motivating example for a de Vries algebra is the complete Boolean algebra of the regular open sets of a compact Hausdorff space, with $U \prec V$ if the topological closure of U is contained in V. Isbell [95] developed a duality between *compact regular frames*, that is, complete distributive lattices satisfying additional conditions and compact Hausdorff spaces. The lattice of open sets of a compact Hausdorff space is an example of a compact regular frame. Other dualities for compact Hausdorff spaces include Gelfand-Naimark duality [76] using commutative C^*-algebras, Kakutani-Yosida duality [101, 102, 157] using vector lattices and Jung-Sünderhauf duality [100] using proximity lattices.

[7] A *Stone space* is a compact and Hausdorff spaces with a basis of clopen (both closed and open) sets.

[8] A *Priestley space* is a partially ordered Stone space $(X, \leq)$ in which, whenever $x \leq y$, there is a clopen up-set U such that $x \in U$ and $y \notin U$.

[9] An *Esakia space* is a Priestley space in which the down-set of each clopen set is clopen.

Stone duality was extended in a different direction to a duality between modal algebras and *modal spaces* [98, 83, 84, 137]. Modal spaces are Stone spaces with a special binary relation. This duality is commonly referred to as the Jónsson-Tarski duality. From the algebraic completeness for modal logic, it is well-known that modal logics are complete with respect to modal algebras (see, e.g., [30, Chapter 5]). As a consequence of the Jónsson-Tarski duality, every axiomatic system of modal logic is complete with respect to modal spaces. This duality is central to modal logic, and moreover, many problems in the area of modal logic have been resolved using the aforementioned duality e.g., Kripke incompleteness [31, 32], interpolation [122] .

In Chapter 7 of the thesis we prove a "modal-like" alternative to de Vries duality for compact Hausdorff spaces. In particular, we show that the category of de Vries algebras is dual to the category of *Gleason spaces*, which are extremely disconnected spaces with a closed irreducible equivalence relation. The proof involves the characterization of properties of the relation on compact Hausdorff spaces corresponding to the axioms on de Vries algebras. It also leads to the problem of developing a *topological correspondence theory* — similar to modal correspondence theory — which characterizes formulas which have an elementary frame correspondent over topological spaces.

The research program of extending the dualities for compact Hausdorff spaces to the modal case was initiated in [24]. To this end, *modal compact Hausdorff spaces* (MKH-space) — which are compact Hausdorff spaces with a binary relation satisfying additional conditions — are defined as natural generalizations of modal spaces. In the same paper, modal operators on de Vries algebras and compact regular frames are introduced, and both Isbell and de Vries duality are extended to the modal cases. In [23], a choice-free proof of the equivalence between modal compact regular frames and modal de Vries algebras is presented. The authors also study logical aspects of MKH-spaces and prove a Sahlqvist canonicity theorem in [24]. In [27] a Sahlqvist correspondence and canonicity result is proved for the modal mu-calculus in the zero-dimensional setting of modal spaces. We extend this line of research in Chapter 8 by studying canonicity for Sahlqvist formulas in the language of positive modal fixed-point logic over MKH-spaces. The proof involves Esakia's lemma for compact Hausdorff spaces and follows the Sambin-Vaccaro proof strategy as discussed earlier. The study of fixed point semantics over topological spaces can be used for applications in in topological epistemic logic [19, 28] or region-based theories of space [7]. Finally, it would also be interesting to develop unified correspondence methods for MKH-spaces.

We mention two other related areas of research which are not covered in this thesis. The first concerns the relationship between canonicity and elementarity. A logic is *determined* by a class of frames if it is sound and complete with respect to that class. Fine's theorem [66] states that if a modal logic is determined by some elementary class of frames, then it is canonical. It was shown by Goldblatt,

$$\begin{tikzcd}
\mathbf{DeV}^{\mathbf{op}} \arrow[r, bend right] & \mathbf{KHauS} \arrow[l, bend right, "\text{de Vries duality}"'] \arrow[r, bend left, "\text{Isbell duality}"] & \mathbf{KRegFr}^{\mathbf{op}} \arrow[l, bend left]
\end{tikzcd}$$

Figure 1.4: Dualities for **KHauS**

Hodkinson and Venema [82, 81], however, that the converse of Fine's theorem does not hold.

The second topic of research related to Sahlqvist theory is *Kracht's theory* [106], or inverse correspondence. Kracht's theory is aimed at characterizing classes of first-order formulas each of which corresponds to some Sahlqvist formula. Kracht's theorem states that any Sahlqvist formula locally corresponds to a *Kracht formula*; and conversely, every Kracht formula is a local first-order correspondent of some Sahlqvist formula which can be effectively obtained from the Kracht formula (cf. [30, Theorem 3.59]). This result is improved in [103] where a syntactic characterization is given to the first-order formulas which correspond to *inductive formulas*, defined in [89]. Recently, inverse correspondence has been used for query answering in database theory [160].

1.1 Outline of chapters

We present a brief overview of the contents of the thesis below.

Chapter 2. We give preliminaries on basics of modal logic, Sahlqvist correspondence theory including syntactic classes and algorithmic strategies, and duality and canonicity.

Chapter 3. We present an introductory algebraic treatment of the well-known Sahlqvist correspondence theory for classical modal logic. The crucial feature of our algebraic account is that it highlights the order-theoretic conditions that guarantee the applicability of the reduction strategies for the elimination of the second-order variables. Further related to this, the algebraic proofs throw light in particular on adjunction as a fundamental ingredient of the order-theoretic conditions — a fact which is not easily recognizable in the model-theoretic account of Sahlqvist correspondence. This chapter is based on [52].

Chapter 4. We extend unified correspondence theory to Kripke frames with impossible worlds and their associated *regular* modal logics. These are logics the modal connectives of which are not required to be normal: only the weaker properties of additivity $\Diamond x \vee \Diamond y = \Diamond(x \vee y)$ and multiplicativity $\Box x \wedge \Box y = \Box(x \wedge y)$ are required. Conceptually, it has been argued that the lack of necessitation makes regular modal logics better suited than normal modal logics for the formalization of epistemic and deontic settings. Our contributions include: the definition of Sahlqvist inequalities for regular modal logics on a distributive lattice propositional base; the adaptation of the algorithm ALBA to the setting of regular modal logics on two non-classical (distributive lattice and intuitionistic) bases; the proof that the adapted ALBA is guaranteed to succeed on a syntactically defined class which properly includes the Sahlqvist one; finally, the application of the previous results so as to obtain proofs, alternative to Kripke's, of the strong completeness of Lemmon's epistemic logics E2-E5 w.r.t. elementary classes of Kripke frames with impossible worlds. This chapter is based on [130].

Chapter 5. We extend unified correspondence theory to modal mu-calculi with a non-classical base. We focus in particular on the language of bi-intuitionistic modal mu-calculus, and we enhance ALBA [50] for the elimination of monadic second order variables, so as to guarantee its success over a class including the Sahlqvist mu-formulas defined in [17].

The soundness of this enhancement follows from the order-theoretic properties of the algebraic interpretation of the fixed point operators. We define the class of recursive mu-inequalities based on a positive order-theoretic classification of connectives, and justify that it projects onto the Sahlqvist mu-formulas [17]. Finally, we prove that the enhanced ALBA is successful on all recursive mu-inequalities, and hence that each of them has a frame correspondent in first-order logic with least fixed points ($\mathsf{FO} + \mathsf{LFP}$). This chapter is based on [45]. The part on mu-calculus dealing with order-theoretic properties of fixed point operators is a thoroughly revised and amended version of the story appearing in the master thesis of Yves Fomatati [68], which turned out to be either too weak, or with gaps.

Chapter 6. We generalize Venema's proof of the canonicity of the additivity of positive terms [154], from classical modal logic to the logic of distributive lattice expansions (DLE). Moreover, using a suitable enhancement of the algorithm ALBA, we prove the canonicity of certain syntactically defined classes of DLE-inequalities (called the *meta-inductive inequalities*), relative to the structures in which the formulas asserting the additivity of some given terms are valid. This chapter is based on [53]

Chapter 7. We introduce the concept of a subordination, which is dual to the concept of the so-called precontact relation on a Boolean algebra. We develop a full categorical duality between Boolean algebras with a subordination and

Stone spaces with a closed relation, thus generalizing the results of [58]. We introduce the concept of an irreducible equivalence relation, and that of a Gleason space, which is a pair (X, R), where X is an extremally disconnected compact Hausdorff space and R is an irreducible equivalence relation on X. We prove that the category of Gleason spaces is equivalent to the category of compact Hausdorff spaces, and is dually equivalent to the category of de Vries algebras, thus providing a "modal-like" alternative to de Vries duality. This chapter is based on [70].

Chapter 8. We study *topological fixed point logic*, by which we mean a family of fixed-point logics that admit topological interpretations, and where the fixed-point operators are evaluated with respect to these topological interpretations. Our topological models are modal compact Hausdorff spaces (MKH-spaces), which were introduced in [24] as concrete realizations of the Vietoris functor on a compact Hausdorff space. We investigate topological semantics of the least fixed-point operator in the framework of MKH-spaces.

We give an interpretation of the least fixed-point operator on compact regular frames and show that the duality between compact Hausdorff spaces and compact regular locales extends to the language with the least fixed-point operator. We introduce a new topological semantics for the least fixed-point operator as the intersection of topological pre-fixed-points. In the new semantics, we prove that Esakia's lemma holds for the class of shallow fixed-point formulas which do not have any nesting of fixed-point operators. As a consequence, we obtain a Sahlqvist preservation result for MKH-spaces analogous to the one in [27] for Stone spaces. We also show that the Sahlqvist sequents in our language have a frame correspondent in $\mathsf{FO} + \mathsf{LFP}$, which is first-order language extended with fixed-point operators with topological interpretations. This chapter is based on [29].

For the interested reader, we would like to mention two other papers which were a part of the author's PhD project: [129] which presents an ALBA-aided Jónsson-style proof of canonicity of inductive inequalities, and [63] in which we introduce and study generalized Vietoris bisimulation for Stone coalgebras.

Chapter 2

Sahlqvist correspondence and canonicity

In the present chapter, we collect the formal preliminaries to Sahlqvist correspondence and canonicity theory for basic modal logic. We assume that the reader is familiar with basics of set theory, category theory, universal algebra, propositional logic, first-order logic and modal logic. The reader who is familiar with the textbook approach to Sahlqvist correspondence and canonicity theory (see, e.g., [30]) can skip this chapter, and move on to Chapter 3.

2.1 Modal logic

Syntax. The *basic modal language*, denoted ML, is defined using a countably infinite set Prop of propositional variables, also called atomic propositions, propositional constant $\bot$, propositional connectives $\neg$, $\vee$, and a unary modal operator $\Diamond$ ('diamond'). The well-formed *formulas* of this language are given by the rule:

$$\varphi ::= p \mid \bot \mid \neg\varphi \mid \varphi \vee \psi \mid \Diamond\varphi,$$

where $p \in \mathsf{Prop}$. The propositional constant $\top$ is defined as $\top := \neg\bot$, and the propositional connective $\wedge$ is defined as $\varphi \wedge \psi := \neg(\neg\varphi \vee \neg\psi)$. The unary modal connective $\Box$ ('box'), the dual of $\Diamond$, is defined as $\Box\varphi := \neg\Diamond\neg\varphi$. For modal formulas φ and ψ, we define $\varphi \to \psi := \neg\varphi \vee \psi$, and $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$.

2.1.1. Definition. A *normal modal logic* Λ is a set of formulas that contains all the propositional tautologies and the following axioms:

(K)	$\Box(p \to q) \to (\Box p \to \Box q)$
(Dual)	$\Box p \leftrightarrow \neg\Diamond\neg p$

and is closed under the following inference rules:

Modus Ponens (MP) : From φ and $\varphi \to \psi$, infer ψ.

Substitution (Subs) : From $\varphi(p_1, \ldots, p_n)$, infer $\varphi(\psi_1, \ldots, \psi_n)$.

Necessitation (N) : From φ, infer $\Box\varphi$.

The smallest normal modal logic is called **K**. Given a normal modal logic Λ and a modal formula φ, let $\Lambda + \varphi$ be the logic obtained by adding φ to Λ as an axiom.

Semantics. We interpret the formulas in our language on Kripke frames and models. A *Kripke frame* is a structure $\mathcal{F} = (W, R)$ where W is a non-empty set and R is a binary relation on W. The elements of the set W are called *worlds*. For $w \in W$, we define $R[w] = \{v \in W : wRv\}$ and $R^{-1}[w] = \{v \in W : vRw\}$. Also, for $S \subseteq W$, $R[S] = \{w \in W : R^{-1}[w] \cap S \neq \varnothing\}$ and $R^{-1}[S] = \{w \in W : R[w] \cap S \neq \varnothing\}$. Augmenting $\mathcal{F}$ with a *valuation*, or an *assignment*, which is a map $V : \mathsf{Prop} \to \mathcal{P}(W)$ we obtain a *Kripke model* $\mathcal{M} = (W, R, V)$. The truth of a formula $\varphi \in \mathrm{ML}$ at a world $w \in W$ (notation: $\mathcal{M}, w \Vdash \varphi$) is defined using the following induction.

- $\mathcal{M}, w \Vdash \top$, $\mathcal{M}, w \nVdash \bot$,
- $\mathcal{M}, w \Vdash p$ iff $w \in V(p)$,
- $\mathcal{M}, w \Vdash \neg\varphi$ iff $\mathcal{M}, w \nVdash \varphi$,
- $\mathcal{M}, w \Vdash \varphi \vee \psi$ iff $\mathcal{M}, w \Vdash \varphi$ or $\mathcal{M}, w \Vdash \psi$,
- $\mathcal{M}, w \Vdash \varphi \wedge \psi$ iff $\mathcal{M}, w \Vdash \varphi$ and $\mathcal{M}, w \Vdash \psi$,
- $\mathcal{M}, w \Vdash \Diamond\varphi$ iff there exists a world v such that wRv, and $\mathcal{M}, v \Vdash \varphi$,
- $\mathcal{M}, w \Vdash \Box\varphi$ iff for all worlds v such that wRv, $\mathcal{M}, v \Vdash \varphi$.

If $\mathcal{M}, w \Vdash \varphi$, then we say that the formula φ is *true* at w, or w *satisfies* φ in the model $\mathcal{M}$. We say a formula φ is *true* in a model $\mathcal{M}$ (notation: $\mathcal{M} \Vdash \varphi$) if $\mathcal{M}, w \Vdash \varphi$, for all $w \in W$. We say a formula φ is *valid* in a frame $\mathcal{F}$ (notation: $\mathcal{F} \Vdash \varphi$) if $\mathcal{M} \Vdash \varphi$ for every valuation V on $\mathcal{F}$.

A normal modal logic Λ is *sound* with respect to a class $\mathcal{C}$ of Kripke frames if every member of $\mathcal{C}$ is a Λ-frame, i.e. validates all Λ-theorems. A normal modal logic Λ is *complete* with respect to $\mathcal{C}$ if any formula that is valid in all members of $\mathcal{C}$ is a Λ-theorem. A normal modal logic Λ is *frame-incomplete* if there exists a formula that is valid in all Λ-frames but is not a theorem of Λ.

For Kripke frames, (W, R) and (W', R'), a function $f : W \to W'$ is a *p-morphism* if (i) wRw' implies $f(w)Rf(w')$, and (ii) $f(w)Rv$ implies there is $u \in W$ with wRu and $f(u) = v$. The truth of modal formulas is invariant under p-morphisms.

General frames, admissible valuations, and canonicity. Appealing as they are, Kripke frames are not adequate to provide uniform completeness results for *all* modal logics (the first examples of a frame-incomplete modal logic were given by Thomason [149]. This issue has been further clarified by Blok [31]). For uniform completeness, Kripke frames need to be equipped with extra structure.

2.1.2. DEFINITION. A *general frame* is a triple $\mathcal{G} = (W, R, \mathcal{A})$, such that $\mathcal{G}^\sharp = (W, R)$ is a Kripke frame, and $\mathcal{A}$ is a set of subsets of W, i.e. $\mathcal{A} \subseteq \mathcal{P}(W)$ such that

1. $\varnothing, W \in \mathcal{A}$,
2. If $A_1, A_2 \in \mathcal{A}$, then $A_1 \cap A_2 \in \mathcal{A}$,
3. If $A \in \mathcal{A}$, then $(W \setminus A) \in \mathcal{A}$,
4. If $A \in \mathcal{A}$, then $R^{-1}[A] \in \mathcal{A}$.

An *admissible valuation* on $\mathcal{G}$ is a map $v : \mathsf{Prop} \longrightarrow \mathcal{A}$. Satisfaction and validity of modal formulas w.r.t. general frames are defined as in the case of Kripke frames, but by restricting to admissible valuations.

In fact, the desired uniform completeness can be given in terms of the following proper subclass of general frames.

2.1.3. DEFINITION. Let $\mathcal{G} = (W, R, \mathcal{A})$ be a general frame.

1. $\mathcal{G}$ is called *differentiated* if for each $w, v \in W$,
$$w \neq v \text{ implies there is } A \in \mathcal{A} \text{ such that } w \in A \text{ and } v \notin A.$$
2. $\mathcal{G}$ is called *tight* if for each $w, v \in W$,
$$\neg(wRv) \text{ implies there is a } A \in \mathcal{A} \text{ such that } v \in A \text{ and } u \notin R^{-1}[A].$$
3. $\mathcal{G}$ is called *refined* if it is differentiated and tight.
4. $\mathcal{G}$ is called *compact* if for every $\mathcal{B} \subseteq \mathcal{A}$ with the finite intersection property we have $\bigcap \mathcal{B} \neq \varnothing$.
5. $\mathcal{G}$ is called *descriptive* if it is refined and compact.

In the light of the uniform completeness w.r.t. descriptive general frames, to prove that a given modal logic is frame-complete, it is sufficient to show that its axioms are valid on a given descriptive general frame $\mathcal{G}$ if, and only if, they are valid on its underlying Kripke frame $\mathcal{G}^\sharp$. Formulas the validity of which is preserved in this way are called *canonical*.

Modal algebras and Jónsson-Tarski representation. In this section, we recall the algebraic semantics for modal logic. An advantage of the algebraic perspective is that it allows us to use techniques from universal algebra to solve problems in modal logic. Moreover, modal algebras provide a uniform completeness result for every modal logic, unlike Kripke frames. For further details on modal algebras we refer to [30, Chapter 5] and [155].

2.1.4. DEFINITION. A *modal algebra* (also called *Boolean algebra with an operator*) is a tuple $\mathbb{B} = (B, \wedge, \vee, \neg, \top, \bot, \Diamond)$, where $(B, \wedge, \vee, \neg, \top, \bot)$ is a Boolean algebra, and $\Diamond : B \to B$ is a map satisfying:

1. $\Diamond(a \vee b) = \Diamond(a) \vee \Diamond(b)$,
2. $\Diamond(\bot) = \bot$.

A *modal homomorphism* is a Boolean homomorphism $h : \mathbb{B}_1 \to \mathbb{B}_2$ such that $h(\Diamond_1 a) = \Diamond_2 h(a)$

An *assignment* is a map $v : \mathsf{Prop} \to \mathbb{B}$. Given an assignment v, the interpretation of a formula $\varphi \in$ ML is defined recursively as follows:

- $v(\bot) = \bot^{\mathbb{B}}$, $v(\top) = \top^{\mathbb{B}}$,
- $v(\neg\varphi) = \neg^{\mathbb{B}} v(\varphi)$,
- $v(\varphi \vee \psi) = v(\varphi) \vee^{\mathbb{B}} v(\psi)$,
- $v(\varphi \wedge \psi) = v(\varphi) \wedge^{\mathbb{B}} v(\psi)$,
- $v(\Diamond\varphi) = \Diamond^{\mathbb{B}} v(\varphi)$,

where $\bot^{\mathbb{B}}$, $\top^{\mathbb{B}}$, $\neg^{\mathbb{B}}$, $\wedge^{\mathbb{B}}$, $\vee^{\mathbb{B}}$, $\Diamond^{\mathbb{B}}$ denote the operations on the modal algebra $\mathbb{B}$. A formula is said to be *true* under an assignment v in $\mathbb{B}$ if $v(\varphi) \approx \top^{\mathbb{B}}$. A formula is said to be *valid* on $\mathbb{B}$ if it is true under all assignments into $\mathbb{B}$.

2.1.5. EXAMPLE. The *complex algebra* of a frame $\mathcal{F} = (W, R)$ is the modal algebra

$$\mathcal{F}^+ = (\mathcal{P}(W), \cap, \cup, \setminus_W, \varnothing, W, \langle R \rangle)$$

where $\setminus_W$ denotes set complementation relative to W, and $\langle R \rangle$ the unary operation on $\mathcal{P}(W)$ defined by the assignment $X \mapsto R^{-1}[X]$ (cf. Page 18 for notation). The perspective we develop in Chapter 3, Section 3.2 is based on $(\mathcal{P}(W), \subseteq)$ being a partial order and the operations of $\mathcal{F}^+$ enjoying certain properties w.r.t. this order.

The completeness of basic modal logic with respect to modal algebras can be shown using the standard Lindenbaum-Tarski construction; for a proof see, e.g., [30, Theorem 5.27].

2.1.6. Theorem. $\mathbf{K} \vdash \varphi$ *iff* φ *is valid in every modal algebra.*

The following theorem shows that every modal algebra can be represented using the admissible subsets of a descriptive general frame. We refer to [30, Theorem 5.43] for a proof.

2.1.7. Theorem. *For every modal algebra* $\mathbb{B}$*, there exists a descriptive general frame* $\mathcal{G} = (W, R, \mathcal{A})$ *such that* $\mathbb{B}$ *is isomorphic to* $(\mathcal{A}, \cup, \cap, \setminus, \varnothing, \langle R \rangle)$.

The standard translation. When interpreted on models, modal logic is essentially a fragment of first-order logic, into which we can effectively and straightforwardly translate it using the so-called *standard translation.* In order to introduce it, we need some preliminary definitions.

Let L_0 be the first-order language with $=$ and a binary relation symbol R, over a set of denumerably many individual variables $\mathsf{VAR} = \{x_0, x_1, \ldots\}$. Also, let L_1 be the extension of L_0 with a set of unary predicate symbols $P, Q, P_0, P_1 \ldots$, corresponding to the propositional variables $p, q, p_0, p_1 \ldots$ of Prop. The language L_2 is the extension of L_1 with universal second-order quantification over the unary predicates $P, Q, P_0, P_1 \ldots$

Clearly, Kripke frames are structures for *both* L_0 and L_2. Moreover, (modal) models $\mathcal{M} = (W, R, V)$ can be regarded as structures for L_1, by interpreting the predicate symbol P associated with a given atomic proposition $p \in \mathsf{Prop}$ as the subset $V(p) \subseteq W$.

ML-formulas are translated into L_1 by means of the following *standard translation* ST_x from [30]. Given a first-order variable x and a modal formula φ, this translation yields a first-order formula $\mathrm{ST}_x(\varphi)$ in which x is the only free variable. $\mathrm{ST}_x(\varphi)$ is given inductively by

$$\begin{array}{rcl}
\mathrm{ST}_x(p) & = & Px, \\
\mathrm{ST}_x(\bot) & = & x \neq x, \\
\mathrm{ST}_x(\neg\varphi) & = & \neg(ST_x(\varphi)), \\
\mathrm{ST}_x(\varphi \vee \psi) & = & ST_x(\varphi) \vee ST_x(\psi), \\
\mathrm{ST}_x(\Diamond\varphi) & = & \exists y(xRy \wedge ST_y(\varphi)), \text{ where } y \text{ is any fresh variable,} \\
\mathrm{ST}_x(\Box\varphi) & = & \forall y(xRy \to ST_y(\varphi)), \text{ where } y \text{ is any fresh variable.}
\end{array}$$

The *standard second-order translation* of a modal formula φ is the L_2-formula obtained by universal second-order quantification over all predicates corresponding to proposition letters occurring in φ, that is, the formula $\forall P_1 \ldots \forall P_n \mathrm{ST}_x(\varphi)$. The following proposition is well known and easy to check (see, e.g., [30, Proposition 3.12]).

2.1.8. Proposition. *Let* $\mathcal{F} = (W, R)$ *be a Kripke frame,* V *be a valuation on* $\mathcal{F}$*, and* $w \in W$.

1. $(\mathcal{F}, V), w \Vdash \varphi$ *iff* $(\mathcal{F}, V) \models \mathrm{ST}_x(\varphi)[x := w]$

2. $\mathcal{F}, w \Vdash \varphi$ *iff* $\mathcal{F} \models \forall P_1 \ldots \forall P_n \mathrm{ST}_x(\varphi)[x := w]$
3. $(\mathcal{F}, V) \Vdash \varphi$ *iff* $(\mathcal{F}, V) \models \forall x \mathrm{ST}_x$
4. $\mathcal{F} \Vdash \varphi$ *iff* $\mathcal{F} \models \forall P_1 \ldots \forall P_n \forall x \mathrm{ST}_x(\varphi)$.

2.2 Correspondence

As seen in the previous subsection, the correspondence between modal languages and predicate logic depends on where one focusses in the multi-layered hierarchy of relational semantics notions. At the bottom of this hierarchy lies the model. At this level, the question of correspondence, at least when approached from the modal side, is trivial: all modal formulas define first-order conditions on these structures. This can be made more precise: we have the following elegant theorem by van Benthem:

2.2.1. Theorem (cf. [13]). *Modal logic is exactly the bisimulation invariant fragment of* L_1.

At the top of the hierarchy, the interpretation of modal languages over relational structures via the notion of validity turns them into fragments of monadic second-order logic, and rather expressive fragments at that. Indeed, as Thomason [151] has shown, second-order consequence may be effectively reduced to the modal consequence over relational structures.

On the other hand, as already indicated, some modal formulas actually define first-order conditions on Kripke frames. For instance, in the standard second-order translation of any formula φ which contains no propositional variables (called a *constant formula*), the second-order quantifier prefix is empty. Hence, to mention a concrete example, the standard second-order translation $\mathrm{ST}_x(\Box\bot)$ is $\forall y(xRy \to y \neq y) \equiv \forall y(\neg xRy)$.

We refer to $\Box\bot$ and $\forall y(\neg xRy)$ as *local frame correspondents*, since for all Kripke frames $\mathcal{F}$ and states w,

$$\mathcal{F}, w \Vdash \Box\bot \quad \text{iff} \quad \mathcal{F} \models \forall y(\neg xRy)[x := w].$$

A modal formula φ and a first-order sentence α are *global frame correspondents* if $\mathcal{F} \Vdash \varphi$ iff $\mathcal{F} \models \alpha$ for all Kripke frames $\mathcal{F}$.

But formulas need not be constant to define first-order conditions: indeed, $p \to \Diamond p$ and xRx are local frame correspondents. A short proof of this fact might be instructive: Let $\mathcal{F} = (W, R)$ and $w \in W$. Suppose wRw, and let V be any valuation such that $(\mathcal{F}, V), w \Vdash p$. Then, since wRw, the state w has a successor satisfying p, and hence $(\mathcal{F}, V), w \Vdash \Diamond p$. Since V was arbitrary, we conclude that $\mathcal{F}, w \Vdash p \to \Diamond p$. Conversely, suppose $\neg(wRw)$, and let V be *some* valuation such that $V(p) = \{w\}$. Then $(\mathcal{F}, V), w \Vdash p$ but $(\mathcal{F}, V), w \nVdash \Diamond p$, hence $\mathcal{F}, w \nVdash p \to \Diamond p$.

The latter direction is an instance of the so-called *minimal valuation argument*, which pivots on the fact that some "first-order definable" minimal element exists in the class of valuations which make the antecedent of the formula true at w. We will take full stock of this observation in Section 3.2.1.

According to the picture emerging from the facts collected so far, it is the correspondence of modal logic and first-order logic on frames which is most interesting. It is here where our efforts are needed in order to try and rescue as much of modal logic as we can from the computational disadvantages of second-order logic. Indeed, there is much that can be salvaged.

In [13], van Benthem provides an elegant model-theoretic characterization of the modal formulas which have global first-order correspondents. The constructions involved in this characterization (viz. ultrapowers) are infinitary. So it would be useful to couple this result with a theorem providing an effective way to check whether a formula is elementary. This would, of course, be much too good to be true, and indeed, our skepticism is confirmed by Chagrova's theorem:

2.2.2. THEOREM ([40]). *It is algorithmically undecidable whether a given modal formula is elementary.*

An effective characterization is, therefore, impossible, but if we are willing to be satisfied with approximations, all is not lost. Various large and interesting, syntactically defined classes of (locally) elementary formulas are known.

2.3 Syntactic classes

A large part of the study of the correspondence between modal and first-order logic has traditionally consisted of the identification of syntactically specified classes of modal formulas which have local frame correspondents. The aim of the present section is to give precise definitions of the syntactic classes that we are going to treat in Chapter 3, Section 3.2. For the reader's convenience, we will do this in a hierarchical form which is conceptually akin to the treatment in [30], although it is slightly different from it, to better fit our account in the following section. We confine our presentation to the basic modal language.

Formulas without nesting. These are the modal formulas in which no nesting of modal operators occurs. Their elementarity was proved by van Benthem [12].

Closed and Uniform formulas ([30]). The *closed* modal formulas are those that contain no proposition letter. An occurrence of a proposition letter p in a formula φ is *positive* (*negative*) if it is under the scope of an even (odd) number of negation signs. (To apply this definition correctly one of course has to bear in mind the negation signs introduced by the defined connectives $\to$ and $\leftrightarrow$.) A formula φ is *positive* in p (*negative* in p) if all occurrences of p in φ are positive (negative).

A proposition letter occurs *uniformly* in a formula if it occurs only positively or only negatively. A modal formula is *uniform* if all the propositional letters it contains occur uniformly.

Sahlqvist formulas ([30]) This is the archetypal class of elementary modal formulas, due to Sahlqvist [136].

Very simple Sahlqvist formula. A *very simple Sahlqvist formula* is a formula built up from $\top$, $\bot$, and proposition letters, using $\wedge$ and $\Diamond$. A *very simple Sahlqvist formula* is an implication $\varphi \to \psi$ in which φ is a very simple Sahlqvist antecedent and ψ is a positive formula.

Simple Sahlqvist formula. A *boxed atom* is a propositional variable preceded by a (possibly empty) string of boxes, i.e. a formula of the form $\Box^n p$ where $n \in \mathbb{N}$ and $p \in \mathsf{Prop}$. A *simple Sahlqvist antecedent* is any formula constructed from $\top, \bot$, and boxed atoms by applying $\wedge$ and $\Diamond$. A *simple Sahlqvist* formula is an implication with a simple Sahlqvist antecedent as antecedent, and a positive formula as consequent.

Sahlqvist implications. A *Sahlqvist antecedent* is a formula built up from $\top$, $\bot$, negative formulas and boxed atoms, using $\vee$, $\wedge$ and $\Diamond$. A *Sahlqvist implication* is an implication $\varphi \to \psi$ in which ψ is positive and φ is a Sahlqvist antecedent.

Sahlqvist formulas. A *Sahlqvist formula* is a formula that is built up from Sahlqvist implications by freely applying boxes, conjunctions and disjunctions.

Most of the well-known axioms in modal logic such as the axioms for reflexivity ($p \to \Diamond p$), transitivity ($\Diamond\Diamond p \to \Diamond p$), symmetry ($p \to \Box\Diamond p$), seriality ($\Box p \to \Diamond p$), and the Churh-Rosser property ($\Diamond\Box p \to \Box\Diamond p$), are examples of Sahlqvist formulas. A well-known example of an axiom which is not Sahlqvist is the McKinsey axiom $\Box\Diamond p \to \Diamond\Box p$. Over the years, many extensions, variations and analogues of this result have appeared, including alternative proofs (e.g. [138]), generalizations to arbitrary modal signatures (e.g. [135]), variations of the correspondence language (e.g. [126] and [15]), or Sahlqvist results for hybrid logics (e.g. [37]).

Apart from being elementary, the Sahlqvist formulas have the added virtue of being canonical (i.e., of being valid in the canonical, or Henkin, frames of the logics axiomatized by them), and hence, of axiomatizing complete normal modal logics. In other words, logics axiomatized using Sahlqvist formulas are sound and complete with respect to elementary classes of Kripke frames.

Inductive formulas. The class of inductive formulas, introduced by Goranko and Vakarelov [89], is a proper generalization of the Sahlqvist class. It was also shown in [89] that inductive formulas are both elementary and canonical.

Inductive implications ([86, Definition 122]). Let $\sharp$ be a symbol not belonging to ML. A *box-form of* $\sharp$ in ML is defined recursively as follows:

1. for every $k \in \mathbb{N}$, $\Box^k\sharp$ is a box-form of $\sharp$;
2. If $B(\sharp)$ is a box-form of $\sharp$, then for any positive formula φ, $\Box(\varphi \to B(\sharp))$ is a box-form of $\sharp$.

Thus, box-forms of $\sharp$ are of the type

$$\Box(\varphi_0 \to \Box(\varphi_1 \to \ldots \Box(\varphi_n \to \Box^k\sharp)\ldots)),$$

where $\varphi_0, \ldots, \varphi_n$ are positive formulas.

By substituting a propositional variable $p \in \mathsf{Prop}$ for $\sharp$ in a box-form $B(\sharp)$ we obtain a *box-formula of* p, denoted $B(p)$. The last occurrence of the variable p is the *head* of $B(p)$ and every other occurrence of a variable in $B(p)$ is *inessential* there. A *regular antecedent* is a formula built up from $\top$, $\bot$, negative formulas, and box-formulas, using $\vee$, $\wedge$ and $\Diamond$.

The *dependency digraph* of a set of box-formulas $\mathcal{B} = \{B_1(p_1), \ldots, B_n(p_n)\}$ is the directed graph $G_{\mathcal{B}} = \langle V, E\rangle$ where $V = \{p_1, \ldots, p_n\}$ is the set of heads of members of $\mathcal{B}$, and E is a binary relation on V such that p_iEp_j iff p_i occurs as an inessential variable in a box formula in $\mathcal{B}$ with head p_j. A digraph is *acyclic* if it contains no directed cycles or loops. Note that the transitive closure of the edge relation E of an acyclic digraph is a strict partial order, i.e. it is irreflexive and transitive, and consequently also antisymmetric. The dependency digraph of a formula φ is the dependency digraph of the set of box-formulas that occur as subformulas of φ.

An *inductive antecedent* is a regular antecedent with an acyclic dependency digraph. A *regular (resp. inductive) implication* is an implication $\varphi \to \psi$ in which ψ is positive and φ is a regular (resp. inductive) antecedent.

2.3.1. Example. Consider the following formulas:

$$\begin{aligned}
\varphi_1 &:= p \wedge \Box(\Diamond p \to \Box q) \to \Diamond\Box\Box q,\\
\varphi_2 &:= \Diamond\Box p \wedge \Diamond(\Box(p \to q) \vee \Box(p \to \Box\Box r)) \to \Diamond\Box(q \vee \Diamond r)\\
\varphi_3 &:= \Diamond(\Box(p \to \Box\Box q) \vee \Box(q \to \Box p)) \to \Diamond\Box p.
\end{aligned}$$

The formula φ_1 is an inductive implication, which is not a Sahlqvist implication. It was shown in [89] that φ_1 is not frame equivalent to any

Sahlqvist formula. The antecedent is the conjunction of the box-formulas p and $\Box(\Diamond p \to \Box q)$. The dependency digraph over the set of heads $\{p, q\}$ has only one edge, from p to q, and thus linearly orders the variables.

The formula φ_2 is an inductive implication. Its dependency digraph has three vertices p, q, and r, and arcs from p to q and from p to r.

The formula φ_3 is a regular but not inductive implication. Its dependency digraph contains a 2-cycle on the vertices p and q.

Inductive formulas. An *inductive formula* is a formula that is built up from atomic inductive implications by freely applying boxes, conjunctions, and disjunctions.

Modal reduction principles. A *modal reduction principle* is an ML-formula of the form $Q_1 Q_2 \ldots Q_n p \to Q_{n+1} Q_{n+2} \ldots Q_{n+m} p$ where $0 \leq n, m$ and $Q_i \in \{\Box, \Diamond\}$ for $1 \leq i \leq n + m$. Many well-known modal axioms take this form, e.g. reflexivity, transitivity, symmetry, and the McKinsey axiom. In [10], van Benthem provides a complete classification of the modal reduction principles that define first-order properties on frames. For example, as we have already seen, $\Diamond\Box p \to \Box\Diamond p$ defines such a property and $\Box\Diamond p \to \Diamond\Box p$ does not. In [10], it is also shown that, when interpreted over transitive frames, *all* modal reduction principles define first-order properties.

Complex formulas. This class was introduced by Vakarelov [152]. Complex formulas can be seen as substitution instances of Sahlqvist formulas obtained through the substitution of certain elementary disjunctions for propositional variables. The resulting formulas may violate the Sahlqvist definition.

2.4 Algorithmic strategies

In this section we give a brief overview of algorithmic strategies for correspondence.

Sahlqvist-van Benthem algorithm. For a detailed exposition of the Sahlqvist-van Benthem algorithm, we refer to [30, Chapter 3].

2.4.1. Lemma ([30], Lemma 3.53). *Let $\varphi, \psi \in$ ML.*

1. *If φ and $\alpha(x)$ are local correspondents, then so are $\Box\varphi$ and $\forall y(xRy \to [y/x]\alpha)$.*

2. *If φ (locally) corresponds to α, and ψ (locally) corresponds to β, then $\varphi \wedge \psi$ (locally) corresponds to $\alpha \wedge \beta$.*

3. *If φ locally corresponds to α, ψ locally corresponds to β, and φ and ψ have no proposition letters in common, then $\varphi \vee \psi$ locally corresponds to $\alpha \vee \beta$.*

2.4.2. Theorem (Sahlqvist correspondence theorem [136, 13]). *For every Sahlqvist formula φ, there exists a corresponding first-order-sentence χ such that φ is valid in a frame iff χ holds in the frame.*

Proof. In order to simplify our presentation, which closely follows [30, Section 3.6], we work with a Sahlqvist implication $\varphi \to \psi$. It follows from Lemma 2.4.1 that in order to compute the frame correspondent of a Sahlqvist formula, it is sufficient to compute the frame correspondents of Sahlqvist implications which occur as subformulas. The following algorithm effectively computes the first-order frame correspondent χ of $\varphi \to \psi$.

Step 1. Let $ST_x(\varphi)$ be the standard translation of φ. Using the following equivalences, we move all the existential quantifiers in $ST_x(\varphi)$ at the front of the implication $ST_x(\varphi \to \psi)$.

$$(\exists x_i \alpha(x_i) \wedge \beta) \leftrightarrow \exists x_i(\alpha(x_i) \wedge \beta),$$
$$(\exists x_i \alpha(x_i) \to \beta) \leftrightarrow \forall x_i(\alpha(x_i) \wedge \beta).$$

Step 2. Let $\mathcal{F} = (W, R)$ be a Kripke frame, and fix $w \in W$. Let $p_1, \ldots, p_n \in \mathsf{Prop}$ be the set of propositional variables occurring in φ, and $\mathcal{F}, w \Vdash \varphi \to \psi$. We compute the *minimal valuation* $V_0(p_i)$, $1 \leq i \leq n$, for each propositional variable as follows: let $\beta_1, \ldots, \beta_{m_i}$ be the boxed atoms in φ which contain p_i, with $\beta_j = \Box^{d_j} p_i, 1 \leq j \leq m_i$ and $d_j \geq 0$. Let $R^0[w] = \{w\}$ and $R^n[w] = \{w' \in W : \exists w_1, \ldots, w_n$ such that $wRw_1R \ldots Rw_n$ and $w_n = w'\}$ for $n \geq 1$. The minimal valuation for p_i is: $V_0(p_i) = R^{d_1}[w] \cup \ldots \cup R^{d_{m_i}}[w]$.

Step 3. Let V_0 be the minimal valuation computed in Step 2. The syntactic shape of the Sahlqvist implication ensures the following:

If $\varphi \to \psi$ is a Sahlqvist implication, then the following are equivalent:

1. $\mathcal{F}, w \Vdash \varphi \Rightarrow \mathcal{F}, w \Vdash \psi$,
2. $\mathcal{F}, V_0, w \Vdash \varphi \Rightarrow \mathcal{F}, V_0, w \Vdash \psi$.

Since $\mathcal{F}, w \Vdash \varphi \to \psi$, it is clear that $(1 \Rightarrow 2)$. We prove the converse using a contrapositive argument. Suppose $\mathcal{F}, w \Vdash \varphi$ and $\mathcal{F}, w \nVdash \psi$. Then there exists a valuation V such that $\mathcal{F}, V, w \Vdash \varphi$ and $\mathcal{F}, V, w \nVdash \psi$. We show that the minimal valuation V_0 is such that $\mathcal{F}, V_0, w \Vdash \varphi$ and $\mathcal{F}, V, w \nVdash \varphi$, using an induction on the complexity of φ.

The base case with $\varphi = \bot$ is trivial. If $\varphi = \Box^n p$, it is easy to check that $\mathcal{F}, V, w \Vdash \Box^n p$ if, and only if $\mathcal{F}, V_0, w \Vdash \Box^n p$, where $V_0(p) = R^n[w]$ is the minimal valuation computed in Step 1. Since ψ is a positive formula and $V_0(p) \subseteq V(p)$, it follows that $\mathcal{F}, V_0, w \nVdash \psi$. If $\varphi = \varphi_1 \wedge \varphi_2$, using induction hypothesis, there exist minimal valuations $V_0'(p) \subseteq V(p)$ and $V_0''(p) \subseteq V(p)$ for φ_1 and φ_2, respectively. Let $V_0(p) = V_0'(p) \cup V_0''(p)$, which implies $V_0(p) \subseteq V(p)$. Hence, $\mathcal{F}, V_0, w \nVdash w$ If $\varphi = \Diamond\varphi_1$, the minimal valuation V_0 such that $\mathcal{F}, V_0, w \Vdash \varphi$ and $\mathcal{F}, V, w \nVdash \psi$, is the same as the minimal valuation for φ_1.

Step 4. We showed in Step 3 that a Sahlqvist implication is valid under an arbitrary valuation if and only if it is valid under a minimal valuation. As it is shown below, the minimal assignment V_0 computed in Step 1 is first-order definable. Hence, it ensures that the frame-condition corresponding to a Sahlqvist implication is first-order definable.

Let $\chi' := \forall P_1 \ldots \forall P_n \forall x\ ST_x(\varphi \to \psi)$ be the second-order translation of $\varphi \to \psi$. Suppose $V_0(p_i) = R^{d_1}[w] \cup \ldots \cup R^{d_{m_i}}[w]$ for $p_i \in \mathsf{Prop}$. The first-order frame condition χ is obtained from χ' by replacing $\forall P_i$ with $\forall z_i$, where z_i is a fresh first-order variable, and each atomic formula of the form $P_i(v)$ with a first-order formula $\theta_i := \exists y_0, \ldots, y_n[z_i = y_0 \wedge \bigwedge_{j=0}^{n-1} Ry_jy_{j+1} \wedge y_n = v]$, which says 'there exists an R-path from z_i to v in n steps'.

Finally, we show that the first-order sentence χ is a frame condition for $\varphi \to \psi$. The minimal valuations for all the propositional variables in φ computed in Step 2 above are first-order definable. By Proposition 2.1.8, χ is equivalent to $\varphi \to \psi$. Hence, it follows that χ is the first-order frame condition corresponding to $\varphi \to \psi$. $\square$

2.4.3. Example. We illustrate the algorithm by means of an example. Consider the Church-Rosser axiom $\Diamond\Box p \to \Box\Diamond p$. Its second-order translation is:

$$\forall P \forall x \Big(\exists y(xRy \wedge \forall z(yRz \to Pz)) \to \forall u(xRy \to \exists v(uRv \wedge Pv)) \Big).$$

On moving the existential quantifier in the antecedent to the front of the implication we get:

$$\forall P \forall x \forall y((xRy \wedge \forall z(yRz \to Pz)) \to \forall u(xRu \to \exists v(uRv \wedge Pv))).$$

Since p occurs in the scope of a $\Box$ in the antecedent, its minimal valuation is $V_0(p) := \{z : yRz\}$. On substituting Pz and Pv with the minimal valuation we obtain:

$$\forall x \forall y((xRy \wedge \forall z(yRz \to yRz)) \to \forall u(xRu \to \exists v(uRv \wedge yPv))),$$

which simplifies to the familiar confluence condition:

$$\forall x \forall y \forall u(xRy \wedge xRu \to \exists v(uRv \wedge yPv)).$$

2.4.4. Remark. From the proof of the Sahlqvist correspondence theorem, it is evident that it is an ad hoc proof for modal logic. Since the proof does not explain the underlying mechanism which makes Sahlqvist formulas correspond to a first-order condition, it is not immediately clear how can we systematically extend this theory to other non-classical logics. Moreover, it does not provide any intuition regarding the syntactic definition of Sahlqvist formulas themselves. An algebraic analysis of this proof, as presented in Chapter 3 of this thesis, uncovers the mathematical ingredients which constitute the correspondence phenomenon. It also paves the way for the development of this theory in the later chapters.

The algorithm SQEMA. The algorithm SQEMA, introduced in [48], computes first-order frame correspondents for all inductive (and hence Sahlqvist) formulas, among others. It has been extended and developed in a series of papers by the same authors. It also guarantees the canonicity of all formulas on which it succeeds. The algorithm is based on the following version of Ackermann's lemma (see Lemma 4.2.1 for a modal version of Ackermann's lemma). The Ackermann's lemma uses a monotonicity argument to eliminate variables. Other than its role in SQEMA and ALBA, it also plays an important role in the substitution-rewrite approaches to second-order quantifier elimination such as the DLS algorithm [147].

2.4.5. Lemma (Ackermann's Lemma, [2]). *Let P be an n-ary predicate variable and $A(\overline{z}, \overline{x})$ a first-order formula not containing P. Then, if $B(P)$ is negative in P, the equivalence*

$$\exists P \forall \overline{x}((\neg A(\overline{z}, \overline{x}) \vee P(\overline{x})) \wedge B(P)) \equiv B[A(\overline{z}, \overline{x})/P]$$

holds, with $B[A(\overline{z}, \overline{x})/P]$ the formula obtained by substituting $A(\overline{z}, \overline{x})$ for all occurrences P in B, the actual arguments of each occurrence of P being substituted for $\overline{x}$ in $A(\overline{z}, \overline{x})$ every time. If $B(P)$ is positive in P, then the following equivalence holds:

$$\exists P \forall \overline{x}((\neg P(\overline{x}) \vee A(\overline{z}, \overline{x})) \wedge B(P)) \equiv B[A(\overline{z}, \overline{x})/P].$$

A sequent $\varphi \Rightarrow \psi$ with $\varphi, \psi \in$ ML is called a *SQEMA-sequent*, and a finite set of *SQEMA-sequents* is called a *SQEMA-system*. For a SQEMA-system Sys, let Form(Sys) be the conjunction of all $\varphi_i \Rightarrow \psi_i$ for all sequents $\varphi_i \Rightarrow \psi_i \in$ Sys. Given a modal formula φ as input, the algorithm reduces to a first-order formula using the following steps. In the rules below, the letters $A, B, ..$ denote modal formulas.

Phase 1 (Preprocessing). The negation of φ is converted to a negation normal form, and $\Diamond$ and $\wedge$ are distributed over $\vee$ using the equivalences: $\Diamond(\varphi \vee \psi) \equiv \Diamond\varphi \vee \Diamond\psi$ and $\varphi \wedge (\psi \vee \gamma) \equiv (\varphi \wedge \psi) \vee (\varphi \wedge \gamma)$.

Phase 2 (Elimination). For elimination, the modal language is enhanced with a special sort of propositional variables, called *nominals* (denoted by **i**, **j**...) along with semantic condition that any valuations assigns a singleton subset of the domain to a nominal. The following rewrite rules are used to transform each initial system $\mathsf{Sys_i}$ to a stage where the Ackermann rule can be applied to eliminate propositional variables. If all the propositional variables are eliminated, the algorithm proceeds to Phase 3, else it reports failure and terminates.

1.Rules for connectives:

$$\frac{C \Rightarrow (A \wedge B)}{C \Rightarrow A, C \Rightarrow B}\ (\wedge\text{-rule}) \qquad \frac{A \Rightarrow C, B \Rightarrow C}{(A \vee B) \Rightarrow C}\ (\vee\text{-rule})$$

$$\frac{C \Rightarrow (A \vee B)}{(C \wedge \neg A) \Rightarrow B}\ (\text{left-shift } \vee\text{-rule}) \qquad \frac{(C \wedge A) \Rightarrow B}{C \Rightarrow (\neg A \vee C)}\ (\text{right-shift } \vee\text{-rule})$$

$$\frac{A \Rightarrow \Box B}{\Diamond^{-1} A \Rightarrow B}\ (\Box\text{-rule}) \qquad \frac{\Diamond^{-1} A \Rightarrow B}{A \Rightarrow \Box B}\ (\text{inverse } \Diamond\text{-rule})$$

$$\frac{\mathbf{j} \Rightarrow \Diamond A}{\mathbf{j} \Rightarrow \Diamond \mathbf{k}, \mathbf{k} \Rightarrow A}\ (\Diamond\text{-rule})$$

where $\mathbf{k}$ is a new nominal not occurring in the system.

2. Polarity switching rule: Substitute $\neg p$ for every occurrence of p in the system

3. Ackermann-rule: The soundness of the Ackermann-rule follows from Ackermann's lemma (cf.Lemma 2.4.5).

$$\text{The system} \quad \begin{cases} A \Rightarrow p \\ \\ B_1(p) \\ \vdots \\ B_m(p) \\ \\ C_1 \\ \vdots \\ C_k \end{cases} \quad \text{is replaced by} \quad \begin{cases} B_1(A/p) \\ \vdots \\ B_m(A/p) \\ \\ C_1 \\ \vdots \\ C_k \end{cases}$$

where:

1. p does not occur in $A, C_1, \ldots, C_k$;

2. $B_1 \wedge \ldots \wedge B_m$ is negative in p.

Phase 3 (Translation). The algorithm reaches this stage only if all the propositional variables have been eliminated. Let Sys be the system obtained after the elimination stage.The frame correspondent is then returned as the standard translation of the conjunction of sequents in the system.

2.5 Duality and Canonicity

In an influential paper [142], Stone established a duality between Boolean algebras and *Boolean spaces* (also called *Stone spaces*). A Stone space is a compact, Hausdorff and zero-dimensional space[1]. A *filter* of a Boolean algebra is a subset $F \subseteq \mathbb{B}$ such that (i) $\top \in \mathbb{B}$, (ii) If $a \in F$ and $b \geq a$, then $b \in F$, (iii) If $a, b \in F$, then $a \wedge b \in F$. An *ultrafilter* is a filter F such that for all $a \in \mathbb{B}$, either $a \in F$ or $\neg a \in F$. Given a Boolean algebra $\mathbb{B}$, define $\mathbb{B}_*$ to be the space of its ultrafilters $Ult(\mathbb{B})$ topologized by the basis $\{u \in Ult(\mathbb{B}) : a \in u\}$, for each $a \in \mathbb{B}$. Then $\mathbb{B}_*$ is a Stone space. For a Boolean homomorphism $h : \mathbb{A} \to \mathbb{B}$, the map $h_* : \mathbb{B}_* \to \mathbb{A}_*$ defined as $h^* = h^{-1}$ is a continuous map. Conversely, given a Stone space X, the algebra X^* of its clopen subsets with union, intersection and negation as the algebraic operations is a Boolean algebra. If $f : X \to Y$ is a continuous map, then $f^* : X^* \to Y^*$ defined as $f^* = f^{-1}$ is a homomorphism. Based on these constructions, we have the following result.

2.5.1. Theorem (Stone duality [142]). *The category of Boolean algebras and homomorphisms is dually equivalent to the category of Stone spaces and continuous maps.*

Jónsson and Tarski [98] proved a representation theorem for modal algebras. This was later extended by Goldblatt [83, 84] to a duality between modal algebras and modal homomorphisms, and descriptive general frames and p-morphisms. The topological version of this duality that we present below is due to Sambin and Vaccaro [137]. This duality is often referred to as the Jónsson-Tarski duality, even though in [98], Jónsson and Tarski did not prove a full categorical duality for modal algebras. However, their use of canonical extenstions of modal algebras for the representation of a modal algebra can be seen to encode the topological duality in an algebraic form. The categorical duality was later developed by Goldblatt, and Sambin and Vaccaro.

2.5.2. Definition. A *modal space* is a pair (W, R) where W is a Stone space and R is a binary relation on W satisfying (i) $R[x]$ is closed for each $x \in W$ and (ii) $R^{-1}[U]$ is clopen for each clopen $U \subseteq W$. For modal spaces, (W, R) and (W', R'), a function $f : W \to W'$ is a *p-morphism* if (i) wRw' implies $f(w)Rf(w')$ and (ii) $f(w)Rv$ implies there is $u \in W$ with wRu and $f(u) = v$. Let **MS** be the category of modal spaces and continuous p-morphisms.

Let **MA** be the category of modal algebras modal homomorphisms. Given a modal algebra $(\mathbb{B}, \Diamond)$, the dual modal space is $(\mathbb{B}_*, R)$ where $\mathbb{B}_*$ is the dual Stone space of $\mathbb{B}$ and R is a binary relation defined as: uRv iff $\Diamond v \subseteq u$. For a modal homomorphism $h : (\mathbb{B}_1, \Diamond_1) \to (\mathbb{B}_2, \Diamond_2)$, the map $h_* : ((\mathbb{B}_2)_*, R_2) \to ((\mathbb{B}_1)_*, R_1)$

[1]A topological space is *zero-dimensional* if has a basis of clopen (both closed and open) sets.

defined as $h^* = h^{-1}$ is a continuous p-morphism. For a modal space (W, R), the dual modal algebra is $(W^*, \Diamond)$ where W^* is the Boolean algebra of clopen subsets of X, and $\Diamond$ is a unary map on X^* defined as: $\Diamond U = R^{-1}(U)$. If $f : (W_1, R_1) \to (W_2, R_2)$ is a continuous p-morphism, then $f^* : ((W_2)^*, \Diamond_2) \to ((W_1)^*, \Diamond_1)$ defined as $f^* = f^{-1}$ is a modal homomorphism. This leads to the following result.

2.5.3. Theorem (Jónsson-Tarski duality [98, 83, 84, 137]). *The categories* **MA** *and* **MS** *are dually equivalent.*

It is important to note that modal spaces are in one-to-one correspondence with descriptive general frames introduced earlier (cf. Definition 2.1.3). For a modal space (W, R), the triple $(W, R, \mathsf{Clop}(X))$ is a descriptive general frame. Conversely, given a descriptive general frame $(W, R, \mathcal{A})$, define a topology on W by letting $\mathcal{A}$ be a basis for the topology. Then (W, R) is a modal space. We refer to [30, Chapter 5] for a proof. Sambin and Vaccaro [138] used modal spaces to prove canonicity of Sahlqvist formulas.

2.5.4. Theorem (Sahlqvist canonicity theorem [136, 138]). *If φ is a Sahlqvist formula, then $\Lambda + \varphi$ is canonical whenever Λ is a canonical logic.*

Proof. We give a brief sketch of the proof. For canonicity, we need to show that for any Sahlqvist formula φ, if φ is valid on a modal space $\mathcal{X} = (W, R)$ under clopen valuations, then it is valid on the underlying Kripke frame $\mathcal{X}_\#$ of the modal space under arbitrary valuations. Suppose $\mathcal{X}_\# \not\models \varphi$, then there is a minimal valuation which refutes φ. The key observation is that the minimal valuation is topologically closed, and this is guaranteed by the syntactic shape of Sahlqvist formulas. Now using the following Esakia's lemma, we can find a clopen valuation which refutes φ on the modal space $\mathcal{X}$. $\square$

2.5.5. Lemma (Esakia's Lemma [64]). *Let (W, R) be a modal space, $F \subseteq W$ a closed set, and $\varphi(p)$ be a modal formula positive in p. Then,*

$$[\![\varphi(F)]\!] = \bigcap \{[\![\varphi(U)]\!] : U \in \mathsf{Clop}(W), F \subseteq U\}$$

where, for $X \subseteq W$, $[\![\varphi(X)]\!]$ denotes the set of worlds in W where φ is true under the valuation which maps p to X[2].

Recently Sahlqvist correspondence and canonicity theorems were extended to modal mu-calculus in [17] and [27], respectively. In Chapter 8 of this thesis, we generalize the above results to the setting of compact Hausdorff spaces.

[2] cf. Section 3.1.1 for a detailed definition of $[\![\varphi]\!]$

Chapter 3

Basic algebraic modal correspondence

In this chapter, which is a revised version of [52], we propose a new introductory treatment of the well-known Sahlqvist correspondence theory for classical modal logic as presented in the previous chapter. The first motivation for the present treatment is *pedagogical*: classical Sahlqvist correspondence is presented in a smooth and modular way. The second motivation is *methodological*: indeed, the present treatment aims at highlighting the *algebraic* and *order-theoretic* nature of the correspondence mechanism, which also plays an important role in the later chapters of this thesis. The exposition remains elementary and does not presuppose any previous knowledge or familiarity with the algebraic approach to logic.

The connections between the algebraic and the relational semantics of modal logic and other propositional logics form a mathematically rich and deep theory which dates back to Stone [142], Jónsson and Tarski [98], and more recently to Goldblatt [79]. For the sake of keeping the presentation elementary, in the present chapter we only focus on Sahlqvist correspondence and do not treat Sahlqvist canonicity in any form. On the other hand, the systematic algebraic treatment of Sahlqvist correspondence is much newer than the algebraic treatment of Sahlqvist canonicity (which goes back to Jónsson and Tarski [98, 99]). While making it possible to consider and reason about properties with a distinct algebraic and order-theoretic flavour, the environment of complex algebras retains and supports our set-theoretic intuitions coming from Kripke frames.

The present exposition is closely related to but also very different from the standard textbook treatments (cf. e.g. [30, 41, 107]), and, without introducing technicalities such as the Ackermann lemma, explains in elementary terms the conceptual foundations of unified correspondence theory. We believe that the present treatment can be useful in making unified correspondence theory accessible to a wider community of logicians than the experts in algebraic methods in logic.

The present chapter is organized as follows: in Section 3.1, we introduce preliminaries on the meaning function of a formula and definite implications. In Section 3.2, which is the core section of this chapter, we illustrate the algebraic correspondence methodology for different syntactic classes, namely, uniform and closed formulas, very simple Sahlqvist implications, and Sahlqvist implications. Finally in Section 3.3, we present our conclusions. In addition to the above contents, the article [52] contains an algebraic treatment of correspondence for inductive formulas, which is not included in this chapter.

3.1 Preliminaries

We now define the semantics of formulas in the modal language ML on models and frames via the following *meaning function*. This formulation will be convenient later on in Section 3.2.1, when we will develop the discussion on the reduction strategies.

3.1.1 Meaning Function

For a formula $\varphi \in$ ML we write $\varphi = \varphi(p_1, \ldots, p_n)$ to indicate that *at most* the atomic propositions $p_1, \ldots p_n$ occur in φ. Every such φ induces an n-ary operation on $\mathcal{P}(W)$,

$$[\![\varphi]\!] : \mathcal{P}(W)^n \longrightarrow \mathcal{P}(W),$$

inductively given by:

$$\begin{array}{rl} [\![\bot]\!] & \text{is the constant function } \varnothing \\ [\![p]\!] & \text{is the identity map } Id_{\mathcal{P}(W)} \\ [\![\neg\varphi]\!] & \text{is the complementation } W \setminus [\![\varphi]\!] \\ [\![\varphi \vee \psi]\!] & \text{is the union } [\![\varphi]\!] \cup [\![\psi]\!] \\ [\![\Diamond\varphi]\!] & \text{is the semantic diamond } R^{-1}([\![\varphi]\!]). \end{array}$$

It follows that

$$\begin{array}{rl} [\![\top]\!] & \text{is the constant function } W \\ [\![\varphi \wedge \psi]\!] & \text{is the intersection } [\![\varphi]\!] \cap [\![\psi]\!] \\ [\![\Box\varphi]\!] & \text{is the semantic box } l_R([\![\varphi]\!]), \end{array}$$

where, $l_R : \mathcal{P}(W) \to \mathcal{P}(W)$ is defined as $l_R(X) := \{w \in W |$ for all v if wRv then $v \in X\}$, or equivalently, $l_R(X) := W \setminus (R^{-1}(W \setminus X)$. For every formula φ, the n-ary operation $[\![\varphi]\!]$ can be also regarded as a map that takes valuations as arguments and gives subsets of $\mathcal{P}(W)$ as its output. Indeed, for every $\varphi \in$ ML, let

$$[\![\varphi]\!](V) := [\![\varphi]\!](V(p_1), \ldots, V(p_n)). \tag{3.1}$$

Then $[\![\varphi]\!](V)$ is the *extension* of φ under the valuation V, i.e. the set of the states of $(\mathcal{F}, V)$ at which φ is true. Since this happens for all valuations, we can think of $[\![\varphi]\!]$ as the *meaning* function of φ. We can now define the notion of *truth* of a formula φ at a point w in a model $\mathcal{M} = (W, R, V)$, denoted $\mathcal{M}, w \Vdash \varphi$, by

$$\mathcal{M}, w \Vdash \varphi \quad \text{iff} \quad w \in [\![\varphi]\!](V).$$

Similarly, *validity* at a point in a frame is given by

$$\mathcal{F}, w \Vdash \varphi \quad \text{iff} \quad w \in [\![\varphi]\!](V) \text{ for every valuation } V \text{ on } \mathcal{F}.$$

The *global versions* of truth and validity is obtained by quantifying universally over w in the above clauses. Thus we have $\mathcal{M} \Vdash \varphi$ iff $[\![\varphi]\!](V) = W$, and $\mathcal{F}, w \Vdash \varphi$ iff $[\![\varphi]\!](V) = W$ for every valuation V.

3.1.2 Definite implications

Let UF, VSSI, SI and SF, be the class of uniform formulas, very simple Sahlqvist implications, Sahlqvist implications and Sahlqvist formulas, respectively, as defined in Section 2.3. To further set the stage for the treatment in Section 3.2, we will also show that the formulas in these classes can be equivalently rewritten in certain normal forms. The correspondence results for Sahlqvist formulas can be respectively reduced to the correspondence results for Sahlqvist *implications*: this is an immediate consequence of the following proposition:

3.1.1. Proposition. *Every Sahlqvist formula is semantically equivalent to a negated Sahlqvist antecedent, and hence to a Sahlqvist implication.*

Proof. Fix a Sahlqvist formula φ, and let φ' be the formula obtained from $\neg\varphi$ by distributing the negation over all connectives. Since $\varphi \equiv \varphi' \to \bot$, it is enough to show that φ' is a Sahlqvist antecedent, in order to prove the statement. This is done by induction on the construction of φ from Sahlqvist implications. If φ is a Sahlqvist implication $\alpha \to \mathsf{Pos}$, negating and rewriting it as $\alpha \wedge \neg\mathsf{Pos}$ already turns it into a Sahlqvist antecedent. If $\varphi = \Box\psi$, where ψ satisfies the claim, then $\neg\varphi \equiv \Diamond\neg\psi$ hence the claim follows for φ, because Sahlqvist antecedents are closed under diamonds. Likewise, if $\varphi = \psi_1 \wedge \psi_2$, where ψ_1 and ψ_2 satisfy the claim, then $\neg\varphi \equiv \neg\psi_1 \vee \neg\psi_2$ hence the claim follows for φ, because Sahlqvist antecedents are closed under disjunctions. The case of $\varphi = \psi_1 \vee \psi_2$ is completely analogous. $\Box$

In their turn, the latter correspondence results can be respectively reduced to the correspondence results for the subclasses of their *definite implications*. These are defined by forbidding the use of disjunction, except within negative formulas, in the building of antecedents. To be precise:

3.1.2. Definition. A *definite very simple Sahlqvist antecedent* is a formula built up from $\top$, $\bot$, negative formulas and propositional letters, using only $\wedge$ and $\Diamond$. A *definite Sahlqvist antecedent* is a formula built up from $\top$, $\bot$, negative formulas and boxed atoms, using only $\wedge$ and $\Diamond$. Let $\Phi \in \{\mathsf{VSSI}, \mathsf{SI}\}$. Then $\varphi \to \psi \in \Phi$ is a *definite Φ-implication* if φ is a definite Φ-antecedent.

In the next section, we will be able to confine our attention w.l.o.g. to the definite implications in each class, thanks to Fact 3.1.3 and to Proposition 3.1.4 below.

3.1.3. FACT. If $\varphi_i \in \mathrm{ML}$ locally corresponds to $\alpha_i(x) \in L_0$ for $1 \leq i \leq n$, then $\bigwedge_{i=1}^{n} \varphi_i$ locally corresponds to $\bigwedge_{i=1}^{n} \alpha_i(x)$.

3.1.4. PROPOSITION. *Let* $\Phi \in \{\mathsf{VSSI}, \mathsf{SI}\}$. *Every* $\varphi \in \Phi$ *is equivalent to a conjunction of definite implications in* Φ.

Proof. Note that φ can be equivalently rewritten as a conjunction of definite implications in Φ by exhaustively distributing $\Diamond$ and $\wedge$ over $\vee$ in the antecedent, and then applying the equivalence $A \vee B \to C \equiv (A \to C) \wedge (B \to C)$. $\Box$

3.2 Algebraic correspondence

The present section is the heart of the chapter. In it, we will proceed incrementally and give the algebraic correspondence argument for the definite formulas of each class defined in Section 3.1.2.

Methodology: reduction strategies. Both in the model-theoretic and in the algebraic lines of investigation of modal correspondence theory, the stress has shifted somewhat from the quest for new syntactic characterizations and more onto the methodology. Indeed, in the model-theoretic approach, the algorithms provide an effective tool for concretely occurring formulas that can be fed to the algorithm one by one, and the algebraic approach is not linked to any given signature in particular.

Let us now introduce the methodology. The starting point is of course the same: in both approaches, computing the first-order correspondent of a modal formula φ amounts to devising some *reduction strategies* that succeed in eliminating the universal monadic second-order quantification from the clause that expresses the frame validity of φ. From the methodological point of view, the main contribution of the algebraic account of Sahlqvist correspondence theory is that this elimination process can be neatly divided into three stages, that can be then treated independently of one another:

Stage 1. We establish the equivalence between the universal quantification over arbitrary valuations (in the clause that expresses frame validity) and universal quantification over a restricted class of carefully designed valuations; this equivalence will of course not hold in general, but will hold under certain purely order-theoretic conditions on the algebraic operations;

Stage 2. We show that the universal quantification on the designed valuations effectively translates to first-order quantification in the associated frame language;

Stage 3. We formulate the syntactic conditions on the formulas of a given language that will guarantee both the order-theoretic behaviour in Stage 1 and the effective translation in Stage 2.

3.2.1 The general reduction strategy

Our starting point is the well-known fact, already mentioned above, that any modal formula φ *locally* corresponds to its standard second-order translation, i.e.,

$$\mathcal{F}, w \Vdash \varphi \quad \text{iff} \quad \mathcal{F} \models \forall P_1 \dots \forall P_n ST_x(\varphi)[x := w]. \tag{3.2}$$

We are interested in strategies that produce a semantically equivalent first-order condition out of the default local second-order correspondent of φ on the right-hand side of (3.2).

A large and natural class of formulas for which, by definition, this is possible is introduced by van Benthem [12]:

3.2.1. Definition. The class of *van Benthem-formulas*[1] consists of those formulas $\varphi \in$ ML for which $\forall P_1 \dots \forall P_n ST_x(\varphi)$ is equivalent to $\forall P'_1 \dots \forall P'_n ST_x(\varphi)$ where the quantifiers $\forall P'_1 \dots \forall P'_n$ range, not over all subsets of the domain, but only over those that are definable by means of L_0-formulas.

The van Benthem-formulas are the designated targets of the reduction strategy in its most general form. To see this, for every frame $\mathcal{F} = (W, R)$, let

$$\mathsf{Val}_{\mathsf{L_0}}(\mathcal{F}) = \{V' : \mathsf{Prop} \to \mathcal{P}(W) \mid V'(p) \text{ is } L_0\text{-definable for every } p \in \mathsf{Prop}\}.$$

This is the set of the *tame* valuations on $\mathcal{F}$. Using the notation introduced in (3.1), if $\varphi \in$ ML is a van Benthem formula, then the following chain of equivalences holds for every $\mathcal{F}$ and every w:

$$\begin{aligned} \mathcal{F}, w \Vdash \varphi \quad & \text{iff} \quad w \in [\![\varphi]\!](V) \text{ for every } V \text{ on } \mathcal{F} \\ & \text{iff} \quad w \in [\![\varphi]\!](V') \text{ for every } V' \in \mathsf{Val}_{\mathsf{L_0}}(\mathcal{F}). \end{aligned} \tag{3.3}$$

3.2.2. Theorem. *Every van Benthem-formula has a local first-order frame correspondent.*

Proof. Let φ be a van Benthem-formula and let Σ be the set of all L_0 substitution instances of $\mathrm{ST}_x(\varphi)$, i.e. the set of all formulas obtained by substituting L_0-formulas $\alpha(y)$ for occurrences $P(y)$ of predicate symbols in $\mathrm{ST}_x(\varphi)$. Clearly, $\forall \overline{P}\mathrm{ST}_x(\varphi) \models \Sigma[x := w]$, where $\overline{P}$ is the vector of all predicate symbols occurring in $\mathrm{ST}_x(\varphi)$. Also, since φ is a van Benthem-formula, $\Sigma \models \forall \overline{P}\mathrm{ST}_x(\varphi)[x := w]$. Then $\Sigma \models \mathrm{ST}_x(\varphi)[x := w]$, and since this is a first-order consequence, we may

[1] this name first appears in [47].

appeal to the compactness theorem to find some finite subset $\Sigma' \subseteq \Sigma$ such that $\Sigma' \models \mathrm{ST}_x(\varphi)[x := w]$.

We claim that $\Sigma' \models \forall \overline{P}\mathrm{ST}_x(\varphi)[x := w]$. Indeed, let $\mathcal{M}$ be any L_1-model such that $\mathcal{M} \models \Sigma'[x := w]$. Since the predicate symbols in $\overline{P}$ do not occur in Σ', every $\overline{P}$-variant of $\mathcal{M}$ also models Σ', and hence also $\mathrm{ST}_x(\varphi)$. It follows that $\mathcal{M} \models \forall \overline{P}\mathrm{ST}_x(\varphi)[x := w]$. Thus we may take $\bigwedge \Sigma'$ as a local first-order frame correspondent for φ. $\square$

However, relying on compactness, as it does, Theorem 3.2.2 is of little use if we want to explicitly calculate the first-order correspondent for a given $\varphi \in \mathrm{ML}$, or devise an algorithm which produces first-order frame correspondents for each member of a given *class* of modal formulas; therefore a more refined strategy is in order, the development of which is the core of correspondence theory.

Each class of modal formulas of Subsection 3.1.2 is defined so as to guarantee that the second 'iff' (i.e. the non trivial one) of (3.3) can be proved not just for V' ranging arbitrarily over $\mathsf{Val}_{\mathsf{L}_0}(\mathcal{F})$ but rather ranging over a much more restricted and nicely defined subset of it. Moreover, each of these subsets of tame valuations is defined in such a way as to enable the algorithmic generation of the first-order correspondents of the members of the class of formulas it is paired with. More specifically, as we will see next, the following pairings hold between classes of formulas and subsets of tame valuations:

$$\begin{array}{r|l} \mathsf{UF} & V' : \mathsf{Prop} \to \{\varnothing, W\} \\ \mathsf{VSSI} & V' : \mathsf{Prop} \to \mathcal{P}_{fin}(W) \\ \mathsf{SI} & V' : \mathsf{Prop} \to \{R^n[X] \mid n \in \mathbb{N}, X \in \mathcal{P}_{fin}(W)\}, \end{array}$$

where $\mathbb{N}$ denotes the set of natural numbers. So far, our account has been faithful to the textbook exposition, albeit with slightly different notation. The algebraic treatment which we are about to introduce crucially provides an intermediate step which clarifies the textbook account: each class of modal formulas of Section 3.1.2 is defined so as to guarantee that, for every formula φ in the given class, the meaning function $[\![\varphi]\!]$ enjoys certain purely order-theoretic properties that make sure that the second crucial 'iff' can be proved for V' ranging in the corresponding subclass of tame valuations (the definition of which, as we already mentioned, underlies the algorithmic generation of the first-order correspondent of φ). We start to see how this works in the next subsection.

3.2.2 Uniform and Closed formulas

The reduction strategy. Among all the first-order definable valuations V on $\mathcal{F}$, the simplest ones are those which assign W or $\varnothing$ to each propositional variable. Indeed, let V_0 be such a valuation and suppose that the following were equivalent for the modal formula φ:

$$\begin{array}{rll} \mathcal{F}, w \Vdash \varphi & \text{iff} & w \in [\![\varphi]\!](V) \text{ for all } V \text{ on } \mathcal{F} \\ & \text{iff} & w \in [\![\varphi]\!](V_0). \end{array}$$

This would in turn mean that

$$\begin{array}{rl} & \mathcal{F} \models \forall P_1 \ldots \forall P_n ST_x(\varphi)[x := w] \\ \text{iff} & \mathcal{F} \models ST_x(\varphi)[x := w, P_1 := V_0(p_1), \ldots, P_n := V_0(p_n)]. \end{array}$$

Therefore, we could equivalently transform the formula above into a first-order formula by replacing each occurrennce $P_i z$ with either $z \neq z$ if $V_0(p_i) = \varnothing$, or with $z = z$ if $V_0(p_i) = W$. This is enough to effectively generate the first-order correspondent of φ.

Order-theoretic conditions. For which formulas φ would it be possible to implement the reduction strategy outlined above? The answer to this question can be given in purely order-theoretic terms:

3.2.3. Proposition. *Let $\langle X_i, \leq\rangle$, $i = 1, \ldots, n$, and $\langle Y, \leq\rangle$ be posets. Let each X_i have a maximum, $\top_i$, and a minimum, $\bot_i$. Let $f : X_1 \times \cdots \times X_n \longrightarrow Y$. If f is either order preserving or order reversing in each coordinate, then the minimum of f exists and is $f(c_1, \ldots c_n)$, where, for every i, $c_i = \bot_i$ if f is order preserving in the i-th coordinate, and $c_i = \top_i$ if f is order reversing in the i-th coordinate.*

3.2.4. Corollary. For every $\varphi \in$ ML, if $[\![\varphi]\!] : \mathcal{P}(W)^n \to \mathcal{P}(W)$ is order preserving or order reversing in each coordinate, then the following are equivalent:

1. $\forall V[w \in [\![\varphi]\!](V)]$.
2. $w \in [\![\varphi]\!](V_0)$, where $V_0(p_i) = W$ if φ is order reversing in p_i and $V_0(p_i) = \varnothing$ if φ is order preserving in p_i.

Proof. ($1 \Rightarrow 2$) Clear. ($2 \Rightarrow 1$) It follows from Proposition 3.2.3 that $[\![\varphi]\!](V_0)$ is the minimum of $[\![\varphi]\!]$ and hence $[\![\varphi]\!](V_0) \subseteq [\![\varphi]\!](V)$ for every valuation V.

Syntactic conditions. Now that we have the reduction strategy and sufficient order-theoretic conditions for the strategy to apply, it only remains to verify that these conditions are met by the uniform formulas. And indeed, the following proposition can be easily shown by induction on φ:

3.2.5. Proposition. *If $\varphi \in$ ML is a uniform formula, then $[\![\varphi]\!]$ is order preserving (reversing) in those coordinates corresponding to propositional variables in which φ is positive (negative).*

3.2.6. Example. Let us consider the uniform formula $\Box\Diamond p$. The minimal valuation for p is $V_0(p) = \varnothing$, since the formula is positive and hence order-preserving in p. The standard translation of this formula gives

$$\begin{array}{rl} & \mathcal{F} \models \forall P\, \forall y(xRy \to \exists z(yRz \wedge Pz)[x := w] \\ \text{iff} & \mathcal{F} \models\ \forall y(xRy \to \exists z(yRz \wedge P^0 z)[x := w] \end{array}$$

where the predicate $P^0 z$ can be replaced with $z \neq z$ giving a first-order equivalent formula $\forall y(xRy \to \exists z(yRz \wedge z \neq z)$ which simplifies to $\forall y(\neg xRy)$.

To sum up: although the uniform formulas and their accompanying valuations are extremely simple, the key features of our account are already present: first, the subclass of tame valuations is identified, using which the desired first-order correspondent can be effectively computed; second, the order-theoretic properties are highlighted, which guarantee the crucial preservation of equivalence; third, the syntactic specification of the formulas φ of the given class guarantees that their associated meaning functions $[\![\varphi]\!]$ meet the required order-theoretic properties.

Non-uniform formulas and 'minimal valuation' argument. The discussion above also shows that every uniform formula is locally equivalent on frames to some closed formula (which is obtained by replacing every positive variable with $\perp$ and every negative variable with $\top$). This elimination of variables can in fact be applied not only to uniform formulas but also to formulas that are uniform *with respect to some variables*, so as to eliminate those 'uniform' variables separately. Therefore, modulo this elimination, in the following subsections we are going to assume w.l.o.g. that the formulas we consider are non-uniform in each of their variables. Modulo equivalent rewriting, we can assume w.l.o.g. that every such formula is of the form $\varphi \to \psi$, where ψ is positive, and all the variables occurring in ψ also occur in φ. For such formulas, we have:

$$\begin{array}{rll} \mathcal{F}, w \Vdash \varphi \to \psi & \text{iff} & w \in [\![\varphi \to \psi]\!](V) \text{ for all } V \text{ on } \mathcal{F} \\ & \text{iff} & \text{for all } V \text{ on } \mathcal{F}, \text{ if } w \in [\![\varphi]\!](V) \text{ then } w \in [\![\psi]\!](V). \end{array}$$

The textbook heuristics for producing the correspondent of formulas of this form is the 'minimal valuation' method (see Section 2.4): find the (class of) minimal valuation(s) V^* on $\mathcal{F}$ such that $w \in [\![\varphi]\!](V^*)$ (and plug their description in the standard translation of the consequent). The success of this heuristics rests on two conceptually different requirements: first, that 'minimal valuations' exist; second, provided they exists, that they are tame. In the following sections, we present the methodology for correspondence as described in the discussion at the end of Subsection 3.2.1.

3.2.3 Very simple Sahlqvist implications

The reduction strategy. Consider the subclass of the tame valuations which assign finite subsets of some bounded size $m \in \mathbb{N}$ to propositional variables, i.e. valuations $V_1 : \mathsf{Prop} \longrightarrow \mathcal{P}_m(W)$, where

$$\mathcal{P}_m(W) := \{S \subseteq W \mid |S| \leq m\},$$

and suppose the following were equivalent:

1. $\forall V(w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V))$
2. $\forall V_1(w \in [\![\varphi]\!](V_1) \Rightarrow w \in [\![\psi]\!](V_1))$.

This would mean that

$$\begin{array}{rl} & \mathcal{F} \models \forall P_1 \dots \forall P_n ST_x(\varphi \to \psi)[x := w] \\ \text{iff} & \mathcal{F} \models \forall P_1^1 \dots \forall P_n^1 ST_x(\varphi \to \psi)[x := w], \end{array}$$

where the variables P_i^1 would not range over arbitrary subsets of W, but only over those of size at most m. Provided the equivalence between 1 and 2 above holds, we would effectively obtain the local first-order correspondent of $\varphi \to \psi$ by replacing each $\forall P_i^1$ in the prefix with $\forall z_i^1 \forall z_i^2 \dots \forall z_i^m$ and each atomic formula of the form $P_i^1 y$ with $y = z_i^1 \vee y = z_i^2 \vee \dots \vee y = z_i^m$, where all the z's are fresh variables.

Order-theoretic conditions. An operation $f : \mathcal{P}(W)^n \to \mathcal{P}(W)$ is a *complete operator* if f preserves arbitrary joins in each coordinate, i.e., for every $i = 1 \dots n$, every $\mathcal{X} \subseteq \mathcal{P}(W)$, and all $X_1, \dots, X_{i-1}, X_{i+1}, \dots, X_n \in \mathcal{P}(W)$,

$$\begin{array}{rl} & f(X_1, \dots, X_{i-1}, \bigcup \mathcal{X}, X_{i+1}, \dots, X_n) \\ = & \bigcup_{Y \in \mathcal{X}} f(X_1, \dots, X_{i-1}, Y, X_{i+1}, \dots, X_n). \end{array} \tag{3.4}$$

This implies in particular that for all $X_1, \dots, X_n \in \mathcal{P}(W)$,

$$\text{if } X_i = \varnothing \text{ for some } i \in \{1, \dots, n\}, \text{ then } f(X_1, \dots, X_n) = \varnothing. \tag{3.5}$$

Recall that any complete operator is order preserving in each coordinate.

The composition of complete operators will be important for our account: in order to describe their order-theoretic properties, the following definition will be useful.

3.2.7. Definition. Let $g : \mathcal{P}(W)^n \to \mathcal{P}(W)$ be a composition of complete operators.

1. The *degree of g in the ith coordinate*, notation δ_g^i, is defined by induction on g:
   (a) If g is itself a complete operator, then $\delta_g^i = 1$ for every coordinate $1 \leq i \leq n$ whose corresponding variable occurs in g, and $\delta_g^i = 0$ otherwise;
   (b) If $g = f(h_1, h_2, \dots, h_m)$ for some complete operator f and compositions of complete operators, $h_1, \dots, h_m$, then $\delta_g^i = \delta_{h_1}^i + \dots + \delta_{h_m}^i$.
2. The *degree of g*, notation δ_g, is $max\{\delta_g^i \mid 1 \leq i \leq n\}$.

3.2.8. LEMMA. *If $g : \mathcal{P}(W)^n \to \mathcal{P}(W)$ is a composition of complete operators, then*

1. *g is order preserving in each coordinate, and*

2. *for all $X_1, \dots X_n \in \mathcal{P}(W)$, if $X_i = \varnothing$ for some $1 \leq i \leq n$ whose corresponding variable occurs in g, then $g(X_1, \dots, X_n) = \varnothing$.*

Proof. 1. Every complete operator is order preserving and the composition of order preserving maps is order preserving.
2. By induction on δ_g. □

The composition of *unary* complete operators yields complete operators, but that this is not generally the case for non-unary complete operators:

3.2.9. EXAMPLE. Consider the extension map $[\![\varphi]\!]$ for the very simple Sahlqvist antecedent $\varphi(p) = \Diamond p \wedge \Diamond\Diamond p$, defined on the complex algebra of the frame $\mathcal{F} = (W, R)$ with $W = \{w, v, u\}$ and $R = \{(w, v), (v, u)\}$. Then,

$$\begin{array}{rcl}
\varphi(\{v\} \cup \{u\}) & = & R^{-1}(\{v, u\}) \cap R^{-1}(R^{-1}(\{v, u\})) \\
& = & \{w, v\} \cap \{w\} \\
& = & \{w\}. \\
\varphi(\{v\}) \cup \varphi(\{u\}) & = & (R^{-1}(\{v\}) \cap R^{-1}(R^{-1}(\{v\}))) \cup R^{-1}(\{u\}) \cap R^{-1}(R^{-1}(\{u\})) \\
& = & (\{w\} \cap \varnothing) \cup (\{v\} \cap \{w\}) \\
& = & \varnothing \cup \varnothing = \varnothing.
\end{array}$$

However, compositions of complete operators do retain a certain semblance of the join-preservation of the operators from which they are built, as the next lemma shows. We will write $Y \subseteq_k X$ or $Y \in \mathcal{P}_k(X)$ to indicate that $Y \subseteq X$ and $|Y| \leq k$, for $k \in \mathbb{N}$.

3.2.10. LEMMA. *If $g : \mathcal{P}(W)^n \to \mathcal{P}(W)$ is a composition of complete operators, and $X_1, \dots, X_n \in \mathcal{P}(W)$, then*

$$g(X_1, \dots, X_n) = \bigcup \{g(S_1, \dots, S_n) \mid S_i \subseteq_{\delta_g^i} X_i,\ 1 \leq i \leq n\}.$$

Proof. By induction on the degree of g. If $\delta_g = 1$, then g is a complete operator f and hence

$$\begin{array}{rcl}
f(X_1, \dots, X_n) & = & f(\bigcup_{x_1 \in X_1} \{x_1\}, \dots, \bigcup_{x_n \in X_n} \{x_n\}) \\
& = & \bigcup \{f(\{x_1\}, \dots, \{x_n\}) \mid \{x_i\} \subseteq_1 X_i,\ 1 \leq i \leq n\}.
\end{array}$$

If $\delta_g > 1$, then g is of the form $f(h_1, \ldots, h_m)$ where f is a complete operator and each h_i is a composition of complete operators. Then

$$\begin{aligned}
g(X_1, \ldots, X_n) &= f(h_1(X_1, \ldots, X_n), \ldots, h_m(X_1, \ldots, X_n)) \\
&= f(\bigcup\{h_i(S_1^i, \ldots, S_n^i) \mid S_j^i \subseteq_{\delta^j_{h_i}} X_j,\ 1 \leq j \leq n\})_{i=1}^m \\
&= \bigcup\{f(h_i(S_1^i, \ldots, S_n^i))_{i=1}^m \mid S_j^i \subseteq_{\delta^j_{h_i}} X_j,\ 1 \leq j \leq n\} \\
&\subseteq \bigcup\{f(h_i(S_1, \ldots, S_n))_{i=1}^m \mid S_j \subseteq_{\delta^j_{h_1}+\cdots+\delta^j_{h_m}} X_j,\ 1 \leq j \leq n\} \\
&= \bigcup\{g(S_1, \ldots, S_n) \mid S_j \subseteq_{\delta^j_g} X_j,\ 1 \leq j \leq n\}.
\end{aligned}$$

Here the second equality holds by the inductive hypothesis, and the third since f is a complete operator. The inclusion holds since the set of which the union is taken in the third line is a subset of the corresponding set in the fourth line. The last equality holds by the assumptions on g and by definition of δ_g.
The converse inclusion follows from g being order preserving (Lemma 3.2.8). $\square$

Consider the following conditions on $[\![\varphi]\!]$:

(a) $[\![\varphi(p_1, \ldots, p_n)]\!] = [\![\varphi']\!](p_1, \ldots, p_n, [\![\gamma_1]\!], \ldots, [\![\gamma_\ell]\!])$, where

(b) $[\![\varphi'(p_1, \ldots, p_n, s_1, \ldots, s_l)]\!]$ is a composition of complete operators on $\mathcal{P}(W)$,

(c) $[\![\gamma_1]\!]$ to $[\![\gamma_\ell]\!]$ are order reversing in each coordinate.

For every frame $\mathcal{F}$ and every $m \in \mathbb{N}$, let $\mathsf{Val}_1(\mathcal{F})$ be the set of valuations on $\mathcal{F}$ of type $V_1 : \mathsf{Prop} \to \mathcal{P}_m(W)$.

3.2.11. Proposition. *Let $\varphi \to \psi \in$ ML be such that $[\![\varphi]\!]$ verifies the conditions (a)-(c) above and $[\![\psi]\!]$ is order-preserving. Let $m = max_{i=1}^n m_i$ where m_i is the degree of $[\![\varphi]\!]$ relative to its ith coordinate. Then the following are equivalent for every frame $\mathcal{F}$:*

1. $(\forall V \in \mathsf{Val}(\mathcal{F}))[w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V)]$

2. $(\forall V_1 \in \mathsf{Val}_1(\mathcal{F}))[w \in [\![\varphi]\!](V_1) \Rightarrow w \in [\![\psi]\!](V_1)]$.

Proof. ($1 \Rightarrow 2$) Clear. ($2 \Rightarrow 1$) Let m_i be the degree of $[\![\varphi]\!]$ in the i-th coordinate, for $1 \leq i \leq n$. Fix V and let $w \in [\![\varphi]\!](V)$. Hence,

$$\varnothing \neq [\![\varphi]\!](V) = [\![\varphi']\!](V(p_1), \ldots V(p_n), [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)).$$

By Lemma 3.2.8(2), this implies that $V(p_i) \neq \varnothing$ for every $i = 1, \ldots, n$. By Lemma 3.2.10,

$$[\![\varphi]\!](V) = \bigcup\{[\![\varphi']\!](S_1, \ldots, S_n, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \mid S_i \subseteq_{m_i} V(p_i),\ 1 \leq i \leq n\}.$$

Hence, $w \in [\![\varphi]\!](V)$ implies that $w \in [\![\varphi']\!](T_1, \ldots, T_n, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V))$ for some $T_i \subseteq_{m_i} V(p_i)$, $1 \leq i \leq n$. Let V_1 be the valuation that maps any $q \in \mathsf{Prop} \setminus \{p_i \mid 1 \leq i \leq n\}$ to $\varnothing$ and such that $V_1(p_i) = T_i$: clearly, $V_1 \in \mathsf{Val}_1(\mathcal{F})$; moreover $w \in [\![\varphi]\!](V_1)$: indeed,

$$\begin{array}{rcl} w & \in & [\![\varphi']\!](T_1, \ldots, T_n, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \\ & \subseteq & [\![\varphi']\!](T_1, \ldots, T_n, [\![\gamma_1]\!](V_1), \ldots, [\![\gamma_\ell]\!](V_1)) \\ & = & [\![\varphi']\!](V_1(p_1), \ldots, V_1(p_n), [\![\gamma_1]\!](V_1), \ldots, [\![\gamma_\ell]\!](V_1)) \\ & = & [\![\varphi]\!](V_1); \end{array} \tag{3.6}$$

the inclusion in the chain above follows since $V_1(p) \subseteq V(p)$ for every $p \in \mathsf{Prop}$ and the extensions of the γ's are reversing. Hence, by assumption (2), $w \in [\![\psi]\!](V_1)$. Since $[\![\psi]\!]$ is order preserving in every coordinate, and again $V_1(p) \subseteq V(p)$ for every $p \in \mathsf{Prop}$, we get $w \in [\![\psi]\!](V_1) \subseteq [\![\psi]\!](V)$, which concludes the proof. □

Syntactic conditions. It remains to verify that the very simple Sahlqvist implications verify the assumptions of Proposition 3.2.11. The assumptions on ψ are verified because of Proposition 3.2.5. As to the assumptions on φ:

3.2.12. PROPOSITION. *If $\varphi = \varphi(p_1, \ldots, p_n)$ is a very simple Sahlqvist antecedent then it verifies the assumptions (a)-(c) of Proposition 3.2.11. In particular, the maps $[\![\gamma]\!]$'s are exactly the ones induced by the negative formulas occurring in the construction of φ, the map $[\![\varphi']\!]$ is induced by the compound occurrences of $\wedge$ and $\Diamond$, and for every $1 \leq i \leq n$, the degree of $[\![\varphi']\!]$ in the ith coordinate is the number of positive occurrences of p_i in φ.*

Proof. For the first part, note that the identity map $id : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$, the intersection $\cap : \mathcal{P}(W) \times \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ ($\langle X, Y \rangle \mapsto X \cap Y$) and the semantic diamond operations $R^{-1} : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ ($X \mapsto R^{-1}(X)$) are complete operators. The second part is proven by induction on φ. □

3.2.4 Sahlqvist implications

The reduction strategy. Another promising subclass of tame valuations is formed by those $V_2 \in \mathsf{Val}(\mathcal{F})$ such that for every $p \in \mathsf{Prop}$, $V_2(p) = R[z]$ for some $z \in W$. Indeed, suppose that the following were equivalent:

1. $\forall V(w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V))$

2. $\forall V_2(w \in [\![\varphi]\!](V_2) \Rightarrow w \in [\![\psi]\!](V_2))$.

This would mean that

$$\mathcal{F} \models \forall P_1 \ldots \forall P_n ST_x(\varphi \to \psi)[\![w]\!] \quad \text{iff} \quad \mathcal{F} \models \forall P_1^2 \ldots \forall P_n^2 ST_x(\varphi \to \psi)[\![w]\!]$$

where the variables P_i^2 would not range over $\mathcal{P}(W)$, but only over $\{R[z] \mid z \in W\}$. Therefore, the formula above on the right-hand side can be transformed into a first-order formula by replacing each $\forall P_i^2$ in the prefix with $\forall z_i$, and each atomic formula of the form $P_i^2 y$ with $z_i Ry \vee z_i^1 Ry \vee \ldots \vee z_i^m Ry$, where all the z's are fresh variables.

Actually, this argument can be refined and extended to valuations V_2 such that for every $p \in \mathsf{Prop}$, $V_2(p) = R^k[z]$ for some $z \in W$, and some $k \in \mathbb{N}$ relation on W (Notice that the valuations V_1 ranging over singletons are the special case of V_2 where $k = 0$). In this case, the formula above on the right-hand side can be equivalently transformed into a first-order formula by replacing each $\forall P_i^2$ in the prefix with $\forall z_i$, and each formula of the form $P_i^2 y$ with an L_0-formula which says 'there exists an R-path from z_i to y in k_i steps', such as:

$$\exists v_0, \ldots v_{k_i}[(z_i = v_0 \wedge \bigwedge_{j=0}^{k_i-1} v_j R v_{j+1} \wedge v_{k_i} = y]$$

This time we are after some conditions on φ and ψ that guarantee that the universal quantification $\forall V$ can be equivalently replaced with the universal quantification $\forall V_2$.

Order-theoretic conditions. The maps $f, g : \mathcal{P}(W) \to \mathcal{P}(W)$ form an *adjoint pair*[2] (notation: $f \dashv g$) iff for every $X, Y \in \mathcal{P}(W)$,

$$f(X) \subseteq Y \ \text{ iff } \ X \subseteq g(Y).$$

Whenever $f \dashv g$, f is the *left adjoint* of g and g is *the right adjoint* of f. One important property of adjoint pairs of maps is that if a map admits a left (resp. right) adjoint, the adjoint is unique and can be computed pointwise from the map itself and the order (which in our case is the inclusion). This means that admitting a left (resp. right) adjoint is an *intrinsically* order-theoretic property of maps.

3.2.13. PROPOSITION. *1. Right adjoints between complete lattices are exactly the completely meet-preserving maps, i.e., in the concrete case of powerset algebras $\mathcal{P}(W)$ they are exactly those maps g such that $g(\bigcap S) = \bigcap\{g(X) \mid X \in S\}$ for all $S \subseteq \mathcal{P}(W)$;*

[2] For a more general definition see Section 4.1.6

2. *Right adjoints on a powerset algebra $\mathcal{P}(W)$ are exactly maps of the form $l_{\mathcal{S}}$ for some binary relation $\mathcal{S}$ on W.*

3. *For any binary relation $\mathcal{S}$ on W, the left adjoint of $l_{\mathcal{S}}$ is the map $m_{\mathcal{S}^{-1}}$, defined by the assignment $X \mapsto \mathcal{S}[X]$.*

Proof. 1. See [56, Proposition 7.34]. 2. We leave to the reader to verify that every map of the form $l_{\mathcal{S}}$ is completely meet preserving, hence it is a right adjoint. Conversely, let $g : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ be a right adjoint. Then by item 1 above, g is completely meet preserving. Define $\mathcal{S} \subseteq W \times W$ as follows: for every $x, z \in W$,

$$x\mathcal{S}z \quad \text{iff} \quad x \notin g(W \setminus \{z\}).$$

We claim that $g = l_{\mathcal{S}}$. Hence,

$$x \in l_{\mathcal{S}}(W \setminus \{z\}) \text{ iff } \mathcal{S}[x] \subseteq (W \setminus \{z\}) \text{ iff } z \notin \mathcal{S}[x] \text{ iff } x \in g(W \setminus \{z\}),$$

which shows our claim for all the special subsets of W of type $W \setminus \{z\}$. In order to show it in general, fix $X \in \mathcal{P}(W)$ and notice that $X = \bigcap_{z \notin X}(W \setminus \{z\})$. Using the fact that g is completely meet preserving and the special case shown above, we get:

$$\begin{array}{rcl}
g(X) & = & g(\bigcap\{(W \setminus \{z\}) \mid z \notin X\}) \\
& = & \bigcap\{g(W \setminus \{z\}) \mid z \notin X\} \\
& = & \bigcap\{l_{\mathcal{S}}(W \setminus \{z\}) \mid z \notin X\} \\
& (*) & \\
& = & l_{\mathcal{S}}(\bigcap\{(W \setminus \{z\}) \mid z \notin X\} \\
& = & l_{\mathcal{S}}(X).
\end{array}$$

The marked equality can be verified directly, but also follows from the more general fact that $l_{\mathcal{S}}$ is completely meet preserving for every $\mathcal{S}$.
3. Left to the reader. □

Consider the following conditions on $[\![\varphi]\!]$:

(a) $\varphi(p_1, \ldots, p_n) = \varphi'(\chi_1(p_1), \ldots, \chi_n(p_n), \gamma_1, \ldots, \gamma_\ell)$, moreover

(b) $[\![\varphi']\!]$ is a complete operator;

(c) for $1 \leq i \leq n$, $[\![\chi_i]\!] : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ is a right adjoint, i.e. there exists some $f_i : \mathcal{P}(W) \longrightarrow \mathcal{P}(W)$ such that for every $X, Y \in \mathcal{P}(W)$, $f_i(X) \subseteq Y$ iff $X \subseteq [\![\chi_i]\!](Y)$;

(d) for every $1 \leq i \leq n$, f_i is defined by $X \mapsto R^{k_i}[X]$

(e) for every $1 \leq j \leq \ell$, $[\![\gamma_j]\!]$ is order reversing in each coordinate.

Notice that, by Proposition 3.2.13, condition (c) already guarantees that for every i, f_i is defined by $X \mapsto \mathcal{S}_i[X]$ for some *arbitrary* binary relation $\mathcal{S}_i$ on W; however, since $\mathcal{S}_i$ is arbitrary, this is not yet enough to guarantee that valuations defined by $p_i \mapsto f_i(\{z_i\})$ be L_0-definable. Condition (d) above guarantees this last point.

For every frame $\mathcal{F}$, let $\mathsf{Val}_2(\mathcal{F})$ be the set of valuations on $\mathcal{F}$ such that, for every $p \in \mathsf{Prop}$, $V_2(p) = R^k[x]$ for some $x \in W$ and some $k \in \mathbb{N}$.

3.2.14. PROPOSITION. *Let $\varphi \to \psi \in$ ML be such that $[\![\varphi]\!]$ verifies the conditions (a)-(d) above and $[\![\psi]\!]$ is order preserving in each coordinate. Then the following are equivalent:*

1. $(\forall V \in \mathsf{Val}(\mathcal{F}))[w \in [\![\varphi]\!](V) \Rightarrow w \in [\![\psi]\!](V)]$

2. $(\forall V_2 \in \mathsf{Val}_2(\mathcal{F}))[w \in [\![\varphi]\!](V_2) \Rightarrow w \in [\![\psi]\!](V_2)]$.

Proof. $(1 \Rightarrow 2)$ Clear.
$(2 \Rightarrow 1)$ Fix V and let $w \in [\![\varphi]\!](V)$. Hence,

$$\varnothing \neq [\![\varphi]\!](V) = [\![\varphi']\!]([\![\chi_1]\!](V(p_1)), \ldots, [\![\chi_n]\!](V(p_n)), [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)),$$

and since by assumption (b) $[\![\varphi']\!]$ is a complete operator, $[\![\chi_i]\!](V(p_i)) \neq \varnothing$ for every $1 \leq i \leq n$. Moreover, because every set is the union of the singletons of its elements and complete operators preserve arbitrary unions in each coordinate, the following chain of equalities holds:

$$\begin{aligned} & w \in [\![\varphi]\!](V) \\ & \quad = [\![\varphi']\!]([\![\chi_1]\!](V(p_1)), \ldots, [\![\chi_n]\!](V(p_n)), [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \\ & \quad = \bigcup \{[\![\varphi']\!](\{x_1\}, \ldots, \{x_n\}, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \mid \{x_i\} \subseteq [\![\chi_i]\!](V(p_i))\}_{i=1}^n \\ & \quad = \bigcup \{[\![\varphi']\!](\{x_1\}, \ldots, \{x_n\}, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \mid f_i(\{x_i\}) \subseteq V(p_i)\}_{i=1}^n \end{aligned}$$

where the last equality is a consequence of assumption (c). Then
$w \in [\![\varphi']\!](\{z_1\}, \ldots, \{z_n\}, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)))$ for some $z_i \in W$, $1 \leq i \leq n$, such that $f_i(\{z_i\}) \subseteq V(p_i)$. Let V_2 be the valuation that maps any $q \in \mathsf{Prop} \setminus \{p_i \mid 1 \leq i \leq n\}$ to $\varnothing$ and such that $V_2(p_i) = f_i(\{z_i\})$. By assumption (d), $V_2 \in \mathsf{Val}_2(\mathcal{F})$.

Let us show that $w \in [\![\varphi]\!](V_2)$: indeed,

$$
\begin{aligned}
w &\in [\![\varphi']\!](\{z_1\}, \ldots, \{z_n\}, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \\
&\subseteq \bigcup\{[\![\varphi']\!](\{x_1\}, \ldots, \{x_n\}, [\![\gamma_1]\!](V), \ldots, [\![\gamma_\ell]\!](V)) \mid \mathcal{F}_i(\{x_i\}) \subseteq V_2(p_i)\}_{i=1}^n \\
&\subseteq \bigcup\{[\![\varphi']\!](\{x_1\}, \ldots, \{x_n\}, [\![\gamma_1]\!](V_2), \ldots, [\![\gamma_\ell]\!](V_2)) \mid f_i(\{x_i\}) \subseteq V_2(p_i)\}_{i=1}^n \\
&= \bigcup\{[\![\varphi']\!](\{x_1\}, \ldots, \{x_n\}, [\![\gamma_1]\!](V_2), \ldots, [\![\gamma_\ell]\!](V_2)) \mid \{x_i\} \subseteq [\![\chi_i]\!](V_2(p_i))\}_{i=1}^n \\
&= [\![\varphi']\!]([\![\chi_1]\!](V_2(p_1)), \ldots, [\![\chi_n]\!](V_2(p_n)), [\![\gamma_1]\!](V_2), \ldots, [\![\gamma_\ell]\!](V_2)) \\
&= [\![\varphi]\!](V_2).
\end{aligned}
$$

The second inclusion holds since $V_2(p) \subseteq V(p)$ for every $p \in \mathsf{Prop}$ and the $[\![\gamma]\!]'s$ are order reversing. By assumption (2), we can conclude that $w \in [\![\psi]\!](V_2)$. Now, since $[\![\psi]\!]$ is order preserving in every coordinate, and $V_2(p_i) = f_i(\{z_i\}) \subseteq V(p_i)$ for every $1 \leq i \leq n$, we get $w \in [\![\psi]\!](V_2) \subseteq [\![\psi]\!](V)$, which concludes the proof. $\square$

Syntactic conditions.

3.2.15. PROPOSITION. *If $\varphi \to \psi \in$ ML is a Sahlqvist implication, then $\varphi \to \psi$ verifies the hypotheses of Proposition 3.2.14. In particular, the maps $[\![\chi_i]\!]$ are exactly the ones induced by the boxed atoms.*

Proof. It follows from Propositions 3.2.12 and 3.2.13, and the additional fact that, for every $R_1, R_2 \subseteq W \times W$, $l_{R_2} \circ l_{R_1} = l_{R_1 \circ R_2}$. $\square$

3.2.16. EXAMPLE. The (definite) Sahlqvist implication $\Diamond\Box p \wedge \Box q \to \Box\Diamond(p \wedge q)$ has the following standard second-order frame equivalent

$$\forall P \forall Q[(\exists y(xRy \wedge \forall u(yRu \to P(u))) \wedge \forall v(xRv \to Q(v))) \to$$

$$\forall w(xRw \to \exists s(Rws \wedge P(s) \wedge Q(s)))].$$

The reduction strategy prescribes that in the above we replace $\forall P \forall Q$ with $\forall z_1 \forall z_2$, and substitute $P(y)$ and $Q(y)$ with $\exists v_0 \exists v_1(v_0 = z_1 \wedge v_0 R v_1 \wedge v_1 = y)$ and $\exists v_0 \exists v_1(v_0 = z_2 \wedge v_0 R v_1 \wedge v_1 = y)$, respectively, which simplify to $z_1 R y$ and $z_2 R y$, respectively. Doing this we obtain the first-order frame equivalent

$$\forall z_1 \forall z_2[(\exists y(xRy \wedge \forall u(yRu \to z_1 R u)) \wedge \forall v(xRv \to z_2 R v)) \to$$

$$\forall w(xRw \to \exists s(Rws \wedge z_1 R s \wedge z_2 R s))].$$

Using the well-known fact that for any first-order formula $\beta(x, y)$ it holds that $\forall x \forall y \beta(x, y) \models \forall x \forall x \beta(x, x)$, we see (by pulling out quantifiers and setting $z_1 = y$ and $z_2 = x$) that the above has as consequence

$$\forall y \forall w(xRy \wedge xRw \rightarrow \exists s(xRs \wedge yRs \wedge wRs)).$$

An easy semantic argument shows that the converse also holds, and hence that the last formula is actually a local first-order frame correspondent for $\Diamond\Box p \wedge \Box q \rightarrow \Box\Diamond(p \wedge q)$.

3.3 Conclusions

In this chapter, the Sahlqvist-style syntactic identification of classes of modal formulas that are endowed with local first-order correspondents has been explained in terms of certain order-theoretic properties of the extension maps corresponding to the formulas of these classes. Our treatment is modular: in particular, we neatly divided the correspondence proof for each class of formulas in three stages. Although, for simplicity, we confined our treatment to the basic modal signature, the most important stage, i.e. the one referred to as 'order-theoretic conditions' is intrinsically independent of any algebraic signature. Therefore, it can be applied to any one, and in particular to any modal signature.

The statements and proofs about the order-theoretic conditions only use the following two features of powerset algebras: that they are complete distributive lattices and that they are completely join-generated by their completely join prime elements[3] (the singleton subsets). Therefore, these proofs go through virtually unchanged in the more general setting of distributive lattices enjoying these two properties.

Although it was not strictly needed for our exposition, in this chapter correspondence is regarded as a by-product of the duality between Kripke frames and complete and atomic BAO's. For instance, results such as Proposition 3.2.13 are essentially characterizations of objects across a duality. Moreover, in general, the relational interpretation of modal logic can be obtained by dualizing its canonical algebraic interpretation on BAO's. This modus operandi is not confined to modal logic: any duality involving the class of algebras canonically associated with a given propositional logic provides the appropriate setting for correspondence results.

[3] An element $c \neq \bot$ of a complete lattice L is *completely join prime* if, for every $S \subseteq L$, $c \leq s$ for some $s \in S$ whenever $c \leq \bigvee S$.

Chapter 4

Algorithmic correspondence and canonicity for regular modal logic

In the present chapter, which is a revised version of [130], we apply the unified correspondence approach to obtain Sahlqvist-type correspondence and canonicity results about regular modal logic on classical, intuitionistic and distributive-lattice propositional bases.

The formalization of situations in which logical impossibilities are thinkable and sometimes even believable has been a key topic in modal logic since its onset, and has attracted the interest of various communities of logicians over the years. This specific imperfection of cognitive agency can be directly translated in the language of modal logic by stipulating that, for a given agent a, the formula $\Diamond_a \bot$ is not a contradiction, and hence, that the necessitation rule is not admissible. Impossible worlds have been introduced by Kripke in [111] in the context of his relational semantic account of modal logics, as an elegant way to invalidate the necessitation rule while retaining all other axioms and rules of normal modal logic, and hence to provide complete semantics for important non-normal modal logics such as Lemmon's systems E2-E5 [115]. More recently, impossible worlds have been used in close connection with counterfactual reasoning, paraconsistency (e.g. to model inconsistent databases, cf. [8]). The reader is referred to [125] for a comprehensive survey on impossible worlds.

The logics E2-E5 mentioned above are prominent examples of regular modal logics, which are classical modal logics (cf. [42]) in which the necessitation rule is not valid (equivalently, modal logics that do not contain $\Box\top$ as an axiom) but such that $\Box$ distributes over conjunction. Arguably, their lacking necessitation makes regular modal logics better suited than normal modal logics at the formalization of epistemic and deontic settings. To briefly expand on the type of objections against normality raised in these settings, we mention Lemmon's argument in [115], the same paper in which the systems E2-E5 have been introduced together with other logics. The rule of necessitation is not included in them since it causes the presence in the logic of theorems of the form $\Box\varphi$. In the context of the interpretation of the $\Box$-operator as moral obligation or scientific but not logical necessity, Lemmon's systems are in line with the view that *nothing should be a scientific law or a moral obligation as a matter of logic.*

Notwithstanding the fact that the two variants of Kripke relational models (namely with and without impossible worlds) appeared almost at the same time, the state of development of their mathematical theory is not the same. In particular, although unsystematic correspondence results exist (viz. the ones in [111]), no Sahlqvist-type results are available.

We introduce an adaptation, referred to as ALBA^r, of the calculus ALBA to regular modal logic (on weaker than classical bases). We define the class of inductive inequalities in the regular setting. Again, this definition follows the principles of unified correspondence, and is given in terms of the order-theoretic properties of the algebraic interpretations of the logical connectives. Similar to the inductive inequalities defined in other settings, inductive DLR-inequalities properly and significantly extend Sahlqvist inequalities, while sharing their most important properties, namely the fact that the (regular) modal logics generated by them are strongly complete w.r.t. the class of Kripke frames defined by their first-order correspondent.

The chapter is organized as follows: in Section 4.1 we introduce the necessary preliminaries on impossible worlds and non-normal modal logics. In Section 4.2, we illustrate the algebraic-algorithmic correspondence mechanism. In Section 4.2.1, we introduce the calculus ALBA^r for regular modal logic, and inductive DLR-inequalities. Finally, in Section 4.5 the strong completeness of Lemmon's logics E2-E5 w.r.t. elementary classes of Kripke frames with impossible worlds is obtained as a consequence of the theory developed, and the defining first-order conditions are effectively computed via ALBA^r. In addition to the above contents, the first part of the paper [130], on which this chapter is based on, studies Jónsson-style canonicity, and shows that Jónsson's strategy for proving canonicity goes through all the same under weaker assumptions of the modal operators being regular.

4.1 Preliminaries

In this section, we collect preliminaries on Kripke frames with impossible worlds and their complex algebras. We also report on historically important examples of non-normal modal logics, namely Lemmon's E2-E4, which have been given a semantic interpretation in terms of Kripke frames with impossible worlds. We briefly mention how the discrete duality on objects between usual Kripke frames and Boolean algebras with operators can be extended to Kripke frames with impossible worlds and perfect regular Boolean algebra expansions (r-BAEs, cf. Definition 4.1.7). Finally, we outline the generalization of this discrete duality to the distributive lattice-based counterparts of r-BAEs and the poset-based counterparts of Kripke frames with impossible worlds.

4.1.1 Regular modal logics

Classical modal logics (cf. [42], Definition 8.1, and [139]) are weaker than normal modal logics, and are only required to contain the axiom $\Diamond A \leftrightarrow \neg\Box\neg A$ and be closed under the following rule:

$$\text{(RE)}\ \frac{A \leftrightarrow B}{\Box A \leftrightarrow \Box B}$$

Monotonic modal logics are required to contain the axiom above and be closed under the following rule:

$$\text{(RM)}\ \frac{A \rightarrow B}{\Box A \rightarrow \Box B}$$

Regular modal logics (cf. [42], Definition 8.8) are required to contain the axiom above and be closed under the following rule:

$$\text{(RR)}\ \frac{(A \wedge B) \rightarrow C}{(\Box A \wedge \Box B) \rightarrow \Box C}$$

Notice that (RE) can be derived from either (RR) or (RM), and hence both monotonic modal logics and regular modal logics are classical. Notice also that (RR) can be derived from (RM) in the presence of the axiom $(\Box p \wedge \Box q) \rightarrow \Box(p \wedge q)$. Hence, regular modal logics can be equivalently defined as monotonic modal logics which contain the axiom above.

In what follows, we will consider as running examples some historically important modal logics which have been given a semantic interpretation in terms of Kripke frames with impossible worlds. These are Lemmon's logics E2-E5.

Lemmon's epistemic logics. In [115], the systems E1-E5 have been introduced as epistemic counterparts of Lewis' modal systems S1-S5. The E-systems E2-E4 are examples of regular but not normal modal logics, while E5 turns out to coincide with the normal modal logic S5 (as mentioned in [111, p. 209]). A semantic proof of this fact is given in Section 4.5. With the E-systems, Lemmon intended to capture the principle that "nothing is a scientific law as a matter of logic". In particular, Lemmon argues that the necessitation rule is not plausible if the modal operator $\Box$ is to be interpreted as "scientific but not logical necessity". The following axioms and rules are reported here with the same names as in [115]. The notation has been changed to the currently standard one.

Axioms:

(P)	Propositional tautologies		
(1)	$\Box(p \to q) \to \Box(\Box p \to \Box q)$	(1')	$\Box(p \to q) \to (\Box p \to \Box q)$
(2)	$\Box p \to p$		
(4)	$\Box p \to \Box\Box p$	(5)	$\neg\Box p \to \Box\neg\Box p$.

Rules:

(PCa) If α is a propositional tautology, then $\vdash \alpha$.

(PCb) Substitution for proposition variables.

(PCc) Modus Ponens.

(Eb) If $\vdash \alpha \to \beta$, then $\vdash \Box\alpha \to \Box\beta$.

The logics E2-E5 are defined as follows:

PC: (P)+(PCa)+(PCb)+(PCc)	
E2: PC+(Eb)+(1')+(2)	E3: PC+(Eb)+(1)+(2)
E4: E2+(4)	E5: E2+(5)

4.1.1. FACT. In the presence of *(Eb)* and *(1')*, we have $\vdash \Box(\alpha \wedge \beta) \leftrightarrow \Box\alpha \wedge \Box\beta$.

Proof. The left-to-right implication is obtained by applying (Eb) to the propositional tautologies $\alpha \wedge \beta \to \alpha$ and $\alpha \wedge \beta \to \beta$, which yields $\vdash \Box(\alpha \wedge \beta) \to \Box\alpha$ and $\vdash \Box(\alpha \wedge \beta) \to \Box\beta$. From these, PC derives $\vdash \Box(\alpha \wedge \beta) \to \Box\alpha \wedge \Box\beta$.

As to the right-to-left implication, it is enough to show that $\vdash (\Box\alpha) \to (\Box\beta \to \Box(\alpha \wedge \beta))$. Indeed:

$$\vdash^{(1)} (\Box\alpha) \to (\Box(\beta \to (\alpha \wedge \beta))).$$

The entailment $\vdash^{(1)}$ can be derived by applying (Eb) to the propositional tautology $\vdash \alpha \to (\beta \to (\alpha \wedge \beta))$. Finally, using a suitable instantiation of (1'), we obtain

$$\vdash^{(2)} \Box(\beta \to (\alpha \wedge \beta)) \to (\Box\beta \to \Box(\alpha \wedge \beta)).$$

$\Box$

4.1.2. COROLLARY. *Lemmon's logics E2-E5 are regular.*

Proof. Notice preliminarily that (1) and (2) entail $(1')$. Indeed,

$$\vdash^{(1)} \Box(\alpha \to \beta) \to \Box(\Box\alpha \to \Box\beta).$$

The entailment $\vdash^{(1)}$ immediately follows from the assumption (1). Using a suitable instantiation of (2), we obtain

$$\vdash^{(2)} \Box(\Box\alpha \to \Box\beta) \to \Box\alpha \to \Box\beta.$$

Hence, the regularity of E2-E5 immediately follows from Fact 4.1.1. $\Box$

In [111], Kripke proved the completeness of E2 and E3 w.r.t. certain Kripke frames with impossible worlds, and stated that E5 coincides with the well-known logic S5. In Section 4.5, we are going to obtain these and other results as instances of correspondence theory for Kripke frames with impossible worlds, which are defined in the following subsection.

4.1.2 Kripke frames with impossible worlds and their complex algebras

4.1.3. Definition. [cf. [111, 20]] A *Kripke frame with impossible worlds* is a triple $\mathbb{F} = (W, S, N)$ such that $W \neq \varnothing$, $N \subseteq W$ and $S \subseteq N \times W$. The set N is regarded as the collection of so called *normal worlds*, i.e. $W \setminus N$ is the collection of impossible worlds. A *Kripke model with impossible worlds* is a pair $\mathbb{M} = (\mathbb{F}, V)$ such that $\mathbb{F}$ is a Kripke frame with impossible worlds, and $V : \mathsf{Prop} \to \mathcal{P}(W)$ is an assignment.

The satisfaction of atomic formulas and of formulas the main connective of which is a Boolean connective is defined as usual. As to $\Box$- and $\Diamond$-formulas,

- $\mathbb{M}, w \Vdash \Box\varphi$ iff $w \in N$ and for all $v \in W$ if wSv then $\mathbb{M}, v \Vdash \varphi$.
- $\mathbb{M}, w \Vdash \Diamond\varphi$ iff $w \notin N$ or there exists some $v \in W$ s.t. wSv, and $\mathbb{M}, v \Vdash \varphi$.

Validity of formulas is defined as usual.

The definition above is tailored to make the necessitation rule fail. Indeed, $\top$ is valid at every point in every model, however $\mathcal{M}, w \not\Vdash \Box\top$ whenever $w \notin N$.

4.1.4. Fact. Axiom $(\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$ is valid on any Kripke frame with impossible worlds.

Proof. Let $\mathbb{F} = (W, S, N)$ be a Kripke frame with impossible worlds. Fix a valuation V and let $w \in W$. if $w \notin N$, then $w \not\Vdash \Box p$, $w \not\Vdash \Box q$, $w \not\Vdash \Box(p \wedge q)$. So $w \Vdash (\Box p \wedge \Box q) \leftrightarrow \Box(p \wedge q)$. If $w \in N$, the following chain of equivalences holds:

$$\begin{array}{ll}
 & w \Vdash \Box p \wedge \Box q \\
\text{iff} & w \Vdash \Box p \text{ and } w \Vdash \Box q \\
\text{iff} & \text{for all } v \in W \text{ s.t. } wSv,\ v \Vdash p \text{ and for all } v \in W \text{ s.t. } wSv,\ v \Vdash q \\
\text{iff} & \text{for all } v \in W \text{ s.t. } wSv,\ v \Vdash p \text{ and } v \Vdash q \\
\text{iff} & w \Vdash \Box(p \wedge q).
\end{array}$$

□

4.1.5. DEFINITION. For any Kripke frame with impossible worlds $\mathbb{F} = (W, S, N)$, the *complex algebra* associated with $\mathbb{F}$ is $\mathbb{F}^+ := (\mathcal{P}(W), f_S)$ where $f_S : \mathcal{P}(W) \to \mathcal{P}(W)$ is defined by the following assignment:

$$X \mapsto \{w \in W \mid w \notin N \text{ or } wSv \text{ for some } v \in X\} = N^c \cup S^{-1}[X].$$

4.1.6. FACT. For any Kripke frame with impossible worlds $\mathbb{F} = (W, S, N)$ and any $\mathcal{X} \subseteq \mathcal{P}(W)$, if $\mathcal{X} \neq \varnothing$, then

$$f_S(\bigcup \mathcal{X}) = \bigcup \{f_S(X) \mid X \in \mathcal{X}\}.$$

Proof. Indeed, the following chain of identities holds:

$$\begin{array}{rcl} f_S(\bigcup \mathcal{X}) & = & N^c \cup S^{-1}[\bigcup \mathcal{X}] \\ & = & N^c \cup \bigcup \{S^{-1}[X] \mid X \in \mathcal{X}\} \\ & = & \bigcup \{N^c \cup S^{-1}[X] \mid X \in \mathcal{X}\} \\ & = & \bigcup \{f_S(X) \mid X \in \mathcal{X}\}. \end{array}$$

□

The fact above shows that the diamond-type operation of the complex algebra associated with any Kripke frame with impossible worlds is completely additive. Facts 4.1.4 and 4.1.6 witness that Kripke frames with impossible worlds are a natural semantic environment for the so-called regular modal logics.

4.1.3 Algebraic semantics

In this subsection, we collect basic facts about algebraic semantics for (Boolean-based) regular modal logics.

4.1.7. DEFINITION. A *regular Boolean algebra expansion* (from now on abbreviated as r-BAE) is a tuple $\mathbb{A} = (\mathbb{B}, f)$ such that $\mathbb{B}$ is a Boolean algebra, and f is a unary and additive operation, i.e. f preserves finite non-empty joins of $\mathbb{B}$. An r-BAE is *perfect* if $\mathbb{B}$ is complete and atomic, and f is completely additive, i.e. f preserves arbitrary non-empty joins of $\mathbb{B}$.

The complex algebra associated with every Kripke frame with impossible worlds is a perfect r-BAE (cf. Fact 4.1.6).

Formulas in the language of regular modal logic are interpreted into r-BAEs via assignments in the usual way. An *assignment* is a function $h : \mathsf{Prop} \to \mathbb{A}$, which has a unique homomorphic extension to the algebra of formulas over Prop in the usual way. An *equation* is a pair of formulas (s, t), usually written as $s \approx t$, which is valid on an r-BAE $\mathbb{A}$ (notation: $\mathbb{A} \models s \approx t$) if $h(s) = h(t)$ for all assignments h. An *inequality* is a pair of formulas (s, t), usually written as $s \leq t$, which is valid on an r-BAE $\mathbb{A}$ (notation: $\mathbb{A} \models s \leq t$) if $h(s) \leq^{\mathbb{A}} h(t)$ for all assignments h, where $\leq^{\mathbb{A}}$ is the lattice order on $\mathbb{A}$. If K is a class of r-BAEs, then $s \approx t$ is valid on K (notation: $\mathsf{K} \models s \approx t$) if $s \approx t$ is valid in all $\mathbb{A} \in \mathsf{K}$, and the validity of $s \leq t$ on K is defined similarly.

It is well known (cf. [93, Theorem 7.5]) that the basic regular modal logic is sound and complete w.r.t. the class of r-BAEs.

Discrete duality on objects. The well-known duality on objects between Kripke frames and complete atomic Boolean algebras with complete operators generalizes to the setting of perfect r-BAEs and Kripke frames with impossible worlds. Indeed, Definition 4.1.5 provides half of the connection. As to the other half:

4.1.8. DEFINITION. For every perfect r-BAE $\mathbb{A} = (\mathbb{B}, f)$, the *atom structure with impossible worlds* associated with $\mathbb{A}$ is $\mathbb{A}_+ := (At(\mathbb{A}), S_f, N)$, where $At(\mathbb{A})$ is the collection of atoms of $\mathbb{A}$, $N := \{x \in At(\mathbb{A}) \mid x \nleq f(\bot)\}$ and for all $x, y \in At(\mathbb{A})$ such that $x \nleq f(\bot)$,

$$xS_f y \quad \text{iff} \quad x \leq f(y).$$

4.1.9. PROPOSITION. *For every Kripke frame with impossible worlds $\mathbb{F}$ and every perfect r-BAE $\mathbb{A}$,*

$$(\mathbb{F}^+)_+ \cong \mathbb{F} \quad \textit{and} \quad (\mathbb{A}_+)^+ \cong \mathbb{A}.$$

4.1.4 The distributive setting

In this subsection, we briefly outline the Bounded Distributive Lattice (BDL)-versions of the definitions and facts of the previous subsection. The reason for this generalization is that most of the treatment in the following sections is presented in the setting of distributive lattices.

We recall that a distributive lattice is *perfect* if it is complete, completely distributive and completely join-generated by the collection of its completely join-prime elements. Equivalently, a distributive lattice is perfect iff it is isomorphic to the lattice of upsets of some poset. For a distributive lattice $\mathbb{A}$, an $x \in \mathbb{A}$ is *completely join-irreducible* (resp. *completely join-prime*) if $x \neq \bot$ and for every $A \subseteq \mathbb{A}$, if $x = \bigvee S$ (resp. $x \leq \bigvee S$) then $x = s$ (resp. $x \leq s$) for some $s \in S$. In the setting of perfect distributive lattices, completely join-irreducibles coincide with completely join-primes.

4.1.10. DEFINITION. A *regular distributive lattice expansion*[1] (abbreviated as DLR in the remainder of the section) is a tuple $\mathbb{A} = (\mathbb{B}, f)$ such that $\mathbb{B}$ is a bounded

[1]Actually, this name refers much more generally to bounded distributive lattices with an arbitrary but finite number of additional operations of any finite arity. Hence, the definition given above captures only a very restricted subclass of distributive lattice expansions which is sufficient for the sake of the present section.

distributive lattice, and f is a unary and additive operation, i.e. it preserves finite non-empty joins of $\mathbb{B}$. An DLR is *perfect* if $\mathbb{B}$ is a perfect distributive lattice, and f is completely additive, i.e. it preserves arbitrary non-empty joins of $\mathbb{B}$.

4.1.11. Definition. A *distributive Kripke frame with impossible worlds* is a quadruple $\mathbb{F} = (W, \leq, S, N)$ such that $W \neq \varnothing$, $\leq\ \subseteq W \times W$ is a partial order, $N \subseteq W$ is a downset and $S \subseteq N \times W$ is such that $\geq_{\upharpoonright(N\times N)} \circ S \circ \geq\ \subseteq\ S$. The set N is regarded as the collection of so-called *normal worlds*.

For every perfect DLR $\mathbb{A} = (\mathbb{B}, f)$, let $\mathbb{A}_+ := (J^\infty(\mathbb{A}), \geq, S_f, N)$ where $J^\infty(\mathbb{A})$ is the collection of completely join-irreducible elements of $\mathbb{A}$,the symbol $\geq$ denotes the converse of the lattice order restricted to $J^\infty(\mathbb{A})$, $N := \{x \in At(\mathbb{A}) \mid x \nleq f(\bot)\}$, and for all $x, y \in At(\mathbb{A})$ such that $x \nleq f(\bot)$,

$$xS_f y \quad \text{iff} \quad x \leq f(y).$$

For every distributive Kripke frame with impossible worlds $\mathbb{F} = (W, \leq, S, N)$, let $\mathbb{F}^+ := (\mathcal{P}^\uparrow(W), f)$ where $f : \mathcal{P}^\uparrow(W) \to \mathcal{P}^\uparrow(W)$ is defined by the following assignment:

$$X \mapsto \{w \in W \mid w \notin N \text{ or } wSv \text{ for some } v \in X\} = N^c \cup S^{-1}[X].$$

4.1.12. Proposition. *For every distributive Kripke frame with impossible worlds $\mathbb{F}$ and every perfect DLR $\mathbb{A}$,*

$$(\mathbb{F}^+)_+ \cong \mathbb{F} \quad \textit{and} \quad (\mathbb{A}_+)^+ \cong \mathbb{A}.$$

4.1.5 Canonical extension

In this section, we briefly recall the theory of canonical extension for bounded distributive lattices (BDLs), and define the canonical extension of a regular distributive lattice expansion (DLR).

For any BDL $\mathbb{A} = (A, \vee, \wedge, \bot, \top)$, let $\mathbb{A}^\partial := (A, \wedge, \vee, \top, \bot)$ be its *dual* BDL. Moreover, let $\mathbb{A}^1 := \mathbb{A}$. An *$n$-order-type* ε is an element of $\{1, \partial\}^n$, and its i-th coordinate is denoted ε_i. We omit n when it is clear from the context. Let ε^∂ denote the *dual order-type* of ε, that is, $\varepsilon^\partial_i := 1$ (resp. $\varepsilon^\partial_i := \partial$) if $\varepsilon_i = \partial$ (resp. $\varepsilon_i = 1$). Given an order-type ε, we let $\mathbb{A}^\varepsilon$ be the BDL $\mathbb{A}^{\varepsilon_1} \times \ldots \times \mathbb{A}^{\varepsilon_n}$.

4.1.13. Definition. The *canonical extension* of a BDL $\mathbb{A}$ is a complete BDL $\mathbb{A}^\delta$ containing $\mathbb{A}$ as a sublattice, such that:

1. *(denseness)* every element of $\mathbb{A}^\delta$ can be expressed both as a join of meets and as a meet of joins of elements from $\mathbb{A}$;
2. *(compactness)* for all $S, T \subseteq \mathbb{A}$ with $\bigwedge S \leq \bigvee T$ in $\mathbb{A}^\delta$, there exist some finite sets $F \subseteq S$ and $G \subseteq T$ s.t. $\bigwedge F \leq \bigvee G$.

It is well known that the canonical extension of a BDL is unique up to isomorphism (cf. [75, Section 2.2]), and that the canonical extension of a BDL is a perfect BDL (cf. [75, Definition 2.14]). An element $x \in \mathbb{A}^\delta$ is *closed* (resp. *open*) if it is the meet (resp. join) of some subset of $\mathbb{A}$. Let $K(\mathbb{A}^\delta)$ (resp. $O(\mathbb{A}^\delta)$) be the set of closed (resp. open) elements of $\mathbb{A}^\delta$. It is easy to see that the denseness condition in Definition 4.1.13 implies that $J^\infty(\mathbb{A}^\delta) \subseteq K(\mathbb{A}^\delta)$ and $M^\infty(\mathbb{A}^\delta) \subseteq O(\mathbb{A}^\delta)$ (cf. [75], page 9).

Let A, B be BDLs. An order-preserving map $f : A \to B$ can be extended to a map : $A^\delta \to B^\delta$ in two canonical ways. Let f^σ and f^π respectively denote the σ and π-*extension* of f defined as follows:

4.1.14. Definition. [cf. Remark 2.17 in [75]] If $f : \mathbb{A} \to \mathbb{B}$ is order-preserving, then for all $u \in A^\delta$,

$$f^\sigma(u) = \bigvee\{\bigwedge\{f(a) : x \leq a \in A\} : u \geq x \in K(\mathbb{A}^\delta)\}$$

$$f^\pi(u) = \bigwedge\{\bigvee\{f(a) : y \geq a \in A\} : u \leq y \in O(\mathbb{A}^\delta)\}.$$

4.1.15. Definition. The canonical extension of a DLR $\mathbb{A} = (A, f)$ is defined as the tuple $\mathbb{A}^\delta = (A^\delta, f^\sigma)$, where A^δ is the canonical extension of the BDL A and f^σ is defined according to Definition 4.1.14.

4.1.6 Adjoints and residuals

In this section, we give the relevant preliminaries on adjoints and residuals, which play a crucial role in the algebraic-algorithmic correspondence theory. In this section, $\mathbb{C}$ and $\mathbb{C}'$ are complete lattices, and we will often use subscripts to indicate the underlying lattice/algebra of the order/meet/join. Given a lattice $\mathbb{C} = (C, \wedge, \vee, \bot, \top)$, we denote the *dual lattice* $(C, \vee, \wedge, \top, \bot)$ as $\mathbb{C}^\partial$. For the proofs of the propositions in this section, see [56].

4.1.16. Definition. The monotone maps $f : \mathbb{C} \to \mathbb{C}'$ and $g : \mathbb{C}' \to \mathbb{C}$ form an *adjoint pair* (notation: $f \dashv g$), if for every $x \in \mathbb{C}, y \in \mathbb{C}'$,

$$f(x) \leq_{\mathbb{C}'} y \text{ iff } x \leq_{\mathbb{C}} g(y).$$

Whenever $f \dashv g$, f is called the *left adjoint* of g and g the *right adjoint* of f. We also say f is a left adjoint and g is a right adjoint.

An important property of adjoint pairs of maps is that if a map has a left (resp. right) adjoint, then the adjoint is unique and can be computed pointwise from the map itself and the order relation on the lattices. This means that having a left (resp. right) adjoint is an intrinsically order-theoretic property of maps.

4.1.17. PROPOSITION. *For monotone maps $f : \mathbb{C} \to \mathbb{C}'$ and $g : \mathbb{C}' \to \mathbb{C}$ such that $f \dashv g$, for every $x \in \mathbb{C}, y \in \mathbb{C}'$,*

1. $f(x) = \bigwedge_{\mathbb{C}'}\{y \in \mathbb{C}' : x \leq_{\mathbb{C}} g(y)\}$;
2. $g(y) = \bigvee_{\mathbb{C}}\{x \in \mathbb{C} : f(x) \leq_{\mathbb{C}'} y\}$.

4.1.18. PROPOSITION. *For any map $f : \mathbb{C} \to \mathbb{C}'$,*

1. *f is completely join-preserving iff it has a right adjoint;*
2. *f is completely meet-preserving iff it has a left adjoint.*

4.1.19. DEFINITION. For n-ary maps $f : \mathbb{C}^n \to \mathbb{C}$ and $g : \mathbb{C}^n \to \mathbb{C}$, they form a *residual pair* in the ith coordinate (notation: $f \dashv_i g$), if for all $x_1, \ldots, x_n, y \in \mathbb{C}$,

$$f(x_1, \ldots, x_i, \ldots, x_n) \leq_{\mathbb{C}} y \ \text{ iff } \ x_i \leq_{\mathbb{C}} g(x_1, \ldots, y, \ldots, x_n).$$

Whenever $f \dashv_i g$, f is called the *left residual* of g in the ith coordinate and g the *right residual* of f in the ith coordinate. We also say f is a left residual and g is a right residual.

4.1.20. PROPOSITION. *For $f : \mathbb{C}^n \to \mathbb{C}$ and $g : \mathbb{C}^n \to \mathbb{C}$ such that $f \dashv_i g$, for all $x_1, \ldots, x_n, y \in \mathbb{C}$,*

1. $f(x_1, \ldots, x_i, \ldots, x_n) = \bigwedge_{\mathbb{C}}\{y \in \mathbb{C} : x_i \leq_{\mathbb{C}} g(x_1, \ldots, y, \ldots, x_n)\}$;
2. $g(x_1, \ldots, y, \ldots, x_n) = \bigvee_{\mathbb{C}}\{x_i \in \mathbb{C} : f(x_1, \ldots, x_i, \ldots, x_n) \leq_{\mathbb{C}} y\}$.

4.1.21. PROPOSITION. *For any n-ary map $f : \mathbb{C}^n \to \mathbb{C}$,*

1. *f is completely join-preserving in the i-th coordinate iff it has a right residual in that same coordinate;*
2. *f is completely meet-preserving in the i-th coordinate iff it has a left residual in that same coordinate.*

4.1.22. EXAMPLE. Given a perfect BAO $\mathbb{A}$, the operations $\Diamond : \mathbb{A} \to \mathbb{A}$ and $\Box : \mathbb{A} \to \mathbb{A}$ are completely join- and meet-preserving, respectively, and, therefore, are left and right adjoints, respectively. Hence there are operations $\blacksquare : \mathbb{A} \to \mathbb{A}$ and $\blacklozenge : \mathbb{A} \to \mathbb{A}$ such that for every $x, y \in \mathbb{A}$,

$$\Diamond x \leq_{\mathbb{A}} y \text{ iff } x \leq_{\mathbb{A}} \blacksquare y,$$
$$x \leq_{\mathbb{A}} \Box y \text{ iff } \blacklozenge x \leq_{\mathbb{A}} y.$$

Therefore, by Proposition 4.1.17, $\blacklozenge x = \bigwedge_{\mathbb{A}}\{y \in \mathbb{A} : x \leq_{\mathbb{A}} \Box y\}$ and $\blacksquare y = \bigvee_{\mathbb{A}}\{x \in \mathbb{A} : \Diamond x \leq_{\mathbb{A}} y\}$.

4.1.23. Example. Given a perfect distributive lattice $\mathbb{A}$, the binary operations $\wedge : \mathbb{A}\times\mathbb{A} \to \mathbb{A}$ and $\vee : \mathbb{A}\times\mathbb{A} \to \mathbb{A}$ are completely join- and meet-preserving in each coordinate, respectively, and therefore have right and left residuals, respectively. Hence there are binary operations $\to: \mathbb{A}^\partial \times \mathbb{A} \to \mathbb{A}$ and $- : \mathbb{A} \times \mathbb{A}^\partial \to \mathbb{A}$ such that for every $x, y, z \in \mathbb{A}$,

$$x \wedge y \leq_{\mathbb{A}} z \text{ iff } x \leq_{\mathbb{A}} y \to z,$$
$$x - y \leq_{\mathbb{A}} z \text{ iff } x \leq_{\mathbb{A}} y \vee z.$$

Therefore, by Proposition 4.1.20, $y \to z = \bigvee_{\mathbb{A}}\{x \in \mathbb{A} : x \wedge y \leq_{\mathbb{A}} z\}$ and $x - y = \bigwedge_{\mathbb{A}}\{z \in \mathbb{A} : x \leq_{\mathbb{A}} y \vee z\}$.

4.1.24. Example. Consider the *diagonal map* $\Delta : \mathbb{C} \to \mathbb{C} \times \mathbb{C}$, defined by the assignment $x \mapsto (x, x)$. The defining clauses of the least upper bound and greatest lower bound can be equivalently restated by saying that the operations $\vee$ and $\wedge$ are the left and the right adjoints of the diagonal map Δ, respectively:

$$x \leq_{\mathbb{C}} y \wedge z \text{ iff } x \leq_{\mathbb{C}} y \text{ and } x \leq_{\mathbb{C}} z, \text{ iff } \Delta(x) \leq_{\mathbb{C}\times\mathbb{C}} (y, z),$$
$$y \vee z \leq_{\mathbb{C}} x \text{ iff } y \leq_{\mathbb{C}} x \text{ and } z \leq_{\mathbb{C}} x, \text{ iff } (y, z) \leq_{\mathbb{C}\times\mathbb{C}} \Delta(x).$$

4.2 Algebraic-algorithmic correspondence

The contribution of the present section is set in the context of order-theoretic algorithmic correspondence theory [50, 46]. As illustrated in Chapter 3, the strategy of instantiating propositional variables with first-order definable 'minimal valuations' can be developed in the context of the algebraic semantics of modal logic, and then generalized to various other logics. The algebraic setting helps to distill the essentials of this strategy. Before giving a more detailed account of this theory, we will guide the reader through the main principles which make it work, by means of an example.

The algorithm ALBA illustrated. Let us start with one of the best-known examples in correspondence theory, namely $\Diamond\Box p \to \Box\Diamond p$. It is well known that for every Kripke frame $\mathcal{F} = (W, R)$,

$$\mathcal{F} \Vdash \Diamond\Box p \to \Box\Diamond p \quad \text{iff} \quad \mathcal{F} \models \forall xyz\,(xRy \wedge xRz \to \exists u(yRu \wedge zRu)).$$

As is discussed at length in [50], every piece of argument used to prove this correspondence on frames can be translated by duality to complex algebras (see Example 2.1.5). As is well known, complex algebras are characterized in purely algebraic terms as complete and atomic BAOs where the modal operations are completely join-preserving. These are also known as *perfect* BAOs.

First of all, the condition $\mathcal{F} \Vdash \Diamond\Box p \to \Box\Diamond p$ translates to the complex algebra $\mathbb{A} = \mathcal{F}^+$ of $\mathcal{F}$ as $[\![\Diamond\Box p]\!] \subseteq [\![\Box\Diamond p]\!]$ for every assignment of p into $\mathbb{A}$, so this validity clause can be rephrased as follows:

$$\mathbb{A} \models \forall p[\Diamond\Box p \leq \Box\Diamond p], \tag{4.1}$$

where the order $\leq$ is interpreted as set inclusion in the complex algebra. In perfect BAOs every element is both the join of the completely join-prime elements (the set of which is denoted $J^\infty(\mathbb{A})$) below it and the meet of the completely meet-prime elements (the set of which is denoted $M^\infty(\mathbb{A})$) above it[2]. Hence, taking some liberties in our use of notation, the condition above can be equivalently rewritten as follows:

$$\mathbb{A} \models \forall p[\bigvee\{i \in J^\infty(\mathbb{A}) \mid i \leq \Box\Diamond p\} \leq \bigwedge\{m \in M^\infty(\mathbb{A}) \mid \Box\Diamond p \leq m\}].$$

By elementary properties of least upper bounds and greatest lower bounds in posets (cf. [56]), this condition is true if and only if every element in the join is less than or equal to every element in the meet; thus, condition (4.1) above can be rewritten as:

$$\mathbb{A} \models \forall p\forall\mathbf{i}\forall\mathbf{m}[(\mathbf{i} \leq \Diamond\Box p \ \ \& \ \ \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}], \tag{4.2}$$

where the variables $\mathbf{i}$ and $\mathbf{m}$ range over $J^\infty(\mathbb{A})$ and $M^\infty(\mathbb{A})$ respectively. Since $\mathbb{A}$ is a perfect BAO, the element of $\mathbb{A}$ interpreting $\Box p$ is the join of the completely join-prime elements below it. Hence, if $i \in J^\infty(\mathbb{A})$ and $i \leq \Diamond\Box p$, because $\Diamond$ is completely join-preserving on $\mathbb{A}$, we have that

$$i \leq \Diamond(\bigvee\{j \in J^\infty(\mathbb{A}) \mid j \leq \Box p\}) = \bigvee\{\Diamond j \mid j \in J^\infty(\mathbb{A}) \text{ and } j \leq \Box p\},$$

which implies that $i \leq \Diamond j_0$ for some $j_0 \in J^\infty(\mathbb{A})$ such that $j_0 \leq \Box p$. Hence, we can equivalently rewrite the validity clause above as follows:

$$\mathbb{A} \models \forall p\forall\mathbf{i}\forall\mathbf{m}[(\exists\mathbf{j}(\mathbf{i} \leq \Diamond\mathbf{j} \ \ \& \ \ \mathbf{j} \leq \Box p) \ \ \& \ \ \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}], \tag{4.3}$$

and then use standard manipulations from first-order logic to pull out quantifiers:

$$\mathbb{A} \models \forall p\forall\mathbf{i}\forall\mathbf{m}\forall\mathbf{j}[(\mathbf{i} \leq \Diamond\mathbf{j} \ \ \& \ \ \mathbf{j} \leq \Box p \ \ \& \ \ \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{4.4}$$

Now we observe that the operation $\Box$ preserves arbitrary meets in the perfect BAO $\mathbb{A}$. By the general theory of adjunction in complete lattices, this is equivalent to $\Box$ being a right adjoint (cf. Proposition 4.1.17). It is also well known that the left or lower adjoint of $\Box$ is the operation $\blacklozenge$, which can be recognized

[2]In BAOs the completely join-prime elements, the completely join-irreducible elements and the atoms coincide. Moreover, the completely meet-prime elements, the completely meet-irreducible elements and the co-atoms coincide.

as the backward-looking diamond P, interpreted with the converse R^{-1} of the accessibility relation R of the frame $\mathcal{F}$ in the context of tense logic (cf. [30, Example 1.25] and [56, Exercise 7.18] modulo translating the notation). Hence the condition above can be equivalently rewritten as:

$$\mathbb{A} \models \forall p \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \& \ \blacklozenge \mathbf{j} \leq p \ \& \ \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}], \tag{4.5}$$

and then as follows:

$$\mathbb{A} \models \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \& \ \exists p(\blacklozenge \mathbf{j} \leq p \ \& \ \Box\Diamond p \leq \mathbf{m})) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{4.6}$$

At this point we are in a position to eliminate the variable p and equivalently rewrite the previous condition as follows:

$$\mathbb{A} \models \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{j}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \& \ \Box\Diamond\blacklozenge \mathbf{j} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{4.7}$$

Let us justify this equivalence: for the direction from top to bottom, fix an interpretation V of the variables $\mathbf{i}, \mathbf{j}$, and $\mathbf{m}$ such that $\mathbf{i} \leq \Diamond \mathbf{j}$ and $\Box\Diamond\blacklozenge \mathbf{j} \leq \mathbf{m}$. To prove that $\mathbf{i} \leq \mathbf{m}$ holds under V, consider the variant V^* of V such that $V^*(p) = \blacklozenge \mathbf{j}$. Then it can be easily verified that V^* witnesses the antecedent of (4.6) under V; hence $\mathbf{i} \leq \mathbf{m}$ holds under V. Conversely, fix an interpretation V of the variables $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{m}$ such that $\mathbf{i} \leq \Diamond \mathbf{j} \ \& \ \exists p(\blacklozenge \mathbf{j} \leq p \ \& \ \Box\Diamond p \leq \mathbf{m})$. Then, by monotonicity, the antecedent of (4.7) holds under V, and hence so does $\mathbf{i} \leq \mathbf{m}$, as required. This is an instance of the following result, known as *Ackermann's lemma* ([2], see also [48]):

4.2.1. LEMMA. *Fix an arbitrary propositional language L. Let $\alpha, \beta(p), \gamma(p)$ be L-formulas such that α is p-free, β is positive and γ is negative in p. For any assignment V on an L-algebra $\mathbb{A}$, the following are equivalent:*

1. $\mathbb{A}, V \models \beta(\alpha/p) \leq \gamma(\alpha/p)$;
2. *there exists a p-variant V^* of V such that $\mathbb{A}, V^* \models \alpha \leq p$ and $\mathbb{A}, V^* \models \beta(p) \leq \gamma(p)$,*

where $\beta(\alpha/p)$ and $\gamma(\alpha/p)$ denote the result of uniformly substituting α for p in β and γ, respectively.

The proof is essentially the same as [50, Lemma 4.2]. Whenever, in a reduction, we reach a shape in which the lemma above (or its order-dual) can be applied, we say that the condition is in *Ackermann shape.* Taking stock, we note that we have equivalently transformed (4.1) into (4.7), which is a condition in which all propositional variables (corresponding to monadic second-order variables) have been eliminated, and all remaining variables range over completely join- and meet-prime elements. Via the duality, the latter correspond to singletons and complements of singletons, respectively, in Kripke frames. Moreover, $\blacklozenge$ is interpreted on Kripke frames using the converse of the same accessibility relation used to interpret $\Box$. Hence, clause (4.7) translates equivalently into a condition in the first-order correspondence language. To facilitate this translation we first rewrite (4.7) as follows, by reversing the reasoning that brought us from (4.1) to (4.2):

$$\mathbb{A} \models \forall \mathbf{j}[\Diamond \mathbf{j} \leq \Box \Diamond \blacklozenge \mathbf{j}].$$

By again applying the fact that $\Box$ is a right adjoint we obtain

$$\mathbb{A} \models \forall \mathbf{j}[\blacklozenge \Diamond \mathbf{j} \leq \Diamond \blacklozenge \mathbf{j}]. \tag{4.8}$$

Recalling that $\mathbb{A}$ is the complex algebra of $\mathcal{F} = (W, R)$, this gives $\forall w(R[R^{-1}[w]] \subseteq R^{-1}[R[w]]$. Notice that $R[R^{-1}[w]]$ is the set of all states $x \in W$ which have a predecessor z in common with w, while $R^{-1}[R[w]]$ is the set of all states $x \in W$ which have a successor in common with w. This can be spelled out as

$$\forall x \forall w(\exists z(zRx \wedge zRw) \rightarrow \exists y(xRy \wedge wRy))$$

or, equivalently,

$$\forall z \forall x \forall w((zRx \wedge zRw) \rightarrow \exists y(xRy \wedge wRy))$$

which is the familiar Church-Rosser condition.

4.2.1 The basic calculus for correspondence

The example in Section 4.2 illustrated the main strategy for the elimination of second order variables. We transformed the initial validity condition into a shape to which Ackermann's lemma was applicable (i.e., into Ackermann shape). Two order-theoretic ingredients were used to reach Ackermann shape, namely:

(a) The ability to *approximate* elements of the algebra from below or from above using completely join-prime and completely meet-prime elements;

(b) the fact that $\Box$ is a right *adjoint*. Moreover, in general, in perfect distributive lattices with operators, all the operations interpreting the logical connectives are either residuals or adjoints.

We can repackage these two observations, together with Ackermann's lemma, in the form of *proof rules*, grouped in the following types: the *approximation rules*, *residuation/adjunction rules* and *Ackermann rules*. These rules, together with the strategy governing the order of their application, as illustrated in Section 4.2, constitute the algorithm ALBA, a rigorous specification of which can be found in [50, Section 6]. ALBA takes an inequality in input, preprocesses it and transforms it into one or more expressions known as quasi-inequalities: given a propositional language $\mathcal{L}$, an *$\mathcal{L}$-quasi-inequality* is an expression of the form $\varphi_1 \leq \psi_1 \,\&\, \cdots \,\&\, \varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi$ where the φ_i, ψ_i, φ and ψ are $\mathcal{L}$-formulas.

ALBA's goal is to transform all the obtained quasi-inequalities into (sets of) *pure* quasi-inequalities, i.e., into quasi-inequalities in which no propositional variables occur. If such a state is reached, we say ALBA *succeeds* on the input inequality.

First approximation rule. This rule is applied only once to transform an inequality into a quasi-inequality (as in (4.2)) after some possible preprocessing.

$$\frac{\varphi \leq \psi}{\forall \mathbf{j} \forall \mathbf{m}[(\mathbf{j} \leq \varphi \ \& \ \psi \leq \mathbf{m}) \Rightarrow \mathbf{j} \leq \mathbf{m}]} \text{(FA)}$$

Approximation rules. Each of the following rules can be proved sound with an argument similar to that used in Section 4.2 to justify the transition from (4.2) to (4.3). For more details, see [50, Lemma 8.4].

$$\frac{\Box\psi \leq \mathbf{m}}{\exists \mathbf{n}(\Box\mathbf{n} \leq \mathbf{m} \ \& \ \psi \leq \mathbf{n})} (\Box\text{Appr}) \qquad \frac{\mathbf{j} \leq \Diamond\psi}{\exists \mathbf{i}(\mathbf{j} \leq \Diamond\mathbf{i} \ \& \ \mathbf{i} \leq \psi)} (\Diamond\text{Appr})$$

$$\frac{\chi \to \varphi \leq \mathbf{m}}{\exists \mathbf{j}(\mathbf{j} \to \varphi \leq \mathbf{m} \ \& \ \mathbf{j} \leq \chi)} (\to\text{Appr}_1) \qquad \frac{\chi \to \varphi \leq \mathbf{m}}{\exists \mathbf{n}(\chi \to \mathbf{n} \leq \mathbf{m} \ \& \ \varphi \leq \mathbf{n})} (\to\text{Appr}_2)$$

$$\frac{\mathbf{i} \leq \chi - \varphi}{\exists \mathbf{j}(\mathbf{i} \leq \mathbf{j} - \varphi \ \& \ \mathbf{j} \leq \chi)} (-\text{Appr}_1) \qquad \frac{\mathbf{i} \leq \chi - \varphi}{\exists \mathbf{n}(\mathbf{i} \leq \chi - \mathbf{n} \ \& \ \varphi \leq \mathbf{n})} (-\text{Appr}_2)$$

Adjunction and residuation rules. Each of the following rules can be proved sound with an argument similar to that used in Section 4.2 to justify the transition from (4.4) to (4.5), cf. [50, Lemma 8.4].

$$\frac{\varphi \vee \chi \leq \psi}{\varphi \leq \psi \quad \chi \leq \psi} (\vee\text{LA}) \qquad \frac{\varphi \leq \chi \vee \psi}{\varphi - \chi \leq \psi} (\vee\text{RR}) \qquad \frac{\varphi \leq \chi \to \psi}{\varphi \wedge \chi \leq \psi} (\to\text{RR})$$

$$\frac{\psi \leq \varphi \wedge \chi}{\psi \leq \varphi \quad \psi \leq \chi} (\wedge\text{RA}) \qquad \frac{\chi \wedge \psi \leq \varphi}{\chi \leq \psi \to \varphi} (\wedge\text{LR}) \qquad \frac{\chi - \psi \leq \varphi}{\chi \leq \psi \vee \varphi} (-\text{LR})$$

Specifically, the rules in the left column above are justified by the fact that $\vee$ and $\wedge$ are respectively the left and the right adjoint of the diagonal map Δ, defined by the assignment $a \mapsto (a, a)$; the ones in the middle and right hand columns above are justified by $\vee$ and $\wedge$ being respectively the right residual of $-$ and the left residual of $\to$. For $\Diamond$ and $\Box$ we have:

$$\frac{\Diamond\varphi \leq \psi}{\varphi \leq \blacksquare\psi} (\Diamond\text{LA}) \qquad \frac{\varphi \leq \Box\psi}{\blacklozenge\varphi \leq \psi} (\Box\text{RA})$$

Ackermann rules. The soundness of the following rules is justified by Lemma 4.2.1 and its symmetric version.

$$\frac{\exists p[\&_{i=1}^{n} \alpha_i \leq p \ \& \ \&_{j=1}^{m} \beta_j(p) \leq \gamma_j(p)]}{\&_{j=1}^{m} \beta_j(\bigvee_{i=1}^{n} \alpha_i/p) \leq \gamma_j(\bigvee_{i=1}^{n} \alpha_i/p)}\ (\mathrm{RA}) \qquad \frac{\exists p[\&_{j=1}^{m} \beta_j(p) \leq \gamma_j(p)]}{\&_{j=1}^{m} \beta_j(\bot/p) \leq \gamma_j(\bot/p)}\ (\bot)$$

$$\frac{\exists p[\&_{i=1}^{n} p \leq \alpha_i \ \& \ \&_{j=1}^{m} \gamma_j(p) \leq \beta_j(p)]}{\&_{j=1}^{m} \gamma_j(\bigwedge_{i=1}^{n} \alpha_i/p) \leq \beta_j(\bigwedge_{i=1}^{n} \alpha_i/p)}\ (\mathrm{LA}) \qquad \frac{\exists p[\&_{j=1}^{m} \gamma_j(p) \leq \beta_j(p)]}{\&_{j=1}^{m} \gamma_j(\top/p) \leq \beta_j(\top/p)}\ (\top)$$

The rules above are subject to the restrictions that the α_i are p-free, and that the β_j and the γ_j are respectively positive and negative in p. Notice that the rules $(\bot)$ and $(\top)$ can be regarded as the special case of (RA) and (LA) in which $\alpha := \bot$ and $\alpha := \top$, respectively.

Unlike the rules given in the previous paragraphs which apply locally and rewrite individual inequalities, the Ackermann rules involve the set of inequalities in the antecedent of a quasi-inequality as a whole. A quasi-inequality to which one of these rules is applicable is said to be in Ackermann shape. In particular, this requires that either all positive occurrences of p occur *in display* in inequalities of the form $\alpha_i \leq p$ (in the case of (RA)), or that all negative occurrences of p occur in display in inequalities of the form $p \leq \alpha_i$ (in the case of (LA)).

4.3 ALBA on regular BDL and HA expansions

We adapt the calculus for correspondence ALBA to the setting of regular modal logic, and we consider both the case in which the propositional base is given by the logic of bounded distributive lattices (BDLs) and by intuitionistic logic, the algebras associated with which are Heyting algebras (HAs). Unlike the the distributive modal logic setting, the modal operators in the setting of DLR/HAR setting are only additive, and not normal. In order to compute adjoints, we define their normal approximations below.

As we will discuss below, all rules of ALBA^r are sound with respect to all perfect DLRs. In what follows, we define ALBA^r for DLR and HAR simultaneously and from first principles. We use the symbol $\mathcal{L}$ to refer indifferently to DLR or to HAR.

4.3.1 The expanded language $\mathcal{L}^+$

Analogously to what has been done in [50], we need to introduce the expanded language $\mathcal{L}^+$ that the calculus ALBA^r will manipulate. The language $\mathcal{L}^+$ will be shaped on the perfect $\mathcal{L}$-algebras. Indeed, as usual, $\mathcal{L}^+$ will be built on three pairwise disjoint sets of variables: proposition variables in Prop (denoted by p, q, r), *nominal* variables in Nom (denoted by $\mathbf{i}, \mathbf{j}, \mathbf{k}$) and *conominal* variables in CNom (denoted by $\mathbf{m}, \mathbf{n}, \mathbf{o}$). Nominals and conominals are to be interpreted as completely join-irreducible and completely meet-irreducible elements of perfect DLRs.

With each additive map $f : \mathbb{A} \to \mathbb{B}$ between perfect BDLs, we may associate its *normalization*, that is a map

$$\Diamond_f u = \bigvee\{j \in J^\infty(\mathbb{B}) \mid j \leq f(i) \text{ for some } i \in J^\infty(\mathbb{A}) \text{ such that } i \leq u\}.$$

Order-dually, with each multiplicative map $g : \mathbb{A} \to \mathbb{B}$ between perfect BDLs, we may associate its *normalization*, that is a map $\Box_g : \mathbb{A} \to \mathbb{B}$ such that, for every $u \in \mathbb{A}$,

$$\Box_g u = \bigwedge\{n \in M^\infty(\mathbb{B}) \mid g(m) \leq n \text{ for some } m \in M^\infty(\mathbb{A}) \text{ such that } u \leq m\}.$$

Intuitively, the above definition also states that the normalizations of the maps f and g are the diamond and box operators associated with the dual binary relations defined by f and g, respectively (cf. Section 4.1.4). By definition, the normalizations of f and g are completely join-preserving and completely meet-preserving respectively. Since perfect lattices are complete, this implies that the normalizations are adjoints, i.e., there exist maps[3] $\blacksquare_f, \blacklozenge_g : \mathbb{B} \to \mathbb{A}$ such that for every $u \in \mathbb{A}$ and $v \in \mathbb{B}$,

$$\Diamond_f u \leq v \text{ iff } u \leq \blacksquare_f v \qquad \blacklozenge_g v \leq u \text{ iff } v \leq \Box_g u.$$

The discussion above motivates the following recursive definition of the formulas in the expanded language $\mathcal{L}^+$ (which is the same for both DLR and HAR):

$$\varphi ::= \ \bot \mid \top \mid p \mid \mathbf{j} \mid \mathbf{m} \mid \varphi \vee \varphi \mid \varphi \wedge \varphi \mid \varphi - \varphi \mid \varphi \to \varphi \mid f(\varphi) \mid g(\varphi) \mid \tag{4.9}$$

$$\begin{cases} \blacksquare_f \varphi & \text{if } \eta(f) = 1 \\ \blacktriangleleft_f \varphi & \text{if } \eta(f) = \partial \end{cases} \mid \begin{cases} \blacklozenge_g \varphi & \text{if } \eta(g) = 1 \\ \blacktriangleright_g \varphi & \text{if } \eta(g) = \partial \end{cases} \tag{4.10}$$

where $p \in \mathsf{Prop}$, $\mathbf{j} \in \mathsf{Nom}$, $\mathbf{m} \in \mathsf{CNom}$, $\eta \in \{1, \partial\}$ is a 1 order-type (cf. Page 58), and f, g are unary connectives.

The inclusion of the additional propositional connectives $-$ and $\to$ in the signature above is motivated by the well known fact that perfect DLRs have a natural structure of bi-Heyting algebras. In what follows, we will also use the

[3] Using the alternative notation, there exist maps $\blacktriangleleft_f, \blacktriangleright_g : \mathbb{B} \to \mathbb{A}$ such that for every $u \in \mathbb{A}$ and $v \in \mathbb{B}$,

$$\lhd_f u \leq v \text{ iff } \blacktriangleleft_f v \leq u \qquad v \leq \rhd_g u \text{ iff } u \leq \blacktriangleright_g v$$

connective $\parr$ to denote (meta-)disjunction in the context of quasi-inequalities (i.e., disjunction of a system of quasi-inequalities).

The interpretation of the modal operators is the natural one suggested by the notation, and indeed we are using the same symbols to denote both the logical connectives and their algebraic interpretations.

4.3.2 The algorithm ALBA^r

In what follows, we illustrate we introduce the rules of our calculus ALBA^r. For examples of execution of the algorithm, we refer to Section 4.5.

ALBA^r manipulates input $\mathcal{L}$-inequalities $\varphi \leq \psi$ and proceeds in three stages:

First stage: preprocessing and first approximation. ALBA^r preprocesses the input inequality $\varphi \leq \psi$ by performing the following steps exhaustively:

1. (a) Push down the occurrences of $+\wedge$, $+f$ for $\eta_f = 1$, $-g$ for $\eta_g = \partial$ towards variables, by distributing them over nodes labelled with $+\vee$, and

   (b) Push down the occurrences of $-\vee$, $-g$ for $\eta_g = 1$, $+f$ for $\eta_f = \partial$ towards variables, by distributing them over nodes labelled with $-\wedge$.

2. Apply the splitting rules (applied from top-to-bottom) :

$$\frac{\alpha \leq \beta \wedge \gamma}{\alpha \leq \beta \quad \alpha \leq \gamma} \qquad \frac{\alpha \vee \beta \leq \gamma}{\alpha \leq \gamma \quad \beta \leq \gamma}$$

3. Apply the monotone and antitone variable-elimination rules (applied from top-to-bottom):

$$\frac{\alpha(p) \leq \beta(p)}{\alpha(\bot) \leq \beta(\bot)} \qquad \frac{\beta(p) \leq \alpha(p)}{\beta(\top) \leq \alpha(\top)}$$

   for $\beta(p)$ positive in p and $\alpha(p)$ negative in p.

Let $\mathsf{Preprocess}(\varphi \leq \psi)$ be the finite set $\{\varphi_i \leq \psi_i \mid 1 \leq i \leq n\}$ of inequalities obtained after the exhaustive application of the previous rules. We proceed separately on each of them, and hence, in what follows, we focus only on one element $\varphi_i \leq \psi_i$ in $\mathsf{Preprocess}(\varphi \leq \psi)$, and we drop the subscript. Next, the following *first approximation rule* is applied *only once* to every inequality in $\mathsf{Preprocess}(\varphi \leq \psi)$:

$$\frac{\mathbf{i}_0 \leq \varphi \quad \psi \leq \mathbf{m}_0}{\varphi \leq \psi}$$

Here, $\mathbf{i}_0$ and $\mathbf{m}_0$ are a nominal and a co-nominal respectively. The first-approximation step gives rise to systems of inequalities $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$ for each inequality in $\mathsf{Preprocess}(\varphi \leq \psi)$. Each such system is called an *initial system*, and is now passed on to the reduction-elimination cycle.

Second stage: reduction-elimination cycle. The goal of the reduction-elimination cycle is to eliminate all propositional variables from the systems which it receives from the preprocessing phase. The elimination of each variable is effected by an application of one of the Ackermann rules given below. In order to apply an Ackermann rule, the system must have a specific shape. The adjunction, residuation, approximation, and splitting rules are used to transform systems into this shape. The rules of the reduction-elimination cycle, viz. the adjunction, residuation, approximation, splitting, and Ackermann rules, will be collectively called the *reduction* rules.

Residuation rules.

$$\frac{\varphi \wedge \psi \leq \chi}{\psi \leq \varphi \rightarrow \chi} \qquad \frac{\varphi \leq \psi \vee \chi}{\varphi - \chi \leq \psi}$$

In the HAR setting, the rule on the left-hand side above is allowed to be executed bottom-to-top.

Adjunction rules.

$$\frac{f(\varphi) \leq \psi}{f(\bot) \leq \psi \quad \varphi \leq \blacksquare_f \psi} \text{ (if } \eta_f = 1) \qquad \frac{\varphi \leq g(\psi)}{\varphi \leq g(\top) \quad \blacklozenge_g \varphi \leq \psi} \text{ (if } \eta_g = 1)$$

$$\frac{f(\varphi) \leq \psi}{f(\top) \leq \psi \quad \blacktriangleleft_\lambda \psi \leq \varphi} \text{ (if } \eta_f = \partial) \qquad \frac{\varphi \leq g(\psi)}{\varphi \leq g(\bot) \quad \psi \leq \blacktriangleright_\rho \varphi} \text{ (if } \eta_g = \partial)$$

In a given system, each of these rules replaces an instance of the upper inequality with the corresponding instances of the two lower inequalities.

The leftmost inequalities in each rule above will be referred to as the *side condition*.

Approximation rules. The following rules are applicable if $\eta_f = \eta_g = 1$:

$$\frac{\mathbf{i} \leq f(\varphi)}{[\mathbf{i} \leq f(\bot)] \ \parr \ [\mathbf{j} \leq \varphi \quad \mathbf{i} \leq f(\mathbf{j})]} \qquad \frac{g(\psi) \leq \mathbf{m}}{[g(\top) \leq \mathbf{m}] \ \parr \ [\psi \leq \mathbf{n} \quad g(\mathbf{n}) \leq \mathbf{m}]}$$

If $\eta_f = \eta_g = \partial$, the following approximation rules are applicable:

$$\frac{\mathbf{i} \leq f(\varphi)}{[\mathbf{i} \leq f(\top)] \ \parr \ [\varphi \leq \mathbf{m} \quad \mathbf{i} \leq f(\mathbf{m})]} \qquad \frac{g(\psi) \leq \mathbf{m}}{[g(\bot) \leq \mathbf{m}] \ \parr \ [\mathbf{i} \leq \psi \quad g(\mathbf{i}) \leq \mathbf{m}]}$$

The leftmost inequalities in each rule above will be referred to as the side condition.

Each approximation rule transforms a given system $S \cup \{s \leq t\}$ into systems $S \cup \{s_1 \leq t_1\}$ and $S \cup \{s_2 \leq t_2, s_3 \leq t_3\}$, the first of which containing only the side condition (in which no propositional variable occurs), and the second one containing the instances of the two remaining lower inequalities.

The nominals and co-nominals introduced by the approximation rules must be *fresh*, i.e. must not already occur in the system before applying the rule.

Additional approximation rules for the HAR setting.

$$\frac{\varphi \rightarrow \chi \leq \mathbf{m}}{\mathbf{j} \leq \varphi \ \&\ \chi \leq \mathbf{n} \ \&\ \mathbf{j} \rightarrow \mathbf{n} \leq \mathbf{m}}\ (\rightarrow\text{Appr}_1)$$

Ackermann rules. These rules are the core of ALBA^r, since their application eliminates proposition variables. As mentioned earlier, all the preceding steps are aimed at equivalently rewriting the input system into one or more systems, each of which is of a shape in which the Ackermann rules can be applied. An important feature of Ackermann rules is that they are executed on the full set of inequalities in which a given variable occurs, and not on a single inequality.

$$\frac{\exists p \left[\&_{i=1}^{n}\{\alpha_i \leq p\} \ \ \&\ \ \&_{j=1}^{m}\{\beta_j(p) \leq \gamma_j(p)\} \right]}{\&_{j=1}^{m}\{\beta_j(\bigvee_{i=1}^{n}\alpha_i) \leq \gamma_j(\bigvee_{i=1}^{n}\alpha_i)\}} (RAR)$$

where p does not occur in $\alpha_1, \ldots, \alpha_n$, the formulas $\beta_1(p), \ldots, \beta_m(p)$ are positive in p, and $\gamma_1(p), \ldots, \gamma_m(p)$ are negative in p. Here below is the left-Ackermann rule:

$$\frac{\exists p \left[\&_{i=1}^{n}\{p \leq \alpha_i\} \ \ \&\ \ \&_{j=1}^{m}\{\beta_j(p) \leq \gamma_j(p)\} \right]}{\&_{j=1}^{m}\{\beta_j(\bigwedge_{i=1}^{n}\alpha_i) \leq \gamma_j(\bigwedge_{i=1}^{n}\alpha_i)\}} (LAR)$$

where p does not occur in $\alpha_1, \ldots, \alpha_n$, the formulas $\beta_1(p), \ldots, \beta_m(p)$ are negative in p, and $\gamma_1(p), \ldots, \gamma_m(p)$ are positive in p.

Third stage: output. If there was some system in the second stage from which not all occurring propositional variables could be eliminated through the application of the reduction rules, then ALBA^r reports failure and terminates. Else, each system $\{\mathbf{i}_0 \leq \varphi_i, \psi_i \leq \mathbf{m}_0\}$ obtained from $\mathsf{Preprocess}(\varphi \leq \psi)$ has been reduced to a system, denoted $\mathsf{Reduce}(\varphi_i \leq \psi_i)$, containing no propositional variables. Let $\mathsf{ALBA}^r(\varphi \leq \psi)$ be the set of quasi-inequalities

$$\&[\mathsf{Reduce}(\varphi_i \leq \psi_i)] \Rightarrow \mathbf{i}_0 \leq \mathbf{m}_0$$

for each $\varphi_i \leq \psi_i \in \mathsf{Preprocess}(\varphi \leq \psi)$.

Notice that all members of $\mathsf{ALBA}^r(\varphi \leq \psi)$ are free of propositional variables. Hence, translating them as discussed in Section 4.5 produces sentences in the language of the Kripke frames with impossible worlds dual to the perfect r-BAEs which hold simultaneously on those Kripke frames with impossible worlds iff the input inequality is valid on them. ALBA^r returns $\mathsf{ALBA}^r(\varphi \leq \psi)$ and terminates. An inequality $\varphi \leq \psi$ on which ALBA^r succeeds will be called an ALBA^r-*inequality*.

4.3.3 Soundness and canonicity of ALBA^r

In this section we give an outline of the proof that ALBA^r is sound, i.e. for any ALBA^r-inequality $\varphi \leq \psi$, for any DLR $\mathbb{A}$,

$$\mathbb{A}^\delta \models \varphi \leq \psi \qquad \text{iff} \qquad \mathbb{A}^\delta \models \mathsf{ALBA}^r(\varphi \leq \psi),$$

where $\mathbb{A}^\delta$ is the canonical extension of $\mathbb{A}$ (cf. Definition 4.1.15).

4.3.1. Theorem (Soundness). *For any ALBA^r-inequality $\varphi \leq \psi$, for any perfect $\mathcal{L}$-algebra $\mathbb{A}$,*

$$\mathbb{A} \models \varphi \leq \psi \textit{ iff } \mathbb{A} \models \mathsf{ALBA}^r(\varphi \leq \psi).$$

Proof. The rules that deserve discussion are the approximation and the adjunction rules, the soundness of which can be proved, by deriving them from a set of rules which includes the following ones, which, for the sake of conciseness, are given as formula-rewriting rules:

$$\eta(f) = 1\ \frac{f(p)}{f(\bot) \vee \Diamond_f(p)} \qquad \frac{g(p)}{g(\top) \wedge \Box_g(p)}\ \eta(g) = 1$$

$$\eta(f) = \partial\ \frac{f(p)}{f(\top) \vee \lhd_f(p)} \qquad \frac{g(p)}{g(\bot) \wedge \rhd_g(p)}\ \eta(g) = \partial$$

Notice that in any perfect DLR $\mathbb{A}$, the interpretations of the connectives f and g are completely additive and multiplicative, respectively. Hence, it is easy to see that

$$\mathbb{A} \models f(p) = f(\bot) \vee \Diamond_f(p) \text{ if } \eta(f) = 1 \qquad \mathbb{A} \models g(p) = g(\top) \wedge \Box_g(p) \text{ if } \eta(g) = 1$$

$$\mathbb{A} \models f(p) = f(\top) \vee \lhd_f(p) \text{ if } \eta(f) = \partial \qquad \mathbb{A} \models g(p) = g(\bot) \wedge \rhd_g(p) \text{ if } \eta(g) = \partial$$

which proves the soundness and invertibility of the rules above on any perfect DLR $\mathbb{A}$. Let us give two derivations as examples: $\eta(f) = \partial$ in the left-hand one and $\eta(f) = 1$ in the right-hand one.

$$\dfrac{\dfrac{\dfrac{f(\varphi) \leq \psi}{f(\top) \vee \lhd_f(\varphi) \leq \psi}}{f(\top) \leq \psi \quad \lhd_f(\varphi) \leq \psi}}{f(\top) \leq \psi \quad \blacktriangleleft_\lambda \psi \leq \varphi} \qquad \dfrac{\dfrac{\dfrac{\dfrac{\mathbf{i} \leq f(\psi)}{\mathbf{i} \leq f(\bot) \vee \Diamond_f(\psi)}}{[\mathbf{i} \leq f(\bot)] \ ⅋ \ [\mathbf{i} \leq \Diamond_f(\psi)]}}{[\mathbf{i} \leq f(\bot)] \ ⅋ \ [\mathbf{j} \leq \psi \quad \mathbf{i} \leq \Diamond_f(\mathbf{j})]}}{[\mathbf{i} \leq f(\bot)] \ ⅋ \ [\mathbf{j} \leq \psi \quad \mathbf{i} \leq f(\mathbf{j})]}$$

□

4.3.2. Definition. An execution of ALBA^r is *safe* if no side conditions (cf. Page 69) introduced by applications of adjunction rules for the new connectives are further modified, except for receiving Ackermann substitutions.

4.3.3. Theorem. *All inequalities on which* ALBA^r *safely succeeds are canonical.*

Proof. Let $\varphi \leq \psi$ be an inequality such that there exists a safe and successful execution of ALBA^r on it. In order to prove canonicity of $\varphi \leq \psi$, we follow the U-shaped argument illustrated below:

$$\begin{array}{ccc}
\mathbb{A} \models \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\
\Updownarrow & & \\
\mathbb{A}^\delta \models_A \varphi \leq \psi & & \Updownarrow \\
\Updownarrow & & \\
\mathbb{A}^\delta \models_A \mathsf{ALBA}^r(\varphi \leq \psi), & \Leftrightarrow & \mathbb{A}^\delta \models \mathsf{ALBA}^r(\varphi \leq \psi),
\end{array}$$

where the notation $\mathbb{A}^\delta \models_A \varphi \leq \psi$ denotes that the inequality is valid in the canonical extension $\mathbb{A}^\delta$ for all admissible assignments (i.e., assignments whose co-domain is restricted to the original DLR $\mathbb{A}$). In order to complete the proof, we need to argue that under the assumption of the theorem, all the rules of ALBA^r are sound on $\mathbb{A}^\delta$, both under arbitrary and under admissible assignments. The soundness of the approximation and adjunction rules for the new connectives has been discussed in Section 4.3.1, and the argument is entirely similar for arbitrary and for admissible valuations. The only rule which needs to be expanded on is the Ackermann rule under admissible assignments. This soundness follows from Propositions B.2.5 and Lemma B.2.4. □

Before moving on, let us briefly discuss the specific HAR-setting w.r.t. canonicity. The only difference between HAR and DLR is that the logical connective $\rightarrow$ is part of the original signature. Hence, in the HAR setting, the connective $\rightarrow$ has better topological/order-theoretic properties than it has in the DLR setting, namely that an HAR is closed with respect to the operation $\rightarrow$.

4.4 Sahlqvist and Inductive DLR- and HAR- inequalities

In this section, we define the class of inductive DLR- and HAR-inequalities (from now on abbreviated as inductive inequalities, unless we need to distinguish the two languages), and prove that ALBA^r succeeds on each of them with a safe run. In the light of Theorems 4.3.1 and 4.3.3, this will show that inductive inequalities are elementary and canonical. Unlike the corresponding definitions in [50, Section 3], the definitions below are given in terms of a positive classification which is based on the order-theoretic properties of the algebraic interpretations of the logical connectives..

We recall that an order-type over $n \in \mathbb{N}$ is an n-tuple $\varepsilon \in \{1, \partial\}^n$. For every order-type ε, let ε^∂ be the *opposite* order-type, i.e., $\varepsilon_i^\partial = 1$ iff $\varepsilon_i = \partial$ for every $1 \leq i \leq n$. A DLR- or HAR- term s can be associated with its positive (resp. negative) *signed generation tree* defined below.

4.4.1. Definition. The *positive (resp. negative) signed generation tree* for a term s is defined as follows:

- The root node $+s$ (resp. $-s$) is the root node of the positive (resp. negative) generation tree of s signed with $+$ (resp. $-$).
- If a node is labelled with $\vee, \wedge$, assign the same sign to its child node(s).
- If a node is labelled with f, g, assign the same sign to its child node with order-type 1 and different sign to its child node with order-type ∂.

We say that a node in the signed generation tree is *positive* (resp. *negative*), if it is signed $+$ (resp. $-$). We note that a subterm α of a term s generates a subtree of s, namely the one which has α as its root. We will also make use of the *sub-tree relation* $\gamma \prec \varphi$, which extends to signed generation trees.

For any term $s(p_1, \ldots p_n)$, any order-type ε over n, and any $1 \leq i \leq n$, an *ε-critical node* in a signed generation tree of s is a leaf node $+p_i$ with $\varepsilon_i = 1$ or $-p_i$ with $\varepsilon_i = \partial$. An *ε-critical branch* in the tree is a branch ending in an ε-critical node.

For every term $s(p_1, \ldots p_n)$ and every order-type ε, we say that $*s$ ($* \in \{+, -\}$) is *ε-uniform*, or that $*s$ *agrees with* ε, if every leaf in the signed generation tree of $*s$ is ε-critical. A signed term $*s$ is *uniform* if it is ε-uniform for some order-type ε. We will write $\varepsilon(\gamma) \prec *\varphi$ to indicate that γ, regarded as a sub- (signed generation) tree of $*\varphi$, agrees with ε.

4.4.2. Example. We illustrate the notion of a signed generation tree by means of an example. Consider the axiom $\Box(p \to q) \to \Box(\Box p \to \Box q)$. The positive and

Figure 4.1: Signed generation tree for $\Box(p \to q) \to \Box(\Box p \to \Box q)$

the negative signed generation trees for the terms $\Box(p \to q)$ and $\Box(\Box p \to \Box q)$, respectively, are shown in Figure 4.1. In the negative signed generation tree of $\Box(\Box p \to \Box q)$, since $+\Box p$ occurs as a sub- (signed generation) tree of $-\Box(\Box p \to \Box q)$, we say that $+\Box p \prec -\Box(\Box p \to \Box q)$. For the order-type $\varepsilon_p = \partial$, and $\varepsilon_q = \partial$, the ε-critical branches are the branch ending in $-p$ in positive signed generation tree of $\Box(p \to q)$, and the branch ending in $-q$ in the negative signed generation tree of $\Box(\Box p \to \Box q)$. The signed term $+\Box(p \to q)$ is ε-uniform with respect to the order-type $\varepsilon_p = \partial$ and $\varepsilon_q = 1$, since all the branches in the positive signed generation tree of $\Box(p \to q)$ are ε-critical for this order-type.

Skeleton	PIA
Δ-adjoints	SMP
$+$ $\vee$ $\wedge$	$+$ $\wedge$ g
$-$ $\wedge$ $\vee$	$-$ $\vee$ f
SAC	SRR
$+$ $\vee$ $\wedge$ f	$+$ $\vee$ $\rightarrow$
$-$ $\rightarrow$ $\wedge$ $\vee$ g	$-$ $\wedge$

Table 4.1: Classification of nodes for HAR and DLR.

4.4.3. Definition. Nodes in signed generation trees will be called Δ-*adjoints*, *syntactically additive coordinate-wise (SAC)*, *syntactically p-multiplicative (SMP)*, *syntactically right residual (SRR)* according to the specification given in Table 4.1. The acronym PIA in the above table stands for Positive Implies Atomic (cf. Remark 4.4.4). For $* \in \{+, -\}$, a branch in a signed generation tree $*s$ is:

- a *good branch* if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes, and P_2 consists (apart from variable nodes) only of Skeleton-nodes.

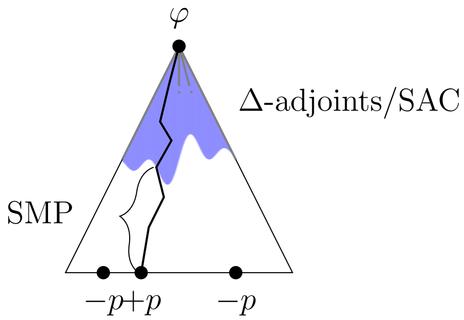

Figure 4.2: Signed generation tree for a regular Sahlqvist antecedent

- an *excellent branch* if it is good, and moreover P_1 consists (apart from variable nodes) only of SMP-nodes.

4.4.4. Remark. Let us expand on the criteria motivating the classification of nodes in the Table 4.1. Firstly, we are using two types of names: "meaningful" names such as *syntactically right residual*, and more "cryptic" names, such as Skeleton and PIA. Hence, the resulting classification has two layers, one of which accounts for what the connectives are order theoretically, and the other for what they are for, in the context of ALBA. Specifically, the "meaningful" names explicitly refer to intrinsic order-theoretic properties of the interpretation of the logical connectives. For example, the SAC connectives will be interpreted as operations which are additive in each coordinate (modulo order-type). The more "cryptic" names classify connectives in terms of the kind of rules which will be applied to them by ALBA. The idea is that only approximation/splitting rules are to be applied to skeleton nodes, with the aim of surfacing the PIA subterms containing the ε-critical occurrences of propositional variables, and only residuation/adjunction rules are to be applied to PIA nodes, with the aim of computing the "minimal valuations". The fact that these rules can be soundly applied is guaranteed by the intrinsic order-theoretic properties. Sometimes, the intrinsic order-theoretic properties of a connective are such that other rules are also soundly applicable to it. However, the Skeleton/PIA classification indicates that this is not required in order to reach Ackermann shape. The term Skeleton formula comes from [17]. The acronym PIA was introduced by van Benthem in [14]. The analysis of PIA-formulas conducted in [14, 17] can be summarized in the slogan "PIA formulas provide minimal valuations", which is precisely the role of those terms which we call PIA-terms here. Again, this choice of terminology is not based on the original syntactic description of van Benthem, but rather on which rules are best being applied to them in order to be guaranteed success of the execution of the algorithm. In this respect, the crucial property possessed by PIA-formulas in the setting of normal modal logic is the *intersection property*, isolated by van Benthem in [14], which is enjoyed by those formulas which, seen as operations on the complex algebra of a frame, preserve arbitrary intersections of subsets. The order-theoretic import of this property is clear: a formula has the intersection property iff the term function associated with it is completely meet-preserving. In the complete lattice setting, this is equivalent to it being a right adjoint; this is exactly the order-theoretic property guaranteeing the soundness of adjunction/residuation rules in the setting of normal modal logic. This terminology is also used later Chapters 5 and 6 of this thesis to establish a connection with analogous terminology in [17].

4.4.5. Definition. Given an order-type ε, the signed generation tree $*s$ (for $* \in \{-,+\}$) of a term $s(p_1, \ldots p_n)$ is *ε-regular Sahlqvist* (ε-DLR-Sahlqvist) if for all $1 \leq i \leq n$, every ε-critical branch with leaf labelled p_i is excellent.

An inequality $s \leq t$ is *ε-regular Sahlqvist* if the trees $+s$ and $-t$ are both ε-regular Sahlqvist. An inequality $s \leq t$ is *regular Sahlqvist* (DLR-Sahlqvist) if it is ε-regular Sahlqvist for some ε.

4.4.6. Example. Lemmon's axioms (cf. Subsection 4.1.1) are examples of DLR-Sahlqvist formulas/inequalities. Indeed, the following axioms are DLR-Sahlqvist for the order-type $\varepsilon_p = 1$:

$$(2)\ \Box p \rightarrow p, \qquad (4)\ \Box p \rightarrow \Box\Box p.$$

For the axiom $\Box p \rightarrow p$, the positive signed generation tree of $\Box p$ contains the PIA node $+\Box$, which is an SMP node, on the ε-critical branch (i.e., branch ending in $+p$). Therefore, it is an excellent branch, and hence the axiom $\Box p \rightarrow p$ is Sahlqvist. Since the formula $\Box p \rightarrow \Box\Box p$ has $\Box p$ as an antecedent, it is also Sahlqvist for the same reason.

The following axiom is DLR-Sahlqvist for the order-type $\varepsilon_p = \partial$:

$$(5)\ \neg\Box p \rightarrow \Box\neg\Box p.$$

For the order-type $\varepsilon_p = \partial$, the positive signed generation tree of $\neg\Box p$ contains the only SAC node $-\Box$. Hence, the ε-critical branch is excellent, which implies that the axiom $\neg\Box p \rightarrow \Box\neg\Box p$ is Sahlqvist.

The following axioms are DLR-Sahlqvist for the order-type $\varepsilon_p = 1, \varepsilon_q = \partial$:

$$(1)\ \Box(p \rightarrow q) \rightarrow \Box(\Box p \rightarrow \Box q) \text{ and } (1')\ \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q).$$

For the axiom $\Box(p \rightarrow q) \rightarrow \Box(\Box p \rightarrow \Box q)$, and the order-type $\varepsilon_p = 1, \varepsilon_q = \partial$, the critical branches are the branches ending in $+p$ and $-q$ in the negative signed generation tree of the term $\Box(\Box p \rightarrow \Box q)$ (cf. Figure 4.4.1). The path from leaf in the branch ending in $+p$ contains an SMP node ($+\Box$) followed by SAC nodes ($-\rightarrow$ and $-\Box$), which makes it excellent. The other branch ending in $-q$ is also an excellent branch, since it contains only SAC nodes ($-\Box$, $-\rightarrow$ and $-\Box$). Therefore, the axiom (1) is Sahlqvist. A similar reasoning shows that the axiom ($1'$) is Sahlqvist.

Note that the axioms (1) and (1') are not DLR-Sahlqvist for the natural order-type $\varepsilon_p = 1, \varepsilon_q = 1$. For the axiom (1), and the order-type $\varepsilon_p = 1, \varepsilon_q = 1$, the branch ending in $+q$ in the positive signed generation tree of $\Box(p \rightarrow q)$ is a critical branch. However, this branch is not an excellent branch as it contains an SRR node ($+\rightarrow$). Therefore, the axiom (1) is not Sahlqvist for the natural order-type.

4.4.7. Definition. [Inductive DLR-inequalities] For any order-type ε and any strict partial order $<_\Omega$ on $p_1, \ldots p_n$, the signed generation tree $*s$, $* \in \{-, +\}$, of a term $s(p_1, \ldots p_n)$ is (Ω, ε)-*inductive* if for all $1 \leq i \leq n$ every ε-critical branch with leaf p_i is good, and moreover, every binary SRR node occurring in it is of the form $*(\alpha \circledast \beta)$, where:

1. $\varepsilon^\partial(\alpha) \prec *s$ (cf. Definition 4.4.1), and

2. $p_k <_\Omega p_i$ for every p_k occurring in α.

We will refer to $<_\Omega$ as the *dependency order* on the variables. An inequality $s \leq t$ is (Ω, ε)-*inductive* if the trees $+s$ and $-t$ are both (Ω, ε)-inductive. An inequality $s \leq t$ is *inductive* if it is (Ω, ε)-inductive for some Ω and ε.

Notice that, in the DLR-signature, SRR nodes can be either $+\vee$ or $-\wedge$, whereas in the HAR-signature, SRR nodes can also be $+\rightarrow$, and hence the corresponding subtree of $*s$ is either $+(\alpha \vee \beta)$ or $-(\alpha \wedge \beta)$ or $+(\alpha \rightarrow \beta)$. In each of the two settings, since excellent branches are in particular good, it is easy to see that Sahlqvist inequalities are special inductive inequalities.

4.4.8. Example. Regarded as HAR-formulas/inequalities, the following axioms from Section 4.1,

$$(1)\ \Box(p \rightarrow q) \rightarrow \Box(\Box p \rightarrow \Box q) \quad \text{and} \quad (1')\ \Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$$

which are Sahlqvist for the order-type $\varepsilon_p = 1, \varepsilon_q = \partial$, are inductive, but not Sahlqvist, for the natural order-type $\varepsilon_p = \varepsilon_q = 1$ and dependency order $p <_\Omega q$. As mentioned earlier (cf. Example 4.4.6), for the axiom (1), with the natural order-type $\varepsilon_p = \varepsilon_q = 1$, the positive signed generation tree of $\Box(p \rightarrow q)$ contains an SRR node ($+\rightarrow$). Hence, it is not a Sahlqvist formula for that order-type. However, in the positive signed generation tree of $\Box(p \rightarrow q)$, we have that $-p \prec +\Box(p \rightarrow q)$, i.e., $\varepsilon^\partial(p) \prec +\Box(p \rightarrow q)$. Moreover, $p <_\Omega q$ from our assumption. Hence, the term $\Box(p \rightarrow q)$ satisfies both the conditions in the Definition 4.4.7, and is therefore inductive for the natural order-type.

Due to its technical nature, the proof of Theorem 4.4.9 below is included in the Appendix A (see Section A.1). The intuitive idea of the proof is that each rule in the algorithm (i.e., approximation rule, adjunction rule, Ackermann rule) transforms the syntactic shape of the input inductive inequality to a different syntactically defined shape. This eventually ensures that all the propositional variables in the input inductive inequality are eliminated after the algorithm terminates.

We illustrate the proof by means of an example. Consider the axiom $\Box p \rightarrow \Box\Box p$, which is Sahlqvist (cf. Example 4.4.6). For a detailed reduction of the axiom to its first-order correspondent, see Proposition 4.5.2. The axiom does not need any preprocessing, and hence its signed generation is *definite* (Ω, ε)*-inductive* (cf. Definition A.1.3), that is, there are no occurrences of $+\vee$ or $-\wedge$ nodes on ε-critical branches. The first approximation produces the system consisting of inequalities $\mathbf{i} \leq \Box p$, $\Box\Box p \leq \mathbf{m}$ and $\mathbf{i} \leq \mathbf{m}$. Each of these inequalities are in *definite* (Ω, ε)*-good shape* (cf. Definition A.1.5), that is, an inequality $s \leq t$ where s is pure and $+t$ is definite (Ω, ε)-inductive, or t is pure and $-s$ is definite (Ω, ε)-inductive. The next step in the algorithm is the application of the adjunction rule for $\Box$ which produces the inequalities $\mathbf{i} \leq \Box\top$, $\blacklozenge\mathbf{i} \leq p$, $\Box\Box p \leq \mathbf{m}$ and $\mathbf{i} \leq \mathbf{m}$. This system is in the (Ω, ε)*-Ackermann form* (cf. Definition A.1.7) with respect to p, that is, for each inequality $s \leq t$ in the system, either s is pure and $t = p$, or s is positive in p and t is negative in p. Finally, the application of the Ackermann rule produces the pure system consisting of inequalities $\mathbf{i} \leq \Box\top$ and $\Box\Box\Diamond\mathbf{i} \leq \mathbf{m}$ and $\mathbf{i} \leq \mathbf{m}$.

4.4.9. Theorem. *Each inductive inequality admits a safe and successful execution of* ALBA^r.

As a corollary of Theorems 4.3.1, 4.3.3, and 4.4.9 we obtain:

4.4.10. Theorem. *All inductive* $\mathcal{L}$*-inequalities are elementary and canonical.*

4.5 Applications to Lemmon's logics

In this section, we apply the theory developed so far to Lemmon's logics E2-E5 (cf. Definition 54). As to E2-E4, we will show that they are strongly complete with respect to elementary classes of Kripke models with impossible worlds. Moreover, we will give a semantic proof to Kripke's statement in [111] that E5 coincides with S5. We have already seen (cf. Example 4.4.6) that each axiom involved in the axiomatization of these logics is Sahlqvist. By Theorem 4.5.3, this implies that E2-E5 are strongly complete and elementary. In Subsection 4.5.1, we adapt the definition of standard translation given in [50, Subsection 2.5.2] to the setting of Kripke frames with impossible worlds. In Subsection 4.5.2, we effectively compute the first-order conditions defining their associated classes of Kripke frames with impossible worlds by providing a successful and safe run of ALBA^r on each axiom. We would like to point out that our setting is Boolean for what follows in this section.

4.5.1 Standard translation

Let L_1 be the first-order language with equality with binary relation symbols R, and unary predicate symbols $P, Q, \ldots$ corresponding to the propositional variables $p, q, \ldots \in \mathsf{Prop}$ and unary predicate symbol N for normal worlds. As usual, we let L_1 contain a denumerable infinity of individual variables. We will further assume that L_1 contains denumerably infinite individual variables $i, j, \ldots$ corresponding to the nominals $\mathbf{i}, \mathbf{j}, \ldots \in \mathsf{Nom}$ and $n, m, \ldots$ corresponding to the co-nominals $\mathbf{n}, \mathbf{m} \in \mathsf{CNom}$. Let L_0 be the sub-language which does not contain the unary predicate symbols $P, Q, \ldots$ corresponding to the propositional variables. Let us now define the *standard translation* of $\mathcal{L}^+$ into L_1 recursively:

$$
\begin{array}{lcl}
\mathrm{ST}_x(\bot) & := & x \not\equiv x \\
\mathrm{ST}_x(\top) & := & x \equiv x \\
\mathrm{ST}_x(p) & := & P(x) \\
\mathrm{ST}_x(\mathbf{j}) & := & j \equiv x \\
\mathrm{ST}_x(\mathbf{m}) & := & x \not\equiv m \\
\mathrm{ST}_x(\neg\varphi) & := & \neg\mathrm{ST}_x(\varphi)
\end{array}
\qquad
\begin{array}{lcl}
\mathrm{ST}_x(\varphi \to \psi) & := & \mathrm{ST}_x(\varphi) \to \mathrm{ST}_x(\psi) \\
\mathrm{ST}_x(\varphi \vee \psi) & := & \mathrm{ST}_x(\varphi) \vee \mathrm{ST}_x(\psi) \\
\mathrm{ST}_x(\varphi \wedge \psi) & := & \mathrm{ST}_x(\varphi) \wedge \mathrm{ST}_x(\psi) \\
\mathrm{ST}_x(\Diamond\varphi) & := & \neg Nx \vee \exists y(Rxy \wedge \mathrm{ST}_y(\varphi)) \\
\mathrm{ST}_x(\Box\varphi) & := & Nx \wedge \forall y(Rxy \to \mathrm{ST}_y(\varphi)) \\
\mathrm{ST}_x(\blacklozenge\varphi) & := & \exists y(Ryx \wedge \mathrm{ST}_y(\varphi)) \\
\mathrm{ST}_x(\blacksquare\varphi) & := & \forall y(Ryx \to \mathrm{ST}_y(\varphi))
\end{array}
$$

ST_x extends to inequalities and quasi-inequalities as follows: for inequalities, $\mathrm{ST}_x(\varphi \leq \psi) := \mathrm{ST}_x(\varphi) \to \mathrm{ST}_x(\psi)$, and for quasi-inequalities, $\mathrm{ST}_x(\varphi_1 \leq \psi_1 \,\&\, \cdots \,\&\, \varphi_n \leq \psi_n \Rightarrow \varphi \leq \psi) := [\mathrm{ST}_x(\varphi_1 \leq \psi_1) \wedge \cdots \wedge \mathrm{ST}_x(\varphi_n \leq \psi_n)] \to \mathrm{ST}_x(\varphi \leq \psi)$. We also extend ST_x to finite sets of inequalities by declaring $\mathrm{ST}_x(\{\varphi_1 \leq \psi_1, \ldots, \varphi_n \leq \psi_n\}) := \bigwedge_{1 \leq i \leq n} \mathrm{ST}_x(\varphi_i \leq \psi_i)$.

Observe that if ST_x is applied to pure terms, inequalities, or quasi-inequalities, it produces formulas in the sublanguage L_0. The following lemma is proved by a routine induction.

4.5.1. Lemma. *For any state w in a Kripke frame with impossible worlds $\mathcal{F}$ and for every formula, inequality or quasi-inequality ξ in the language $\mathcal{L}^+$,*

1. *$\mathcal{F}, w \Vdash \xi$ iff $\mathcal{F} \models \forall\overline{P}\forall\overline{j}\forall\overline{m}\mathrm{ST}_x(\xi)[x := w]$, and*

2. *$\mathcal{F} \Vdash \xi$ iff $\mathcal{F} \models \forall x\forall\overline{P}\forall\overline{j}\forall\overline{m}\mathrm{ST}_x(\xi)$,*

where $\overline{P}$, $\overline{j}$, and $\overline{m}$ are, respectively, the vectors of all predicate symbols corresponding to propositional variables, individual variables corresponding to nominals, and individual variables corresponding to co-nominals, occurring in $\mathrm{ST}_x(\xi)$.

4.5.2 Strong completeness and elementarity of E2-E5

In this subsection, we provide successful and safe runs of the algorithm ALBA^r on each of the axioms in E2-E5. Thereafter, we use the standard translation defined in the previous subsection to give an interpretation of the pure inequalities on Kripke frames with impossible worlds.

4.5.2. Proposition. *The following axioms in Lemmon's system are canonical.*

(1)	$\Box(p \to q) \to \Box(\Box p \to \Box q)$	*(1')*	$\Box(p \to q) \to (\Box p \to \Box q)$
(2)	$\Box p \to p$		
(4)	$\Box p \to \Box\Box p$	*(5)*	$\neg\Box p \to \Box\neg\Box p.$

Proof. As discussed in Example 4.4.6, the above axioms are DLR-Sahlqvist. Hence, by Theorem 4.3.3, they are canonical. $\Box$

In what follows, we compute the first-order frame correspondents of the above axioms. By Theorem 4.3.1, each of the logics E2-E5 is strongly complete with respect to the class of frames defined by the first-order frame correspondents of its axioms (see Tables 4.3 and 4.4).

- (2) $\Box p \to p$

  $\forall p(\Box p \leq p)$
  $\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box p \ \&\ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$
  $\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box\top \ \&\ \blacklozenge\mathbf{i} \leq p \ \&\ p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$ (adjunction rule for $\Box$)
  $\forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box\top \ \&\ \blacklozenge\mathbf{i} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$ (Ackermann rule)
  $\forall \mathbf{i}[\mathbf{i} \leq \Box\top \Rightarrow \mathbf{i} \leq \blacklozenge\mathbf{i}]$.

  Using the standard translation, the last clause above translates to the sentence below, which is then further simplified:

  $\forall i \forall x((i \equiv x \to Nx) \to (i \equiv x \to \exists y(Ryx \land i \equiv y))$
  $\forall i \forall x(Ni \to \exists y(Ryi \land i \equiv y))$
  $\forall i(Ni \to Rii)$.

- (4) $\Box p \to \Box\Box p$

  $\forall p(\Box p \leq \Box\Box p)$
  $\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box p \ \&\ \Box\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$
  $\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box\top \ \&\ \blacklozenge\mathbf{i} \leq p \ \&\ \Box\Box p \leq \mathbf{m}) \Rightarrow (\mathbf{i} \leq \mathbf{m})]$ (adjunction rule for $\Box$)
  $\forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box\top \ \&\ \Box\Box\blacklozenge\mathbf{i} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$ (Ackermann rule)
  $\forall \mathbf{i}[\mathbf{i} \leq \Box\top \Rightarrow \mathbf{i} \leq \Box\Box\blacklozenge\mathbf{i}]$.

  Using the standard translation and after some simplifying steps, the clause above we get

$\forall i(Ni \to \forall y(Riy \to Ny \wedge \forall z(Ryz \to Riz))))$,

which is equivalent to

$\forall i \forall y \forall z(Ni \wedge Ny \wedge Riy \wedge Ryz \to Riz) \wedge \forall i \forall y(Ni \wedge Riy \to Ny)$.

- (5) $\neg\Box p \to \Box\neg\Box p$

  The validity of (5) $\neg\Box p \to \Box\neg\Box p$ is equivalent to the validity of $\Diamond p \to \Box\Diamond p$, which can be reduced as follows.

  $\forall p(\Diamond p \to \Box\Diamond p)$

  $\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Diamond p \ \&\ \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$

  $\forall p \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m}[((\mathbf{i} \leq \Diamond\bot \ ⅋\ (\mathbf{i} \leq \Diamond\mathbf{j} \ \&\ \mathbf{j} \leq p)) \ \&\ \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$ (*)

  $\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Diamond\bot \ \&\ \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\ \&$
  $\forall p \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m}[(\mathbf{i} \leq \Diamond\mathbf{j} \ \&\ \mathbf{j} \leq p \ \&\ \Box\Diamond p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$

  $\forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Diamond\bot \ \&\ \Box\Diamond\bot \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\ \&$
  $\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m}[(\mathbf{i} \leq \Diamond\mathbf{j} \ \&\ \Box\Diamond\mathbf{j} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$ (**)

  $[\Diamond\bot \leq \Box\Diamond\bot]\ \&\ \forall \mathbf{j}[(\Diamond\mathbf{j} \leq \Box\Diamond\mathbf{j}]$.

  In the step (*) above, adjunction rule for $\Diamond$ is applied, and in the step (**), Ackermann rule is applied eliminate p. Using the standard translation and after some simplifying steps, the clause above can be rewritten as:

  $\forall x Nx \wedge \forall x \forall y \forall z(Nx \wedge Ny \wedge Rxy \wedge Rxz \to Ryz)$.

  Hence, the frame correspondent of $\neg\Box p \to \Box\neg\Box p$ is $\forall x Nx \wedge \forall x \forall y \forall z(Nx \wedge Ny \wedge Rxy \wedge Rxz \to Ryz)$.

- (1) $\Box(p \to q) \to \Box(\Box p \to \Box q)$

  In what follows, we run the algorithm ALBA^r with respect to the natural order-type $\varepsilon_p = 1, \varepsilon_q = 1$. Notice that the axioms (1) above is inductive but not DLR-Sahlqvist w.r.t. this order-type.

  $\forall p \forall q \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box(p \to q) \ \&\ \Box(\Box p \to \Box q) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$

  $\forall p \forall q \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box\top \ \&\ \blacklozenge\mathbf{i} \leq p \to q \ \&\ \Box(\Box p \to \Box q) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$

  $\forall p \forall q \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box\top \ \&\ \blacklozenge\mathbf{i} \wedge p \leq q \ \&\ \Box(\Box p \to \Box q) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$

  $\forall p \forall q \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{n}[(\mathbf{i} \leq \Box\top \ \&\ \blacklozenge\mathbf{i} \wedge p \leq q \ \&\ ((\Box\top \leq \mathbf{m}) \ ⅋\ (\Box\mathbf{n} \leq \mathbf{m} \ \&$
  $\Box p \to \Box q \leq \mathbf{n}))) \Rightarrow \mathbf{i} \leq \mathbf{m}]$

  $\forall p \forall q \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box\top \ \&\ \blacklozenge\mathbf{i} \wedge p \leq q \ \&\ \Box\top \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]\ \&$
  $\forall p \forall q \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{n}[(\mathbf{i} \leq \Box\top \ \&\ \blacklozenge\mathbf{i} \wedge p \leq q \ \&\ \Box\mathbf{n} \leq \mathbf{m} \ \&\ \Box p \to \Box q \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$.

  Notice that the first of the two quasi-inequalities above is a tautology: indeed, $\mathbf{i} \leq \Box\top$ and $\Box\top \leq \mathbf{m}$ imply $\mathbf{i} \leq \mathbf{m}$. Hence, the clause above simplifies to:

$$\forall p \forall q \forall \mathbf{i} \forall \mathbf{m} \forall \mathbf{n}[(\mathbf{i} \leq \Box\top \ \& \ \blacklozenge\mathbf{i} \wedge p \leq q \ \& \ \Box\mathbf{n} \leq \mathbf{m} \ \& \ \Box p \to \Box q \leq \mathbf{n}) \Rightarrow (\mathbf{i} \leq \mathbf{m})]$$

$$\forall p \forall q \forall \mathbf{i} \forall \mathbf{i}_0 \forall \mathbf{m} \forall \mathbf{n} \forall \mathbf{n}_0[(\mathbf{i} \leq \Box\top \ \& \ \blacklozenge\mathbf{i} \wedge p \leq q \ \& \ \Box\mathbf{n} \leq \mathbf{m} \ \& \ \mathbf{i}_0 \leq \Box p \ \& \ \Box q \leq \mathbf{n}_0 \ \& \ \mathbf{i}_0 \to \mathbf{n}_0 \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$$

$$\forall p \forall q \forall \mathbf{i} \forall \mathbf{i}_0 \forall \mathbf{m} \forall \mathbf{n} \forall \mathbf{n}_0[(\mathbf{i} \leq \Box\top \ \& \ \blacklozenge\mathbf{i} \wedge p \leq q \ \& \ \Box\mathbf{n} \leq \mathbf{m} \ \& \ \mathbf{i}_0 \leq \Box\top \ \& \ \blacklozenge\mathbf{i}_0 \leq p \ \& \ \Box q \leq \mathbf{n}_0 \ \& \ \mathbf{i}_0 \to \mathbf{n}_0) \leq \mathbf{n} \Rightarrow \mathbf{i} \leq \mathbf{m}].$$

By applying the right-hand Ackermann rule to p, with $\alpha = \blacklozenge\mathbf{i}_0$, $\beta(p) = \blacklozenge\mathbf{i} \wedge p$, and $\gamma(p) = q$, we have:

$$\forall q \forall \mathbf{i} \forall \mathbf{i}_0 \forall \mathbf{m} \forall \mathbf{n} \forall \mathbf{n}_0[(\mathbf{i} \leq \Box\top \ \& \ \blacklozenge\mathbf{i} \wedge \blacklozenge\mathbf{i}_0 \leq q \ \& \ \Box\mathbf{n} \leq \mathbf{m} \ \& \ \mathbf{i}_0 \leq \Box\top \ \& \ \Box q \leq \mathbf{n}_0 \ \& \ \mathbf{i}_0 \to \mathbf{n}_0 \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$$

By applying the right-hand Ackermann rule to q, with $\alpha = \blacklozenge\mathbf{i} \wedge \blacklozenge\mathbf{i}_0$, $\beta(q) = \Box q$, and $\gamma(q) = \mathbf{n}_0$, we have:

$$\forall \mathbf{i} \forall \mathbf{i}_0 \forall \mathbf{m} \forall \mathbf{n} \forall \mathbf{n}_0[(\mathbf{i} \leq \Box\top \ \& \ \Box\mathbf{n} \leq \mathbf{m} \ \& \ \mathbf{i}_0 \leq \Box\top \ \& \ \Box(\blacklozenge\mathbf{i} \wedge \blacklozenge\mathbf{i}_0) \leq \mathbf{n}_0 \ \& \ \mathbf{i}_0 \to \mathbf{n}_0 \leq \mathbf{n}) \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

which simplifies to:

$$\forall \mathbf{i} \forall \mathbf{i}_0[(\mathbf{i} \leq \Box\top \ \& \ \mathbf{i}_0 \leq \Box\top \ \Rightarrow (\blacklozenge\mathbf{i} \wedge \mathbf{i}_0) \leq \Box(\blacklozenge\mathbf{i} \wedge \blacklozenge\mathbf{i}_0)].$$

Using the standard translation, it simplifies to the following first-order condition:

$$\forall i \forall i_0 \forall y (Ni \wedge Ni_0 \wedge Rii_0 \wedge Ri_0 y \to Riy).$$

- $(1')$ $\Box(p \to q) \to (\Box p \to \Box q)$

  We note that the axiom (1') above is inductive but not DLR-Sahlqvist with respect to the natural order-type $\varepsilon_p = 1, \varepsilon_q = 1$. In what follows, we show that the axiom (1') is valid on each Kripke frame with impossible worlds.

$$\forall p \forall q \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box(p \to q) \ \& \ \Box p \to \Box q \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$$

$$\forall p \forall q \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box\top \ \& \ \blacklozenge\mathbf{i} \leq p \to q \ \& \ \Box p \to \Box q \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$$

$$\forall p \forall q \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \Box\top \ \& \ \blacklozenge\mathbf{i} \wedge p \leq q \ \& \ \Box p \to \Box q \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$$

$$\forall p \forall q \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} \forall \mathbf{n}[(\mathbf{i} \leq \Box\top \ \& \ \blacklozenge\mathbf{i} \wedge p \leq q \ \& \ \Box q \leq \mathbf{n} \ \& \ \mathbf{j} \leq \Box p \ \& \ \mathbf{j} \to \mathbf{n} \leq \mathbf{m}) \Rightarrow (\mathbf{i} \leq \mathbf{m})]$$

$$\forall p \forall q \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} \forall \mathbf{n}[(\mathbf{i} \leq \Box\top \ \& \ \blacklozenge\mathbf{i} \wedge p \leq q \ \& \ \Box q \leq \mathbf{n} \ \& \ \blacklozenge\mathbf{j} \leq p \ \& \ \mathbf{j} \leq \Box\top \ \& \ \mathbf{j} \to \mathbf{n} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}].$$

  By applying the right-hand Ackermann rule to p, with $\alpha = \blacklozenge\mathbf{j}$, $\beta(p) = \blacklozenge\mathbf{i} \wedge p$, and $\gamma(p) = q$, we have:

$$\forall q \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} \forall \mathbf{n}[(\mathbf{i} \leq \Box\top \ \& \ \blacklozenge \mathbf{i} \wedge \blacklozenge \mathbf{j} \leq q \ \& \ \Box q \leq \mathbf{n} \ \& \ \mathbf{j} \leq \Box\top \ \& \ \mathbf{j} \rightarrow \mathbf{n} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$$

By applying the right-hand Ackermann rule to q, with $\alpha = \blacklozenge \mathbf{i} \wedge \blacklozenge \mathbf{j}$, $\beta(q) = \Box q$, and $\gamma(q) = \mathbf{n}$, we have:

$$\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} \forall \mathbf{n}[(\mathbf{i} \leq \Box\top \ \& \ \Box(\blacklozenge \mathbf{i} \wedge \blacklozenge \mathbf{j}) \leq \mathbf{n} \ \& \ \mathbf{j} \leq \Box\top \ \& \ \mathbf{j} \rightarrow \mathbf{n} \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]$$

In order to show that the clause above is a tautology, let us further simplify it by rewriting it as follows:

$$\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{n}[(\mathbf{i} \leq \Box\top \ \& \ \Box(\blacklozenge \mathbf{i} \wedge \blacklozenge \mathbf{j}) \leq \mathbf{n} \ \& \ \mathbf{j} \leq \Box\top) \Rightarrow \mathbf{i} \leq \mathbf{j} \rightarrow \mathbf{n}]$$

$$\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{n}[(\mathbf{i} \leq \Box\top \ \& \ \Box(\blacklozenge \mathbf{i} \wedge \blacklozenge \mathbf{j}) \leq \mathbf{n} \ \& \ \mathbf{j} \leq \Box\top) \Rightarrow \mathbf{i} \wedge \mathbf{j} \leq \mathbf{n}]$$

$$\forall \mathbf{i} \forall \mathbf{j}[(\mathbf{i} \leq \Box\top \ \& \ \mathbf{j} \leq \Box\top) \Rightarrow \mathbf{i} \wedge \mathbf{j} \leq \Box(\blacklozenge \mathbf{i} \wedge \blacklozenge \mathbf{j})]$$

$$\forall \mathbf{i} \forall \mathbf{j}[(\mathbf{i} \leq \Box\top \ \& \ \mathbf{j} \leq \Box\top) \Rightarrow \mathbf{i} \wedge \mathbf{j} \leq \Box\blacklozenge \mathbf{i} \wedge \Box\blacklozenge \mathbf{j}]$$

$$\forall \mathbf{i} \forall \mathbf{j}[(\mathbf{i} \leq \Box\top \ \& \ \mathbf{j} \leq \Box\top) \Rightarrow (\mathbf{i} \wedge \mathbf{j} \leq \Box\blacklozenge \mathbf{i} \ \ \& \ \ \mathbf{i} \wedge \mathbf{j} \leq \Box\blacklozenge \mathbf{j})]$$

$$\forall \mathbf{i} \forall \mathbf{j}[(\mathbf{i} \leq \Box\top \ \& \ \mathbf{j} \leq \Box\top) \Rightarrow \mathbf{i} \wedge \mathbf{j} \leq \Box\blacklozenge \mathbf{i}] \ \& \ \forall \mathbf{i} \forall \mathbf{j}[(\mathbf{i} \leq \Box\top \ \& \ \mathbf{j} \leq \Box\top) \Rightarrow \mathbf{i} \wedge \mathbf{j} \leq \Box\blacklozenge \mathbf{j}].$$

The two quasi-inequalities above are respectively implied by

$$\forall \mathbf{i}(\mathbf{i} \leq \Box\top \Rightarrow \mathbf{i} \leq \Box\blacklozenge \mathbf{i}) \ \text{ and } \ \forall \mathbf{j}(\mathbf{j} \leq \Box\top \Rightarrow \mathbf{j} \leq \Box\blacklozenge \mathbf{j}).$$

By applying the adjunction rule to the inequality in the consequent, we get:

$$\forall \mathbf{i}(\mathbf{i} \leq \Box\top \Rightarrow \mathbf{i} \leq \Box\top \ \& \ \blacklozenge \mathbf{i} \leq \blacklozenge \mathbf{i}),$$

which is implied by the tautology

$$\forall \mathbf{i}(\blacklozenge \mathbf{i} \leq \blacklozenge \mathbf{i}),$$

and similarly for $\mathbf{j}$, by the tautology

$$\forall \mathbf{j}(\blacklozenge \mathbf{j} \leq \blacklozenge \mathbf{j}).$$

The correspondence results obtained above are summarized in the Tables 4.2 and 4.3.

4.5.3. Theorem. *Each of E2, E3, E4, E5 is strongly complete w.r.t. the class of frames specified in Table 4.4.*

Proof. The canonicity of axioms (1), (1'), (2), (4) and (5) is shown in Proposition 4.5.2. As is shown above, ALBA^r (safely) succeeds on each of them. Hence, by Theorem 4.3.1, the statement follows. □

Notice that any Kripke frame with impossible worlds which satisfies pre-normal reflexivity, pre-normal euclideanness and normality (cf. Tables 4.2 and 4.3) can be uniquely associated with a standard Kripke frame, the binary relation of which is an equivalence relation. Conversely, any such standard Kripke frame can be uniquely associated with a Kripke frame with impossible worlds which satisfies pre-normal reflexivity, pre-normal euclideanness and normality. These observations provide a semantic proof of Kripke's statement that E5 coincides with S5.

4.6 Conclusions

In this chapter, we extended the theory of unified correspondence to regular distributive modal logics (DLRs), i.e., non-normal modal logics the modal connectives of which preserve binary conjunctions or disjunctions, and the propositional base of which is the logic of distributive lattices. The core technical tool was an adaptation of ALBA, referred to as ALBA^r. We defined the class of inductive DLR-inequalities. The calculus ALBA^r is shown to succeed on every inductive DLR-inequality. As a result we obtain that the (regular) modal logics generated by inductive DLR-inequalities are strongly complete with respect to the class of Kripke frames defined by their first-order correspondent.

A natural extension of the results in the present chapter concerns the development of the regular counterpart of the inductive inequalities of [50]. We expect that this direction will further develop the techniques and facts introduced in [159].

Elementary frame condition	**First-order formula**
Normality	$\forall x Nx$
Closure under normality	$\forall x \forall y (Nx \wedge Rxy \rightarrow Ny)$
Pre-normal reflexivity	$\forall x (Nx \rightarrow Rxx)$
Pre-normal transitivity	$\forall x \forall y \forall z (Nx \wedge Ny \wedge Rxy \wedge Ryz \rightarrow Rxz)$
Pre-normal euclideanness	$\forall x \forall y \forall z (Nx \wedge Ny \wedge Rxy \wedge Rxz \rightarrow Ryz)$

Table 4.2: Elementary frame conditions

Modal axiom	**Elementary frame condition**
$\Box p \rightarrow p$	Pre-normal reflexivity
$\Box p \rightarrow \Box\Box p$	Pre-normal transitivity and closure under normality
$\neg\Box p \rightarrow \Box\neg\Box p$	Normality and pre-normal euclideanness
$\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$	$\top$
$\Box(p \rightarrow q) \rightarrow \Box(\Box p \rightarrow \Box q)$	Pre-normal transitivity

Table 4.3: Lemmon's modal axioms and their elementary frame conditions

Lemmon's system	**Elementary class of frames**
E2	Pre-normal reflexivity
E3	Pre-normal reflexivity and pre-normal transitivity
E4	Pre-normal reflexivity, pre-normal transitivity and closure under normality
E5	Pre-normal reflexivity, pre-normal euclideanness and normality

Table 4.4: Lemmon's systems and their elementary classes of frames

Chapter 5

Algorithmic correspondence for intuitionistic modal mu-calculus

In the present chapter the algorithmic correspondence theory is extended to modal mu-calculi with a non-classical base. We focus in particular on the language of bi-intuitionistic modal mu-calculus. We enhance the algorithm ALBA so as to guarantee its success on the class of recursive mu-inequalities, which we introduce in this chapter.

Modal mu-calculus [105] is a logical framework combining simple modalities with fixed point operators, enriching the expressivity of modal logic so as to deal with infinite processes like recursion. It has a simple syntax, an easily given semantics, and is decidable. Modal mu-calculus has become a fundamental logical tool in theoretical computer science and has been extensively studied [34], and applied for instance in the context of temporal properties of systems, and of infinite properties of concurrent systems. Many expressive modal and temporal logics such as PDL, CTL, CTL* can be seen as fragments of the modal mu-calculus [34, 86]. It provides a unifying framework connecting modal and temporal logics, automata theory and the theory of games, where fixed point constructions can be used to talk about the long-term strategies of players, as discussed in [94].

Sahlqvist-style frame-correspondence theory for modal mu-calculus has recently been developed in [17]. Such analysis strengthens the general mathematical theory of the mu-calculus, facilitates the transfer of results from first-order logic with fixed points, and aids in understanding the meaning of mu-formulas interpreted over frames, which is often difficult to grasp. The correspondence results in [17] are developed purely model-theoretically. However, they can be naturally encompassed within the existing *algebraic* approach to correspondence theory, which we saw in the previous chapter, and generalized to mu-calculi on a weaker-than-classical (and, particularly, *intuitionistic*) base.

There are three types of reasons for studying (bi-)intuitionistic mu-calculi. Firstly, the correspondence results obtained in this setting project onto those obtainable in the classical setting of [17]. Conceptually, this means that the correspondence mechanisms for mu-calculi are intrinsically independent of their being set in classical logic, and hence the non-classical mu-calculi provide clearer insights into their nature, by abstracting from unneeded assumptions.[1] Secondly, these mu-calculi also bring practical advantages, since their greater generality means of course wider applicability. Finally, it can be argued that such a study is now timely, given that closely related areas of logic such as constructive modal logics and type theory are of increasing foundational and practical relevance in such fields as semantics of programming languages [127], and intuitionistic modal mu-calculi can be a valuable tool to these investigations. It is worth stressing that all the results and, in particular, all the practical reductions developed for bi-intuitionistic modal mu-calculus are immediately applicable to the classical case.

This chapter is organized as follows. Within Section 5.1, we collect some details about the syntax, and algebraic and relational semantics of bi-intuitionistic modal logic. In Section 5.2, the stage is set for extending correspondence theory to mu-calculus by proving the relevant order-theoretic preservation properties of fixed points, and introducing approximation and adjunction rules for fixed point binders. In 5.1.1, the language and semantics of the bi-intuitionistic modal mu-calculus are introduced. In Section 5.3, the recursive mu-inequalities are defined on the basis of order-theoretic properties of the algebraic interpretation of the logical connectives. In Section 5.4, we define certain syntactic shapes of formulas, the (*normal*) *inner formulas*, which guarantee the applicability of the approximation rules as stated in Section 5.2.2, In Section 5.6, we show the execution of the algorithm on two examples. Finally, in Appendix A.2, we show that the enhanced version of ALBA is successful on all recursive inequalities.

5.1 Preliminaries

5.1.1 The bi-intuitionistic modal mu-language and its semantics

Syntax. Let Prop and FVar be disjoint sets of propositional variables and of fixed point variables (the elements of which are respectively denoted by p, q, r and by X, Y, Z). Let x, y, z be general purpose variables, which can be either used as place-holder variables, or as generic variables ranging in $\mathsf{Prop} \cup \mathsf{FVar}$. Let us define, by simultaneous recursion,
(a) the set $\mathcal{L}$ of bi-intuitionistic modal mu-formulas[2] over Prop and FVar,

[1]In particular, the bi-intuitionistic setting accounts for the projection over the classical setting more naturally than the intuitionistic one, for various technical reasons which will be expanded upon in Remark 5.3.6.

[2]Henceforth we will sometimes refer to bi-intuitionistic modal mu-formulas as modal mu-formulas, mu-formulas, or simply formulas.

(b) the set $FV(\varphi)$ of their free variables,
as follows: $\top$ and $\bot$ are bi-intuitionistic modal mu-formulas; $FV(\top) = FV(\bot) = \varnothing$. Any $x \in \mathsf{Prop} \cup \mathsf{FVar}$ is a bi-intuitionistic modal mu-formula; $FV(x) = \{x\}$. If φ and ψ are modal mu-formulas, then so are $\varphi \wedge \psi, \varphi \vee \psi, \varphi \to \psi, \varphi - \psi, \Box\varphi, \Diamond\varphi$; for $\odot \in \{\Box, \Diamond\}$, we let $FV(\odot\varphi) = FV(\varphi)$, and for $\odot \in \{\wedge, \vee, \to, -\}$, we let $FV(\varphi \odot \psi) = FV(\varphi) \cup FV(\varphi)$. If every free occurrence of X in the positive generation tree of φ is labelled positively, then $\mu X.\varphi$ and $\nu X.\varphi$ are modal mu-formulas; we let $FV(\odot.\varphi) = FV(\varphi) \setminus \{X\}$. An occurrence of X in φ is *bound* if $X \notin FV(\varphi)$. A *sentence* is a modal mu-formula with no free fixed point variables. The symbol $\varphi(p_1, ..., p_n, X_1, ..., X_m)$ indicates that the propositional variables and free fixed point variables in φ are among $p_1, ..., p_n$ and $X_1, ..., X_m$ respectively; in $\varphi(p_1, ..., p_n, X_1, ..., X_m)$, which we will typically abbreviate as $\varphi(\overline{p}, \overline{X})$, the variables $p_1, ..., p_n, X_1, ..., X_m$ will be understood as pairwise distinct. For modal mu-formulas φ and ψ and $x \in \mathsf{Prop} \cup \mathsf{FVar}$, the symbol $\varphi(\psi/x)$ denotes the mu-formula obtained by replacing all free occurrences of x in φ by ψ.

Semantics and the expanded language $\mathcal{L}^+$. The non-fixed point fragment of this language can be interpreted on several types of relational structures such as those described in the Section 5.1.2; each interpretation yields a different corresponding definition of the complex algebra. Irrespective of these differences, the complex algebras of these relational structures are always *perfect modal bi-Heyting algebras* (see Definition 5.1.2). Each operation in such a perfect algebra is either a *residual* or an *adjoint* (see e.g. Section 4.1.6). The core of the theory presented in this chapter can (and will) be developed only on the basis of these properties, hence independently of any particular choice of relational dual semantics.

Term functions are associated with $\mathcal{L}$-formulas in the usual way, see e.g., 4.3.1,[35, Definition 10.2]. For interpretation of fixed-point connectives, we first recall the Knaster-Tarski theorem.

5.1.1. Theorem (Knaster-Tarski Theorem). *Let $\mathbb{C}$ be a complete lattice and $f : \mathbb{C} \to \mathbb{C}$ be a monotone map, that is, for each $a, b \in \mathbb{C}$, with $a \leq b$ we have $f(a) \leq f(b)$. The Knaster-Tarski theorem states that the map f has a least fixed-point $LFP(f)$ and a greatest fixed point $GFP(f)$ which can be computed as*

$$LFP(f) = \bigwedge\{a \in \mathbb{C} : f(a) \leq a\},$$

$$GFP(f) = \bigvee\{a \in \mathbb{C} : a \leq f(a)\}.$$

We recall that the least fixed-point of f or $LFP(f)$ can be computed in a constructive way as follows (see e.g., [61]). For an ordinal α, let $f^0(\bot) = \bot$,

$f^\alpha(0) = f(f^\beta(\bot))$ if $\alpha = \beta + 1$, and $f^\alpha(\bot) = \bigvee_{\beta \leq \alpha} f^\beta(\bot)$, if α is a limit ordinal. Then $LFP(f) = f^\alpha(\bot)$, for some ordinal α such that $f^{\alpha+1}(0) = f^\alpha(\bot)$. The greatest fixed point can be defined dually.

In particular, as to the interpretation of fixed point binders, if $\varphi(\overline{p}, Y, \overline{X})$ is positive in Y, then its associated term function is monotone in Y and hence, by the Knaster-Tarski theorem, for every given assignment of elements to $\overline{p}$ and $\overline{X}$, the resulting function in Y has a greatest and a least fixed point, which are, respectively, the values for $\nu Y.\varphi(\overline{p}, Y, \overline{X})$ and $\mu Y.\varphi(\overline{p}, Y, \overline{X})$ under the given assignment.

For the case of distributive and intuitionistic modal logic, the special properties of perfect (distributive) lattices make it possible to define an interpretation for the following *expanded modal mu-language* $\mathcal{L}^+$, which is built over $\mathsf{Prop} \cup \mathsf{FVar} \cup \mathsf{Nom} \cup \mathsf{CNom}$, where the variables $\mathbf{i}, \mathbf{j} \in \mathsf{Nom}$ (called *nominals*) and $\mathbf{m}, \mathbf{n} \in \mathsf{CNom}$ (called *co-nominals*) are respectively interpreted in any perfect bi-Heyting algebra $\mathbb{C}$ as elements of $J^\infty(\mathbb{C})$ and of $M^\infty(\mathbb{C})$ (see Section 5.1.2), additionally closing under the modal operators $\blacklozenge$ and $\blacksquare$ (respectively interpreted in $\mathbb{C}$ as the left adjoint of $\Box^\mathbb{C}$ and as the right adjoint of $\Diamond^\mathbb{C}$). A formula of $\mathcal{L}^+$ is *pure* if it contains no $p \in \mathsf{Prop}$.

Notational conventions. For every formula φ, let $\neg\varphi$ and $\sim\varphi$ abbreviate $\varphi \to \bot$ and $\top - \varphi$ respectively. For every order-type ε, let ε^∂ be its *opposite* order-type, i.e., $\varepsilon^\partial_i = 1$ iff $\varepsilon_i = \partial$ for every $1 \leq i \leq n$. In what follows we will find it convenient to use the following conventions: we write $\top^1$ and $\top^\partial$ for $\top$ and $\bot$ respectively; likewise, we write $\bot^1$ and $\bot^\partial$ for $\bot$ and $\top$ respectively. Analogous conventions will hold for $\wedge, \vee, \mu, \nu, \leq$; in particular, $\wedge^\partial, \vee^\partial, \mu^\partial, \nu^\partial, \leq^\partial$ will respectively denote $\vee, \wedge, \nu, \mu, \geq$. The exponent in these conventions will typically be a generic ε_i for some order-type ε. Hence, for instance, $\bot^{\varepsilon_i}$ will denote $\bot$ if $\varepsilon_i = 1$ and $\top$ if $\varepsilon_i = \partial$. Similarly, $\mathbf{j}^{\varepsilon_i}$ denotes a nominal if $\varepsilon_i = 1$ and a conominal if $\varepsilon_i = \partial$, and dually, $\mathbf{n}^{\varepsilon_i}$ denotes a conominal if $\varepsilon_i = 1$ and a nominal if $\varepsilon_i = \partial$. We will use the symbols $\&$, $⅋$, and $\Rightarrow$, interpreted as conjunction, disjunction, and implication, respectively, to combine $\mathcal{L}^+$-inequalities into quasi-inequalities. Given two tuple of variables $\overline{x}$ and $\overline{y}$, denote by $\overline{x} \oplus \overline{y}$ their concatenation.

A glimpse at the first-order correspondence language. Pure formulas can be equivalently translated over the relational semantics (see Section 5.1.2) via a well known standard translation process, similar to the one defined in [49], see also [50]. This translation targets the associated first-order correspondence language augmented with least fixed points (see [61]). Since there are many options when it comes to relational dual semantics for non-classical logics of this type, we have chosen not to commit to a specific translation, but to focus only on the reduction process up to the elimination of propositional variables, as this remains invariant, irrespective of the choice of the specific relational semantics.

Depending on this choice, the final propositional variable-free clause above will then receive different translations.

5.1.2 Perfect modal bi-Heyting algebras

A *bi-Heyting algebra* is an algebra $(A, \wedge, \vee, \rightarrow, -, \top, \bot)$ such that both $(A, \wedge, \vee, \rightarrow, \top, \bot)$ and $(A, \wedge, \vee, -, \top, \bot)^\partial$ are Heyting algebras. In particular, the operation $-$ (referred to as 'subtraction', 'exclusion' or 'disimplication') is uniquely identified by the following property holding for every $a, b, c \in A$:

$$a - b \leq c \ \text{ iff } \ a \leq c \vee b.$$

In the special case of Boolean algebras, $a - b = a \wedge \neg b$. A *modal bi-Heyting algebra* is an algebra $(A, \wedge, \vee, \rightarrow, -, \top, \bot, \Box, \Diamond)$ such that $(A, \wedge, \vee, \rightarrow, -, \top, \bot)$ is a bi-Heyting algebra and $\Box$ and $\Diamond$ preserve finite meets and joins, respectively.

5.1.2. Definition. A *perfect bi-Heyting algebra* is a bi-Heyting algebra the lattice reduct of which is a perfect distributive lattice. A *perfect modal bi-Heyting algebra* is a modal bi-Heyting algebra the bi-Heyting reduct of which is a perfect bi-Heyting algebra, and moreover such that $\Box$ and $\Diamond$ preserve arbitrary meets and joins, respectively.

A Stone-type duality on objects (extending the finite *Birkhoff* duality) holds between perfect bi-Heyting algebras and *posets*, which is defined as follows: every poset X is associated with the lattice $\mathcal{P}^\uparrow(X)$ of the upward-closed[3] subsets of X, on which the implication and the subtraction are defined as $Y \rightarrow Z = (Y^c \cup Z){\downarrow}^c$ and $Y - Z = (Y \cap Z^c){\uparrow}$ for all $Y, Z \in \mathcal{P}^\uparrow(X)$; here $(\cdot)^c$ denotes the complement relative to W; conversely, every perfect bi-Heyting algebra $\mathbb{C}$ is associated with $(J^\infty(\mathbb{C}), \geq)$ where $\geq$ is the reverse lattice order in $\mathbb{C}$, restricted to $J^\infty(\mathbb{C})$.

Just in the same way in which the duality between complete atomic Boolean algebras and sets can be expanded to a duality between complete atomic modal algebras and Kripke frames, the duality between perfect bi-Heyting algebras and posets can be expanded to a duality between perfect modal bi-Heyting algebras and posets endowed with arrays of relations, each of which dualizes one additional operation in the usual way, i.e., n-ary operations give rise to $n+1$-ary relations, and the assignments between operations and relations are defined as in the classical setting. We are not going to report on this duality in full detail (we refer e.g. to [141, 75, 50]), but we limit ourselves to mention that dual frames to perfect modal bi-Heyting algebras can be defined as relational structures $\mathcal{F} = (W, \leq, R_\Diamond, R_\Box)$ such that $(W, \leq)$ is a nonempty poset, $R_\Diamond$ and $R_\Box$ are binary relations on W, and the following inclusions hold:

[3] A subset Y of a poset X is *upward-closed* if $x \in Y$ and $x \leq y \in X$ implies $y \in Y$. We write $Y{\uparrow} = \{x \in X \mid \exists y(y \in Y \ \&\ y \leq x)\}$. Dually for *downward-closed* subsets and $Y{\downarrow}$.

$$\geq \circ R_\Diamond \circ \geq \ \subseteq \ R_\Diamond \qquad\qquad \leq \circ R_\Box \circ \leq \ \subseteq \ R_\Box.$$

The *complex algebra* of any such relational structure $\mathcal{F}$ (cf. [75, Sec. 2.3]) is

$$\mathcal{F}^+ = (\mathcal{P}^\uparrow(W), \cup, \cap, \varnothing, W, \langle R_\Diamond\rangle, [R_\Box]),$$

where, for every $X \subseteq W$,

$$\begin{array}{lclcl} [R_\Box]X & := & \{w \in W \mid R_\Box[w] \subseteq X\} & = & (R_\Box^{-1}[X^c])^c \\ \langle R_\Diamond\rangle X & := & \{w \in W \mid R_\Diamond[w] \cap X \neq \varnothing\} & = & R_\Diamond^{-1}[X]. \end{array}$$

Here $R[x] = \{w \mid w \in W \text{ and } xRw\}$ and $R^{-1}[x] = \{w \mid w \in W \text{ and } wRx\}$. Moreover, $R[X] = \bigcup\{R[x] \mid x \in X\}$ and $R^{-1}[X] = \bigcup\{R^{-1}[x] \mid x \in X\}$.

5.2 ALBA for bi-intuitionistic modal mu-calculus: setting the stage

Our aim in this chapter is to extend the algorithmic-algebraic techniques for correspondence to a larger class of mu-inequalities, including those defined in [17]In order to do this we will need to

1. analyze the order-theoretic properties of the term functions associated with mu-calculus formulas, which we do in Section 5.2.1.

2. define the bi-intuitionistic syntactic and semantic settings for mu-calculus, which we do in Section 5.1.1.

3. on the basis of the analysis in Section 5.2.1 formulate approximation and adjunction rules for fixed point binders. This we do in Sections 5.2.2 and 5.2.3.

To complete our account it would be sufficient to define a syntactic class of bi-intuitionistic mu-inequalities, as is done in Section 5.3, and then show that the algorithm enhanced with the rules above succeeds on all its members. However, some non-trivial further justification needs to be given as to how the rules of Sections 5.2.2 and 5.2.3 are applicable to the syntactic specifications of Section 5.3. This is further discussed in Section 5.2.5.

5.2.1 Preservation and distribution properties of extremal fixed points

For L and M complete lattices and $G : M \times L \to L$, let $\mu y.G : M \to L$ and $\nu y.G : M \to L$ be the maps respectively given by $b \mapsto LFP(G(b, y))$ and $b \mapsto GFP(G(b, y))$ for each $b \in M$ such that $LFP(G(b, y))$ and $GFP(G(b, y))$

are defined, where $LFP(G(b,y))$ and $GFP(G(b,y))$ denote the least and greatest fixed points of the map $G(b,y) : L \to L$, respectively.

For each such G, and every ordinal κ, let $G^{\kappa\downharpoonleft}(b,y)$ be defined by the following induction: $G^{0\downharpoonleft}(b,y) = y$, $G^{\kappa+1\downharpoonleft}(b,y) = G(b, G^{\kappa\downharpoonleft}(b,y))$ and for λ a limit ordinal, $G^{\lambda\downharpoonleft}(b,y) = \bigwedge_{\kappa<\lambda} G^{\kappa\downharpoonleft}(b,y)$. Also, for a map $F : L \to L$ we define $F^{\kappa\upharpoonleft}(x)$ for all ordinals κ by induction as follows: $F^{0\upharpoonleft}(x) = x$, $F^{\kappa+1\upharpoonleft}(x) = F(F^{\kappa\upharpoonleft}(x))$ and $F^{\lambda\upharpoonleft}(y) = \bigvee_{\kappa<\lambda} F^{\kappa\upharpoonleft}(x)$.

5.2.1. LEMMA. *Let L, M and G be as above.*

1. *If $G : M \times L \to L$ is completely meet-preserving, then the map $g^\kappa : M \to L$ defined by the assignment $b \mapsto G^{\kappa\downharpoonleft}(b, \top)$ is completely meet-preserving for every ordinal κ.*

2. *If $F : L \to M \times L$ is the left adjoint of G, and $F_1 : L \to M$ and $F_2 : L \to L$ are such that $F = (F_1, F_2)$, then F_1 and F_2 are completely join-preserving.*

3. *If F, F_1 and F_2 are as in the previous item, then for every ordinal κ, the left adjoint of g^κ is the map defined by the assignment $a \mapsto F_1(a \vee \bigvee_{\kappa'<\kappa} F_2^{\kappa'\upharpoonleft}(a))$.*

4. *If G is completely meet-preserving, then $\nu y.G : M \to L$ is defined everywhere on M, and is completely meet-preserving.*

5. *If G is completely meet-preserving, then the left adjoint of $\nu y.G$ is the map defined by the assignment $a \mapsto F_1(a \vee \mu y.F_2(a \vee y))$.*

Proof.

1. Let $S \subseteq M$. We proceed by induction on κ: we have $G^{1\downharpoonleft}(\bigwedge S, \top) = G(\bigwedge S, \top) = G(\bigwedge\{(s, \top) \mid s \in S\}) = \bigwedge\{G(s,\top) \mid s \in S\} = \bigwedge\{G^{1\downharpoonleft}(s,\top) \mid s \in S\}$, where the penultimate equality holds by the assumption that $G : M \times L \to L$ is completely meet-preserving and the fact that the second coordinate is $\top$.

   Assume the claim holds for κ and consider the case for $\kappa + 1$:

$$
\begin{array}{lll}
& G^{(\kappa+1)\downharpoonleft}(\bigwedge S, \top) & \\
= & G(\bigwedge S, G^{\kappa\downharpoonleft}(\bigwedge S, \top)) & \\
= & G(\bigwedge S, \bigwedge\{G^{\kappa\downharpoonleft}(s,\top) \mid s \in S\}) & \text{(Induction hypothesis)} \\
= & G(\bigwedge\{(s, G^{\kappa\downharpoonleft}(s,\top)) \mid s \in S\}) & \\
= & \bigwedge\{G(s, G^{\kappa\downharpoonleft}(s,\top)) \mid s \in S\} & \text{(G completely meet-preserving)} \\
= & \bigwedge\{G^{(\kappa+1)\downharpoonleft}(s,\top) \mid s \in S\} & \\
. & &
\end{array}
$$

   If λ is a limit ordinal, then

$$
\begin{array}{lll}
& G^{\lambda\downarrow}(\bigwedge S, \top) & \\
= & \bigwedge_{\kappa<\lambda} G^{\kappa\downarrow}(\bigwedge S, \top) & \\
= & \bigwedge_{\kappa<\lambda} \bigwedge\{G^{\kappa\downarrow}(s, \top) \mid s \in S\} & \text{(Induction hypothesis)} \\
= & \bigwedge_{s\in S} \bigwedge\{G^{\kappa\downarrow}(s, \top) \mid \kappa < \lambda\} & \text{(Associativity and commutativity)} \\
= & \bigwedge_{s\in S} G^{\lambda\downarrow}(s, \top) & .
\end{array}
$$

2. Let $S \subseteq L$. The inequality $\bigvee\{F_1(s) \mid s \in S\} \leq F_1(\bigvee S)$ follows immediately from the fact that F is order-preserving and hence F_1 and F_2 are. Conversely, fix $b \in M$ arbitrarily, suppose that $\bigvee\{F_1(s) \mid s \in S\} \leq b$ and let us show that $F_1(\bigvee S) \leq b$:

$$
\begin{array}{rll}
\bigvee\{F_1(s) \mid s \in S\} \leq b & \text{iff} & F_1(s) \leq b \text{ for each } s \in S \\
& \text{iff} & F(s) \leq (b, \top) \text{ for each } s \in S \\
& \text{iff} & s \leq G(b, \top) \text{ for each } s \in S \\
& \text{iff} & \bigvee S \leq G(b, \top) \\
& \text{iff} & F(\bigvee S) \leq (b, \top) \\
& \text{iff} & F_1(\bigvee S) \leq b.
\end{array}
$$

The case for F_2 can be proved similarly.

3. We proceed by induction on κ. If $\kappa = 1$, then for every $a \in L$ and $b \in M$, we have that $a \leq G(b, \top)$ iff $F(a) \leq (b, \top)$ iff $F_1(a) \leq b$ and $F_2(a) \leq \top$ iff $F_1(a) \leq b$, which proves the base case.

Assume the claim holds for κ and consider the case for $\kappa + 1$:

$$
\begin{array}{ll}
& a \leq G^{(\kappa+1)\downarrow}(b, \top) \\
\text{iff} & a \leq G(b, G^{\kappa\downarrow}(b, \top)) \\
\text{iff} & F(a) \leq (b, G^{\kappa\downarrow}(b, \top)) \\
\text{iff} & F_1(a) \leq b \text{ and } F_2(a) \leq G^{\kappa\downarrow}(b, \top) \\
\text{iff} & F_1(a) \leq b \text{ and } F_1(F_2(a) \vee \bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(F_2(a))) \leq b \\
\text{iff} & F_1(a) \vee F_1(F_2(a) \vee \bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(F_2(a))) \leq b \\
\text{iff} & F_1(a) \vee F_1(F_2(a) \vee \bigvee_{2\leq\kappa'<\kappa+1} F_2^{\kappa'\uparrow}(a)) \leq b \\
\text{iff} & F_1(a \vee F_2(a) \vee \bigvee_{2\leq\kappa'<\kappa+1} F_2^{\kappa'\uparrow}(a)) \leq b \\
\text{iff} & F_1(a \vee \bigvee_{\kappa'<\kappa+1} F_2^{\kappa'\uparrow}(a)) \leq b.
\end{array}
$$

Let λ be a limit ordinal and assume that the claim holds for all $\kappa < \lambda$:

$$
\begin{array}{lll}
& a \leq G^{\lambda\downarrow}(b, \top) & \\
\text{iff} & a \leq \bigwedge_{\kappa<\lambda} G^{\kappa\downarrow}(b, \top) & \\
\text{iff} & a \leq G^{\kappa\downarrow}(b, \top) \text{ for every } \kappa < \lambda & \\
\text{iff} & F_1(a \vee \bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(a)) \leq b \text{ for every } \kappa < \lambda & \\
\text{iff} & \bigvee_{\kappa<\lambda} F_1(a \vee \bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(a)) \leq b & \\
\text{iff} & F_1(a \vee \bigvee_{\kappa<\lambda} \bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(a)) \leq b & \text{(by item 2 above)} \\
\text{iff} & F_1(a \vee \bigvee_{\kappa<\lambda} F_2^{\kappa\uparrow}(a)) \leq b &
\end{array}
$$

.

4. Since G is completely meet-preserving, G is monotone in each coordinate. Hence, by the Knaster-Tarski theorem, $\nu y.G$ is everywhere defined. By the general theory of fixed points, for all $b \in M$, we have $\nu y.G(b, y) = \bigwedge_{\kappa\geq 1} G^{\kappa\downarrow}(b, \top)$. Hence,

$$\begin{array}{lll} & \nu y.G(\bigwedge S, y) & \\ = & \bigwedge_{\kappa\geq 1} G^{\kappa\downarrow}(\bigwedge S, \top) & \\ = & \bigwedge_{\kappa\geq 1} \bigwedge\{G^{\kappa\downarrow}(s, \top) \mid s \in S\} & \text{(item 1 above)} \\ = & \bigwedge_{s\in S} \bigwedge\{G^{\kappa\downarrow}(s, \top) \mid \kappa \geq 1\} & \text{(Associativity and commutativity)} \\ = & \bigwedge_{s\in S} \nu y.G(s, y). & \end{array}$$

5. For all $a \in L$ and $b \in M$,

$$\begin{array}{lll} & a \leq \nu y.G(b, y) & \\ \text{iff} & a \leq \bigwedge_{\kappa\geq 1} G^{\kappa\downarrow}(b, \top) & \\ \text{iff} & a \leq G^{\kappa\downarrow}(b, \top) \text{ for every } \kappa \geq 1 & \\ \text{iff} & F_1(a \vee \bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(a)) \leq b \text{ for every } \kappa \geq 1 & \text{(by item 3 above)} \\ \text{iff} & \bigvee_{\kappa\geq 1} F_1(a \vee \bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(a)) \leq b & \\ \text{iff} & F_1(a \vee \bigvee_{\kappa\geq 1} \bigvee_{\kappa'<\kappa} F_2^{\kappa'\uparrow}(a)) \leq b & \text{(by item 2 above)} \\ \text{iff} & F_1(a \vee \bigvee_{\kappa\geq 1} F_2^{\kappa\uparrow}(a)) \leq b & \\ \text{iff} & F_1(a \vee \mu y.F_2(a \vee y)) \leq b. & \end{array}$$

□

5.2.2. Remark. In the following sections we will use the lemma above with $M = L^\varepsilon$ for some order-type ε over n. In such a setting the map $F_1 : L \to L^\varepsilon$ takes the form $(F_{1,1}, \ldots, F_{1,n})$ where $F_{1,i} : L \to L^{\varepsilon_i}$ for each $1 \leq i \leq n$. Hence the left adjoint of $\nu y.G(\overline{x}, y) : L^\varepsilon \to L$ is the map defined by the assignment $a \mapsto (F_{1,1}(a \vee \mu y.F_2(a \vee y)), \ldots, F_{1,n}(a \vee \mu y.F_2(a \vee y)))$, i.e., for all $a \in L$ and $\overline{b} \in L^\varepsilon$,

$$a \leq \nu y.G(\overline{b}, y) \quad \text{iff} \quad \mathop{\&}_{1\leq i\leq n} F_{1,i}(a \vee \mu y.F_2(a \vee y)) \leq^{\varepsilon_i} b_i.$$

5.2.3. Lemma. *Let L, M_1 and M_2 be complete lattices.*

1. *If $f : L \to L$ preserves all finite non-empty joins and $g_i : M_i \to L$, $i \in \{1, 2\}$, then*

$$\mu x.[f(x) \vee (g_1(x_1) \vee g_2(x_2))] = \mu x.[f(x) \vee g_1(x_1)] \vee \mu x.[f(x) \vee g_2(x_2)].$$

2. *If $f : L \to L$ preserves all finite non-empty meets and $g_i : M_i \to L$, $i \in \{1, 2\}$, then*

$$\nu x.[f(x) \wedge (g_1(x_1) \wedge g_2(x_2))] = \nu x.[f(x) \wedge g_1(x_1)] \wedge \nu x.[f(x) \wedge g_2(x_2)].$$

Proof. We only prove item 1, item 2 being order dual.

$$\begin{array}{rl} & \mu x.[f(x) \vee (g_1(x_1) \vee g_2(x_2))] \\ = & \bigvee_{\kappa \geq 0} (f^{\kappa+1}(\bot) \vee f^{\kappa}(g_1(x_1) \vee g_2(x_2))) \\ = & \bigvee_{\kappa \geq 0} (f^{\kappa+1}(\bot) \vee f^{\kappa}(g_1(x_1))) \vee \bigvee_{\kappa \geq 0} (f^{\kappa+1}(\bot) \vee f^{\kappa}(g_2(x_2))) \\ = & \mu x.[f(x) \vee g_1(x_1)] \vee \mu x.[f(x) \vee g_2(x_2)]. \end{array}$$

□

In applying the lemma above, M_1 and M_2 will typically be powers of L. Accordingly, x_1 and x_2 will tuples of variables which we will write as $\overline{x}_1$ and $\overline{x}_2$.

5.2.2 Approximation rules and their soundness

Let ε be an order-type on an n-tuple $\overline{x}$.We let $\bar{\mathbf{i}}_i^{\varepsilon}$ be the n-tuple whose i-th coordinate is $\mathbf{i}^{\varepsilon_i}$ and whose j-th coordinate is $\bot^{\varepsilon_j}$ for all $j \neq i$. Dually, we let $\overline{\mathbf{n}}_i^{\varepsilon}$ be the n-tuple whose i-th coordinate is $\mathbf{n}^{\varepsilon_i}$ and whose j-th coordinate is $\top^{\varepsilon_j}$ for all $j \neq i$.[4] The approximation rules are:

$$\frac{\mathbf{i} \leq \mu X.\psi(\overline{\varphi}/\overline{x}, X, \overline{z})}{\mathop{\large\wp}_{i=1}^{n}(\exists \mathbf{j}^{\varepsilon_i}[\mathbf{i} \leq \mu X.\psi(\bar{\mathbf{j}}_i^{\varepsilon}/\overline{x}, X, \overline{z}) \ \&\ \mathbf{j}^{\varepsilon_i} \leq^{\varepsilon_i} \varphi_i])} \ (\mu^{\varepsilon}\text{-A})$$

$$\frac{\nu X.\varphi(\overline{\psi}/\overline{x}, X, \overline{z}) \leq \mathbf{m}}{\mathop{\large\wp}_{i=1}^{n}(\exists \mathbf{n}^{\varepsilon_i}[\nu X.\varphi(\overline{\mathbf{n}}_i^{\varepsilon}/\overline{x}, X, \overline{z}) \leq \mathbf{m} \ \&\ \psi_i \leq^{\varepsilon_i} \mathbf{n}^{\varepsilon_i}])} \ (\nu^{\varepsilon}\text{-A})$$

where

1. in each rule, the tuples $\overline{x}$ and $\overline{z}$ are disjoint, and the variables $\overline{x} \in Var$ do not occur in any formula in $\overline{\psi}$ or in $\overline{\varphi}$;
2. in (μ^{ε}-A) the associated term function of $\psi(\overline{x}, X, \overline{z})$ is completely $\bigvee$-preserving in $(\overline{x}, X) \in \mathbb{C}^{\varepsilon} \times \mathbb{C}$, for any perfect modal bi-Heyting algebra $\mathbb{C}$;
3. in (ν^{ε}-A) the associated term function of $\varphi(\overline{x}, X, \overline{z})$ is completely $\bigwedge$-preserving in $(\overline{x}, X) \in \mathbb{C}^{\varepsilon} \times \mathbb{C}$, for any perfect modal bi-Heyting algebra $\mathbb{C}$.

The soundness of (μ^{ε}-A) is proven in the following proposition and that of (ν^{ε}-A) is dual.

5.2.4. PROPOSITION. *Let $\psi(\overline{x}, X, \overline{z}), \overline{\varphi} \in \mathcal{L}^{+}$ such that $\overline{x}$ and $\overline{z}$ are disjoint and the term function associated with $\psi(\overline{x}, X, \overline{z})$ is completely $\bigvee$-preserving in $(\overline{x}, X) \in \mathbb{C}^{\varepsilon} \times \mathbb{C}$. Let V be an assignment on $\mathbb{C}$. Then the following are equivalent:*

[4] Of course, if we fix a value for $\mathbf{i}^{\varepsilon_i}$, then $\bar{\mathbf{i}}_i^{\varepsilon}$ denotes the element in $J^{\infty}(\mathbb{A}^{\varepsilon})$ corresponding to $\mathbf{i}^{\varepsilon_i}$ in the i-th coordinate. Dually, $\overline{\mathbf{n}}_i^{\varepsilon}$ ranges in $M^{\infty}(\mathbb{A}^{\varepsilon})$ in an analogous way.

1. $\mathbb{C}, V \models \mathbf{i} \leq \mu X.\psi(\overline{\varphi}/\overline{x}, X, \overline{z})$,
2. $\mathbb{C}, V' \models \mathbf{i} \leq \mu X.\psi(\overline{\mathbf{j}}_i^{\varepsilon}/\overline{x}, X, \overline{z})$ *and* $\mathbb{C}, V' \models \mathbf{j}^{\varepsilon_i} \leq^{\varepsilon_i} \varphi_i$ *for some* $\mathbf{j}^{\varepsilon_i}$*-variant* V' *of* V*, and some* $1 \leq i \leq n$.

Proof. (2) $\Rightarrow$ (1) follows by ε-monotonicity. Conversely, assume that $\mathbb{C}, V \models \mathbf{i} \leq \mu X.\psi(\overline{\varphi}/\overline{x}, X, \overline{z})$. The assumption implies, by the order dual of Lemma 5.2.1.1 with $M = \mathbb{C}^{\varepsilon}$ and $L = \mathbb{C}$, that the term function associated with $\mu X.\psi(\overline{x}, X, \overline{z})$ obtained by fixing $\overline{z}$ according to V is completely join-preserving in $\mathbb{C}^{\varepsilon}$. Since $\mathbb{C}$, and hence $\mathbb{C}^{\varepsilon}$, is a perfect modal bi-Heyting algebra, we have:

$$\mu X.\psi(\overline{\varphi}/\overline{x}, X, \overline{z}) = \bigvee\{\mu X.\psi(j, X, \overline{z}) \mid j \in J^{\infty}(\mathbb{C}^{\varepsilon}) \,\&\, j \leq \overline{\varphi}\}.$$

Since $V(\mathbf{i}) \in J^{\infty}(\mathbb{C})$, this implies that $V(\mathbf{i}) \leq \mu X.\psi(j_0, X, \overline{z})$ for some $j_0 \in J^{\infty}(\mathbb{C}^{\varepsilon})$ such that $j_0 \leq \overline{\varphi}$. Notice that j_0 is an n-tuple which is equal to $\perp^{\mathbb{C}^{\varepsilon}}$ except for exactly one coordinate, the i-th say, which is equal to some $j_{0i} \in J^{\infty}(\mathbb{C}^{\varepsilon_i})$. Let V' be the $\mathbf{j}^{\varepsilon_i}$-variant of V which sends $\mathbf{j}^{\varepsilon_i}$ to $j_{0i} \in J^{\infty}(\mathbb{C}^{\varepsilon_i})$. Then (2) holds under this choice of i and V'. $\square$

5.2.3 Adjunction rules and their soundness

$$\frac{\chi \leq \nu X.\varphi(\overline{\varphi}/\overline{x}, X, \overline{z})}{\&_{i=1}^{n} F_{1,i}(\chi \vee \mu Y.F_2(\chi \vee Y, \overline{z}), \overline{z}) \leq^{\varepsilon_i} \varphi_i}\ (\nu^{\varepsilon}\text{-Adj})$$

where $\varphi, \overline{\varphi}, \chi \in \mathcal{L}^+$, the arrays of variables $\overline{x}$ and $\overline{z}$ are disjoint, $\overline{x}$ has arity n, the term function associated with $\varphi(\overline{x}, X, \overline{z})$ is a right adjoint in $(\overline{x}, X) \in \mathbb{C}^{\varepsilon} \times \mathbb{C}$ for any perfect modal bi-Heyting algebra $\mathbb{C}$, and $F = ((F_{1,i}(y, \overline{z}))_{i=1}^{n}, F_2(y, \overline{z})) : \mathbb{C} \to \mathbb{C}^{\varepsilon} \times \mathbb{C}$ is its left adjoint.

$$\frac{\mu X.\psi(\overline{\psi}/\overline{x}, X, \overline{z}) \leq \chi}{\&_{i=1}^{n} \psi_i \leq^{\varepsilon_i} G_{1,i}(\chi \wedge \nu Y.G_2(\chi \wedge Y, \overline{z}), \overline{z})}\ (\mu^{\varepsilon}\text{-Adj})$$

where $\psi, \overline{\psi}, \chi \in \mathcal{L}^+$, the arrays of variables $\overline{x}$ and $\overline{z}$ are disjoint, $\overline{x}$ has arity n, the term function associated with $\psi(\overline{x}, X, \overline{z})$ is a left adjoint in $(\overline{x}, X) \in \mathbb{C}^{\varepsilon} \times \mathbb{C}$ for any perfect modal bi-Heyting algebra $\mathbb{C}$, and $G = ((G_{1,i}(y, \overline{z}))_{i=1}^{n}, G_2(y, \overline{z})) : \mathbb{C} \to \mathbb{C}^{\varepsilon} \times \mathbb{C}$ is its right adjoint.

The next proposition formally states and proves the soundness of (ν^{ε}-Adj). The soundness of the rule (μ^{ε}-Adj) can be proven similarly using an order-dual version of Lemma 5.2.1.

5.2.5. PROPOSITION. *Let* $\varphi(\overline{x}, X, \overline{z})$*,* χ*, and* F *be as in (*ν^{ε}*-Adj). Let* $\mathbb{C}$ *be a complete modal bi-Heyting algebra and let* V *be an assignment on* $\mathbb{C}$*. Then the following are equivalent:*

1. $\mathbb{C}, V \models \chi \leq \nu X.\varphi(\overline{\varphi}/\overline{x}, X, \overline{z})$,

2. $\mathbb{C}, V \models \mathop{\&}_{i=1}^{n} F_{1,i}(\chi \vee \mu Y.F_2(\chi \vee Y, \overline{z}), \overline{z}) \leq^{\varepsilon_i} \varphi_i$.

Proof. The statement immediately follows from Lemma 5.2.1.5 with $M = \mathbb{C}^{\varepsilon}$, $L = \mathbb{C}$, and G the term function $\varphi(\overline{x}, X, V(\overline{z}))$, cf. Remark 5.2.2. ☐

5.2.4 Recursive Ackermann rules

If we relax the requirement that p does not occur in α in Ackermann lemma (see Lemma 4.2.1) and are willing to admit fixed point operators in our (correspondence) language, we can formulate the following more general version of the Ackermann lemma (see also [49]):

5.2.6. Lemma. *Let $\alpha(p)$, $\beta(p)$, and $\gamma(p)$ be L-formulas, with $\alpha(p)$ and $\beta(p)$ positive in p, and $\gamma(p)$ negative in p. For any assignment V on a complete L-algebra $\mathbb{A}$, the following are equivalent:*

1. *$\mathbb{A}, V \models \beta(\mu p.\alpha(p)/p) \leq \gamma(\mu p.\alpha(p))/p)$;*

2. *there exists a p-variant V^* of V such that $\mathbb{A}, V' \models \alpha(p) \leq p$, and $\mathbb{A}, V' \models \beta(p) \leq \gamma(p)$,*

where $\mu p.\alpha(p)$ denotes the least fixed point of $\alpha(p)$, and need not be an expression in the language L.

Proof. We begin by noting that, since we are working in a complete lattice, least fixed points of monotone operations exist by the Knaster-Tarski theorem. As regards '1 ⇒ 2', let $V'(p) := V(\mu p.\alpha(p))$. As regards '2 ⇒ 1', $\mathbb{A}, V' \models \alpha(p) \leq p$ implies that $V'(p)$ is a pre-fixed point of $\alpha(\cdot)$,[5] and hence $\mu p.\alpha(p) \leq V'(p)$. Therefore, $\beta(\mu p.\alpha(p)/p) \leq \beta(V'(p)) \leq \gamma(V'(p)) \leq \gamma(\mu p.\alpha(p)/p)$. ☐

Lemma 5.2.6 proves the soundness of the following more general *recursive Ackermann rules*, which allow us to eliminate a propositional variable p even if the α_i are not p-free. The recursive Ackermann lemma can be incorporated into ALBA executions as following rules:

$$\frac{\exists p[\mathop{\&}_{i=1}^{n} \alpha_i(p) \leq p \ \& \ \mathop{\&}_{j=1}^{m} \beta_j(p) \leq \gamma_j(p)]}{\mathop{\&}_{j=1}^{m} \beta_j(\mu X.[\bigvee_{i=1}^{n} \alpha_i(X)]/p) \leq \gamma_j(\mu X.[\bigvee_{i=1}^{n} \alpha_i(X)]/p)} \ (\mathrm{RA}_{rec})$$

$$\frac{\exists p[\mathop{\&}_{i=1}^{n} p \leq \alpha_i(p) \ \& \ \mathop{\&}_{j=1}^{m} \gamma_j(p) \leq \beta_j(p)]}{\mathop{\&}_{j=1}^{m} \gamma_j(\nu X.[\bigwedge_{i=1}^{n} \alpha_i(X)]/p) \leq \beta_j(\nu X.[\bigwedge_{i=1}^{n} \alpha_i(X)]/p)} \ (\mathrm{LA}_{rec})$$

The rules are applicable subject to the restrictions that the $\alpha_i(p)$ and β_j are positive in p, that the γ_j are negative in p, and X is a fresh fixed point variable.

[5] Here $\alpha(\cdot)$ is obtained from the term function α by leaving p free and fixing all other variables to the values prescribed by V.

5.2.5 From semantic to syntactic rules

The conditions of applicability of the rules defined in Sections 5.2.2 and 5.2.3 are given in terms of the order-theoretic properties of the term functions associated with the argument of the fixed point binder. This makes the present formulation of these rules unsuitable for inclusion in an extended calculus for correspondence, which is supposed to be a purely syntactic tool. This also makes the practical application of these rules very inconvenient, since the order-theoretic properties have to be verified each time. These difficulties are further compounded by the fact that, unlike other approximation and adjunction rules that apply to a single connective at a time, here we need to consider an entire subformula as a whole. Another serious difficulty is posed by the conclusions of the adjunction rules, which give no information as to how the F_i and G_i are to be computed, or whether they are expressible as $\mathcal{L}^+$-term functions at all. It is therefore highly desirable to have syntactic versions of these rules. To this end, in Section 5.4, a syntactic class of formulas, called the inner formulas, is defined which is shown to verify the assumptions for the applicability of the approximation and adjunction rules.

5.3 Recursive mu-inequalities

In the present section, the definition of recursive inequalities for the signature of bi-intuitionistic modal mu-calculus is introduced. The style of this definition is grounded on a certain classification of the nodes in the signed generation trees of formulas (cf. Table 5.1). However, one major difference with [50] is that the classification of nodes adopted in the present chapter is based on the order-theoretic properties which the operations interpreting the logical connectives enjoy, rather than on those they lack. This is reflected in the names of the groupings in Table 5.1: recall that SLA, SRA, SLR and SRR stand for syntactically left adjoint, syntactically right adjoint, syntactically left residual and syntactically right residual, respectively. In order to establish connections with the model-theoretic analysis conducted in [17], nodes are firstly classified as *inner* and *outer skeleton nodes* and *PIA nodes*, cf. Table 5.1. This order-theoretic classification is then applied within these categories.

Note that in Table 5.1 an array of signed connectives wider than that of the language of bi-intuitionistic modal mu-calculus is classified. This serves as a template for extending the definition of ε-recursive inequalities to different languages. Specifically, the extra connectives $\circ$, $\star$, $\lhd$, and $\rhd$ serve as generic connectives which respectively are (completely) join-preserving in each coordinate, (completely) meet-preserving in each coordinate, (completely) meet-reversing[6],

[6]For any complete lattices P, Q, a map $f : P \to Q$ is completely join-reversing if $f(\bigvee S) = \bigwedge\{f(s) \mid s \in S\}$ for any $S \subseteq P$, and completely meet-reversing if $f(\bigwedge S) = \bigvee\{f(s) \mid s \in S\}$ for

Outer Skeleton	Inner Skeleton	PIA
Δ-adjoints	Binders	Binders
$+$ $\vee$ $\wedge$	$+$ μ	$+$ ν
$-$ $\wedge$ $\vee$	$-$ ν	$-$ μ
SLR	SLA	SRA
$+$ $\Diamond$ $\lhd$ $\circ$ $-$	$+$ $\Diamond$ $\lhd$ $\vee$	$+$ $\Box$ $\rhd$ $\wedge$
$-$ $\Box$ $\rhd$ $\star$ $\rightarrow$	$-$ $\Box$ $\rhd$ $\wedge$	$-$ $\Diamond$ $\lhd$ $\vee$
	SLR	SRR
	$+$ $\wedge$ $\circ$ $-$	$+$ $\vee$ $\star$ $\rightarrow$
	$-$ $\vee$ $\star$ $\rightarrow$	$-$ $\wedge$ $\circ$ $-$

Table 5.1: Skeleton and PIA nodes.

and (completely) join-reversing. Notice, in particular, that the order-theoretic behaviour of the defined connectives $\sim$ and $\neg$ matches that of $\lhd$ and $\rhd$, respectively, and hence they will be classified in the same way as $\lhd$ and $\rhd$.

5.3.1 Recursive mu-inequalities

Every term in the language of bi-intuitionistic modal mu-calculus can be associated with a signed generation tree: for terms without fixed point connectives, we follow Definition 4.4.1); for $\odot \in \{\mu X, \nu X\}$, the $*$-signed generation tree of $\odot.\varphi$ consists of a root node, labelled by $*\odot$, whose only child is the root of the $*$-signed generation tree of φ. We recall the sub-tree relation $\gamma \prec \varphi$, which extends to signed generation trees, and we will write $\varepsilon(\gamma) \prec *\varphi$ to indicate that γ, regarded as a sub- (signed generation) tree of $*\varphi$, agrees with ε.

5.3.1. Definition. Nodes in signed generation trees will be called *skeleton nodes* and *PIA nodes* according to the specification given in Table 5.1. A branch in a signed generation tree $*\varphi$, for $* \in \{+, -\}$, ending in a propositional variable is an *ε-good branch* if, apart from the leaf, it is the concatenation of three paths P_1, P_2, and P_3, each of which may possibly be of length 0, such that P_1 is a path from the leaf consisting only of PIA-nodes, P_2 consists only of inner skeleton-nodes, and P_3 consists only of outer skeleton-nodes. Moreover,

1. The formula corresponding to the uppermost node on P_1 is a mu-sentence.

2. On any SRR-node in P_1 of the form $\gamma \odot \beta$, where β is the side where the branch lies, γ is a mu-sentence and $\varepsilon^\partial(\gamma) \prec *\varphi$ (see above for this notation). Unravelling the condition $\varepsilon^\partial(\gamma) \prec *\varphi$ specifically to the $\mathcal{L}$-signature, we obtain:

any $S \subseteq P$.

a) if $\gamma \odot \beta$ is $+(\gamma \star \beta)$, $+(\gamma \vee \beta)$, $+(\beta \to \gamma)$, or $-(\beta - \gamma)$, then $\varepsilon^{\partial}(+\gamma)$;
b) if $\gamma \odot \beta$ is $+(\gamma \to \beta)$, $-(\gamma \wedge \beta)$, $-(\gamma \circ \beta)$, or $-(\gamma - \beta)$, then $\varepsilon(+\gamma)$, i.e., $\varepsilon^{\partial}(-\gamma)$.

3. On any SLR-node in P_2 of the form $\gamma \odot \beta$, where β is the side where the branch lies, γ is a mu-sentence and $\varepsilon^{\partial}(\gamma) \prec *\varphi$ (see above for this notation). Unravelling the condition $\varepsilon^{\partial}(\gamma) \prec *\varphi$ specifically to the $\mathcal{L}$-signature, we obtain:
a) if $\gamma \odot \beta$ is $-(\gamma \star \beta)$,$-(\gamma \vee \beta)$, $-(\beta \to \gamma)$, or $+(\beta - \gamma)$, then $\varepsilon(+\gamma)$, i.e., $\varepsilon^{\partial}(-\gamma)$;
b) if $\gamma \odot \beta$ is $-(\gamma \to \beta)$, $+(\gamma \wedge \beta)$, $+(\gamma \circ \beta)$, or $+(\gamma - \beta)$, then $\varepsilon^{\partial}(+\gamma)$.

5.3.2. Definition. Given an order-type ε, the signed generation tree $*\varphi$, with $* \in \{-, +\}$, of an $\mathcal{L}$-sentence $\varphi(p_1, \dots p_n)$ is *ε-recursive* if every ε-critical branch is ε-good. Such a signed generation is *non-trivially ε-recursive* if contains at least one ε-critical branch.

An $\mathcal{L}$-inequality $\varphi \leq \psi$ is *ε-recursive* if the signed generation trees $+\varphi$ and $-\psi$ are both ε-recursive. An $\mathcal{L}$-inequality $\varphi \leq \psi$ is *recursive* if it is ε-recursive for some order-type ε.

The signed generation tree $*\varphi$, with $* \in \{-, +\}$, is *ε-PIA* if it is ε-recursive and all ε-critical branches consist only of PIA-nodes. Such a signed generation is *non-trivially ε-PIA* if contains at least one ε-critical branch.

5.3.3. Example. The inequality $\nu X.\Box(p \wedge X) \leq p$, corresponding to the formula $\nu X.\Box(p \wedge X) \to p$ from [17, Section 5.3] was discussed at the beginning of Section 5.2. This inequality is ε-recursive for $\varepsilon = (1)$ and $\varepsilon = (\partial)$. In Section 5.2 we gave the ALBA-reduction according to $\varepsilon = (\partial)$. In Section 5.4.2 we discuss how to do a reduction according to $\varepsilon = (1)$.

5.3.4. Example. The inequality $\nu X.\neg(p \wedge \neg X) \leq \Diamond\Box p$ is not ε-recursive for any order-type ε. Indeed, if $\varepsilon_p = \partial$ then, on the critical branch in $+\nu X.\neg(p \wedge \neg X)$, the $-\wedge$ is an SRR node which separates the p and the fixed point variable X. If $\varepsilon_p = 1$ then the critical branch in $-\Diamond\Box p$ is clearly not good. On the other hand, the unfolding of the fixed point stabilizes after the first step as $\top$, hence the inequality is equivalent to $\top \leq \Diamond\Box p$ which is ε-recursive for $\varepsilon_p = \partial$. In fact, the first-order definability of $\top \leq \Diamond\Box p$ already follows from the fact that it is monotone in p.

5.3.5. Example. Consider the inequality

$$\Diamond\mu X.[(p \vee X) \vee {\sim}\nu Y.[\Diamond(X \vee {\sim}((Y \wedge p) \wedge \mu Z.{\sim}(\Box p \wedge \neg Z))) \to \Diamond\Box p]] \leq \Diamond\Box p.$$

This is ε-recursive with $\varepsilon_p = 1$. Indeed, in the positive generation tree of the left-hand side, there are two critical branches, respectively corresponding to the first and third occurrences of p in the formula, counting from the left. The branch leading from the first is $+p, +\vee, +\vee, +\mu X, +\Diamond$, and partitioning this as $P_1 = \varnothing$, $P_2 = +\vee, +\vee, +\mu X$, and $P_3 = +\Diamond$ satisfies the requirements of Definition 5.3.2. The branch leading from the third occurrence of p is

$$+p, \underbrace{+\Box, +\wedge, -\sim, -\mu Z}_{P_1}, \underbrace{-\wedge, +\sim, +\vee, +\Diamond, -\rightarrow, -\nu Y, +\sim, +\vee, +\mu X}_{P_2}, \underbrace{+\Diamond}_{P_3},$$

and partitioning it as indicated satisfies the requirements of Definition 5.3.2. In particular, there are no SRR nodes, and the only occurring SLR node is $-\rightarrow$, which satisfies condition 3(a) of the definition since $\Diamond\Box p$ is a sentence and $\varepsilon^\partial(-\Diamond\Box p)$.

5.3.6. Remark. Definition 5.3.2 implies that on a good branch, within P_2 and within P_1, occurrences of nodes $\rightarrow$ and $-$ where the branch goes through the child corresponding to the antitone coordinate need to be in strict alternation. This can be seen, e.g., in Example 5.3.5 in the P_2-part of the displayed branch. This implies that, if we restrict to the signature of intuitionistic modal logic by removing $-$, we would be able to change polarity at most once within the P_2 and P_1 parts of a good branch. Given the further restrictions imposed by Definition 5.3.1.2 and 5.3.1.3, this would imply that no good branch could go through the antitone coordinate of $\rightarrow$ within the scope of a fixed point binder, thus severely restricting the diversity of order-theoretic behaviour within the resulting class of recursive mu-inequalities. This brings with it the added inconvenience that, when projecting onto the classical setting (see Section 5.3.2 below) we would have to restrict the range to formulas in negation normal form.

5.3.2 General syntactic shapes and a comparison with existing Sahlqvist-type classes

The aim of the present subsection is to position the ε-recursive mu-inequalities with respect to the general syntactic shape of Sahlqvist/Inductive/Recursive inequalities discussed in [46, Subsections 36.6.1 and 36.7.2], and to compare them with the Sahlqvist mu-formulas defined in [17, Definition 3.4].

Recursive mu-inequalities and the general Sahlqvist/Inductive/Recursive shape

It can be straightforwardly checked that the outer-skeleton nodes (see Table 5.1) of an ε-recursive mu-inequality satisfy the same order-theoretic requirements of the nodes of an ε-Sahlqvist inequality [46, definitions 36.6.2 and 36.6.3] in which the length of the P_1 paths of ε-critical branches is 0. It is also straightforward to see that, in any ε-recursive mu-inequality, the ε-PIA subtrees are defined in such a way that at most one ε-critical branch may pass through any given SRR-node; as discussed in [46, Subsection 36.7.2], this is the defining feature of ε-Recursive inequalities across languages. The specific definition of the PIA-subtrees for mu-languages incorporates extra conditions regulating the relative positions of free fixed point variables and ε-critical variables in each subtree; as we will see further, these conditions ensure that formulas in the scope of binders have the appropriate order-theoretic properties, ultimately guaranteeing the applicability of the μ- and ν-adjunction rules.

The inner skeleton essentially arises by the addition of fixed point binders, in the appropriate polarity, to the 'outer skeleton' shape. This introduction blocks the application of the Δ-rules ($\wedge$ RA) and ($\vee$ LA) (and, more generally, also the possibility of applying rules to single connectives), leaving us with only μ- and ν-approximation rules. Hence, in inner skeletons, all the nodes are reclassified according to the properties which they enjoy and which are now relevant. Similar to the PIA-subtrees, the inner-skeleton shape incorporates extra conditions regulating the relative positions of free fixed point variables and ε-critical variables; as we will see in the remainder of the chapter, these conditions ensure that formulas in the scope of binders have the appropriate order-theoretic properties guaranteeing the applicability of the μ- and ν-approximation rules.

The shape of ε-recursive mu-inequalities provides a uniform 'winning strategy' for the success of ALBA, analogous to the one described for ε-inductive and ε-Sahlqvist inequalities in [50, Section 10] and [46, Subsection 36.6.1]. Indeed, as we will show in Section A.2, the order-type ε tells us which occurrences of a given variable we need to solve for so as to reach Ackermann shape, and the ε-recursive shape guarantees that this is always possible. Specifically, going down a critical branch, we can surface the PIA-subtree, containing the ε-critical occurrences of propositional variables, by means of applications of approximation rules to the skeleton nodes. Then adjunction/residuation rules such as (μ-Adj) and (ν-A) are applied to the PIA-subtrees so as to display the ε-critical occurrences, and to simultaneously calculate the minimal valuation for them. Finally, notice that the remaining occurrences of variables are of the opposite order-type: this guarantees that they have the right polarity to receive the calculated minimal valuations, as prescribed by (LA), (RA) or their recursive counterparts. An exhaustive and formal account of this procedure will be given in Section A.2.

Finally, as hinted above, notice that the winning strategy outlined so far does not provide information about which version of the Ackermann rule will actually be applied in the reduction procedure. Should we want to guarantee that either (LA) or (RA) will be applied, and not their recursive counterparts, we need to strengthen Definition 5.3.2 so as to guarantee that, when displaying the critical occurrences in inequalities of the form $\alpha \leq p$ or $p \leq \alpha$, the formula α is p-free. This requirement can be enforced by introducing the (Ω, ε)-*inductive mu-inequalities* along the lines of the (Ω, ε)-inductive DML/IML inequalities of

[50, Definition 3.1]: namely, by imposing a partial ordering Ω upon the variables in Recursive inequalities, and demanding not only that at most one ε-critical branch pass through any given SRR-node, but also that if an ε-critical branch passes through an SRR-node, all variables occurring on other branches passing through it have to be strictly Ω-smaller than the variable on the critical branch.

Recursive mu-inequalities and Sahlqvist mu-formulas

In [17], the following notions are introduced in the language of classical modal mu-calculus:

5.3.7. Definition. The class of *PIA formulas* is recursively defined as follows:

$$\varphi ::= p \mid X \mid \varphi_1 \wedge \varphi_2 \mid \Box\varphi \mid \nu X.\varphi \mid \neg\pi \vee \varphi,$$

where $p \in \mathsf{Prop}$, $X \in \mathsf{FVar}$, and π is a positive sentence. The class of *Sahlqvist mu-formulas* is recursively defined as follows:

$$\chi ::= X \mid \pi \mid \neg\varphi \mid \chi_1 \wedge \chi_2 \mid \Box\chi \mid \nu X.\chi \mid \pi \vee \chi \mid \sigma_1 \vee \sigma_2,$$

where $p \in \mathsf{Prop}$, $X \in \mathsf{FVar}$, φ is a PIA sentence, π is a positive sentence, and σ_1 and σ_2 are Sahlqvist mu-formulas which are sentences.

Let us consider the mapping τ from the language of classical modal mu-calculus to the language of bi-intuitionistic modal mu-calculus recursively defined as expected (in particular, $\tau(\neg\xi) := \tau(\xi) \to \bot$); let us also translate formulas as inequalities by the mapping $\tau'(\xi) := \top \leq \tau(\xi)$. Conversely, consider the mapping λ recursively defined as expected on the connectives which have a primitive classical counterpart, and such that:

$$\begin{array}{rcl} \lambda(\xi_1 \to \xi_2) & = & \neg\lambda(\xi_1) \vee \lambda(\xi_2) \\ \lambda(\xi_1 - \xi_2) & = & \lambda(\xi_1) \wedge \neg\lambda(\xi_2) \\ \lambda(\nu X.\xi(X)) & = & \neg\mu X.\neg\xi(\neg X/X). \end{array}$$

Let us also translate bi-intuitionistic inequalities into classical formulas by the mapping $\lambda'(\xi_1 \leq \xi_2) = \neg\lambda(\xi_1) \vee \lambda(\xi_2)$. We omit the proof of the following proposition, which is straightforward but tedious.

5.3.8. Proposition. *1. Every formula ξ of modal mu-calculus is logically equivalent to $\lambda'(\tau'(\xi))$;*

2. *for every Sahlqvist mu-formula χ, the inequality $\top \leq \tau(\chi)$ is an ε-recursive inequality with $\varepsilon = \overline{1}$;*

3. *for every $\overline{1}$-recursive inequality $\xi_1 \leq \xi_2$, the formula $\neg\lambda(\xi_1) \vee \lambda(\xi_2)$ is a Sahlqvist mu-formula.*

5.4 Inner formulas and their normal forms

As discussed in Section 5.2.5, the aim of the present section is to introduce and study a class of mu-formulas, the *inner formulas*, the syntactic shape of which guarantees that their associated term functions enjoy the order-theoretic properties which in turn guarantee that the approximation and adjunction rules are systematically applicable to them.

This is the most technically involved section of the chapter; for the sake of clarity, it is organized as follows: in Subsection 5.4.1, inner formulas are defined, and is shown that their associated term functions indeed satisfy the mentioned order-theoretic requirements; in Subsection 5.4.2, two case studies are discussed, which focus in particular on how to effectively calculate the adjoints of inner formulas; this discussion motivates the introduction, in Subsection 5.4.3, of the notion of inner formulas in normal form, and its ensuing normalization proposition; finally, in Subsection 5.4.4, a lemma is proven which provides the effective computation of the adjoints of inner formulas in normal form.

5.4.1 Inner formulas

5.4.1. Definition. Let $\overline{y}, \overline{z} \subseteq Var$ and $\overline{X} \subseteq \mathsf{FVar}$ be tuples, each consisting of pairwise different variables, such that $\overline{y}$ and $\overline{z}$ are disjoint. Let $\overline{x} = \overline{y} \oplus \overline{X}$ and let δ be an order-type on $\overline{x} = (x_i)_{i=1}^n$. The *$\delta$-$\Box$ and δ-$\Diamond$ $(\overline{x}, \overline{z})$-inner formulas* (*$(\overline{x}, \overline{z})$-$IF_\delta^\Box$* and *$(\overline{x}, \overline{z})$-$IF_\delta^\Diamond$*), the free variables of which are contained in $(\overline{x}, \overline{z})$, are given by the following simultaneous recursion (for the sake of readability, the parameters $\overline{x}$ and $\overline{z}$ are omitted):

$$\mathrm{IF}_\delta^\Box \ni \varphi \ ::= \ \top \mid x_i \mid \Box\varphi \mid \varphi_1 \wedge \varphi_2 \mid \nu Y.\varphi' \mid \pi \to \varphi \mid \pi \vee \varphi \mid \psi^c \to \pi$$

$$\mathrm{IF}_\delta^\Diamond \ni \psi \ ::= \ \bot \mid x_i \mid \Diamond\psi \mid \psi_1 \vee \psi_2 \mid \mu Y.\psi' \mid \psi - \pi \mid \pi \wedge \psi \mid \pi - \varphi^c$$

where

1. $\delta_i = 1$ in the base of the recursion, for $1 \leq i \leq n$,
2. π is $\pi(\overline{z}) \in \mathcal{L}^+$,
3. $\varphi' = \varphi'(\overline{y} \oplus \overline{X}', \overline{z})$ and $\psi' = \psi'(\overline{y} \oplus \overline{X}', \overline{z})$ are $\mathrm{IF}_{\delta'}^\Box$ and $\mathrm{IF}_{\delta'}^\Diamond$, respectively, with $\overline{X}' = \overline{X} \oplus Y$ and $\delta' = \delta \oplus 1$,
4. $\psi^c \in (\overline{x}, \overline{z})$-$\mathrm{IF}_{\delta^\partial}^\Diamond$ and $\varphi^c \in (\overline{x}, \overline{z})$-$\mathrm{IF}_{\delta^\partial}^\Box$.
5. All other formulas have their free variables among $(\overline{x}, \overline{z})$.

With similar side conditions we can define the *δ-$\blacksquare$ and δ-$\blacklozenge$ $(\overline{x}, \overline{z})$-inner formulas* ($(\overline{x}, \overline{z})$-$IF_\delta^{\blacksquare}$ and $(\overline{x}, \overline{z})$-$IF_\delta^{\blacklozenge}$) by the following simultaneous recursion:

$$
\begin{array}{llccccccc}
\mathrm{IF}_\delta^{\blacksquare} \ni \varphi & ::= & \top & \mid\ x_i & \mid\ \Box\varphi & \mid\ \blacksquare\varphi & \mid\ \varphi_1 \wedge \varphi_2 & \mid\ \nu Y.\varphi' & \mid \\
 & & \pi \to \varphi & \mid\ \pi \vee \varphi & \mid\ \psi^c \to \pi & & & & \\
\mathrm{IF}_\delta^{\blacklozenge} \ni \psi & ::= & \bot & \mid\ x_i & \mid\ \Diamond\psi & \mid\ \blacklozenge\psi & \mid\ \psi_1 \vee \psi_2 & \mid\ \mu Y.\psi' & \mid \\
 & & \psi - \pi & \mid\ \pi \wedge \psi & \mid\ \pi - \varphi^c & & & &
\end{array}
$$

In what follows, the letter φ and ψ (possibly with superscripts or indexes) will denote $\mathrm{IF}^{\Box}$- and $\mathrm{IF}^{\Diamond}$-formulas, respectively.

Note that every $\mathrm{IF}^{\Box}$-formula is an $\mathrm{IF}^{\blacksquare}$-formula and that every $\mathrm{IF}^{\Diamond}$-formula is a $\mathrm{IF}^{\blacklozenge}$-formula.

5.4.2. Remark. The above definition is tailored to ensure that for any perfect modal bi-Heyting algebra L (cf. Definition 5.1.2), the term function associated with a $\mathrm{IF}_\delta^{\blacksquare}$ (respectively, $\mathrm{IF}_\delta^{\blacklozenge}$) formula is a right (respectively, left) adjoint from $L^\delta \to L$ fixing the variables $\overline{z}$ as parameters (see lemma below).

In particular this requires that in the associated generation tree, on each branch ending in an x_i the nodes corresponding to the negative sides of $\to$ and $-$ are in strict alternation. Moreover, any alternation between $\mathrm{IF}^{\blacksquare}$ and $\mathrm{IF}^{\blacklozenge}$ is accompanied by a change of polarity. Finally, these considerations imply that, in the signature of intuitionistic modal logic, where the subtraction symbol is removed, change of polarity on these 'critical' branches can occur at most once.

5.4.3. Lemma. *For any perfect modal bi-Heyting algebra $\mathbb{C}$,*

1. *the term function associated with any $IF_\delta^{\blacksquare}$-formula $\varphi(\overline{x}, \overline{z})$ is completely meet-preserving as a map $\mathbb{C}^\delta \to \mathbb{C}$, fixing the variables $\overline{z}$, and*
2. *the term function associated with any $IF_\delta^{\blacklozenge}$-formula $\psi(\overline{x}, \overline{z})$ is completely join-preserving as a map $\mathbb{C}^\delta \to \mathbb{C}$, fixing the variables $\overline{z}$.*

Proof. By simultaneous induction on φ and ψ. The base cases are clear, as are the cases corresponding to the third, fourth and fifth columns in the recursive definition above. The case for φ of the form $\nu Y.\varphi'(\overline{y} \oplus \overline{X}', \overline{z})$ follows by the induction hypothesis and Lemma 5.2.1.4. Analogously the case for ψ of the form $\mu Y.\psi'(\overline{y} \oplus \overline{X}', \overline{z})$ follows by the induction hypothesis and the order-dual of Lemma 5.2.1.4. The cases corresponding to the fifth and sixth columns in the recursive definition follow from the induction hypothesis, the fact that $\to$ and $\vee$ are completely meet-preserving in their positive coordinates, while $-$ and $\wedge$ are completely join-preserving in their positive coordinates, and the fact that variables from $\overline{x}$ appear in at most one coordinate of each, which are moreover positive. Similarly, the cases corresponding to the last column follow from the fact that $-$ and $\rightarrow$ are respectively completely meet and join-reversing in their negative coordinates, and the fact that variables from $\overline{x}$ appear in at most their negative coordinates. $\square$

5.4.4. Remark. We note that inner formulas provide sufficient conditions for the term functions to be either completely join-preserving or meet-preserving. In [69], the authors provide a syntactic characterization of fragments of mu-calculus enjoying such order-theoretic properties.

5.4.2 Towards syntactic adjunction rules

The lemma above guarantees that the approximation rules (μ^δ-A) and (ν^δ-A) can be respectively applied in particular to inequalities of the form $\mathbf{i} \leq \mu X.\psi(\overline{y}, X, \overline{z})$ and $\nu X.\varphi(\overline{y}, X, \overline{z}) \leq \mathbf{m}$, such that $\mu X.\psi$ and $\nu X.\varphi$ are $(\overline{y}, \overline{z})$-IF$^{\Diamond}_{\delta}$- and $(\overline{y}, \overline{z})$-IF$^{\Box}_{\delta}$-*sentences* respectively. For the same reasons, also the general adjunction rules can be applied to inequalities featuring δ-$\Box$ and δ-$\Diamond$ $(\overline{y}, \overline{z})$-inner sentences as main formulas on the appropriate sides. However, the general adjunction rules do not provide any information as to how the adjoint map can be effectively computed as term functions. Indeed, in what follows, we will work towards new adjunction rules which explicitly incorporate such computations. These new rules will be given in terms of a syntactic refinement of inner formulas, introduced in the next subsection. In order to motivate this refinement, it will be useful to consider the following pair of examples.

Consider the inequality $\nu X.\Box(p \wedge X) \leq p$, which we already solved towards the end of Section 4.2. Notice that $\nu X.\Box(x \wedge X)$ is an $(x, \varnothing)$-IF$^{\Box}_{\delta}$ formula, with $\delta = (1)$. An alternative and more instructive reduction proceeds as follows: after first approximation we get

$$\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \nu X.\Box(p \wedge X) \;\&\; p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}].$$

Trying to solve for the occurrence of p in $\mathbf{i} \leq \nu X.\Box(p \wedge X)$, we *unfold* the fixed point (see, e.g., [61]) and obtain $\mathbf{i} \leq \bigwedge_{\kappa \geq 1} \Box^\kappa p$. This is equivalent to $\mathbf{i} \leq \Box^\kappa p$ for every $\kappa \geq 1$. By general adjunction, each such inequality is equivalent to $\blacklozenge^\kappa \mathbf{i} \leq p$. Hence we have:

$$\mathbf{i} \leq \bigwedge_{\kappa \geq 1} \Box^\kappa p \quad \text{iff} \quad \bigvee_{\kappa \geq 1} \blacklozenge^\kappa \mathbf{i} \leq p.$$

Noticing that $\bigvee_{\kappa \geq 1} \blacklozenge^\kappa \mathbf{i}$ is the unfolding of $\mu X.\blacklozenge(X \vee \mathbf{i})$, the quasi-inequality displayed above is equivalent to

$$\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mu X.\blacklozenge(X \vee \mathbf{i}) \leq p) \;\&\; p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

which is in Ackermann shape and yields

$$\forall \mathbf{i} \forall \mathbf{m}[\mu X.\blacklozenge(X \vee \mathbf{i}) \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

This example illustrates an effective computation of the left adjoint of an $\mathrm{IF}^{\Box}_{\delta}$-formula, with δ constantly 1. Consider now the analogous computation of the adjoint of an $\mathrm{IF}^{\Box}_{\delta}$-formula, where δ is not constantly 1; for instance the left adjoint of $\nu X.\neg\Diamond(p \vee {\sim}X)$. It is easy to see that, unfolding this fixed point, one gets to a conjunction $\bigwedge_{\kappa\geq 0} \rhd_\kappa(p)$, where for every ordinal κ, the symbol $\rhd_\kappa(p)$ denotes an $\mathcal{L}^+$-term which is (completely) join-reversing in p. Hence, proceeding as we did in the previous computation we obtain:

$$\mathbf{i} \leq \bigwedge_{\kappa\geq 0} \rhd_\kappa p \quad \text{iff} \quad p \leq \bigwedge_{\kappa\geq 0} \blacktriangleright_\kappa \mathbf{i}.$$

The main difference between this clause and the analogous clause displayed in the previous computation is that we are not yet in a position to recognize $\bigwedge_{\kappa\geq 1} \blacktriangleright_\kappa \mathbf{i}$ as the unfolding of some fixed point. In particular, for this, we would need to see the parameter κ explicitly as the exponential $()^\kappa$ applied to some term. This term can be calculated either inductively for each κ, or observing that $\nu X.\neg\Diamond(p \vee {\sim}X) = \nu X.[\neg\Diamond p \wedge \neg\Diamond{\sim}X]$, unfolding which yields $\bigwedge_{\kappa\geq 0}(\neg\Diamond{\sim})^\kappa(\neg\Diamond p)$. Now the displayed clause above becomes:

$$\mathbf{i} \leq \bigwedge_{\kappa\geq 0}(\neg\Diamond{\sim})^\kappa(\neg\Diamond p) \quad \text{iff} \quad \bigvee_{\kappa\geq 0}({\sim}\blacksquare\neg)^\kappa(\mathbf{i}) \leq \neg\Diamond p \quad \text{iff} \quad p \leq \blacksquare\neg \bigvee_{\kappa\geq 0}({\sim}\blacksquare\neg)^\kappa(\mathbf{i}).$$

Notice that the term $\nu X.[\neg\Diamond p \wedge \neg\Diamond{\sim}X]$ which was obtained by distributing $\neg\Diamond$ over $\vee$ can be seen as the result of substituting $\neg\Diamond p$ for x in $\nu X.[x \wedge \neg\Diamond{\sim}X]$, and that the latter is an $\mathrm{IF}^{\Box}_{\delta'}$-formula with δ' constantly 1. This neatly breaks the computation of the adjoint into two steps, the first of which calculates the adjoint of the 'right-side-up' fixed point, and the second composes it with the adjoint of the negative term $\neg\Diamond p$. This is the basic idea underlying the notion of normal forms in the following subsection.

5.4.3 Normal forms and normalization

5.4.5. DEFINITION. The *normal* $(\overline{x}, \overline{z})$-$\mathrm{IF}^{\Box}_{\delta}$- and $(\overline{x}, \overline{z})$-$\mathrm{IF}^{\Diamond}_{\delta}$-formulas are given by the same simultaneous recursion as in Definition 5.4.1, subject to the following additional constraints:

1. if φ is of the form $\nu Y.\varphi'(\overline{x}', \overline{z})$, where $\overline{x}' = \overline{y} \oplus \overline{X}'$ and $\overline{X}' = \overline{X} \oplus Y$, then there exists an $(\overline{y}' \oplus \overline{X}', \overline{z})$-$\mathrm{IF}^{\Box}_{\delta'}$-formula φ'', where δ' is the order-type over $\overline{y}' \oplus \overline{X}'$ which is constantly 1 over $\overline{y}'$ and restricts to δ over $\overline{X}'$, such that $\varphi'(\overline{x}', \overline{z}) = \varphi''(\overline{\varphi}/\overline{y}', \overline{X}', \overline{z})$ where the $\overline{\varphi}$ are normal $(\overline{y}, \overline{z})$-$\mathrm{IF}^{\Box}_{\delta''}$-sentences, where δ'' is the restriction of δ to $\overline{y}$.

2. if ψ is of the form $\mu Y.\psi'(\overline{x}', \overline{z})$, where $\overline{x}' = \overline{y} \oplus \overline{X}'$ and $\overline{X}' = \overline{X} \oplus Y$, then there exists an $(\overline{y}' \oplus \overline{X}', \overline{z})$-$\mathrm{IF}^{\Diamond}_{\delta'}$-formula ψ'', where δ' is the order-type over

$\overline{y}' \oplus \overline{X}'$ which is constantly 1 over $\overline{y}'$ and restricts to δ over $\overline{X}'$, such that $\psi'(\overline{x}', \overline{z}) = \varphi''(\overline{\psi}/\overline{y}', \overline{X}', \overline{z})$ where the $\overline{\psi}$ are normal $(\overline{y}, \overline{z})$-IF$^{\Diamond}_{\delta''}$-sentences, where δ'' is the restriction of δ to $\overline{y}$.

5.4.6. Lemma. *1. Every $(\overline{x}, \overline{z})$-IF$^{\Box}_{\delta}$-formula φ with $x_1, x_2 \in \overline{x}$ is equivalent to an $(\overline{x}, \overline{z})$-IF$^{\Box}_{\delta}$-formula of the form $\varphi_1(\overline{x}_1, \overline{z}) \wedge \varphi_2(\overline{x}_2, \overline{z})$, where $\overline{x}_1 = \overline{x} \setminus \{x_2\}$ and $\overline{x}_2 = \overline{x} \setminus \{x_1\}$.*

2. Every $(\overline{x}, \overline{z})$-IF$^{\Diamond}_{\delta}$-formula ψ with $x_1, x_2 \in \overline{x}$ is equivalent to an $(\overline{x}, \overline{z})$-IF$^{\Diamond}_{\delta}$-formula of the form $\psi_1(\overline{x}_1, \overline{z}) \vee \psi_2(\overline{x}_2, \overline{z})$, where $\overline{x}_1 = \overline{x} \setminus \{x_2\}$ and $\overline{x}_2 = \overline{x} \setminus \{x_1\}$.

Proof. By simultaneous induction on φ and ψ. The base cases when $\varphi \in \{x, \top\}$ and $\psi \in \{x, \bot\}$ follow by noting that $\varphi \equiv \varphi \wedge \top$ and $\psi \equiv \psi \vee \bot$, respectively. The case $\varphi = \Box\varphi'$ follows by the induction hypothesis and the distributivity of $\Box$ over $\wedge$. The case $\varphi = \varphi_1 \wedge \varphi_2$ follows by the induction hypothesis and associativity and commutativity of $\wedge$. The case $\varphi = \psi^c \to \pi$ follows by the induction hypothesis on the IF$^{\Diamond}_{\delta^\partial}$-formula ψ^c and the fact that $\to$ turns $\vee$ into $\wedge$ in its first coordinate. Consider the case $\varphi = \nu Y.\varphi'(\overline{x}', \overline{z})$ with $\overline{x}' = \overline{x} \oplus Y$ and $\delta' = \delta \oplus 1$. By induction hypothesis $\varphi'(\overline{x}', \overline{z}) \equiv \varphi_1(\overline{x}_1 \oplus Y, \overline{z}) \wedge \varphi_2(\overline{x}_2 \oplus Y, \overline{z})$, where $\overline{x}_1 = \overline{x} \setminus \{x_2\}$ and $\overline{x}_2 = \overline{x} \setminus \{x_1\}$. By applying the induction hypothesis again to $\varphi_1(\overline{x}_1 \oplus Y, \overline{z})$ (w.r.t. x_1 and Y) and $\varphi_2(\overline{x}_2 \oplus Y, \overline{z})$ (w.r.t. x_2 and Y) we obtain

$$\begin{aligned} & \varphi'(\overline{x}', \overline{z}) \\ \equiv\ & [\varphi_1'(\overline{x}_1' \oplus Y, \overline{z}) \wedge \varphi_1''(\overline{x}_1, \overline{z})] \wedge [\varphi_2'(\overline{x}_2' \oplus Y, \overline{z}) \wedge \varphi_2''(\overline{x}_2, \overline{z})] \\ \equiv\ & [\varphi_1'(\overline{x}_1' \oplus Y, \overline{z}) \wedge \varphi_2'(\overline{x}_2' \oplus Y, \overline{z})] \wedge [\varphi_1''(\overline{x}_1, \overline{z}) \wedge \varphi_2''(\overline{x}_2, \overline{z})], \end{aligned}$$

where $\overline{x}_1' = \overline{x}_1 \setminus \{x_1\}$ and $\overline{x}_2' = \overline{x}_2 \setminus \{x_2\}$. Note that $\varphi_1'(\overline{x}_1' \oplus Y, \overline{z}) \wedge \varphi_2'(\overline{x}_2' \oplus Y, \overline{z})$ is an $(\overline{x} \oplus Y, \overline{z})$-IF$^{\Box}_{\delta'}$-formula, and hence, by Lemma 5.4.3, it is a right adjoint in $\overline{x} \oplus Y$. Therefore, it preserves non-empty joins in Y. Hence, by Lemma 5.2.3, applied to $f(Y) = \varphi_1'(\overline{x}_1' \oplus Y, \overline{z}) \wedge \varphi_2'(\overline{x}_2' \oplus Y, \overline{z})$, $g_1(\overline{x}_1) = \varphi_1''(\overline{x}_1, \overline{z})$, and $g_2(\overline{x}_2) = \varphi_2''(\overline{x}_2, \overline{z})$, we have

$$\begin{aligned} & \nu Y.\varphi'(\overline{x}', \overline{z}) \\ \equiv\ & \nu Y.([\varphi_1'(\overline{x}_1' \oplus Y, \overline{z}) \wedge \varphi_2'(\overline{x}_2' \oplus Y, \overline{z})] \wedge [\varphi_1''(\overline{x}_1, \overline{z}) \wedge \varphi_2''(\overline{x}_2, \overline{z})]) \\ \equiv\ & \nu Y.[\varphi_1'(\overline{x}_1' \oplus Y, \overline{z}) \wedge \varphi_2'(\overline{x}_2' \oplus Y, \overline{z}) \wedge \varphi_1''(\overline{x}_1, \overline{z})] \\ \wedge\ & \nu Y.[\varphi_1'(\overline{x}_1' \oplus Y, \overline{z}) \wedge \varphi_2'(\overline{x}_2' \oplus Y, \overline{z}) \wedge \varphi_2''(\overline{x}_2, \overline{z})], \end{aligned}$$

where x_2 does not occur in the first conjunct, and x_1 does not occur in the second.

The other cases are analogous and are left to the reader. $\Box$

By repeated application of the lemma above we obtain the following Corollary:

5.4.7. Corollary. *1. Every $(\overline{x}, \overline{z})$-IF$^{\Box}_{\delta}$-formula φ is equivalent to an $(\overline{x}, \overline{z})$-IF$^{\Box}_{\delta}$-formula of the form $\varphi_1(\overline{x}_1, \overline{z}) \wedge \varphi_2(\overline{x}_2, \overline{z})$, where $\overline{x}_1$, $\overline{x}_2$ form a partition of $\overline{x}$.*

2. *Every $(\overline{x}, \overline{z})$-IF$^{\Diamond}_{\delta}$-formula ψ is equivalent to an $(\overline{x}, \overline{z})$-IF$^{\Diamond}_{\delta}$-formula of the form $\psi_1(\overline{x}_1, \overline{z}) \vee \psi_2(\overline{x}_2, \overline{z})$, where $\overline{x}_1$, $\overline{x}_2$ form a partition of $\overline{x}$.*

5.4.8. PROPOSITION. *Every IF$^{*}_{\delta}$ formula, $* \in \{\Diamond, \Box\}$, is equivalent to an IF$^{*}_{\delta}$ formula in normal form.*

Proof. Notice that if a $(\overline{x}, \overline{z})$-IF$^{\Box}_{\delta}$-formula φ is non-normal, it must contain a subformula of the form $\nu Y.\varphi'$ which violates Definition 5.4.5.1, and where φ' is an $(\overline{y} \oplus \overline{X} \oplus Y, \overline{z})$-IF$^{\Box}_{\delta'}$-formula. If in φ', every variable $y \in \overline{y}$ occurs only positively, the trivial substitution given by the identity on $\overline{y}$ in φ' itself would witness the normality. This means that, in the positive generation tree $+\varphi'$, there is at least one leaf $-y$ with $y \in \overline{y}$. I.e., on the branch from each such $-y$ to the root there is an odd number of order-reversing nodes which, as per Definition 5.4.1, need to be positive occurrence of $\rightarrow$ and negative occurrences of $-$, in strict alternation. Thus the first order-reversing node above each such leaf $-y$ is a positive occurrence of $\rightarrow$. By the assumption that φ is not in normal form, it follows that at least one of the subformulas rooted at such a node $+\rightarrow$ is not a sentence. (Indeed, if all the subformulas $\zeta = \psi^c \rightarrow \pi$ rooted at such nodes were sentences, then replacing each of them with a fresh variable $y' \in \overline{y}'$ would give us the required formula φ'' of Definition 5.4.5.1). Let a *defect* of φ be an occurrence of a $+\rightarrow$ node in the scope of a $+\nu Y$ node in $+\varphi$ such that the corresponding subformula $\zeta = \psi^c \rightarrow \pi$ is not a sentence and contains negative occurrences of variables in $\overline{y}$. Dually, we define a defect of a $(\overline{x}, \overline{z})$-IF$^{\Diamond}_{\delta}$-formula ψ as a positive occurrence of $-$ in the scope of a $+\mu Y$ node in $+\psi$ such that the corresponding subformula $\zeta = \pi - \varphi^c$ is not a sentence and contains negative occurrences of variables in $\overline{y}$.

The proof now proceeds by induction on the set $(\mathsf{defect}(\chi), \chi)$ ordered lexicographically, where χ is an inner formula and $\mathsf{defect}(\chi)$ is the number of defects occurring in χ. The base case is trivial. As for the induction step, we proceed by cases depending on the form of χ. If the main connective of χ is not a fixed point binder, then the claim follows by the induction hypothesis applied to the immediate subformulas. Now suppose φ is a $(\overline{x}, \overline{z})$-IF$^{\Box}_{\delta}$-formula of the form $\nu Y.\varphi'$. Let $\zeta = \psi^c \rightarrow \pi$ be a defect of φ. Since, by Definition 5.4.1, π must be a sentence, all the free variables of ζ occur only in the IF$^{\Diamond}_{\delta'\partial}$-formula $\psi^c(\overline{y} \oplus \overline{X} \oplus Y, \overline{z})$. By Corollary 5.4.7.2, the formula ψ^c is equivalent to one of the form $\psi_1(\overline{y}, \overline{z}) \vee \psi_2(\overline{X} \oplus Y, \overline{z})$, where ψ_1 and ψ_2 are IF$^{\Diamond}_{\delta'\partial}$-formulas. Hence, ζ is equivalent to — and hence can be replaced by — $(\psi_1(\overline{y}, \overline{z}) \rightarrow \pi) \wedge (\psi_2(\overline{X} \oplus Y) \rightarrow \pi)$. Let φ'' be the formula resulting from this replacement in φ. Notice that $(\psi_1(\overline{y}, \overline{z}) \rightarrow \pi)$ is an IF$^{\Box}_{\delta''}$-sentence, where δ'' is the restriction of δ to $\overline{y}$, and hence, within φ'', no subformula of $(\psi_1(\overline{y}, \overline{z}) \rightarrow \pi) \wedge (\psi_2(\overline{X} \oplus Y) \rightarrow \pi)$ constitutes a defect. Hence φ'' has at least one defect less that φ, so by the inductive hypothesis φ'', and hence φ, is equivalent to a $(\overline{x}, \overline{z})$-IF$^{\Box}_{\delta}$ formula in normal form. $\Box$

5.4.9. Remark. We observe that an effective procedure for transforming any inner formula into an equivalent one in normal form can be extracted from the proof of Proposition 5.4.8. In Section A.2 we will exploit the fact that such a procedure exists, although we will not describe it in any further detail, limiting ourselves to illustrate it by means of the examples below.

5.4.10. Example. The formula $\nu X.\neg(x \vee \sim X)$ is an $(x, \varnothing)$-$\mathrm{IF}^{\Box}_{\delta}$-formula for $\delta = (\partial)$, and it is not in normal form, the subformula $\neg(x \vee \sim X)$ being its only defect. The normalization procedure on this subformula amounts to distributing $\neg$ over $\vee$, so as to obtain $\nu X.[\neg x \wedge \neg\sim X]$, which is in normal form: indeed, the latter is a substitution instance of $\nu X.[y' \wedge \neg\sim X]$ which is a $(y', \varnothing)$-$\mathrm{IF}^{\Box}_{\delta'}$ with $\delta' = (1)$; moreover, y' has been substituted for the $\mathrm{IF}^{\Box}$ sentence $\neg x$.

5.4.11. Example. The formula $\nu X.\Box(X \wedge \neg\mu Y.\Diamond(\sim X \vee (Y \vee x)))$ is an $(x, \varnothing)$-$\mathrm{IF}^{\Box}_{\delta}$ formula for $\delta = (\partial)$, and it is not in normal form, the subformula $\neg\mu Y.\Diamond(\sim X \vee (Y \vee x))$ being its only defect. The normalization procedure on this subformula involves surfacing the innermost $\vee$ node, by applying associativity of $\vee$ and distributivity of $\Diamond$ over $\vee$, so as to obtain $\neg\mu Y.\Diamond Y \vee (\Diamond\sim X \vee \Diamond x)$, to which Lemma 5.2.3.1 applies with $f(Y) := \Diamond Y$, $g_1(X) := \Diamond\sim X$ and $g_2(x) := \Diamond x$, yielding

$$\neg\mu Y.[\Diamond Y \vee (\Diamond\sim X \vee \Diamond x)] = \neg\mu Y.[\Diamond Y \vee \Diamond\sim X] \wedge \neg\mu Y.[\Diamond Y \vee \Diamond x].$$

Hence the original formula can be equivalently rewritten as

$$\nu X.\Box(X \wedge (\neg\mu Y.[\Diamond Y \vee \Diamond\sim X] \wedge \neg\mu Y.[\Diamond Y \vee \Diamond x])),$$

which is in normal form: indeed, it is a substitution instance of the formula $\nu X.\Box(X \wedge (\neg\mu Y.[\Diamond Y \vee \Diamond\sim X] \wedge y'))$ which is a $(y', \varnothing)$-$\mathrm{IF}^{\Box}_{\delta'}$ with $\delta' = (1)$; moreover, y' has been substituted for the $\mathrm{IF}^{\Box}$-sentence $\neg\mu Y.[\Diamond Y \vee \Diamond x]$.

5.4.4 Computing the adjoints of normal inner formulas

By Lemma 5.4.3 we know that the term functions associated with $\mathrm{IF}^{\Box}$- and $\mathrm{IF}^{\Diamond}$-formulas are completely meet- and join-preserving, respectively. In the setting of perfect bi-Heyting algebras this implies that they have left- and right-adjoints, respectively. In the present section we are going to show that these adjoints can be represented, componentwise, as term functions of $\mathrm{IF}^{\blacklozenge}$- and $\mathrm{IF}^{\blacksquare}$-formulas. In fact, in the following lemma we will effectively construct these term functions. To this end, we need to introduce the following notation:

5.4.12. Definition. For any formula $\varphi(\overline{x}, Y, \overline{z})$ we define $\varphi^{\kappa\upharpoonright}(\overline{x}, y, \overline{z})$ and $\varphi^{\kappa\downharpoonright}(\overline{x}, y, \overline{z})$ for every ordinal κ, as follows: $\varphi^{0\upharpoonright}(\overline{x}, y, \overline{z}) = y = \varphi^{0\downharpoonright}(\overline{x}, \bot, \overline{z})$. Assuming that $\varphi^{\kappa\upharpoonright}(\overline{x}, y, \overline{z})$ and $\varphi^{\kappa\downharpoonright}(\overline{x}, y, \overline{z})$ have been defined, we let $\varphi^{(\kappa+1)\upharpoonright}(\overline{x}, y, \overline{z}) = \varphi(\overline{x}, \varphi^{\kappa\upharpoonright}(\overline{x}, y, \overline{z}), \overline{z})$ and $\varphi^{(\kappa+1)\downharpoonright}(\overline{x}, y, \overline{z}) = \varphi(\overline{x}, \varphi^{\kappa\downharpoonright}(\overline{x}, y, \overline{z}), \overline{z})$. Assuming that $\varphi^{\kappa\upharpoonright}(\overline{x}, y, \overline{z})$ and $\varphi^{\kappa\downharpoonright}(\overline{x}, y, \overline{z})$ have been defined for every $\kappa < \lambda$, where λ is a limit ordinal, we let $\varphi^{\lambda\upharpoonright}(\overline{x}, y, \overline{z}) = \bigvee_{\kappa<\lambda} \varphi^{\kappa\upharpoonright}(\overline{x}, y, \overline{z})$ and $\varphi^{\lambda\downharpoonright}(\overline{x}, y, \overline{z}) = \bigwedge_{\kappa<\lambda} \varphi^{\kappa\downharpoonright}(\overline{x}, y, \overline{z})$.

5.4.13. LEMMA. *Let $\varphi(\overline{x},\overline{z})$ and $\psi(\overline{x},\overline{z})$, respectively, be an $(\overline{x},\overline{z})$-$IF_\delta^{\Box}$ and an $(\overline{x},\overline{z})$-$IF_\delta^{\Diamond}$-formula in normal form, where the arity of $\overline{x}$ is n. Then:*

1. *there exists an n-array $\overline{\psi}(u,\overline{z})$, where u is a fresh variable, given componentwise by $\mathcal{L}^+$-formulas $\psi_i(u,\overline{z})$ for any $1\leq i\leq n$, such that*

   (a) *for every $1\leq i\leq n$, the formula $\psi_i(u,\overline{z})$ is an IF_η-formula with η the order-type over 1 with $\eta_1=\delta_i$, and moreover $\psi_i(u,\overline{z})$ is an $IF_\eta^{\blacklozenge}$-formula if $\delta_i=1$ and an $IF_\eta^{\blacksquare}$-formula if $\delta_i=\partial$;*

   (b) *in any perfect modal bi-Heyting algebra $\mathbb{C}$ and for all $\overline{a},b,\overline{c}\in\mathbb{C}$*

$$b\leq\varphi(\overline{a},\overline{c})\quad\textit{iff}\quad\overline{\psi}(b,\overline{c})\leq^{\delta}\overline{a}\quad\textit{iff}\quad\mathop{\&}_{i=1}^{n}\psi_i(b,\overline{c})\leq^{\delta_i}a_i.$$

2. *there exists an n-array $\overline{\varphi}(u,\overline{z})$, where u is a fresh variable, given componentwise by $\mathcal{L}^+$-formulas $\varphi_i(u,\overline{z})$ fo any $1\leq i\leq n$, such that*

   (a) *for every $1\leq i\leq n$, the formula $\varphi_i(u,\overline{z})$ is an IF_η-formula with η the order-type over 1 with $\eta_1=\delta_i$, and moreover $\varphi_i(u,\overline{z})$ is an $IF_\eta^{\blacksquare}$-formula if $\delta_i=1$ and an $IF_\eta^{\blacklozenge}$-formula if $\delta_i=\partial$;*

   (b) *in any perfect modal bi-Heyting algebra $\mathbb{C}$ and for all $\overline{a},b,\overline{c}\in\mathbb{C}$*

$$\psi(\overline{a},\overline{c})\leq b\quad\textit{iff}\quad\overline{a}\leq^{\delta}\overline{\varphi}(b,\overline{c})\quad\textit{iff}\quad\mathop{\&}_{i=1}^{n}a_i\leq^{\delta_i}\varphi_i(b,\overline{c}).$$

Proof. Fix $\overline{a},b,\overline{c}\in\mathbb{C}$. The proof proceeds by simultaneous induction on φ and ψ. As to the base cases: if φ is $\top$, the the claim holds if we let $\psi_j=\bot$ for every $1\leq j\leq n$. Dually, if ψ is $\bot$, then the claim holds if we let $\varphi_j=\top$ for every $1\leq j\leq n$. If φ is x_j for some $1\leq j\leq n$ such that $\delta_j=1$, then the claim holds if we let ψ_j be equal to the variable u, and ψ_i be the constant $\bot^{\delta_i}$ for $i\neq j$. Similarly, if ψ is x_j for some $1\leq j\leq n$ for which $\delta_j=1$, then the claim holds if we let φ_j be equal to the variable u, and φ_i be the constant $\top^{\delta_i}$ for $i\neq j$.

If φ is of the form $\varphi^{(1)}(\overline{x},\overline{z})\wedge\varphi^{(2)}(\overline{x},\overline{z})$ we let $\psi_i=\psi_i^{(1)}(u,\overline{z})\vee^{\delta_i}\psi_i^{(2)}(u,\overline{z})$ for $1\leq i\leq n$. Indeed, we have

$$\begin{array}{lll}
b\leq\varphi^{(1)}(\overline{a},\overline{c})\wedge\varphi^{(2)}(\overline{a},\overline{c}) & \text{iff} & b\leq\varphi^{(j)}(\overline{a},\overline{c}),\ j=1,2\\
& \text{iff} & \psi_i^{(j)}(b,\overline{c})\leq^{\delta_i}a_i,\ j=1,2,\ 1\leq i\leq n\\
& \text{iff} & \psi_i^{(1)}(b,\overline{c})\vee^{\delta_i}\psi_i^{(2)}(b,\overline{c})\leq^{\delta_i}a_i,\ 1\leq i\leq n\\
& \text{iff} & \overline{\psi}(b,\overline{c})\leq^{\delta}\overline{a}.
\end{array}$$

Moreover, if $\delta_i=1$, then by the inductive hypothesis $\psi_i^{(1)}$ and $\psi_i^{(2)}$ are $\mathrm{IF}_\eta^{\blacklozenge}$-formulas with $\eta=(1)$, and hence $\psi_i^{(1)}(u,\overline{z})\vee^{\delta_i}\psi_i^{(2)}(u,\overline{z})$ is $\psi_i^{(1)}(u,\overline{z})\vee\psi_i^{(2)}(u,\overline{z})$ which is an $\mathrm{IF}_\eta^{\blacklozenge}$-formula. If $\delta_i=\partial$, then by the inductive hypothesis $\psi_i^{(1)}$ and $\psi_i^{(2)}$

are $\mathrm{IF}^{\squareulblack}_{\eta}$-formulas with $\eta = (\partial)$, and hence $\psi_i^{(1)}(u,\overline{z}) \vee^{\delta_i} \psi_i^{(2)}(u,\overline{z})$ is $\psi_i^{(1)}(u,\overline{z}) \wedge \psi_i^{(2)}(u,\overline{z})$ which is an $\mathrm{IF}^{\squareulblack}_{\eta}$-formula.

If φ is of the form $\Box\varphi'(\overline{x},\overline{z})$ we let $\psi_i = \psi'_i(\blacklozenge u/u,\overline{z})$ for $1 \leq i \leq n$. Indeed, we have

$$\begin{array}{lll} b \leq \Box\varphi'(\overline{a},\overline{c}) & \text{iff} & \blacklozenge b \leq \varphi'(\overline{a},\overline{c}) \\ & \text{iff} & \psi'_i(\blacklozenge b,\overline{c}) \leq^{\delta_i} a_i,\ 1 \leq i \leq n. \end{array}$$

Moreover, if $\delta_i = 1$, then by the inductive hypothesis $\psi'_i(u,\overline{z})$ is an $\mathrm{IF}^{\diamondtopblack}_{\eta}$-formula with $\eta = (1)$, and then, using Definition 5.4.1, it is not difficult to show that $\psi'_i(\blacklozenge u/u,\overline{z})$ is an $\mathrm{IF}^{\diamondtopblack}_{\eta}$-formula. If $\delta_i = \partial$, then by the inductive hypothesis $\psi'_i(u,\overline{z})$ is an $\mathrm{IF}^{\squareulblack}_{\eta}$-formula with $\eta = (\partial)$, and then, using Definition 5.4.1, it is not difficult to show that $\psi'_i(\blacklozenge u/u,\overline{z})$ is an $\mathrm{IF}^{\squareulblack}_{\eta}$-formula.

Let φ be of the form $\nu Y.\varphi''(\overline{x}',\overline{z})$ with $\overline{x}' = \overline{y} \oplus \overline{X} \oplus Y$, where m and k are the lengths of $\overline{y}$ and $\overline{X}$, respectively. Let $\delta(1)$ and $\delta(2)$ be the restrictions of δ to $\overline{y}$ and $\overline{X}$, respectively. By normality we can assume that $\varphi'' = \varphi'(\overline{\varphi}/\overline{y}',\overline{X},Y,\overline{z})$, where $\overline{y}'$ is an ℓ-tuple of variables, $\varphi'(\overline{y}' \oplus \overline{X} \oplus Y,\overline{z})$ is an $\mathrm{IF}^{\Box}_{\delta'}$-formula with δ' constantly 1 on $\overline{y}'$ and Y, and restricting to δ on $\overline{X}$, and with $\overline{\varphi} = (\varphi_1,\ldots,\varphi_\ell)$ and φ_j a $(\overline{y},\overline{z})$-$\mathrm{IF}^{\Box}_{\delta}$ sentence for every $1 \leq j \leq \ell$. Let $\overline{a}, b, \overline{c} \in \mathbb{C}$ be fixed as above. Let $\overline{a}' = (\varphi_1(\overline{a},\overline{c}),\ldots,\varphi_\ell(\overline{a},\overline{c})) \oplus \top^{\delta(2)}$. Then by induction hypothesis on $\varphi'(\overline{y}' \oplus \overline{X} \oplus Y,\overline{z})$, we have formulas $\psi'_1(y,\overline{z}),\ldots,\psi'_\ell(y,\overline{z})$ and $\psi''_1(y,\overline{z}),\ldots,\psi''_{k+1}(y,\overline{z})$ such that

$$b \leq \varphi'(\overline{a}',\top,\overline{c}) \quad \text{iff} \quad \begin{cases} \psi'_j(b,\overline{c}) \leq \varphi_j(\overline{a},\overline{c}) & \text{for } 1 \leq j \leq \ell, \quad \text{and} \\ \psi''_h(b,\overline{c}) \leq^{\delta(2)_h} \top^{\delta(2)_h} & \text{for } 1 \leq h \leq k, \quad \text{and} \\ \psi''_{k+1}(b,\overline{c}) \leq \top. \end{cases}$$

Moreover, $\psi''_{k+1}(y,\overline{z})$ and $\psi'_j(y,\overline{z})$ for $1 \leq j \leq \ell$ are $\mathrm{IF}^{\diamondtopblack}_{\eta}$-formulas with $\eta = (1)$.

In the following calculation we will abuse notation and write $\varphi^{\kappa}(\overline{a}',\top,\overline{c})$ for $(\varphi(\overline{a}',w,\overline{c}))^{\kappa}[\top/w]$, and $\psi'^{\kappa}(b,\overline{c})$ for $(\psi'(u,\overline{c}))^{\kappa}[b/u]$.

$$\begin{array}{lll}
 & & b \leq \nu Y.\varphi''(\overline{a},Y,\overline{c}) \\
\text{iff} & & b \leq \nu Y.\varphi'(\overline{a}',Y,\overline{c}) \\
\text{iff} & & b \leq \bigwedge_{\kappa\geq 0} \varphi'^{\kappa\downarrow}(\overline{a}',\top,\overline{c}) \\
\text{iff} & & b \leq \varphi'^{\kappa\downarrow}(\overline{a}',\top,\overline{c}) \text{ for all } \kappa \geq 0 \\
\text{iff} & & \psi'_j(b \vee \bigvee_{\kappa'\leq\kappa} \psi''^{\kappa'\uparrow}_{k+1}(b,\overline{c}),\overline{c}) \leq^1 \varphi_j(\overline{a},\overline{c}) \text{ for all } 1 \leq j \leq \ell \text{ and all } \kappa \geq 0 \\
 & \text{and} & \psi'_h(b \vee \bigvee_{\kappa'\leq\kappa} \psi''^{\kappa'\uparrow}_{k+1}(b,\overline{c}),\overline{c}) \leq^{\delta(2)_h} \top^{\delta(2)_h} \text{ for all } 1 \leq h \leq k \text{ and all } \kappa \geq 0 \\
 & \text{and} & \psi'_{k+1}(b \vee \bigvee_{\kappa'\leq\kappa} \psi''^{\kappa'\uparrow}_{k+1}(b,\overline{c}),\overline{c}) \leq^1 \top, \text{ for all } \kappa \geq 0 \\
\text{iff} & & \psi'_j(b \vee \bigvee_{\kappa'\leq\kappa} \psi''^{\kappa'\uparrow}_{k+1}(b,\overline{c}),\overline{c}) \leq^1 \varphi_j(\overline{a},\overline{c}) \text{ for all } 1 \leq j \leq \ell \text{ and all } \kappa \geq 0 \\
\text{iff} & & \bigvee_{\kappa\geq 0} \psi'_j(b \vee \bigvee_{\kappa'\leq\kappa} \psi''^{\kappa'\uparrow}_{k+1}(b,\overline{c}),\overline{c}) \leq^1 \varphi_j(\overline{a},\overline{c}) \text{ for all } 1 \leq j \leq \ell \\
\text{iff} & & \psi'_j(b \vee \bigvee_{\kappa\geq 0}\bigvee_{\kappa'\leq\kappa} \psi''^{\kappa'\uparrow}_{k+1}(b,\overline{c}),\overline{c}) \leq^1 \varphi_j(\overline{a},\overline{c}) \text{ for all } 1 \leq j \leq \ell \qquad (*) \\
\text{iff} & & \psi'_j(b \vee \bigvee_{\kappa\geq 0} \psi''^{\kappa\uparrow}_{k+1}(b,\overline{c}),\overline{c}) \leq^1 \varphi_j(\overline{a},\overline{c}) \text{ for all } 1 \leq j \leq \ell \\
\text{iff} & & \psi'_j(b \vee \mu Y.\psi''_{k+1}(b \vee Y,\overline{c}),\overline{c}) \leq^1 \varphi_j(\overline{a},\overline{c}) \text{ for all } 1 \leq j \leq \ell
\end{array}$$

The second and fourth equivalence hold because of Definition 5.4.12 and Lemma 5.2.1.3, respectively. To see that the starred equivalence holds, recall that $\psi'_j(u,\overline{z})$ is an $\mathrm{IF}^{\blacklozenge}_{\eta}$-formula with $\eta=(1)$, hence by Lemma 5.4.3 its associated term function is completely join-preserving in $\mathbb{C}$. Applying the induction hypothesis to φ_j, we obtain formulas $\psi_{(j,i)}(u,\overline{z})$, for $1\leq i\leq n$, such that

$$\begin{array}{ll} & \psi'_j(b\vee \mu Y.\psi''_{k+1}(b\vee Y,\overline{c}),\overline{c})\leq^1 \varphi_j(\overline{a},\overline{c}) \\ \text{iff} & \psi_{(j,i)}(\psi'_j(b\vee \mu Y.\psi''_{k+1}(b\vee Y,\overline{c}),\overline{c}),\overline{c})\leq^{\delta_i} a_i, \text{ for all } 1\leq i\leq n. \end{array}$$

This shows that, for every $1\leq i\leq n$, we can take $\psi_i(u,\overline{z})$ to be

$$\bigvee\nolimits^{\delta_i}\left\{\psi_{(j,i)}(\psi'_j(u\vee \mu Y.\psi''_{k+1}(u\vee Y,\overline{z}),\overline{z}),\overline{z})\mid 1\leq j\leq \ell\right\}, \tag{5.1}$$

which proves part (b) of the claim. As to part (a), we begin by recalling that $\psi''_{k+1}(y,\overline{z})$ and $\psi'_j(y,\overline{z})$ for $1\leq j\leq \ell$ are $\mathrm{IF}^{\blacklozenge}_{\eta}$-formulas with $\eta=(1)$. Hence $\psi'_j(u\vee \mu Y.\psi''_{k+1}(u\vee Y,\overline{z}),\overline{z})$ is also an $\mathrm{IF}^{\blacklozenge}_{\eta}$-formula. By the induction hypothesis applied to φ_j, each formula $\psi_{(j,i)}(u,\overline{z})$ is an $\mathrm{IF}^{\blacklozenge}_{\eta'}$-formula with $\eta'=(1)$ if $\delta_i=1$, or an $\mathrm{IF}^{\blacksquare}_{\eta'}$-formula with $\eta'=(\partial)$ if $\delta_i=\partial$. Therefore, reasoning about $\bigvee^{\delta_i}$ in a way analogous to the inductive step for $\wedge$ above, we see that (5.1) is an $\mathrm{IF}^{\blacklozenge}_{\eta'}$-formula with $\eta'=(1)$ if $\delta_i=1$, or an $\mathrm{IF}^{\blacksquare}_{\eta'}$-formula with $\eta'=(\partial)$ if $\delta_i=\partial$.
If φ is of the form $\pi(\overline{z})\to\varphi'(\overline{x},\overline{z})$, we let $\psi_i=\psi'_i((u\wedge\pi(\overline{z}))/u,\overline{z})$. Indeed, we have

$$\begin{array}{lll} b\leq \pi(\overline{c})\to\varphi'(\overline{a},\overline{c}) & \text{iff} & b\wedge\pi(\overline{c})\leq\varphi'(\overline{a},\overline{c}) \\ & \text{iff} & \psi'_i(b\wedge\pi(\overline{c}),\overline{c})\leq^{\delta_i} a_i,\ 1\leq i\leq n. \end{array}$$

Moreover, if $\delta_i=1$, then by the inductive hypothesis $\psi'_i(u,\overline{z})$ is an $\mathrm{IF}^{\blacklozenge}_{\eta}$-formula with $\eta=(1)$, and then, using Definition 5.4.1, it is not difficult to show that $\psi'_i((u\wedge\pi(\overline{z}))/u,\overline{z})$ is an $\mathrm{IF}^{\blacklozenge}_{\eta}$-formula. If $\delta_i=\partial$, then by the inductive hypothesis $\psi'_i(u,\overline{z})$ is an $\mathrm{IF}^{\blacksquare}_{\eta}$-formula with $\eta=(\partial)$, and then, using Definition 5.4.1, it is not difficult to show that $\psi'_i((u\wedge\pi(\overline{z}))/u,\overline{z})$ is an $\mathrm{IF}^{\blacksquare}_{\eta}$-formula.
If φ is of the form $\pi(\overline{z})\vee\varphi'(\overline{x},\overline{z})$, we let $\psi_i=\psi'_i((u-\pi(\overline{z}))/u,\overline{z})$. Indeed, we have

$$\begin{array}{lll} b\leq \pi(\overline{c})\vee\varphi'(\overline{a},\overline{c}) & \text{iff} & b-\pi(\overline{c})\leq\varphi'(\overline{a},\overline{c}) \\ & \text{iff} & \psi'_i(b-\pi(\overline{c}),\overline{c})\leq^{\delta_i} a_i,\ 1\leq i\leq n. \end{array}$$

Moreover, if $\delta_i=1$, then by the inductive hypothesis $\psi'_i(u,\overline{z})$ is an $\mathrm{IF}^{\blacklozenge}_{\eta}$-formula with $\eta=(1)$, and then, using Definition 5.4.1, it is not difficult to show that $\psi'_i((u-\pi(\overline{z}))/u,\overline{z})$ is an $\mathrm{IF}^{\blacklozenge}_{\eta}$-formula. If $\delta_i=\partial$, then by the inductive hypothesis $\psi'_i(u,\overline{z})$ is an $\mathrm{IF}^{\blacksquare}_{\eta}$-formula with $\eta=(\partial)$, and then, using Definition 5.4.1, it is not difficult to show that $\psi'_i((u-\pi(\overline{z}))/u,\overline{z})$ is an $\mathrm{IF}^{\blacksquare}_{\eta}$-formula.
If φ is of the form $\psi^c(\overline{x},\overline{z})\to\pi(\overline{z})$, then by clause (4) of Definition 5.4.1 $\psi^c(\overline{x},\overline{z})$ is an $\mathrm{IF}^{\Diamond}_{\delta^\partial}$-formula, and hence by the inductive hypothesis there are formulas $\varphi^c_i(u,\overline{z})$, $1\leq i\leq n$ such that for every $\overline{a}$, $\overline{c}$ and b', we have $\psi^c(\overline{a},\overline{c})\leq b'$ iff $a_i\leq^{\delta_i^\partial}\varphi^c_i(b',\overline{c})$ for $1\leq i\leq n$. We let $\psi_i=\varphi^c_i((u\to\pi(\overline{z}))/u,\overline{z})$. Indeed, we have

$$\begin{array}{rll} b \leq \psi^c(\overline{a}, \overline{c}) \rightarrow \pi(\overline{c}) & \text{iff} & \psi^c(\overline{a}, \overline{c}) \leq b \rightarrow \pi(\overline{c}) \\ & \text{iff} & a_i \leq^{\delta_i{}^\partial} \varphi_i^c(b \rightarrow \pi(\overline{c}), \overline{c}),\ 1 \leq i \leq n \\ & \text{iff} & \varphi_i^c(b \rightarrow \pi(\overline{c}), \overline{c}) \leq^{\delta_i} a_i,\ 1 \leq i \leq n. \end{array}$$

Moreover, if $\delta_i = 1$ (hence $\delta_i^\partial = \partial$), then by the inductive hypothesis applied to $\psi^c(\overline{x}, \overline{z})$, which we recall is an $\mathrm{IF}^{\Diamond}_{\delta^\partial}$-formula, $\varphi_i^c(u, \overline{z})$ is an $\mathrm{IF}^{\Diamondblack}_{\eta^\partial}$-formula with $\eta^\partial = (\partial)$, and then, using Definition 5.4.1, it is not difficult to show that $\varphi_i^c((u \rightarrow \pi(\overline{z}))/u, \overline{z})$ is an $\mathrm{IF}^{\Diamondblack}_{\eta}$-formula with $\eta = (1)$. If $\delta_i = \partial$ (hence $\delta_i^\partial = 1$), then by the inductive hypothesis $\varphi_i^c(u, \overline{z})$ is an $\mathrm{IF}^{\squareulblack}_{\eta^\partial}$-formula with $\eta^\partial = (1)$, and then, using Definition 5.4.1, it is not difficult to show that $\psi_i'((u \rightarrow \pi(\overline{z}))/u, \overline{z})$ is an $\mathrm{IF}^{\squareulblack}_{\eta}$-formula with $\eta = (\partial)$.

Similar proofs can be given in the remaining cases for ψ. $\square$

5.5 Adjunction rules for normal inner formulas

The following definition is extracted from the proof of Lemma 5.4.13.

5.5.1. Definition. For $\overline{x} = \overline{y} \oplus \overline{X}$ of arity n, for each order-type δ over $\overline{x}$, and each $1 \leq i \leq n$, we define maps LA_i^δ and RA_i^δ, sending normal $(\overline{x}, \overline{z})$-$\mathrm{IF}^{\Box}_{\delta}$- and $(\overline{x}, \overline{z})$-$\mathrm{IF}^{\Diamond}_{\delta}$-formulas into $(u, \overline{z})$-$\mathrm{IF}^{\Diamondblack}_{(\delta_i)}$- and $(u, \overline{z})$-$\mathrm{IF}^{\squareulblack}_{(\delta_i)}$-formulas respectively, by the following simultaneous recursion:

$$
\begin{array}{rcl}
\mathsf{LA}_i^\delta(\top) & = & \bot \\
\mathsf{LA}_i^\delta(x_i) & = & u \text{ for } u \in Var - (\overline{x} \cup \overline{z}); \\
\mathsf{LA}_i^\delta(x_j) & = & \bot^{\delta_j} \text{ when } i \neq j; \\
\mathsf{LA}_i^\delta(\Box\varphi(\overline{x},\overline{z})) & = & \mathsf{LA}_i^\delta(\varphi)(\blacklozenge u, \overline{z}); \\
\mathsf{LA}_i^\delta(\varphi_1(\overline{x},\overline{z}) \wedge \varphi_2(\overline{x},\overline{z})) & = & \mathsf{LA}_i^\delta(\varphi_1)(u,\overline{z}) \vee^{\delta_i} \mathsf{LA}_i^\delta(\varphi_2)(u,\overline{z}) \ ; \\
\mathsf{LA}_i^\delta(\nu Y.\varphi(\overline{\varphi}(\overline{x},\overline{z})/\overline{y}', Y, \overline{z})) & = & \bigvee^{\delta_i}\{\mathsf{LA}_i^\delta(\varphi_j)(\mathsf{LA}_j^{\delta'}(\varphi)(u\vee \\
& & \mu Y.\mathsf{LA}_{k+1}^{\delta'}(\varphi)(u \vee Y, \overline{z}), \overline{z}), \overline{z}) \mid 1 \leq j \leq \ell\} \ ; \\
\mathsf{LA}_i^\delta(\pi(\overline{z}) \to \varphi(\overline{x},\overline{z})) & = & \mathsf{LA}_i^\delta(\varphi)(u \wedge \pi(\overline{z}), \overline{z}); \\
\mathsf{LA}_i^\delta(\pi(\overline{z}) \vee \varphi(\overline{x},\overline{z})) & = & \mathsf{LA}_i^\delta(\varphi)(u - \pi(\overline{z}), \overline{z}); \\
\mathsf{LA}_i^\delta(\psi^c(\overline{x},\overline{z}) \to \pi(\overline{z})) & = & \mathsf{RA}_i^{\delta^\partial}(\psi^c)(u \to \pi(\overline{z}), \overline{z}); \\
\\
\mathsf{RA}_i^\delta(\bot) & = & \top \\
\mathsf{RA}_i^\delta(x_i) & = & u \text{ for } u \in Var - (\overline{x} \cup \overline{z}); \\
\mathsf{RA}_i^\delta(x_j) & = & \top^{\delta_j} \text{ when } i \neq j; \\
\mathsf{RA}_i^\delta(\Diamond\psi(\overline{x},\overline{z})) & = & \mathsf{RA}_i^\delta(\psi)(\blacksquare u, \overline{z}); \\
\mathsf{RA}_i^\delta(\psi_1(\overline{x},\overline{z}) \vee \psi_2(\overline{x},\overline{z})) & = & \mathsf{RA}_i^\delta(\psi_1)(u,\overline{z}) \wedge^{\delta_i} \mathsf{RA}_i^\delta(\psi_2)(u,\overline{z}); \\
\mathsf{RA}_i^\delta(\mu Y.\psi(\overline{\psi}(\overline{x},\overline{z})/\overline{y}', Y, \overline{z})) & = & \bigwedge^{\delta_i}\{\mathsf{RA}_i^\delta(\psi_j)(\mathsf{RA}_j^{\delta'}(\psi)(u\wedge \\
& & \nu Y.\mathsf{RA}_{k+1}^{\delta'}(\psi)(u \wedge Y, \overline{z}), \overline{z}), \overline{z}) \mid 1 \leq j \leq \ell\} \ ; \\
\mathsf{RA}_i^\delta(\psi(\overline{x},\overline{z}) - \pi(\overline{z})) & = & \mathsf{RA}_i^\delta(\psi)(\pi(\overline{z}) \vee u, \overline{z}); \\
\mathsf{RA}_i^\delta(\pi(\overline{z}) \wedge \psi(\overline{x},\overline{z})) & = & \mathsf{RA}_i^\delta(\psi)(\pi(\overline{z}) \to u, \overline{z}); \\
\mathsf{RA}_i^\delta(\pi(\overline{z}) - \varphi^c(\overline{x},\overline{z})) & = & \mathsf{LA}_i^{\delta^\partial}(\varphi^c)(\pi(\overline{z}) - u, \overline{z}).
\end{array}
$$

By normality, formulas with νY as main connective are of the form $\nu Y.\varphi(\overline{\varphi}(\overline{y},\overline{z})/\overline{y}', \overline{X}, Y, \overline{z})$ where $\varphi(\overline{y}', \overline{X}, Y, \overline{z})$ is an $\mathrm{IF}_{\delta'}^{\Box}$-formula, the length of $\overline{y}'$ is ℓ, the length of $\overline{y}' \oplus \overline{X} \oplus Y$ is $k+1$, δ' constantly 1 on $\overline{y}'$ and Y and restricting to δ on $\overline{X}$, and $\overline{\varphi}(\overline{y},\overline{z}) = (\varphi_1(\overline{y},\overline{z}), \ldots, \varphi_\ell(\overline{y},\overline{z}))$ is such that $\varphi_j(\overline{y},\overline{z})$ is a $(\overline{y},\overline{z})$-$\mathrm{IF}_{\delta}^{\Box}$-sentence for every $1 \leq j \leq \ell$. Likewise, formulas with μY as main connective are of the form $\mu Y.\psi(\overline{\psi}(\overline{y},\overline{z})/\overline{y}', \overline{X}, Y, \overline{z})$ where $\psi(\overline{y}', \overline{X}, Y, \overline{z})$ is an $\mathrm{IF}_{\delta'}^{\Diamond}$-formula, the length of $\overline{y}'$ is ℓ, the length of $\overline{y}' \oplus \overline{X} \oplus Y$ is $k+1$, δ' constantly 1 on $\overline{y}'$ and Y and restricting to δ on $\overline{X}$, and $\overline{\psi}(\overline{y},\overline{z}) = (\psi_1(\overline{y},\overline{z}), \ldots, \psi_\ell(\overline{y},\overline{z}))$ is such that ψ_j is a $(\overline{y},\overline{z})$-$\mathrm{IF}_{\delta}^{\Diamond}$-sentence for every $1 \leq j \leq \ell$.

The following lemma is just a direct consequence of the definition, but is very useful in simplifying computations.

5.5.2. Lemma. *1. Let φ be a $(\overline{x},\overline{z})$-$IF_{\delta}^{\Box}$-formula for $\overline{x} = \overline{y} \oplus \overline{X}$ of arity n. If x_i does not occur in φ for some i, then $\mathsf{LA}_i^\delta(\varphi) = \bot^{\delta_i}$.*

2. Let ψ be a $(\overline{x},\overline{z})$-$IF_{\delta}^{\Diamond}$-formula for $\overline{x} = \overline{y} \oplus \overline{X}$ of arity n. If x_i does not occur in ψ for some i, then $\mathsf{RA}_i^\delta(\psi) = \top^{\delta_i}$.

Proof. By simultaneous induction on φ and ψ. The base cases hold by definition. If φ is of the form $\nu Y.\varphi(\overline{\varphi}(\overline{x},\overline{z})/\overline{y}', Y, \overline{z})$, then by definition

$$\mathsf{LA}_i^\delta(\varphi) = \bigvee^{\delta_i}\left\{\mathsf{LA}_i^\delta(\varphi_j)(\mathsf{LA}_j^{\delta'}(\varphi)(u \vee \mu Y.\mathsf{LA}_{k+1}^{\delta'}(\varphi)(u \vee Y, \overline{z}), \overline{z}), \overline{z}) \mid 1 \leq j \leq \ell\right\}.$$

Since x_i does not occur in any formula in $\overline{\varphi}$, by induction hypothesis $\mathsf{LA}_i^\delta(\varphi_j) = \bot^{\delta_i}$ for every $1 \leq j \leq \ell$. Hence $\mathsf{LA}_i^\delta(\varphi) = \bigvee^{\delta_i} \bot^{\delta_i} = \bot^{\delta_i}$. The remaining cases are left to the reader.

□

We are now in a position to give versions of the adjunction rules tailored to normal $(\overline{x}, \overline{z})$-IF$_\delta^\Box$- and $(\overline{x}, \overline{z})$-IF$_\delta^\Diamond$-formulas, for which the adjoints are expressible as $\mathcal{L}^+$-term functions:

$$\frac{\eta \leq \varphi(\overline{x}, \overline{z})}{\&_{i=1}^n \mathsf{LA}_i^\delta(\varphi)[\eta/u] \leq^{\delta_i} x_i} \ (\mathrm{IF}_R)$$

where $\varphi \in \mathcal{L}$, $\eta \in \mathcal{L}^+$, the arrays $\overline{x}$ and $\overline{z}$ are disjoint, the arity of $\overline{x}$ is n, and $\varphi \in (\overline{x}, \overline{z})$-IF$_\delta^\Box$.

$$\frac{\psi(\overline{x}, \overline{z}) \leq \eta}{\&_{i=1}^n x_i \leq^{\delta_i} \mathsf{RA}_i^\delta(\psi)[\eta/u]} \ (\mathrm{IF}_L)$$

where $\psi \in \mathcal{L}$, $\eta \in \mathcal{L}^+$, the arrays $\overline{x}$ and $\overline{z}$ are disjoint, the arity of $\overline{x}$ is n, and $\psi \in (\overline{x}, \overline{z})$-IF$_\delta^\Diamond$. The soundness of these rules immediately follows from Lemma 5.4.13.

The rules above are closed under substitution. In particular, the following reformulations are sound for any propositional variables $\overline{p}$ and any sentences $\overline{\gamma}$:

$$\frac{\eta \leq \varphi(\overline{p}/\overline{x}, \overline{\gamma}/\overline{z})}{\&_{i=1}^n \mathsf{LA}_i^\delta(\varphi)[\eta/u, \overline{\gamma}/\overline{z}] \leq^{\delta_i} p_i} \ (\mathrm{IF}_R^\sigma)$$

where $\varphi \in \mathcal{L}$, $\eta \in \mathcal{L}^+$, the arrays $\overline{x}$ and $\overline{z}$ are disjoint, the arity of $\overline{x}$ is n, and φ is a normal $(\overline{x}, \overline{z})$-IF$_\delta^\Box$-sentence, i.e., $\overline{x} = \overline{y}$.

$$\frac{\psi(\overline{p}/\overline{x}, \overline{\gamma}/\overline{z}) \leq \eta}{\&_{i=1}^n p_i \leq^{\delta_i} \mathsf{RA}_i^\delta(\psi)[\eta/u, \overline{\gamma}/\overline{z}]} \ (\mathrm{IF}_L^\sigma)$$

where $\psi \in \mathcal{L}$, $\eta \in \mathcal{L}^+$, the arrays $\overline{x}$ and $\overline{z}$ are disjoint, the arity of $\overline{x}$ is n, and ψ is a normal $(\overline{x}, \overline{z})$-IF$_\delta^\Diamond$-sentence, i.e., $\overline{x} = \overline{y}$.

As discussed earlier, the maps LA_i and RA_i explicitly compute the term functions corresponding to the adjoints of normal $(\overline{x}, \overline{z})$-inner formulas. By construction, this adjunction is parametric in $\overline{z}$. The next lemma states the syntactic version of order-theoretic facts that hold in such situations generally, and which will be useful in the proof that μ-ALBA is successful on all ε-recursive inequalities.

In what follows we will say that is formula φ is δ_i-positive in a variable u if φ is positive in u when $\delta_i = 1$ and negative in u when $\delta_i = \partial$.

5.5.3. LEMMA. *1. Let $\varphi(\overline{x}, \overline{z})$ be a normal IF$^{\Box}_{\delta}$-formula in which each $z \in \overline{z}$ occurs at most once. Then, for each $1 \leq i \leq n$, $\mathsf{LA}^{\delta}_i(\varphi)(u, \overline{z})$ is δ_i-positive in u, and for each $z \in \overline{z}$, the polarity of z in $\mathsf{LA}^{\delta}_i(\varphi)(u, \overline{z})$ is the opposite of (respectively, the same as) its polarity in φ if $\delta_i = 1$ (respectively, if $\delta_i = \partial$).*

2. Let $\psi(\overline{x}, \overline{z})$ be a normal IF$^{\Diamond}_{\delta}$-formula in which each $z \in \overline{z}$ occurs at most once. Then, for each $1 \leq i \leq n$, $\mathsf{RA}^{\delta}_i(\psi)(u, \overline{z})$ is δ_i-positive in u, and for each $z \in \overline{z}$, the polarity of z in $\mathsf{RA}^{\delta}_i(\psi)(u, \overline{z})$ is the opposite of (respectively, the same as) its polarity in ψ if $\delta_i = 1$ (respectively, if $\delta_i = \partial$).

Proof. By simultaneous induction on φ and ψ. The base cases are trivially true. The cases in which the main connective is $\Box$ or $\wedge$ immediately follow from the induction hypothesis. For φ of the form $\nu Y.\varphi'(\overline{\varphi}(\overline{x}, \overline{z}_1)/\overline{y}', Y, \overline{z}_2)$ as in Definition 5.4.5, we have

$$\begin{aligned} & \mathsf{LA}^{\delta}_i(\nu Y.\varphi'(\overline{\varphi}(\overline{x}, \overline{z}_1)/\overline{y}', Y, \overline{z}_2)) \\ = \ & \bigvee\nolimits^{\delta_i} \left\{ \mathsf{LA}^{\delta}_i(\varphi_j)(\mathsf{LA}^{\delta'}_j(\varphi')(u \vee \mu Y.\mathsf{LA}^{\delta'}_{k+1}(\varphi')(u \vee Y, \overline{z}_2), \overline{z}_2), \overline{z}_1) \mid 1 \leq j \leq \ell \right\}, \end{aligned}$$

with δ' constantly 1 on $\overline{y}'$ and Y and restricting to δ on $\overline{X}$, and $\overline{\varphi}(\overline{x}, \overline{z}_1) = (\varphi_1(\overline{x}, \overline{z}_1), \ldots, \varphi_\ell(\overline{x}, \overline{z}_1))$ such that φ_j is a $(\overline{x}, \overline{z})$-IF$^{\Box}_{\delta}$-sentence for every $1 \leq j \leq \ell$. By induction hypothesis, $\mathsf{LA}^{\delta}_i(\varphi_j)(u', \overline{z}_1)$ is δ_i-positive in u', $\mathsf{LA}^{\delta'}_{k+1}(\varphi')(u, \overline{z}_2)$ is δ'_{k+1}-positive (hence positive) in u, and $\mathsf{LA}^{\delta'}_j(\varphi')(u, \overline{z}_2)$ is δ'_j-positive (hence positive) in u for every $1 \leq j \leq \ell$. Hence $\mathsf{LA}^{\delta}_i(\nu Y.\varphi'(\overline{\varphi}(\overline{x}, \overline{z}_1)/\overline{y}', Y, \overline{z}_2))$ is δ_i-positive in u. If $z \in \overline{z}_1$, then z occurs in φ_j for some $1 \leq j \leq \ell$, hence the statement follows by application of the induction hypothesis to φ_j. Let $z \in \overline{z}_2$. Since δ' is constantly 1 on $\overline{y}'$ and Y, by induction hypothesis on φ', it follows that z has the opposite polarity in $\mathsf{LA}^{\delta'}_j(\varphi')(u \vee \mu Y.\mathsf{LA}^{\delta'}_{k+1}(\varphi')(u \vee Y, \overline{z}_2), \overline{z}_2)$ to that which it has in φ'. If $\delta_i = 1$, then $\mathsf{LA}^{\delta}_i(\varphi_j)(u', \overline{z}_1)$ is positive in u', and hence the polarity of z in $\mathsf{LA}^{\delta}_i(\nu Y.\varphi'(\overline{\varphi}(\overline{x}, \overline{z}_1)/\overline{y}', Y, \overline{z}_2))$ is the opposite to that it has in φ'. If $\delta_i = \partial$, then $\mathsf{LA}^{\delta}_i(\varphi_j)(u', \overline{z}_1)$ is negative in u', and hence the polarity of z in $\mathsf{LA}^{\delta}_i(\nu Y.\varphi'(\overline{\varphi}(\overline{x}, \overline{z}_1)/\overline{y}', Y, \overline{z}_2))$ is the same to that it has in φ'.

For φ of the form $\pi(\overline{z}_1) \to \varphi'(\overline{x}, \overline{z}_2)$ we have $\mathsf{LA}^{\delta}_i(\varphi) = \mathsf{LA}^{\delta}_i(\varphi')((u \wedge \pi(\overline{z}_1))/u', \overline{z}_2)$. Then the claims about the polarities of u and $z \in \overline{z}_2$ follows by the inductive hypothesis applied to φ'. If $z \in \overline{z}_1$ then we distinguish two cases: if $\delta_i = 1$ then $\mathsf{LA}^{\delta}_i(\varphi')(u', \overline{z}_2)$ is positive in u' by the induction hypothesis, and since $\pi(\overline{z}_1)$ occurs negatively in φ, the polarity of z in $\mathsf{LA}^{\delta}_i(\varphi')((u \wedge \pi(\overline{z}_1))/u', \overline{z}_2)$ is the opposite of its polarity in φ. If $\delta_i = \partial$ then $\mathsf{LA}^{\delta}_i(\varphi')(u', \overline{z}_2)$ is negative in u' by the induction hypothesis, and since $\pi(\overline{z}_1)$ occurs negatively in φ, the polarity of z in $\mathsf{LA}^{\delta}_i(\varphi')((u \wedge \pi(\overline{z}_1))/u', \overline{z}_2)$ is the same as its polarity in φ.

For φ of the form $\psi^c(\overline{x}, \overline{z}_2) \to \pi(\overline{z}_1)$ we have $\mathsf{LA}^{\delta}_i(\varphi) = \mathsf{RA}^{\delta^{\partial}}_i(\psi^c)((u \to \pi(\overline{z}_1))/u', \overline{z}_2)$. If $\delta_i = 1$, then $\delta^{\partial}_i = \partial$, and by the inductive hypothesis $\mathsf{RA}^{\delta^{\partial}}_i(\psi^c)(u', \overline{z}_2)$ is negative in u', and hence $\mathsf{RA}^{\delta^{\partial}}_i(\psi^c)((u \to \pi(\overline{z}_1))/u', \overline{z}_2)$ is positive in u. If $\delta_i = \partial$, then $\delta^{\partial}_i = 1$, and by the inductive hypothesis $\mathsf{RA}^{\delta^{\partial}}_i(\psi^c)(u', \overline{z}_2)$

is positive in u', and hence $\mathsf{RA}_i^{\delta^\partial}(\psi^c)((u \to \pi(\overline{z}_1))/u', \overline{z}_2)$ is negative in u. If $z \in \overline{z}_2$ and $\delta_i = 1$, then $\delta_i^\partial = \partial$ and hence, by the induction hypothesis, the polarity of z in $\mathsf{RA}_i^{\delta^\partial}(\psi^c)(u', \overline{z}_2)$ is the same as its polarity in ψ^c, and since ψ^c occurs negatively in $\psi^c(\overline{x}, \overline{z}_2) \to \pi(\overline{z}_1)$, we have that the polarity of z in $\mathsf{RA}_i^{\delta^\partial}(\psi^c)(u', \overline{z}_2)$, and hence in $\mathsf{RA}_i^{\delta^\partial}(\psi^c)((u \to \pi(\overline{z}_1))/u', \overline{z}_2)$, is the opposite of its polarity in φ. The case where $z \in \overline{z}_2$ and $\delta_i = \partial$ follows by an order-dual argument. If $z \in \overline{z}_1$ and $\delta_i = 1$ then $\delta_i^\partial = \partial$, and hence by the induction hypothesis, $\mathsf{RA}_i^{\delta^\partial}(\psi^c)(u', \overline{z}_2)$ is negative in u'. So because $\pi(\overline{z}_1)$ occurs positively in φ, it occurs negatively in $\mathsf{RA}_i^{\delta^\partial}(\psi^c)((u \to \pi(\overline{z}_1))/u', \overline{z}_2)$, and hence the polarity of z in $\mathsf{RA}_i^{\delta^\partial}(\psi^c)((u \to \pi(\overline{z}_1))/u', \overline{z}_2)$ is the opposite of its polarity in φ. The case in which $z \in \overline{z}_1$ and $\delta_i = \partial$ follows by an order-dual argument.

The remaining cases are left to the reader. □

5.6 Examples

In the ensuing examples, for the sake of clarity, we will often write $\mathsf{LA}_{x_i}^\delta(\varphi)$ instead of $\mathsf{LA}_i^\delta(\varphi)$ where φ is some $(\overline{x}, \overline{z})$-$\mathrm{IF}_\delta^\Box$-formula. Similarly for $\mathsf{RA}_{x_i}^\delta(\psi)$.

5.6.1. Example. Consider the inequality $\nu X.[\Box(X \wedge \neg\mu Y.[\Diamond(\sim X \vee (Y \vee p))])] \leq \Diamond\Box\neg p$, which is ε-recursive for $\varepsilon_p = \partial$. Its left-hand side has been discussed in Example 5.4.11. After first approximation we have:

$$\forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \nu X.[\Box(X \wedge \neg\mu Y.[\Diamond(\sim X \vee (Y \vee p))])] \ \&\ \Diamond\Box\neg p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \quad (5.2)$$

No approximation rules are applicable, thus we work toward the application of an appropriate adjunction rule to display the p in the first inequality in the antecedent of the quasi-inequality above. As discussed in Example 5.4.11, the left-hand side of this inequality is not in normal form, and its normalization was computed there. We thus apply the adjunction rule (IF_R^σ) to its normalization

$$\varphi = \nu X.[\Box(X \wedge (\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge \neg\mu Y.[\Diamond Y \vee \Diamond p]))].$$

Recall that φ is a substitution instance of the formula $\nu X.\varphi' = \nu X.[\Box(X \wedge (\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge y'))]$, where φ' is a $(y' \oplus X, \varnothing)$-$\mathrm{IF}_\delta^\Box$-formula with $\delta = (1,1)$. Moreover, y' has been substituted for the $(p, \varnothing)$-$\mathrm{IF}_\varepsilon^\Box$-sentence $\neg\psi = \neg\mu Y.[\Diamond Y \vee \Diamond p]$. Thus,

$$\begin{aligned}\mathsf{LA}_p^\varepsilon(\varphi) &= \mathsf{LA}_p^\varepsilon(\neg\psi)[(\mathsf{LA}_{y'}^\delta(\varphi')[(u \vee \mu X.[\mathsf{LA}_X^\delta(\varphi')[(u \vee X)/u']])/u'])/u] \\ &= \mathsf{LA}_p^\varepsilon(\neg\psi)[\mathsf{LA}_{y'}^\delta(\varphi')(u \vee \mu X.\mathsf{LA}_X^\delta(\varphi')(u \vee X))],\end{aligned}$$

where

$$\mathsf{LA}_p^\varepsilon(\neg\psi)(u) = \mathsf{RA}_p^{\varepsilon^\partial}(\psi)(\neg u/u)$$

and $\psi = \mu Y.[\Diamond Y \vee \Diamond p]$ is of the form $\mu Y.\psi'(p/y', Y, \varnothing)$ such that $\psi'(y', Y, \varnothing) = \Diamond Y \vee \Diamond y'$ is an $\mathrm{IF}^{\Diamond}_{\delta'}$-formula with δ' being the order-type constantly 1 on $y' \oplus Y$. Hence ψ is already in normal form. Thus,

$$
\begin{aligned}
\mathsf{RA}^{\varepsilon^\partial}_p(\psi)(u) &= \mathsf{RA}^{\varepsilon^\partial}_p(\mu Y.[\Diamond Y \vee \Diamond p])(u)\\
&= \mathsf{RA}^{\varepsilon^\partial}_p(p)(\mathsf{RA}^{\delta'}_{y'}(\psi')(u \wedge \nu Y.\mathsf{RA}^{\delta'}_Y(\psi')(u \wedge Y)))\\
&= \mathsf{RA}^{\varepsilon^\partial}_p(p)(\mathsf{RA}^{\delta'}_{y'}(\psi')(u \wedge \nu Y.\blacksquare(u \wedge Y)))\\
&= \mathsf{RA}^{\varepsilon^\partial}_p(p)(\blacksquare(u \wedge \nu Y.\blacksquare(u \wedge Y)))\\
&= \blacksquare(u \wedge \nu Y.\blacksquare(u \wedge Y)),
\end{aligned}
$$

and hence,

$$
\mathsf{LA}^{\varepsilon}_p(\neg\psi)(u) \quad = \quad \mathsf{RA}^{\varepsilon^\partial}_p(\psi)(\neg u/u) \quad = \quad \blacksquare(\neg u \wedge \nu Y.\blacksquare(\neg u \wedge Y)).
$$

Next,

$$
\begin{aligned}
& \mathsf{LA}^{\delta}_{y'}(\varphi')(u)\\
= \;& \mathsf{LA}^{\delta}_{y'}(\Box(X \wedge (\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge y')))(u)\\
= \;& \mathsf{LA}^{\delta}_{y'}(X \wedge (\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge y'))(\blacklozenge u/u)\\
= \;& (\mathsf{LA}^{\delta}_{y'}(X) \vee \mathsf{LA}^{\delta}_{y'}(\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge y'))(\blacklozenge u/u)\\
= \;& (\bot \vee (\mathsf{LA}^{\delta}_{y'}(\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X]) \vee \mathsf{LA}^{\delta}_{y'}(y')))(\blacklozenge u/u) && \text{(Lemma 5.5.2)}\\
= \;& (\bot \vee (\bot \vee \mathsf{LA}^{\delta}_{y'}(y')))(\blacklozenge u/u) && \text{(Lemma 5.5.2)}\\
= \;& \mathsf{LA}^{\delta}_{y'}(y')(\blacklozenge u/u)\\
= \;& u(\blacklozenge u/u)\\
= \;& \blacklozenge u.
\end{aligned}
$$

$$
\begin{aligned}
& \mathsf{LA}^{\delta}_{X}(\varphi')(u)\\
= \;& \mathsf{LA}^{\delta}_{X}(\Box(X \wedge (\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge y')))(u)\\
= \;& \mathsf{LA}^{\delta}_{X}(X \wedge (\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge y'))(\blacklozenge u/u)\\
= \;& (\mathsf{LA}^{\delta}_{X}(X) \vee \mathsf{LA}^{\delta}_{X}(\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X] \wedge y'))(\blacklozenge u/u)\\
= \;& (u \vee (\mathsf{LA}^{\delta}_{X}(\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X]) \vee \mathsf{LA}^{\delta}_{X}(y')))(\blacklozenge u/u)\\
= \;& (u \vee (\mathsf{LA}^{\delta}_{X}(\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X]) \vee \bot))(\blacklozenge u/u) && \text{(Lemma 5.5.2)}\\
= \;& (u \vee \mathsf{LA}^{\delta}_{X}(\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X]))(\blacklozenge u/u)\\
= \;& \blacklozenge u \vee \mathsf{LA}^{\delta}_{X}(\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X])(\blacklozenge u/u).
\end{aligned}
$$

$$
\mathsf{LA}^{\delta}_{X}(\neg\psi)(u) \quad = \quad \mathsf{RA}^{\delta^\partial}_{X}(\psi)(\neg u/u)
$$

and $\psi = \mu Y.[\Diamond Y \vee \Diamond{\sim}X]$ is of the form $\mu Y.\psi'({\sim}X/y', Y, \varnothing)$ such that $\psi'(y', Y, \varnothing) = \Diamond Y \vee \Diamond y'$ is an $\mathrm{IF}^{\Diamond}_{\delta'}$-formula with δ' being the order-type constantly 1 on $y' \oplus Y$. Hence ψ is already in normal form. Thus,

$$
\begin{aligned}
\mathsf{RA}^{\delta^\partial}_X(\psi)(u) &= \mathsf{RA}^{\delta^\partial}_X(\mu Y.[\Diamond Y \vee \Diamond{\sim}X])(u)\\
&= \mathsf{RA}^{\delta^\partial}_X({\sim}X)(\mathsf{RA}^{\delta'}_{y'}(\psi')(u \wedge \nu Y.\mathsf{RA}^{\delta'}_Y(\psi')(u \wedge Y)))\\
&= \mathsf{RA}^{\delta^\partial}_X({\sim}X)(\blacksquare(u \wedge \nu Y.\blacksquare(u \wedge Y)))\\
&= ({\sim}u)(\blacksquare(u \wedge \nu Y.\blacksquare(u \wedge Y))/u) && (*)\\
&= {\sim}\blacksquare(u \wedge \nu Y.\blacksquare(u \wedge Y))
\end{aligned}
$$

The starred equality above is justified as follows:

$$\begin{array}{rcl}\mathsf{RA}^{\delta\partial}_X(\sim X)(u) & = & \mathsf{RA}^{\delta\partial}_X(\top - X)(u)\\ & = & \mathsf{LA}^{\delta}_X(X)(\top - u/u)\\ & = & (u)(\top - u/u)\\ & = & \top - u\\ & = & \sim u\end{array}$$

Thus,

$$\begin{array}{rcl}\mathsf{LA}^{\delta}_X(\varphi')(u) & = & \blacklozenge u \vee \mathsf{LA}^{\delta}_X(\neg\mu Y.[\Diamond Y \vee \Diamond{\sim}X])(\blacklozenge u/u)\\ & = & \blacklozenge u \vee (\mathsf{RA}^{\delta\partial}_X(\psi)(\neg u/u))(\blacklozenge u/u)\\ & = & \blacklozenge u \vee ((\sim\blacksquare(u \wedge \nu Y.\blacksquare(u \wedge Y)))(\neg u/u))(\blacklozenge u/u)\\ & = & \blacklozenge u \vee (\sim\blacksquare(\neg u \wedge \nu Y.\blacksquare(\neg u \wedge Y)))(\blacklozenge u/u)\\ & = & \blacklozenge u \vee \sim\blacksquare\neg\blacklozenge u \wedge \nu Y.\blacksquare(\neg\blacklozenge u \wedge Y)\end{array}$$

Finally,

$$\begin{array}{rcl}\mathsf{LA}^{\varepsilon}_p(\varphi)(u) & = & \mathsf{LA}^{\varepsilon}_p(\neg\psi)[\mathsf{LA}^{\delta}_{y'}(\varphi')(u \vee \mu X.\mathsf{LA}^{\delta}_X(\varphi')(u \vee X))]\\ & = & \mathsf{LA}^{\varepsilon}_p(\neg\psi)[\blacklozenge(u \vee \mu X.(\blacklozenge(u \vee X) \vee \sim\blacksquare\neg\blacklozenge(u \vee X)\wedge\\ & & \nu Y.\blacksquare(\neg\blacklozenge(u \vee X) \wedge Y)))]\\ & = & \blacksquare(\neg w \wedge \nu Y.\blacksquare(\neg w \wedge Y))(\blacklozenge(u \vee \mu X.(\blacklozenge(u \vee X)\vee\\ & & \sim\blacksquare\neg\blacklozenge(u \vee X) \wedge \nu Y.\blacksquare(\neg\blacklozenge(u \vee X) \wedge Y)))/w).\end{array}$$

Thus, applying (IF^{σ}_R) to the normalized inequality transforms (5.2) into

$$\forall p\forall \mathbf{i}\forall \mathbf{m}[(\mathsf{LA}^{\varepsilon}_p(\varphi)(\mathbf{i}/u) \leq p \ \&\ \Diamond\Box\neg p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

which is in Ackermann shape, since $\mathsf{LA}^{\varepsilon}_p(\varphi)(\mathbf{i}/u)$ is p-free. Now applying (RA) yields the quasi-inequality

$$\forall \mathbf{i}\forall \mathbf{m}[\Diamond\Box\neg\mathsf{LA}^{\varepsilon}_p(\varphi)(\mathbf{i}/u) \leq \mathbf{m} \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

from which all propositional variables have been eliminated, and which can be further rewritten as

$$\forall \mathbf{i}[\mathbf{i} \leq \Diamond\Box\neg\mathsf{LA}^{\varepsilon}_p(\varphi)(\mathbf{i}/u)].$$

5.6.2. Example. Consider the inequality

$$\Diamond\mu X.[(p \vee X) \vee \sim\nu Y.[\Diamond(X \vee \sim((Y \wedge p) \wedge \mu Z.\sim(\Box p \wedge \neg Z))) \to \Diamond\Box\Box p]] \leq \Diamond\Box p.$$

which, as discussed in Example 5.3.5, is ε-recursive with $\varepsilon_p = 1$. After first approximation we have:

$$\begin{array}{l}\forall p\forall \mathbf{i}\forall \mathbf{m}[(\mathbf{i} \leq \Diamond\mu X.[(p \vee X) \vee \sim\nu Y.[\Diamond(X \vee \sim((Y \wedge p) \wedge \mu Z.\sim(\Box p \wedge \neg Z))) \to\\ \qquad \Diamond\Box\Box p]]\&\ \Diamond\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}].\end{array} \tag{5.3}$$

Applying ($\Diamond$ Appr) to surface the inner skeleton of the first inequality in the antecedent of the quasi-inequality above yields:

$$\forall p \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \& \ \mathbf{j} \leq \psi \ \& \ \Diamond\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}], \tag{5.4}$$

with $\psi = \mu X.[(p \vee X) \vee {\sim}\nu Y.[\Diamond(X \vee {\sim}((Y \wedge p) \wedge \mu Z.{\sim}(\Box p \wedge \neg Z))) \to \Diamond\Box\Box p]]$. Now notice that $\psi = \psi'(\varphi_1/x_1, \varphi_2/x_2, \gamma/z)$, where

$$\begin{array}{rcl}\psi'(x_1, x_2, z) & = & \mu X.[(x_1 \vee X) \vee {\sim}\nu Y.[\Diamond(X \vee {\sim}((Y \wedge p) \wedge x_2)) \to z]], \\ \varphi_1 & = & p, \\ \varphi_2 & = & \mu Z.{\sim}(\Box p \wedge \neg Z), \\ \gamma & = & \Diamond\Box\Box p.\end{array}$$

Moreover, ψ' is an (x_1, x_2, z)-$\mathrm{IF}^{\Diamond}_{(1,\partial)}$ formula. Hence, by Lemma 5.4.3, its associated term function is completely join-preserving as a map $\mathbb{A} \times \mathbb{A}^{\partial} \to \mathbb{A}$, for any perfect modal bi-Heyting algebra $\mathbb{A}$; that is, the inequality $\mathbf{j} \leq \psi$ satisfies the appropriate order-theoretic conditions for the application of the rule (μ-A). Hence, after this application, the inequality $\mathbf{j} \leq \psi$ is equivalently replaced with the following disjunction:

$$\exists \mathbf{j}'[\mathbf{j} \leq \psi'(\mathbf{j}'/x_1, \top/x_2, \gamma/z) \ \& \ \mathbf{j}' \leq \varphi_1] \ \invamp \ \exists \mathbf{n}[\mathbf{j} \leq \psi'(\bot/x_1, \mathbf{n}/x_2, \gamma/z) \ \& \ \varphi_2 \leq \mathbf{n}]$$

At this point we transform the quasi-inequality obtained from (5.4) by performing the replacement above, into the conjunction of two quasi-inequalities, by distributing &s over $\invamp$ in the antecedent so as to make $\invamp$ the main connective of the antecedent, and then distributing $\Rightarrow$ over $\invamp$. This gives us:

$$\begin{array}{l}\forall p \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{j}' \forall \mathbf{m}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \& \ \mathbf{j} \leq \psi'(\mathbf{j}'/x_1, \top/x_2, \gamma/z) \ \& \ \mathbf{j}' \leq \varphi_1 \ \& \ \Diamond\Box p \leq \mathbf{m}) \\ \qquad\qquad \Rightarrow \mathbf{i} \leq \mathbf{m}],\end{array} \tag{5.5}$$

and

$$\begin{array}{l}\forall p \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} \forall \mathbf{n}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \& \ \mathbf{j} \leq \psi'(\bot/x_1, \mathbf{n}/x_2, \gamma/z) \ \& \ \varphi_2 \leq \mathbf{n} \ \& \ \Diamond\Box p \leq \mathbf{m}) \\ \qquad\qquad \Rightarrow \mathbf{i} \leq \mathbf{m}].\end{array} \tag{5.6}$$

Recalling that $\varphi_1 = p$ and $\gamma = \Diamond\Box\Box p$, the quasi-inequality (5.5) is

$$\begin{array}{l}\forall p \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{j}' \forall \mathbf{m}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \& \ \mathbf{j} \leq \psi'(\mathbf{j}'/x_1, \top/x_2, \Diamond\Box\Box p/z) \ \& \ \mathbf{j}' \leq p \ \& \ \Diamond\Box p \leq \mathbf{m}) \\ \qquad\qquad \Rightarrow \mathbf{i} \leq \mathbf{m}],\end{array}$$

which is in Ackermann shape (cf. page 63), hence we can apply (RA) to it and obtain

$$\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{j}' \forall \mathbf{m}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \& \ \mathbf{j} \leq \psi'(\mathbf{j}'/x_1, \top/x_2, \Diamond\Box\Box \mathbf{j}'/z) \ \& \ \Diamond\Box \mathbf{j}' \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

where all propositional variables have been eliminated, and hence can be translated into $\mathsf{FO} + \mathsf{LFP}$ as discussed in Section 4.2. Turning our attention to (5.6), we note that only occurrence of p for which we want to solve is in the inequality $\varphi_2 \leq \mathbf{n}$. Recalling that $\varphi_2 = \mu Z.{\sim}(\Box p \wedge \neg Z)$ we work towards the application of an appropriate adjunction rule. As it stands, φ_2 is not a substitution instance of a normal $\mathrm{IF}^{\Diamond}_{\delta'}$-formula, thus we need to normalize it. Indeed, the normalization consists in distributing $\sim$ over $\wedge$, transforming φ_2 into $\mu Z.[{\sim}\Box p \vee {\sim}\neg Z] = \mu Z.\psi''({\sim}\Box p/y')$ with $\psi''(y', Z) = y' \vee {\sim}\neg Z$ which is a normal $\mathrm{IF}^{\Diamond}_{(1)}$-formula. Hence we may apply (IF^{σ}_{L}) which yields

$$p \leq \mathsf{RA}^{(1)}_{p}(\varphi_2)[\mathbf{n}/u],$$

where

$$\begin{aligned}
\mathsf{RA}^{(1)}_{p}(\varphi_2)(u) &= \mathsf{RA}^{(1)}_{p}(\mu Z.\psi''({\sim}\Box p/y', Z)) \\
&= \mathsf{RA}^{(1)}_{p}({\sim}\Box p)(\mathsf{RA}^{(1)}_{y'}(\psi'')(u \wedge \nu Z.[\mathsf{RA}^{(1)}_{Z}(\psi'')(u \wedge Z)])),
\end{aligned}$$

and

$$\begin{aligned}
\mathsf{RA}^{(1)}_{Z}(\psi'')(u) &= \mathsf{RA}^{(1)}_{Z}(y' \vee {\sim}\neg Z)(u) \\
&= (\mathsf{RA}^{(1)}_{Z}(y') \wedge \mathsf{RA}^{(1)}_{Z}({\sim}\neg Z))(u) \\
&= (\top \wedge \mathsf{RA}^{(1)}_{Z}({\sim}\neg Z))(u) && \text{(Lemma 5.5.2)} \\
&= \mathsf{RA}^{(1)}_{Z}(\top - \neg Z)(u) \\
&= \mathsf{LA}^{(\partial)}_{Z}(\neg Z)((\top - u)/u) \\
&= (\mathsf{RA}^{(1)}_{Z}(Z)(\neg u/u))((\top - u)/u) \\
&= (u(\neg u/u))((\top - u)/u) \\
&= \neg{\sim}u.
\end{aligned}$$

$$\begin{aligned}
\mathsf{RA}^{(1)}_{y'}(\psi'')(u) &= \mathsf{RA}^{(1)}_{y'}(y' \vee {\sim}\neg Z)(u) \\
&= (\mathsf{RA}^{(1)}_{y'}(y') \wedge \mathsf{RA}^{(1)}_{y'}({\sim}\neg Z))(u) \\
&= (u \wedge \top)(u) && \text{(Lemma 5.5.2)} \\
&= u.
\end{aligned}$$

$$\begin{aligned}
\mathsf{RA}^{(1)}_{p}({\sim}\Box p)(u) &= \mathsf{RA}^{(1)}_{p}(\top - \Box p)(u) \\
&= \mathsf{LA}^{(\partial)}_{p}(\Box p)((\top - u)/u) \\
&= (\mathsf{LA}^{(\partial)}_{p}(p)(\blacklozenge u/u))((\top - u)/u) \\
&= (u(\blacklozenge u/u))((\top - u)/u) \\
&= \blacklozenge u((\top - u)/u) \\
&= \blacklozenge{\sim}u.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathsf{RA}^{(1)}_{p}(\varphi_2)(u) &= \mathsf{RA}^{(1)}_{p}({\sim}\Box p)(\mathsf{RA}^{(1)}_{y'}(\psi'')(u \wedge \nu Z.[\mathsf{RA}^{(1)}_{Z}(\psi'')(u \wedge Z)])), \\
&= \blacklozenge{\sim}(u \wedge \nu Z.\neg{\sim}(u \wedge Z)).
\end{aligned}$$

Thus (5.6) becomes

$$\forall p \forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} \forall \mathbf{n}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \&\ \mathbf{j} \leq \psi'(\bot/x_1, \mathbf{n}/x_2, \gamma/z) \ \&\ p \leq \blacklozenge{\sim}(\mathbf{n} \wedge \nu Z.\neg{\sim}(\mathbf{n} \wedge Z)) \\ \&\ \Diamond\Box p \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

which is in Ackermann shape. Applying the Ackermann rule (LA) and recalling that $\gamma = \Diamond\Box\Box p$, we obtain

$$\forall \mathbf{i} \forall \mathbf{j} \forall \mathbf{m} \forall \mathbf{n}[(\mathbf{i} \leq \Diamond \mathbf{j} \ \&\ \mathbf{j} \leq \psi'(\bot/x_1, \mathbf{n}/x_2, \Diamond\Box\Box\blacklozenge{\sim}(\mathbf{n} \wedge \nu Z.\neg{\sim}(\mathbf{n} \wedge Z))/z) \\ \&\ \Diamond\Box\blacklozenge{\sim}(\mathbf{n} \wedge \nu Z.\neg{\sim}(\mathbf{n} \wedge Z)) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}],$$

where all occurring propositional variables have been eliminated.

5.7 Conclusions

In the present chapter, we extended the algorithm ALBA of [50] to the language of bi-intuitionistic modal mu-calculus. We defined the class of recursive inequalities (see Definition 5.3.2) for the bi-intuitionistic modal mu-calculus which is the bi-intuitionistic counterpart of the Sahlqvist mu-formulas defined in [17]. We proved that the enhanced ALBA is successful on all recursive mu-inequalities, and hence that each of them has a frame correspondent in first-order logic with least fixed points ($\mathsf{FO} + \mathsf{LFP}$) [61] .

A future direction of work would be to investigate the canonicity of inequalities in the language of bi-intuitionistic modal mu-calculus. In [44], canonicity results for a class of mu-inequalities has been studied using algorithmic-algebraic techniques. In Chapter 8 of this thesis, we will use order-topological methods to show canonicity of a class of mu-inequalities in the more general setting of compact Hausdorff spaces.

Chapter 6

Pseudocorrespondence and relativized canonicity

In this chapter, which is based on [53], we generalize Venema's result on the canonicity of the additivity of positive terms [154], from classical modal logic to the logic of distributive lattice expansions (DLE). We provide two contrasting proofs for this result: the first is along the lines of Venema's pseudo-correspondence argument but using the rules and methodology associated with the ALBA algorithm; the second is closer to the style of Jónsson. Using insights gleaned from the second proof we define a suitable enhancement of the algorithm ALBA, which we use to prove the canonicity of certain syntactically defined classes of DLE-inequalities (called the *meta-inductive inequalities*), relative to the structures in which the formulas asserting the additivity of some given terms are valid.

The order-theoretic facts underlying this generalization provide the basis for the soundness of additional ALBA rules relative to the classes of structures in which the formulas asserting the additivity of some given terms are valid. These classes do not need to be first-order definable, and in general they are not. Accordingly, an enhanced version of ALBA, which we call ALBA^e, is defined, which is proven to be successful on a certain class of inequalities which extends (see discussion on Section 6.5) the class of inequalities on which the canonicity-via-correspondence argument is known to work. These inequalities are shown to be canonical relative to the subclass defined by the given additivity axioms.

The algorithm ALBA^r for is similar to ALBA^e developed in Chapter 4. However, it is worth mentioning that they are different in important respects. Firstly, the two settings of these algorithms (that is, the present setting and that of Chapter 4) are different: indeed, the present setting is that of a *normal* modal logic (i.e. the primitive modal connectives are normal), but the term functions are assumed to be arbitrary compound formulas. Then, this basic setting is restricted even further to the class of distributive lattice expansions (DLEs) on which the interpretations of the term functions verify additional conditions. In contrast to this, in the regular distributive lattice expansion (r-DLE)-setting of Chapter 4, the primitive connectives are not normal in the first place, but are assumed to be additive or multiplicative. Hence in particular the basic setting of Chapter 4 covers a strictly wider class of algebras than normal DLEs.

Secondly, ALBA^r guarantees all the benefits of classical Sahlqvist correspondence theory for the inequalities on which it succeeds. This is not the case of ALBA^e, which is used to prove relativized canonicity in the absence of correspondence. The reason for this difference is due to the fact that the approximation and the adjunction rules of ALBA^e relative to compound term functions are only sound on perfect DLEs which are canonical extensions of some DLEs, whereas the corresponding rules for ALBA^r concern only primitive regular connectives, and for this reason they can be shown to be sound on arbitrary perfect r-DLEs.

A direction in which the results of the present chapter are useful concerns the investigation of canonicity in the presence of additional axioms (or relativized canonicity, cf. Definition 6.6.1). It is well known that certain modal axioms which are *not* in general canonical (i.e., over the class of all algebras) *are* canonical over some smaller class of algebras. Examples of relativized canonicity are rather rare, canonicity in the presence of transitivity being one example: in [118], Lemmon and Scott prove that the McKinsey formula becomes canonical when th n in conjunction with the transitivity axiom. More generally, all modal reduction principles are canonical in the presence of transitivity, and this can be seen as follows: in [158] Zakharyaschev proves that any extension of **K4** axiomatized with modal reduction principles has the finite model property, and is hence Kripke complete. Combining this fact with the elementarity of the reduction principles over transitive frames as proved by van Benthem [10], the claim follows by Fine's theorem [66]. The problem of relativized canonicity is difficult to tackle directly and in general, and this chapter can be regarded as an ALBA-aided contribution in this direction.

The chapter in organized as follows. In Section 6.1, we provide the necessary preliminaries on the distributive lattice-based logical environment. In Section 6.2 we explain the notion of pseudo-correspondence which we also illustrate by recasting Venema's argument [154] in terms of an ALBA-type reduction and the methodology of unified correspondence while lifting it to the setting of distributive lattice expansions. In Section 6.3 we provide the algebraic and order-theoretic facts, in the setting of distributive lattices, at the basis of the generalization of the results in [154]. We prove the canonicity of the inequalities stating the additivity of ε-positive terms as immediate consequences of the order-theoretic results. In Section 6.4 we introduce the enhanced algorithm ALBA^e and prove the soundness of its new rules on the basis of the results in Section 6.3. In Section 6.5 we prove that ALBA^e succeeds on the class of *meta-inductive* inequalities introduced there; then, in Section 6.6, the relativized canonicity of all meta-inductive inequalities is stated and proved. Section 6.7 presents some examples and in Section 6.8 we draw some conclusions and discuss further directions. Some technical facts are collected in Appendix B.

6.1 Preliminaries

6.1.1 Language, basic axiomatization and algebraic semantics of DLE and DLE*

Our base language is an unspecified but fixed modal-type language DLE, to be interpreted over distributive lattice expansions. For such a language and naturally associated axiomatization, the algorithm ALBA can be deployed to obtain correspondents and a definition of inductive inequalities. In what follows, we will provide a concise account of the main definitions and facts. Moreover, for the sake of the developments in Section 6.5, we will find it useful to work with an expansion DLE* of DLE, obtained by adding 'placeholder modalities'. Since DLE* is itself a member of the DLE family, all the results and notions pertaining to the unified correspondence theory for DLE will apply to DLE* as well.

We fix a set of proposition letters Prop, two sets $\mathcal{F}$ and $\mathcal{G}$ of connectives of arity $n_f, n_g \in \mathbb{N}$ for each $f \in \mathcal{F}$ and $g \in \mathcal{G}$, and define the languages DLE and DLE*, respectively, by the following dependent recursion:

$$\mathrm{DLE} \ni \varphi ::= p \mid \bot \mid \top \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid f(\overline{\varphi}) \mid g(\overline{\varphi})$$

where $p \in \mathsf{Prop}$, $f \in \mathcal{F}$ and $g \in \mathcal{G}$, and

$$\mathrm{DLE}^* \ni \psi ::= \varphi \mid \boxdot\psi \mid \diamonddot\, \psi \mid \lhd\, \psi \mid \rhd \psi$$

where $\varphi \in \mathrm{DLE}$.

We further assume that each $f \in \mathcal{F}$ and $g \in \mathcal{G}$ is associated with some order-type[1] ε_f on n_f (resp. ε_g on n_g). The equational axiomatizations of DLE and DLE* are obtained by adding the following axioms to the equational axiomatization of bounded distributive lattices:

- if $\varepsilon_f(i) = 1$, then

  $$f(p_1, \ldots, p \vee q, \ldots, p_{n_f}) = f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f})$$

  and $f(p_1, \ldots, \bot, \ldots, p_{n_f}) = \bot$;

- if $\varepsilon_f(i) = \partial$, then

  $$f(p_1, \ldots, p \wedge q, \ldots, p_{n_f}) = f(p_1, \ldots, p, \ldots, p_{n_f}) \vee f(p_1, \ldots, q, \ldots, p_{n_f})$$

  and $f(p_1, \ldots, \top, \ldots, p_{n_f}) = \bot$;

- if $\varepsilon_g(j) = 1$, then

  $$g(p_1, \ldots, p \wedge q, \ldots, p_{n_g}) = g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g})$$

  and $g(p_1, \ldots, \top, \ldots, p_{n_g}) = \top$;

[1] Recall that an order-type over $n \in \mathbb{N}$ is an n-tuple $\varepsilon \in \{1, \partial\}^n$.

- if $\varepsilon_g(j) = \partial$, then

$$g(p_1, \ldots, p \vee q, \ldots, p_{n_g}) = g(p_1, \ldots, p, \ldots, p_{n_g}) \wedge g(p_1, \ldots, q, \ldots, p_{n_g})$$

and $g(p_1, \ldots, \bot, \ldots, p_{n_g}) = \top$.

for each $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$) and $1 \leq i \leq n_f$ (resp. for each $1 \leq j \leq n_g$). For DLE* we also add

$$\diamondtimes(p \vee q) = \diamondtimes p \vee \diamondtimes q \quad \diamondtimes \bot = \bot \qquad \boxdot p \wedge \boxdot q = \boxdot(p \wedge q) \quad \top = \boxdot \top$$

$$\lhd (p \wedge q) = \lhd p \vee \lhd q \quad \lhd \top = \bot \qquad \rhd p \wedge \rhd q = \rhd(p \vee q) \quad \top = \rhd \bot.$$

The ALBA algorithm manipulates inequalities in the following expansions of the base languages DLE and DLE*: Let DLE$^+$ be the expansion of DLE with two additional sorts of variables, namely, *nominals* $\mathbf{i}, \mathbf{j}, \ldots$ and *conominals* $\mathbf{m}, \mathbf{n}, \ldots$ (which, as mentioned earlier on, are intended as individual variables ranging over the sets of the completely join-irreducible elements and the completely meet-irreducible elements of perfect DLEs, see below), and with residuals $\rightarrow, -$ of $\wedge$ and $\vee$, and residuals $\underline{f}^{(i)}$ and $\overline{g}^{(j)}$ for each $1 \leq i \leq n_f$ and $1 \leq j \leq n_g$. The language DLE^{*+} is the expansion of DLE$^+$ with the adjoint connectives $\blacklozenge$, $\blacksquare$, $\blacktriangleleft$ and $\blacktriangleright$ for $\boxdot$, $\diamondtimes$, $\lhd$ and $\rhd$, respectively.

6.1.1. Definition. A *distributive lattice expansion* (abbreviated as DLE) is a tuple $A = (D, \mathcal{F}, \mathcal{G})$ such that D is a bounded distributive lattice, and every $f \in \mathcal{F}$ (resp. $g \in \mathcal{G}$) is an n-ary map preserving finite joins (resp. meets) in each coordinate with $\varepsilon_f(i) = 1$ (resp. $\varepsilon_g(i) = 1$) and reversing finite meets (resp. joins) in each coordinate with $\varepsilon_f(i) = \partial$ (resp. $\varepsilon_g(i) = \partial$). A DLE is *perfect* if D is a perfect distributive lattice, and all the preservations and reversions mentioned above are for arbitrary joins and meets.

6.1.2. Definition. Given any DLE $A = (D, \mathcal{F}, \mathcal{G})$, its canonical extension $\mathbb{A}^\delta$ is defined as $\mathbb{A}^\delta = (D^\delta, \mathcal{F}^\delta, \mathcal{G}^\delta)$, where D^δ is the canonical extension of the underlying BDL, the set $\mathcal{F}^\delta$ (resp. $\mathcal{G}^\delta$ consists of the map f^σ (resp. map g^π) for every $f \in \mathcal{F}$.

The canonical extension of a DLE is a perfect DLE (cf. Lemma 2.21 in [75]).

6.1.2 Inductive DLE and DLE* inequalities

In this subsection we define the inductive inequalities in the two languages DLE and DLE* simultaneously. For preliminaries on order-type and signed generation trees, we refer to Section 4.4. The definitions are the same, except that they refer to nodes in table 6.1 and table 6.2, respectively.

Skeleton	PIA
Δ-adjoints	SRA
$+ \quad \vee \quad \wedge$	$+ \quad \wedge \quad g_{(n_g=1)}$
$- \quad \wedge \quad \vee$	$- \quad \vee \quad f_{(n_f=1)}$
SLR	SRR
$+ \quad \wedge \quad f_{(n_f\geq 1)}$	$+ \quad \vee \quad g_{(n_g\geq 2)}$
$- \quad \vee \quad g_{(n_g\geq 1)}$	$- \quad \wedge \quad f_{(n_f\geq 2)}$

Table 6.1: Skeleton and PIA nodes for DLE

Skeleton	PIA
Δ-adjoints	SRA
$+ \quad \vee \quad \wedge$	$+ \quad \wedge \quad \boxdot \quad \rhd \quad g_{(n_g=1)}$
$- \quad \wedge \quad \vee$	$- \quad \vee \quad ⟐ \quad \lhd \quad f_{(n_f=1)}$
SLR	SRR
$+ \quad \wedge \quad ⟐ \quad \lhd \quad f_{(n_f\geq 1)}$	$+ \quad \vee \quad g_{(n_g\geq 2)}$
$- \quad \vee \quad \boxdot \quad \rhd \quad g_{(n_g\geq 1)}$	$- \quad \wedge \quad f_{(n_f\geq 2)}$

Table 6.2: Skeleton and PIA nodes DLE*.

6.1.3. Definition. Nodes in signed generation trees will be called Δ-*adjoints*, *syntactically left residual (SLR)*, *syntactically right residual (SRR)*, and *syntactically right adjoint (SRA)*, according to the specification given in tables 6.1 and 6.2. We will find it useful to group these classes as *Skeleton* and *PIA* as indicated in the table. A branch in a signed generation tree $*s$, with $* \in \{+,-\}$, is called a *good branch* if it is the concatenation of two paths P_1 and P_2, one of which may possibly be of length 0, such that P_1 is a path from the leaf consisting (apart from variable nodes) only of PIA-nodes, and P_2 consists (apart from variable nodes) only of Skeleton-nodes.

6.1.4. Definition. For any order-type ε and any strict partial order $<_\Omega$ on $p_1,\dots p_n$, the signed generation tree $*s$, $* \in \{-,+\}$, of a term $s(p_1,\dots p_n)$ is (Ω,ε)-*inductive* if for all $1 \leq i \leq n$ every ε-critical branch with leaf p_i is good, and moreover, every m-ary SRR node occurring in it is of the form $* \circledast (\gamma_1,\dots,\gamma_{j-1},\beta,\gamma_{j+1}\dots,\gamma_m)$, and where for any $h \in \{1,\dots,m\} \setminus j$:

1. $\varepsilon^\partial(\gamma_h) \prec *s$ (cf. Definition 4.4.1), and

2. $p_k <_\Omega p_i$ for every p_k occurring in γ_h and for every $1 \leq k \leq n$.

We will refer to $<_\Omega$ as the *dependency order* on the variables. An inequality $s \leq t$ is (Ω,ε)-*inductive* if the trees $+s$ and $-t$ are both (Ω,ε)-inductive. An inequality $s \leq t$ is *inductive* if it is (Ω, ε)-inductive for some Ω and ε. Item 2 of definition above and the fact that Ω is a strict partial order imply that the γs are p_i free.

6.2 Pseudo-correspondence and relativized canonicity and correspondence

In this section, we give an account of the proof in [154] of the canonicity of additivity for positive terms, via pseudo-correspondence. Our presentation differs from the one in [154] in some respects. These differences will be useful to motivate the results in the following sections. Namely, our presentation is set in the context of algebras and their canonical extensions, rather than in the original setting of descriptive general frames and their underlying Kripke structures. This makes it possible to establish an explicit link between the proof-strategy of canonicity via pseudo-correspondence in [154] and unified correspondence theory.Specifically, our account of canonicity via pseudo-correspondence is given in terms of an ALBA-type reduction, and the pseudo-correspondent of a given modal formula is defined as a quasi-inequality in the language DLE^{++}, which is the expansion of DLE^{+} with the connectives $\Diamond_\pi$, $\Box_\sigma$, $\lhd_\lambda$ and $\rhd_\rho$, and their respective adjoints $\blacksquare_\pi$, $\Diamondblack_\sigma$, $\blacktriangleleft_\lambda$ and $\blacktriangleright_\rho$. Another difference between [154] and the present account is that here the Boolean setting does not play an essential role. Indeed, the present treatment holds for general DLEs.

6.2.1. Definition. [[154], Definition 1.1] A modal formula φ and a first-order sentence α are *canonical pseudo-correspondents* if the following conditions hold for any descriptive general frame $\mathfrak{g}$ and any Kripke frame $\mathcal{F}$:

1. if $\mathfrak{g} \Vdash \varphi$, then $\mathfrak{g}^\sharp \models \alpha$, where $\mathfrak{g}^\sharp$ denotes the underlying (Kripke) frame of $\mathfrak{g}$;
2. if $\mathcal{F} \models \alpha$ then $\mathcal{F} \Vdash \varphi$.

This definition can be better understood in the context of the familiar canonicity-via-correspondence argument, illustrated by the diagram below:

$$\begin{array}{ccc} \mathfrak{g} \models \varphi & & \mathfrak{g}^\sharp \models \varphi \\ \Updownarrow & & \Updownarrow \\ \mathfrak{g} \models \alpha & \Leftrightarrow & \mathfrak{g}^\sharp \models \alpha, \end{array}$$

where, $\mathfrak{g}$ is a descriptive general frame, and $\mathfrak{g}^\sharp$ denotes the underlying (Kripke) frame of $\mathfrak{g}^\sharp$. Indeed, if α is a first-order frame correspondent of φ, then the U-shaped chain of equivalences illustrated in the diagram above holds,[2] which

[2]The horizontal equivalence in the diagram holds since α is a sentence in the first-order language of Kripke structures, and hence its validity does not depend on assignments of atomic propositions.

implies the canonicity of φ. The observation motivating the definition of pseudo-correspondents is that, actually, less is needed: specifically, the arrows of the following diagram are already enough to guarantee the canonicity of φ:

$$\begin{array}{ccc} \mathfrak{g} \models \varphi & & \mathfrak{g}^\sharp \models \varphi \\ \Downarrow & & \Uparrow \\ \mathfrak{g} \models \alpha' & \Leftrightarrow & \mathfrak{g}^\sharp \models \alpha'. \end{array}$$

The conditions 1 and 2 of the definition above precisely make sure that the implications in the diagram above hold. Thus, we have the following:

6.2.2. Proposition. *If φ and α are canonical pseudo-correspondents, then φ is canonical.*

Definition 6.2.1 above straightforwardly generalizes to the algebraic setting introduced in Subsection 6.1.1 as follows:

6.2.3. Definition. A DLE-inequality $\varphi \leq \psi$ and a pure quasi-inequality α in DLE^{++} (cf. Page 130) are *canonical algebraic pseudo-correspondents* if the following conditions hold for every DLE $\mathbb{A}$ and every perfect DLE $\mathbb{B}$:

1. if $\mathbb{A} \models \varphi \leq \psi$, then $\mathbb{A}^\delta \models \alpha$;
2. if $\mathbb{B} \models \alpha$, then $\mathbb{B} \models \varphi \leq \psi$.

6.2.4. Lemma. *If $\varphi \leq \psi$ and α are canonical algebraic pseudo-correspondents, then $\varphi \leq \psi$ is canonical.*

Proof. Similar to the discussion above, using the following diagram:

$$\begin{array}{ccc} \mathbb{A}^\delta \models_{\mathbb{A}} \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\ \Downarrow & & \Uparrow \\ \mathbb{A}^\delta \models_{\mathbb{A}} \alpha & \Leftrightarrow & \mathbb{A}^\delta \models \alpha. \end{array}$$

In the diagram above, the notation $\models_{\mathbb{A}}$ refers to validity restricted to assignments mapping atomic propositions to elements of the DLE $\mathbb{A}$. $\Box$

In the remainder of the present subsection, we will prove the canonicity of the inequality $\pi(p \vee q) \leq \pi(p) \vee \pi(q)$ for any positive term π, by using the strategy provided by the lemma above. That is, by proving that $\pi(p \vee q) \leq \pi(p) \vee \pi(q)$ and the following pure quasi-inequality are canonical algebraic pseudo-correspondents:

$$C(\pi) = \forall \mathbf{m}[\pi(\bot) \leq \mathbf{m} \Rightarrow \pi(\blacksquare_\pi \mathbf{m}) \leq \mathbf{m}], \tag{6.1}$$

where the new connective $\blacksquare_\pi$ is interpreted in any perfect DLE $\mathbb{B}$ as the operation defined by the assignment $u \mapsto \blacksquare_\pi u := \bigvee\{i \in J^\infty(B) \mid \pi^B(i) \leq u\}$.

Hence, in the light of the discussion in Section 4.2, it is not difficult to see that, under the standard translation, for every conominal variable $\mathbf{m}$, the term $\blacksquare_\pi \mathbf{m}$ denotes a first-order definable set in any Kripke frame. Indeed, if $\mathbf{m}$ is interpreted as $W \setminus \{v\}$ for some state v, then $\blacksquare_\pi \mathbf{m}$ is interpreted as the set of all the states w such that v does not belong to the set defined by the standard translation of π in which the predicate variable P is substituted for the description of the singleton set $\{w\}$.

Hence, via the standard translation, the pure quasi-inequality $C(\pi)$ can be identified with a first-order sentence.

The following proposition is the algebraic counterpart of [154, Proposition 2.1], and shows that $\pi(p \vee q) \leq \pi(p) \vee \pi(q)$ and $C(\pi)$ satisfy item 1 of Definition 6.2.3.

6.2.5. Proposition. *For any algebra* $\mathbb{A}$,

$$\text{if } \mathbb{A} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q), \quad \text{then}$$
$$\mathbb{A}^\delta \models_A C(\pi) = \forall \mathbf{m}[\pi(\bot) \leq \mathbf{m} \Rightarrow \pi(\blacksquare_\pi \mathbf{m}) \leq \mathbf{m}].$$

As in [154], for every $u \in \mathbb{A}^\delta$, let

$$\Diamond_\pi(u) := \bigvee \{\pi(j) \mid j \in J^\infty(\mathbb{A}^\delta) \text{ and } j \leq u\},$$

6.2.6. Lemma. *For every positive term* π *and every algebra* $\mathbb{A}$,

$$\text{if } \mathbb{A} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q), \quad \text{then} \quad \mathbb{A}^\delta \models_A \pi(p) = \Diamond_\pi(p) \vee \pi(\bot).$$

Proof. Let us fix $a \in \mathbb{A}$ and let us show that $\pi(a) = \Diamond_\pi(a) \vee \pi(\bot)$.

The right-to-left inequality immediately follows from the fact that π is monotone. As to the converse, it is enough to show that, if $m \in M^\infty(\mathbb{A}^\delta)$ and $\pi(\bot) \vee \Diamond_\pi(a) \leq m$, then $\pi(a) \leq m$. Consider the set

$$J := \{b \in \mathbb{A} \mid \pi(b) \leq m\}.$$

Note that $J \neq \varnothing$, since $\pi(\bot) \leq \pi(\bot) \vee \Diamond_\pi(a) \leq m$, and that J is a lattice ideal: indeed, J is downward closed by construction, and the join of two elements in J belongs to J, since by assumption $\pi(a \vee b) \leq \pi(a) \vee \pi(b)$.

We claim that $a \leq \bigvee J$. To see this, let $j \in J^\infty(\mathbb{A}^\delta)$ s.t. $j \leq a$ and let us show that there is some $b \in J$ s.t. $j \leq b$. From $j \leq a$, by definition of $\Diamond_\pi(a)$, we have $\pi(j) \leq \Diamond_\pi(a) \leq m$. By applying the intersection lemma [50, 138] to π,

$$\bigwedge \{\pi(b) \mid b \in \mathbb{A} \text{ and } j \leq b\} = \pi(j) \leq m.$$

Since, by assumption, $\pi(b)$ is a clopen hence closed element of $\mathbb{A}^\delta$ for all $b \in \mathbb{A}$, the displayed inequality implies by compactness that $\bigwedge_{i=1}^n \pi(b_i) \leq m$ for some $b_1, \ldots, b_n \in \mathbb{A}$ such that $j \leq b_i$ holds for every $1 \leq i \leq n$. Since π is order-preserving, putting $b := \bigwedge_{i=1}^n b_i$ we get $j \leq b$ and $\pi(b) = \pi(\bigwedge_{i=1}^n b_i) \leq \bigwedge_{i=1}^n \pi(b_i) \leq m$. So we have found some $b \in J$ with $j \leq b$, which finishes the proof of the claim.

From $a \leq \bigvee J$, by compactness, there is some element $b \in J$ such that $a \leq b$. Then $\pi(a) \leq \pi(b) \leq m$, as required. □

As in [154], we are now ready to prove Proposition 6.2.5:

Proof.[Proof of Proposition 6.2.5] We need to show that

$$\mathbb{A}^\delta \models \forall \mathbf{m}[\pi(\bot) \leq \mathbf{m} \Rightarrow \pi(\blacksquare_\pi \mathbf{m}) \leq \mathbf{m}]. \tag{6.2}$$

By assumption, we have that $\mathbb{A} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q)$. Hence, by Lemma 6.2.6,

$$\mathbb{A}^\delta \models_A \forall p[\pi(p) \leq \Diamond_\pi(p) \vee \pi(\bot)]. \tag{6.3}$$

The remainder of the present proof consists in showing that (6.3) is equivalent to (6.2). This equivalence is the counterpart of an analogous equivalence which was proved in [154] between $\mathfrak{g} \Vdash \pi(p) \leq \Diamond_\pi(p) \vee \pi(\bot)$ and $\mathfrak{g}^\sharp \Vdash C(\pi)$ for any descriptive general frame $\mathfrak{g}$.[3] We will prove this equivalence by means of an ALBA-type reduction. By performing a first approximation, we get:

$$\mathbb{A}^\delta \models_A \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \pi(p) \ \&\ \Diamond_\pi(p) \vee \pi(\bot) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{6.4}$$

By applying the splitting rule to the second inequality in the premise in (6.4), we obtain

$$\mathbb{A}^\delta \models_A \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \pi(p) \ \&\ \Diamond_\pi(p) \leq \mathbf{m} \ \&\ \pi(\bot) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{6.5}$$

Notice that the interpretations of $\Diamond_\pi$ and $\blacksquare_\pi$ in any perfect DLE form an adjoint pair (this will be expanded on below). Hence, the syntactic rule corresponding to this semantic adjunction is sound on $\mathbb{A}^\delta$. By applying this new rule, we get:

$$\mathbb{A}^\delta \models_A \forall p \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \pi(p) \ \&\ p \leq \blacksquare_\pi \mathbf{m} \ \&\ \pi(\bot) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{6.6}$$

The quasi-inequality above is in topological Ackermann shape (this will be expanded on below). Hence, by applying the Ackermann rule, we get:

$$\mathbb{A}^\delta \models_A \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \pi(\blacksquare_\pi \mathbf{m}) \ \&\ \pi(\bot) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}], \tag{6.7}$$

which is a pure quasi-inequality, i.e. it is free of atomic propositions. Hence, in particular, its validity does not depend on whether the valuations are admissible or not. Therefore, the condition above can be equivalently rewritten as follows:

$$\mathbb{A}^\delta \models \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \pi(\blacksquare_\pi \mathbf{m}) \ \&\ \pi(\bot) \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{6.8}$$

[3]In [154], $C(\pi)$ is defined in terms of the standard translation, and formulated in the extended language, it would correspond to $\forall \mathbf{i}[\mathbf{i} \leq \pi(\bot) \parr \mathbf{i} \leq \overline{\pi}(\blacklozenge_\pi \mathbf{i})]$, where $\parr$ is disjunction, and the new connective $\blacklozenge_\pi$ is interpreted in any perfect BAO $\mathbb{B}$ as the operation defined by the assignment $u \mapsto \bigvee\{j \in J^\infty(B) \mid i \leq \pi(j)$ for some $i \in J^\infty(B)$ s.t. $i \leq u\}$, and $\overline{\pi}(p) := \neg\pi(\neg p)$.

It is easy to see that the inequality above is equivalent to

$$\mathbb{A}^\delta \models \forall \mathbf{m}[\pi(\bot) \leq \mathbf{m} \Rightarrow \pi(\blacksquare_\pi \mathbf{m}) \leq \mathbf{m}], \tag{6.9}$$

as required. To finish the proof, we need to justify our two claims above. As to the adjunction between $\blacksquare_\pi$ and $\Diamond_\pi$, for all $u, v \in \mathbb{A}^\delta$,

$$\begin{array}{lll} u \leq \blacksquare_\pi(v) & \text{iff} & u \leq \bigvee\{j \in J^\infty(\mathbb{A}^\delta) \mid \pi(j) \leq v\} \\ & \text{iff} & \bigvee\{j \in J^\infty(\mathbb{A}^\delta) \mid j \leq u\} \leq \bigvee\{j \mid \pi(j) \leq v\} \\ & \text{iff} & \text{if } j \in J^\infty(\mathbb{A}^\delta) \text{ and } j \leq u, \text{ then } \pi(j) \leq v \\ & \text{iff} & \pi(j) \leq v \text{ for all } j \in J^\infty(\mathbb{A}^\delta) \text{ s.t. } j \leq u \\ & \text{iff} & \bigvee\{\pi(j) \mid j \in J^\infty(\mathbb{A}^\delta) \text{ and } j \leq u\} \leq v \\ & \text{iff} & \Diamond_\pi(u) \leq v. \end{array}$$

As to the applicability of the topological Ackermann lemma (cf. Appendix B), let us recall that this lemma is the restriction of the general Ackermann lemma to validity w.r.t. descriptive general frames. The algebraic counterpart of this lemma is set in the environment of canonical extensions and requires for its applicability on a given quasi-inequality the additional condition that in every inequality the left-hand side belongs to $K(\mathbb{A}^\delta)$ and the right-hand side belongs to $O(\mathbb{A}^\delta)$. Hence, in order for the topological Ackermann lemma to be applicable to the quasi-inequality (6.6), it remains to be shown that the interpretation of $\blacksquare_\pi \mathbf{m}$ is in $O(\mathbb{A}^\delta)$. This is an immediate consequence of the fact that, for every DLE$^+$-assignment V,

$$V(\blacksquare_\pi \mathbf{m}) = \bigvee\{a \in \mathbb{A} \mid \pi(a) \leq V(\mathbf{m})\}. \tag{6.10}$$

Let $m \in M^\infty(\mathbb{A}^\delta)$ be s.t. $V(\mathbf{m}) = m$. The right-hand side of (6.10) can be equivalently rewritten as $\bigvee\{j \in J^\infty(\mathbb{A}^\delta) \mid j \leq a \text{ for some } a \in \mathbb{A} \text{ s.t. } \pi(a) \leq m\}$. If j is one of the joinands of the latter join, then $\pi(j) \leq \pi(a) \leq m$, hence the right-to-left inequality immediately follows from the fact that $V(\blacksquare_\pi \mathbf{m}) = \bigvee\{i \in J^\infty(A) \mid \pi(i) \leq m\}$.

Conversely, let $i \in J^\infty(\mathbb{A}^\delta)$ s.t. $\pi(i) \leq m$. Since $i = \bigwedge\{a \in \mathbb{A} \mid i \leq a\}$, by the intersection lemma (cf. Appendix B) applied on π, we have $\pi(i) = \pi(\bigwedge\{a \in \mathbb{A} \mid i \leq a\}) = \bigwedge\{\pi(a) \mid a \in \mathbb{A} \text{ and } i \leq a\}$. Hence by compactness, $\bigwedge_{i=1}^n \pi(a_i) \leq m$. Then let $a = a_1 \wedge \ldots \wedge a_n$. Clearly, $\pi(a) = \pi(a_1 \wedge \ldots \wedge a_n) \leq \bigwedge \pi(a_i) \leq m$, which finishes the proof as required. $\square$

The following proposition is the algebraic counterpart of [154, Proposition 2.2]. It proves that $\pi(p \vee q) \leq \pi(p) \vee \pi(q)$ and $C(\pi)$ satisfy item 2 of Definition 6.2.3.

6.2.7. Proposition. *For every positive term π and every perfect algebra $\mathbb{B}$,*

if $\mathbb{B} \models \forall \mathbf{m}[\pi(\bot) \leq \mathbf{m} \Rightarrow \pi(\blacksquare_\pi \mathbf{m}) \leq \mathbf{m}]$, *then* $\mathbb{B} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q)$.

Proof. It is easy to see that the equivalences (6.3)–(6.9) in the proof of Proposition 6.2.5 hold on any perfect algebra $\mathbb{B}$ and for arbitrary valuations on $\mathbb{B}$. Hence, from the assumption we get

$$\mathbb{B} \models \forall p[\pi(p) \leq \Diamond_\pi(p) \vee \pi(\bot)]. \tag{6.11}$$

Notice that the assumption that π is positive implies that the equality holds in the clause above, that is,

$$\mathbb{B} \models \pi(p) = \Diamond_\pi(p) \vee \pi(\bot). \tag{6.12}$$

Since $\Diamond_\pi$ is by definition a complete operator, the condition above immediately implies that the interpretation of π is completely additive. Thus $\mathbb{B} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q)$, as required. $\Box$

6.3 An alternative proof of the canonicity of additivity

In this section, we extract the algebraic and order-theoretic essentials from the account given in Section 6.2 of the proof of the canonicity of the inequality $\pi(p \vee q) \leq \pi(p) \vee \pi(q)$. In the following subsection, we abstract away from any logical signature, and present order-theoretic results on monotone maps $f, g : \mathbb{A}^\delta \to \mathbb{B}^\delta$ defined between the canonical extensions of given bounded distributive lattices (BDLs). The feature that sets apart the results in Subsection 6.3.1 from similar existing results in the theory of canonical extensions is that these maps are not assumed to be the σ- or π-extensions of primitive functions $\mathbb{A} \to \mathbb{B}$. As we will see, this calls for different proof techniques from the ones typically used for σ- and π-extensions.

In Subsection 6.3.2, we apply the results of Subsection 6.3.1 to term functions of given DLE-type modal languages, so as to achieve a generalization of the results in [154] and in Section 6.2 which does not rely anymore on pseudo-correspondence, but which is more similar to the proof strategy sometimes (cf. [129, 130]) referred to as *Jónsson-style canonicity* after Jónsson [97].

6.3.1 A purely order-theoretic perspective

The treatment in this section makes use of some of the notions and proof strategies presented in Section 6.2, and slightly generalizes them.

Throughout the present section, $\mathbb{A}, \mathbb{B}$ will denote bounded distributive lattices (BDLs), and $\mathbb{A}^\delta, \mathbb{B}^\delta$ will be their canonical extensions, respectively. We recall that $J^\infty(\mathbb{A}^\delta)$ (resp. $M^\infty(\mathbb{A}^\delta)$) denotes the set of the completely join-irreducible (resp.

meet-irreducible) elements of $\mathbb{A}^\delta$, and $K(\mathbb{A}^\delta)$ (resp. $O(\mathbb{A}^\delta)$) denotes the set of the closed (resp. open) elements of $\mathbb{A}^\delta$ [4].

6.3.1. Definition. A monotone map $f : \mathbb{A} \to \mathbb{B}$ is *additive* if f preserves non-empty finite joins, and it is *completely additive* if it preserves all (existing) nonempty joins. A monotone map $g : \mathbb{A} \to \mathbb{B}$ is *multiplicative* if g preserves non-empty finite meets, and it is *completely multiplicative* if it preserves all (existing) nonempty meets.

6.3.2. Definition. For all maps $f, g : \mathbb{A}^\delta \to \mathbb{B}^\delta$,

1. f is *closed Esakia* if it preserves down-directed meets of closed elements of $\mathbb{A}^\delta$, that is:
$$f(\bigwedge\{c_i : i \in I\}) = \bigwedge\{f(c_i) : i \in I\}$$
for any downward-directed collection $\{c_i : i \in I\} \subseteq K(A^\delta)$;

2. g is *open Esakia* if it preserves upward-directed joins of open elements of $\mathbb{A}^\delta$, that is:
$$g(\bigvee\{o_i : i \in I\}) = \bigvee\{g(o_i) : i \in I\}$$
for any upward-directed collection $\{o_i : i \in I\} \subseteq O(\mathbb{A}^\delta)$.

Our main aim in this section is proving the following

6.3.3. Theorem. *Let $f, g : \mathbb{A}^\delta \to \mathbb{B}^\delta$ be monotone maps, which are both closed and open Esakia.*

1. *If $f(a) \in K(B^\delta)$ for all $a \in \mathbb{A}$, and $f(a \vee b) \leq f(a) \vee f(b)$ for all $a, b \in \mathbb{A}$, then $f(u \vee v) \leq f(u) \vee f(v)$ for all $u, v \in \mathbb{A}^\delta$.*

2. *If $g(a) \in O(B^\delta)$ for all $a \in \mathbb{A}$, and $g(a \wedge b) \geq g(a) \wedge g(b)$ for all $a, b \in \mathbb{A}$, then $g(u \wedge v) \geq g(u) \wedge g(v)$ for all $u, v \in \mathbb{A}^\delta$.*

Notice that the feature that sets apart the theorem above from similar existing results e.g. in the theory of canonical extensions is that $f, g : \mathbb{A}^\delta \to \mathbb{B}^\delta$ are not assumed to be the σ- or π-extensions of primitive functions $\mathbb{A} \to \mathbb{B}$. As we will see, this calls for different proof techniques from the ones typically used for σ- and π-extensions.

[4]Preliminary versions of Theorem 6.3.3, Lemmas 6.3.5, 6.3.6 and 6.3.9, and Propositions 6.3.7 and 6.3.8 have been developed by Sam van Gool in collaboration with Alessandra Palmigiano in unpublished notes. Proposition 6.3.10 is original to [53].

6.3.4. Definition. For all $f, g : \mathbb{A}^\delta \to \mathbb{B}^\delta$, let $\Diamond_f, \Box_g : \mathbb{A}^\delta \to \mathbb{B}^\delta$ be defined as follows. For any $u \in \mathbb{A}^\delta$,

$$\Diamond_f(u) := \bigvee\{f(j) \mid j \in J^\infty(\mathbb{A}^\delta) \text{ and } j \leq u\};$$

$$\Box_g(u) := \bigwedge\{g(m) \mid m \in M^\infty(\mathbb{A}^\delta) \text{ and } m \geq u\}.$$

The following fact straightforwardly follows from the definition:

6.3.5. Lemma. *For all monotone maps $f, g : \mathbb{A}^\delta \to \mathbb{B}^\delta$,*

1. *$\Diamond_f$ is completely join-preserving, $\Diamond_f(u) \leq f(u)$ for all $u \in \mathbb{A}^\delta$, and $\Diamond_f(j) = f(j)$ for every $j \in J^\infty(\mathbb{A}^\delta)$.*

2. *$\Box_g$ is completely meet-preserving, $\Box_g(u) \geq g(u)$ for all $u \in \mathbb{A}^\delta$, and $\Box_g(m) = g(m)$ for every $m \in M^\infty(\mathbb{A}^\delta)$.*

The following lemma offers the relational perspective as mentioned earlier in Section 4.3.1, while defining the normalization for additive maps.

6.3.6. Lemma. *For all monotone maps $f, g : \mathbb{A}^\delta \to \mathbb{B}^\delta$, and any $u \in \mathbb{A}^\delta$,*

1. *$\Diamond_f(u) = \bigvee\{j \in J^\infty(\mathbb{B}^\delta) \mid \exists i \in J^\infty(\mathbb{A}^\delta) : i \leq u \text{ and } j \leq f(i)\}$;*

2. *$\Box_g(u) = \bigwedge\{m \in M^\infty(\mathbb{B}^\delta) \mid \exists n \in M^\infty(\mathbb{A}^\delta) : u \leq n \text{ and } g(n) \leq m\}$.*

Proof. 1. We first show the inequality from right to left. If $j \in J^\infty(\mathbb{B}^\delta)$ and $j \leq f(i)$ for some $i \in J^\infty(\mathbb{A}^\delta)$ such that $i \leq u$, then $j \leq \Diamond_f(i) \leq \Diamond_f(u)$ by Lemma 6.3.5.1 and the monotonicity of $\Diamond_f$. For the converse direction, it is enough to show that if $j \in J^\infty(\mathbb{B}^\delta)$ and $j \leq \Diamond_f(u)$, then $j \leq f(i)$ for some $i \in J^\infty(\mathbb{A}^\delta)$ such that $i \leq u$; this immediately follows by the definition of $\Diamond_f$ and j being completely join-prime.

2. is an order-variant of 1.

$\Box$

The proof of Theorem 6.3.3 above is an immediate consequence of the following:

6.3.7. Proposition. *Let $f, g : \mathbb{A}^\delta \to \mathbb{B}^\delta$ be monotone maps, which are both closed and open Esakia. Then, for any $u \in \mathbb{A}^\delta$,*

$$f(u) = f(\bot) \vee \Diamond_f(u); \tag{6.13}$$

$$g(u) = g(\top) \wedge \Box_g(u). \tag{6.14}$$

Condition (6.13) implies that the map f is a composition of completely additive maps (cf. Definition 6.3.1), and is therefore completely additive. Similarly, condition (6.14) implies that the map g is a composition of completely multiplicative maps (cf. Definition 6.3.1), and is therefore completely multiplicative.
From Lemma 6.3.5 and the monotonicity of f and g, it immediately follows that for every $u \in \mathbb{A}^\delta$,

$$f(\bot) \vee \Diamond_f(u) \leq f(u) \qquad g(u) \leq g(\top) \wedge \Box_g(u). \tag{6.15}$$

The proof of the converse directions will require two steps. The first one is to show that (6.13) and (6.14) respectively hold for every closed element $k \in K(\mathbb{A}^\delta)$ and every open element $o \in O(\mathbb{A}^\delta)$.

6.3.8. Proposition. *Let $f, g : \mathbb{A}^\delta \to \mathbb{B}^\delta$ be monotone maps, which are both closed and open Esakia.*

1. *$f(k) = f(\bot) \vee \Diamond_f(k)$ for all $k \in K(\mathbb{A}^\delta)$;*
2. *$g(o) = g(\top) \wedge \Box_g(o)$ for all $o \in O(\mathbb{A}^\delta)$.*

Proof.
1. Fix $k \in K(\mathbb{A}^\delta)$. By (6.15), it is enough to show that, if $o \in O(\mathbb{B}^\delta)$ and $f(\bot) \vee \Diamond_f(k) \leq o$, then $f(k) \leq o$. Consider the set

$$J := \{b \in \mathbb{A} \mid f(b) \leq o\}.$$

Note that $J \neq \varnothing$, since $f(\bot) \leq f(\bot) \vee \Diamond_f(k) \leq o$, and that J is a lattice ideal: indeed, J is downward closed by construction, and the join of two elements in J belongs to J, since by assumption $f(a \vee b) \leq f(a) \vee f(b)$.

We claim that $k \leq \bigvee J$. To see this, let $j \in J^\infty(\mathbb{A}^\delta)$ s.t. $j \leq k$ and let us show that there is some $b \in J$ s.t. $j \leq b$. From $j \leq k$, by Lemma 6.3.6.1, we have $f(j) \leq \Diamond_f(k) \leq o$. Since f is closed Esakia,

$$\bigwedge\{f(b) \mid b \in \mathbb{A} \text{ and } j \leq b\} = f(j) \leq o.$$

Since, by assumption, $f(b) \in K(B^\delta)$ for all $b \in \mathbb{A}$, the displayed inequality implies by compactness that $\bigwedge_{i=1}^n f(b_i) \leq o$ for some $b_1, \ldots, b_n \in \mathbb{A}$ such that $j \leq b_i$ holds for every $1 \leq i \leq n$. Since f is order-preserving, putting $b := \bigwedge_{i=1}^n b_i$ we get $j \leq b$ and $f(b) = f(\bigwedge_{i=1}^n b_i) \leq \bigwedge_{i=1}^n f(b_i) \leq o$. So we have found some $b \in J$ with $j \leq b$, which finishes the proof of the claim.

From $k \leq \bigvee J$, by compactness, there is some element $a \in J$ such that $k \leq a$. Then $f(k) \leq f(a) \leq o$, as required.

2. is an order-variant of 1.

$\Box$

The second step will be to show that the inequality proved in the proposition above can be lifted to arbitrary elements of $\mathbb{A}^\delta$. For this, we remark that, by Lemma 6.3.5, the maps $\Diamond_f, \Box_g : \mathbb{A}^\delta \to \mathbb{B}^\delta$ have adjoints, which we respectively denote by $\blacksquare_f, \blacklozenge_g : \mathbb{B}^\delta \to \mathbb{A}^\delta$. We will need the following lemma.

6.3.9. LEMMA. *For all maps $f, g : \mathbb{A}^\delta \to \mathbb{B}^\delta$ as in Theorem 6.3.3, for any $o \in O(\mathbb{B}^\delta)$ and $k \in K(\mathbb{B}^\delta)$,*

1. *if $f(\bot) \leq o$, then $\blacksquare_f(o) = \bigvee\{a \in \mathbb{A} \mid a \leq \blacksquare_f(o)\} \in O(\mathbb{A}^\delta)$.*
2. *if $k \leq g(\top)$, then $\blacklozenge_g(k) = \bigwedge\{a \in \mathbb{A} \mid \blacklozenge_g(k) \leq a\} \in K(\mathbb{A}^\delta)$.*

Proof. 1. To prove the statement, it is enough to show that if $c \in K(\mathbb{A}^\delta)$ and $c \leq \blacksquare_f(o)$, then $c \leq a$ for some $a \in A$ such that $a \leq \blacksquare_f(o)$. By adjunction, $c \leq \blacksquare_f(o)$ is equivalent to $\Diamond_f(c) \leq o$. Then, by assumption, $f(\bot) \vee \Diamond_f(c) \leq o$. Proposition 6.3.8 implies that $f(c) \leq f(\bot) \vee \Diamond_f(c) \leq o$. Since f is closed Esakia, $f(c) = \bigwedge\{f(a) \mid a \in \mathbb{A} \text{ and } c \leq a\}$. Moreover, by assumption, $f(a) \in K(B^\delta)$ for every $a \in \mathbb{A}$. Hence by compactness, $\bigwedge_{i=1}^n f(a_i) \leq o$ for some $a_1, \ldots, a_n \in \mathbb{A}$ s.t. $c \leq a_i$, $1 \leq i \leq n$. Let $a = \bigwedge_{i=1}^n a_i$. Clearly, $c \leq a$ and $a \in \mathbb{A}$; moreover, by the monotonicity of f and Lemma 6.3.5.1, we have $\Diamond_f(a) \leq f(a) \leq \bigwedge_{i=1}^n f(a_i) \leq o$, and hence, by adjunction, $a \leq \blacksquare_f(o)$.

2. is an order-variant of 1.

□

We are now ready to prove Proposition 6.3.7:

Proof.[Proof of identity (6.13).] By (6.15), it is enough to show that, if $o \in O(\mathbb{B}^\delta)$ and $f(\bot) \vee \Diamond_f(u) \leq o$, then $f(u) \leq o$. The assumption implies that $\Diamond_f(u) \leq o$, so $u \leq \blacksquare_f(o)$ by adjunction. Hence $f(u) \leq f(\blacksquare_f(o))$, since f is order-preserving. Since $f(\bot) \leq o$, the following chain holds:

$$
\begin{array}{lll}
\multicolumn{2}{l}{f(u) \leq f(\blacksquare_f(o))} & \\
= & f\left(\bigvee\{a \mid a \in \mathbb{A} \text{ and } a \leq \blacksquare_f(o)\}\right) & \text{(Lemma 6.3.9)} \\
= & \bigvee\{f(a) \mid a \in \mathbb{A} \text{ and } a \leq \blacksquare_f(o)\} & (f \text{ is open Esakia}) \\
= & \bigvee\{f(a) \mid a \in \mathbb{A} \text{ and } \Diamond_f(a) \leq o\} & \text{(adjunction)} \\
= & \bigvee\{f(\bot) \vee \Diamond_f(a) \mid a \in \mathbb{A} \text{ and } \Diamond_f(a) \leq o\} & \text{(Proposition 6.3.8)} \\
\leq & o. & (f(\bot) \leq o)
\end{array}
$$

□

Before moving on, we state and prove the following Esakia-type result. We can call it a *conditional Esakia lemma*, since, unlike other existing versions, it crucially relies on additional assumptions (on $f(\bot)$ and $g(\top)$).

6.3.10. PROPOSITION. *For any $f, g : \mathbb{A}^\delta \to \mathbb{B}^\delta$ as in Theorem 6.3.3, for any upward-directed collection $\mathcal{O} \subseteq O(\mathbb{B}^\delta)$ and any downward-directed collection $\mathcal{C} \subseteq K(\mathbb{B}^\delta)$,*

1. *if* $f(\bot) \leq \bigvee \mathcal{O}$, *then* $\blacksquare_f(\bigvee \mathcal{O}) = \bigvee\{\blacksquare_f o \mid o \in \mathcal{O}\}$. *Moreover, there exists some upward-directed subcollection* $\mathcal{O}' \subseteq \mathcal{O}$ *such that* $\bigvee \mathcal{O}' = \bigvee \mathcal{O}$, *and* $\blacksquare_f o \in O(\mathbb{A}^\delta)$ *for each* $o \in \mathcal{O}'$, *and* $\bigvee\{\blacksquare_f o \mid o \in \mathcal{O}'\} = \bigvee\{\blacksquare_f o \mid o \in \mathcal{O}\}$.

2. *if* $g(\top) \geq \bigwedge \mathcal{C}$, *then* $\blacklozenge_g(\bigwedge \mathcal{C}) = \bigwedge\{\blacklozenge_g c \mid c \in \mathcal{C}\}$. *Moreover, there exists some downward-directed subcollection* $\mathcal{C}' \subseteq \mathcal{C}$ *such that* $\bigwedge \mathcal{C}' = \bigwedge \mathcal{C}$, *and* $\blacklozenge_g c \in K(\mathbb{A}^\delta)$ *for each* $c \in \mathcal{C}'$, *and* $\bigwedge\{\blacklozenge_g c \mid c \in \mathcal{C}'\} = \bigwedge\{\blacklozenge_g c \mid c \in \mathcal{C}\}$.

Proof. 1. It is enough to show that, if $k \in K(\mathbb{A}^\delta)$ and $k \leq \blacksquare_f(\bigvee \mathcal{O})$, then $k \leq \blacksquare_f(o_k)$ for some $o_k \in \mathcal{O}$. The assumption $k \leq \blacksquare_f(\bigvee \mathcal{O})$ can be rewritten as $\Diamond_f k \leq \bigvee \mathcal{O}$, which together with $f(\bot) \leq \bigvee \mathcal{O}$ yields $f(\bot) \vee \Diamond_f k \leq \bigvee \mathcal{O}$. By Proposition 6.3.8, this inequality can be rewritten as $f(k) \leq \bigvee \mathcal{O}$. Since f is closed Esakia and $f(a) \in K(\mathbb{A}^\delta)$ for every $a \in \mathbb{A}$, the element $f(k) \in K(\mathbb{A}^\delta)$. By compactness, $f(k) \leq \bigvee_{i=1}^n o_i$ for some $o_1, \ldots, o_n \in \mathcal{O}$. Since $\mathcal{O}$ is upward-directed, $o_k \geq \bigvee_{i=1}^n o_i$ for some $o_k \in \mathcal{O}$. Then $f(\bot) \vee \Diamond_f k = f(k) \leq o_k$, which yields $\Diamond_f k \leq o_k$, which by adjunction can be rewritten as $k \leq \blacksquare_f(o_k)$ as required.

As to the second part of the statement, notice that the assumption $f(\bot) \leq \bigvee \mathcal{O}$ is too weak to imply that $f(\bot) \leq o$ for each $o \in \mathcal{O}$, and hence we cannot conclude, by way of Lemma 6.3.9, that $\blacksquare_f o \in O(\mathbb{A}^\delta)$ for every $o \in \mathcal{O}$. However, let

$$\mathcal{O}' := \{o \in \mathcal{O} \mid o \geq o_k \text{ for some } k \in K(\mathbb{A}^\delta) \text{ s.t. } k \leq \blacksquare_f(\bigvee \mathcal{O})\}.$$

Clearly, by construction $\mathcal{O}'$ is upward-directed, it holds that $\bigvee \mathcal{O}' = \bigvee \mathcal{O}$, and for each $o \in \mathcal{O}'$, we have that $f(\bot) \leq o_k \leq o$ for some $k \in K(\mathbb{A}^\delta)$ s.t. $k \leq \blacksquare_f(\bigvee \mathcal{O})$, hence, by Lemma 6.3.9, $\blacksquare_f o \in O(\mathbb{A}^\delta)$ for each $o \in \mathcal{O}'$. Moreover, the monotonicity of $\blacksquare_f$ and the previous part of the statement imply that $\bigvee\{\blacksquare_f o \mid o \in \mathcal{O}'\} = \blacksquare_f(\bigvee \mathcal{O}) = \bigvee\{\blacksquare_f o \mid o \in \mathcal{O}\}$.

2. is order-dual.

$\square$

6.3.2 Canonicity of the additivity of DLE-term functions

In this subsection, the canonicity of the additivity for ε-positive term functions in any given DLE-language is obtained as a consequence of Theorem 6.3.3. We recall that for any $n \in \mathbb{N}$, an *order-type on* n is an element $\varepsilon \in \{1, \partial\}^n$.

6.3.11. Lemma. *Let* $\pi(p)$ *be a positive unary term. The map* $\pi^{\mathbb{A}^\delta} : (A^\varepsilon)^\delta \to \mathbb{A}^\delta$ *preserves meets of down-directed collections* $\mathcal{C} \subset K((A^\varepsilon)^\delta)$ *and joins of up-directed collections* $\mathcal{O} \subset O((A^\varepsilon)^\delta)$.

Proof. The proof is analogous to the proof of [159, Lemmas 4.12, 6.10] and is done by induction on π, using the preservation properties of the single connectives, as in e.g. in [50, Esakia Lemma 11.5]. $\square$

6.3.12. THEOREM. *Let ε be an order-type on $n \in \mathbb{N}$, let $\overline{p}, \overline{q}$ be n-tuples of proposition letters, and let $\pi(\overline{p})$ be an n-ary term function which is positive as a map $A^\varepsilon \to A$.*

$$\textit{If } \mathbb{A} \models \pi(\overline{p} \vee \overline{q}) \leq \pi(\overline{p}) \vee \pi(\overline{q}), \quad \textit{then} \quad \mathbb{A}^\delta \models \pi(\overline{p} \vee \overline{q}) \leq \pi(\overline{p}) \vee \pi(\overline{q}).$$

Proof. By Theorem 6.3.3, it is enough to show that π is both closed and open Esakia, and that $\pi(\overline{a}) \in K(\mathbb{A}^\delta)$ for any $\overline{a} \in \mathbb{A}^\varepsilon$. Clearly, the second requirement follows from $\pi : A^\varepsilon \to A \subseteq K(\mathbb{A}^\delta)$. The first requirement is the content of the Lemma 6.3.11. □

6.3.13. EXAMPLE. Let $\overline{p} = (p_1, p_2)$ and $\overline{q} = (q_1, q_2)$ be tuples of propositional letters, and $\pi(\overline{p}) = \Box p_1 \circ \Box p_2$. Since $\pi(\overline{p}) : A^2 \to A$ is a positive map, by Theorem 6.3.12, the inequality

$$\Box(p_1 \vee q_1) \circ \Box(p_2 \vee q_2) \leq (\Box p_1 \circ \Box p_2) \vee (\Box q_1 \circ \Box q_2)$$

is canonical.

The term function $\pi(p) = \rhd\rhd\rhd p$ is positive as a map $A^\partial \to A$. Hence, by Theorem 6.3.12, the following inequality is canonical:

$$\rhd\rhd\rhd(p \wedge q) \leq \rhd\rhd\rhd p \vee \rhd\rhd\rhd q.$$

6.4 Towards extended canonicity results: enhancing ALBA

In this section, we define an enhanced version of ALBA, which we refer to as ALBA^e, which manipulates quasi-inequalities in an expanded language DLE^{++}, and which will be used in Section 6.7 to prove our main result. The results obtained in the previous two sections will be applied to show that the additional rules are sound in the presence of certain additional conditions.

Throughout the present section, we fix unary DLE-terms $\pi(p), \sigma(p), \lambda(p)$ and $\rho(p)$, and we assume that π and σ are positive in p, and that λ and ρ are negative in p.

We recall that the language DLE^{++} is the expansion of DLE$^+$ with the connectives $\Diamond_\pi$, $\Box_\sigma$, $\lhd_\lambda$ and $\rhd_\rho$, and their respective adjoints $\blacksquare_\pi$, $\blacklozenge_\sigma$, $\blacktriangleleft_\lambda$ and $\blacktriangleright_\rho$. The quasi-inequalities in DLE^{++} manipulated by the rules of ALBA^e will have the usual form $\forall\overline{p}\forall\overline{\mathbf{i}}\forall\overline{\mathbf{m}}(\& S \Rightarrow \mathbf{i} \leq \mathbf{m})$, with S being a finite set of DLE^{++}-inequalities, which we will often refer to as a system, and $\overline{p}, \overline{\mathbf{i}}$ and $\overline{\mathbf{m}}$ being the arrays of propositional variables, nominals and conominals occurring in $S \cup \{\mathbf{i} \leq \mathbf{m}\}$. In practice, we will simplify our setting and focus mainly on the system S.

The interpretation of the new connectives is motivated by the specialization of the facts in Section 6.2 and 6.3 to the term functions associated with $\pi(p), \sigma(p), \lambda(p)$ and $\rho(p)$.

6.4.1. Definition. For any term function $\pi^{\mathbb{A}^\delta}, \sigma^{\mathbb{A}^\delta}, \lambda^{\mathbb{A}^\delta}, \rho^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{A}^\delta$ as above, let $\Diamond_\pi, \Box_\sigma, \lhd_\lambda, \rhd_\rho : \mathbb{A}^\delta \to \mathbb{A}^\delta$ be defined as follows. For any $u \in \mathbb{A}^\delta$,

1. $\Diamond_\pi(u) := \bigvee\{\pi^{\mathbb{A}^\delta}(j) \mid j \in J^\infty(\mathbb{A}^\delta) \text{ and } j \leq u\}$.
2. $\Box_\sigma(u) := \bigwedge\{\sigma^{\mathbb{A}^\delta}(m) \mid m \in M^\infty(\mathbb{A}^\delta) \text{ and } m \geq u\}$.
3. $\lhd_\lambda(u) := \bigvee\{\lambda^{\mathbb{A}^\delta}(m) \mid m \in M^\infty(\mathbb{A}^\delta) \text{ and } m \geq u\}$.
4. $\rhd_\rho(u) := \bigwedge\{ \rho^{\mathbb{A}^\delta}(j) \mid j \in J^\infty(\mathbb{A}^\delta) \text{ and } j \leq u\}$.

Each of the functions above has an adjoint (cf. Lemma 6.3.5). Let $\blacksquare_\pi$, $\blacklozenge_\sigma$, $\blacktriangleleft_\lambda$, $\blacktriangleright_\rho$: $\mathbb{A}^\delta \to \mathbb{A}^\delta$ respectively denote the adjoints of the maps $\Diamond_\pi, \Box_\sigma, \lhd_\lambda, \rhd_\rho$. These maps provide a natural interpretation for the new connectives associated with the terms $\pi(p), \sigma(p), \lambda(p)$ and $\rho(p)$.

The algorithm ALBA^e works in an entirely analogous way as ALBA, through the stages of preprocessing, first approximation, reduction/elimination cycle, success, failure and output. Below, we will limit ourselves to mention rules that are additional w.r.t. ALBA on DLE.

Distribution rules. During the preprocessing stage, along with the DLE-distribution rules, the following rules are applicable:

$$\frac{\pi(\varphi \vee \psi) \leq \chi}{\pi(\varphi) \vee \pi(\psi) \leq \chi} \qquad \frac{\lambda(\varphi \wedge \psi) \leq \chi}{\lambda(\varphi) \vee \lambda(\psi) \leq \chi} \qquad \frac{\chi \leq \sigma(\varphi \wedge \psi)}{\chi \leq \sigma(\varphi) \wedge \sigma(\psi)} \qquad \frac{\chi \leq \rho(\varphi \vee \psi)}{\chi \leq \rho(\varphi) \wedge \rho(\psi)}$$

Each of these rules replaces an instance of the upper inequality with the corresponding instance of the lower inequality.

Adjunction rules. During the reduction-elimination stage, the following rules are also applicable:

$$\frac{\pi(\varphi) \leq \psi}{\pi(\bot) \leq \psi \quad \varphi \leq \blacksquare_\pi \psi} \qquad \frac{\varphi \leq \sigma(\psi)}{\varphi \leq \sigma(\top) \quad \blacklozenge_\sigma \varphi \leq \psi}$$

$$\frac{\lambda(\varphi) \leq \psi}{\lambda(\top) \leq \psi \quad \blacktriangleleft_\lambda \psi \leq \varphi} \qquad \frac{\varphi \leq \rho(\psi)}{\varphi \leq \rho(\bot) \quad \psi \leq \blacktriangleright_\rho \varphi}$$

In a given system, each of these rules replaces an instance of the upper inequality with the corresponding instances of the two lower inequalities.

The leftmost inequalities in each rule above will be referred to as the *side condition*.

Approximation rules. During the reduction-elimination stage, the following rules are also applicable:

$$\frac{\mathbf{i} \leq \pi(\psi)}{[\mathbf{i} \leq \pi(\bot)] \;\parr\; [\mathbf{j} \leq \psi \quad \mathbf{i} \leq \Diamond_\pi(\mathbf{j})]} \qquad \frac{\sigma(\varphi) \leq \mathbf{m}}{[\sigma(\top) \leq \mathbf{m}] \;\parr\; [\varphi \leq \mathbf{n} \quad \Box_\sigma(\mathbf{n}) \leq \mathbf{m}]}$$

$$\frac{\mathbf{i} \leq \lambda(\psi)}{[\mathbf{i} \leq \lambda(\top)] \;\parr\; [\psi \leq \mathbf{m} \quad \mathbf{i} \leq \lhd_\lambda(\mathbf{m})]} \qquad \frac{\rho(\varphi) \leq \mathbf{m}}{[\rho(\bot) \leq \mathbf{m}] \;\parr\; [\mathbf{i} \leq \varphi \quad \rhd_\rho(\mathbf{i}) \leq \mathbf{m}]}$$

The leftmost inequalities in each rule above will be referred to as the *side condition*.

Each approximation rule transforms a given system $S \cup \{s \leq t\}$ into the two systems (which respectively correspond to a quasi-inequality) $S \cup \{s_1 \leq t_1\}$ and $S \cup \{s_2 \leq t_2, s_3 \leq t_3\}$, the first of which containing only the side condition (in which no propositional variable occurs), and the second one containing the instances of the two remaining lower inequalities.

6.4.2. Proposition. *The rules given above are sound on any perfect DLE $\mathbb{A}^\delta$ such that*

$$\mathbb{A} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q) \qquad \mathbb{A} \models \sigma(p) \wedge \sigma(q) \leq \sigma(p \wedge q)$$

$$\mathbb{A} \models \lambda(p \wedge q) \leq \lambda(p) \vee \lambda(q) \qquad \mathbb{A} \models \rho(p) \wedge \rho(q) \leq \rho(p \vee q)$$

Proof. The soundness of the distribution rules immediately follows from Lemma 6.4.4. Each of the remaining rules can be derived from standard ALBA rules, plus the following set of rules:

$$\frac{\mathbf{i} \leq \Diamond_\pi(\psi)}{\mathbf{j} \leq \psi \quad \mathbf{i} \leq \Diamond_\pi(\mathbf{j})} \qquad \frac{\Box_\sigma(\varphi) \leq \mathbf{m}}{\varphi \leq \mathbf{n} \quad \Box_\sigma(\mathbf{n}) \leq \mathbf{m}}$$

$$\frac{\mathbf{i} \leq \lhd_\lambda(\psi)}{\psi \leq \mathbf{m} \quad \mathbf{i} \leq \lhd_\lambda(\mathbf{m})} \qquad \frac{\rhd_\rho(\varphi) \leq \mathbf{m}}{\mathbf{i} \leq \varphi \quad \rhd_\rho(\mathbf{i}) \leq \mathbf{m}}$$

$$\frac{\Diamond_\pi \varphi \leq \psi}{\varphi \leq \blacksquare_\pi \psi} \qquad \frac{\varphi \leq \Box_\sigma \psi}{\blacklozenge_\sigma \varphi \leq \psi} \qquad \frac{\lhd_\lambda \varphi \leq \psi}{\blacktriangleleft_\lambda \psi \leq \varphi} \qquad \frac{\varphi \leq \rhd_\rho \psi}{\psi \leq \blacktriangleright_\rho \varphi}$$

For the sake of conciseness, we give the following rules as formula-rewriting rules:

$$\frac{\pi(p)}{\pi(\bot) \vee \Diamond_\pi(p)} \qquad \frac{\sigma(p)}{\sigma(\top) \wedge \Box_\sigma(p)} \qquad \frac{\lambda(p)}{\lambda(\top) \vee \lhd_\lambda(p)} \qquad \frac{\rho(p)}{\rho(\bot) \wedge \rhd_\rho(p)}$$

Indeed, let us give two derivations as examples:

$$\frac{\dfrac{\dfrac{\lambda(\varphi) \leq \psi}{\lambda(\top) \vee \lhd_\lambda(\varphi) \leq \psi}}{\lambda(\top) \leq \psi \quad \lhd_\lambda(\varphi) \leq \psi}}{\lambda(\top) \leq \psi \quad \blacktriangleleft_\lambda \psi \leq \varphi} \qquad \frac{\dfrac{\dfrac{\mathbf{i} \leq \pi(\psi)}{\mathbf{i} \leq \pi(\bot) \vee \Diamond_\pi(\psi)}}{[\mathbf{i} \leq \pi(\bot)] \ \invamp \ [\mathbf{i} \leq \Diamond_\pi(\psi)]}}{[\mathbf{i} \leq \pi(\bot)] \ \invamp \ [\mathbf{j} \leq \psi \quad \mathbf{i} \leq \Diamond_\pi(\mathbf{j})]}$$

So to finish the proof, it is enough to show that the rules given above are sound. The soundness of the first batch of rules follows from the fact that nominals and conominals are respectively interpreted as completely join prime and meet-prime elements of $\mathbb{A}^\delta$, together with Lemma 6.3.5 applied to the term functions $\pi^{\mathbb{A}^\delta}, \sigma^{\mathbb{A}^\delta}, \lambda^{\mathbb{A}^\delta}$ and $\rho^{\mathbb{A}^\delta}$.

The soundness of the second batch of rules follows again from Lemma 6.3.5 applied to the term functions $\pi^{\mathbb{A}^\delta}, \sigma^{\mathbb{A}^\delta}, \lambda^{\mathbb{A}^\delta}$ and $\rho^{\mathbb{A}^\delta}$, since it predicates the existence of the adjoints of the maps $\Diamond_\pi, \Box_\sigma, \lhd_\lambda$ and $\rhd_\rho$.

Finally, the soundness of the third batch of rules directly follows from Lemma 6.4.5. $\Box$

6.4.3. Remark. The proposition above is crucially set on the canonical extension of a given DLE. This implies that the soundness of the approximation rules introducing the additional connectives $\Diamond_\pi, \Box_\sigma, \lhd_\lambda, \rhd_\rho$ has been proved only relative to perfect DLEs which are canonical extensions of some given DLE, and not relative to any perfect DLEs, as is the case of the other rules. The further consequences of this limitation will be discussed in the conclusions.

The following lemma is an immediate application of Theorem 6.3.12:

6.4.4. Lemma. *Let $\pi(p), \sigma(p)$ be positive unary terms and $\lambda(p)$ and $\rho(p)$ be negative unary terms in a given DLE-language.*

1. *if $\mathbb{A} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q)$, then $\mathbb{A}^\delta \models \pi(p \vee q) \leq \pi(p) \vee \pi(q)$;*
2. *if $\mathbb{A} \models \sigma(p \wedge q) \geq \sigma(p) \wedge \sigma(q)$, then $\mathbb{A}^\delta \models \sigma(p \wedge q) \geq \sigma(p) \wedge \sigma(q)$;*
3. *if $\mathbb{A} \models \lambda(p \wedge q) \leq \lambda(p) \vee \lambda(q)$, then $\mathbb{A}^\delta \models \lambda(p \wedge q) \leq \lambda(p) \vee \lambda(q)$;*
4. *if $\mathbb{A} \models \rho(p \vee q) \geq \rho(p) \wedge \rho(q)$, then $\mathbb{A}^\delta \models \rho(p \vee q) \geq \rho(p) \wedge \rho(q)$.*

The following lemma is an immediate application of Lemma 6.3.5 to term functions.

6.4.5. Lemma. *For any term function $\pi^{\mathbb{A}^\delta}, \sigma^{\mathbb{A}^\delta}, \lambda^{\mathbb{A}^\delta}, \rho^{\mathbb{A}^\delta} : \mathbb{A}^\delta \to \mathbb{B}^\delta$ as above,*

1. *if $\mathbb{A} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q)$, then for all $u \in \mathbb{A}^\delta$, $\pi^{\mathbb{A}^\delta}(u) = \pi^A(\bot) \vee \Diamond_\pi(u)$.*
2. *If $\mathbb{A} \models \sigma(p \wedge q) \geq \sigma(p) \wedge \sigma(q)$, then for all $u \in \mathbb{A}^\delta$, $\sigma^{\mathbb{A}^\delta}(u) = \sigma^A(\top) \wedge \Box_\sigma(u)$.*
3. *If $\mathbb{A} \models \lambda(p \wedge q) \leq \lambda(p) \vee \lambda(q)$, then for all $u \in \mathbb{A}^\delta$, $\lambda^{\mathbb{A}^\delta}(u) = \lambda^A(\top) \vee \lhd_\lambda(u)$.*
4. *If $\mathbb{A} \models \rho(p \vee q) \geq \rho(p) \wedge \rho(q)$, then for all $u \in \mathbb{A}^\delta$, $\rho^{\mathbb{A}^\delta}(u) = \rho^A(\bot) \wedge \rhd_\rho(u)$.*

6.5 Meta-inductive inequalities and success of ALBA^e

Recall that DLE* is an expansion of DLE, obtained by closing the set of formulas under the following set of additional connectives $\{\Diamond\!\!\!\cdot\,, \boxdot, \lhd, \rhd\}$. Recall that a DLE*-inequality is an inequality $s \leq t$ such that both s, t are formulas in the language DLE*.

6.5.1. Definition. Let $\Phi : \{\Diamond\!\!\!\cdot\,, \boxdot, \lhd, \rhd\} \to \{\pi, \sigma, \lambda, \rho\}$ such that $\Phi(\Diamond\!\!\!\cdot\,) = \pi$, $\Phi(\boxdot) = \sigma$, $\Phi(\lhd) = \lambda$ and $\Phi(\rhd) = \rho$. A DLE-inequality $\varphi \leq \psi$ is *meta-inductive* with respect to Φ if there exists some inductive DLE*-inequality $s \leq t$, such that $\varphi \leq \psi$ can be obtained from $s \leq t$ by replacing each $\odot \in \{\Diamond\!\!\!\cdot\,, \boxdot, \lhd, \rhd\}$ with $\Phi(\odot)$.

6.5.2. Example. The class of meta-inductive inequalities extends the inductive inequalities. Let $\pi(p) = \Diamond\Box\Diamond(p)$ and $\sigma(p) = \Box(p)$. The McKinsey-type inequality $\Diamond\Box\Diamond\Box p \leq \Box\Diamond\Box\Diamond p$ is meta-inductive with respect to Φ, as it is obtained as a Φ-substitution instance of the Sahlqvist DLE*-inequality $\Diamond\!\!\!\cdot\,\boxdot p \leq \boxdot\Diamond\!\!\!\cdot\, p$, where $\Phi(\Diamond\!\!\!\cdot\,) = \pi$ and, $\Phi(\boxdot) = \sigma$.

6.5.3. Definition. An execution of ALBA^e is *safe* if no side conditions (cf. Page 142) introduced by applications of adjunction rules for the new connectives are further modified, except for receiving Ackermann substitutions.

6.5.4. Theorem. *If $\varphi \leq \psi$ is a meta-inductive inequality, then it admits a safe and successful execution of ALBA^e.*

Proof. Since $\varphi \leq \psi$ is a meta-inductive inequality, there exists some (Ω, ε)-inductive DLE*-inequality $s \leq t$ s.t. $\varphi \leq \psi$ can be obtained from $s \leq t$ by replacing each $\odot \in \{\Diamond\!\!\!\cdot\,, \boxdot, \lhd, \rhd\}$ with $\Phi(\odot)$. Then the version of ALBA on the language DLE* can be successfully executefd on $s \leq t$, following the appropriate Ω solving according to ε. For each rule applied in this execution, the corresponding rule can be applied by ALBA^e on the reduction of $\varphi \leq \psi$. In particular, for each rule applied to $\odot \in \{\Diamond\!\!\!\cdot\,, \boxdot, \lhd, \rhd\}$, the corresponding rule will be applied by ALBA^e to $\chi \in \{\pi, \sigma, \lambda, \rho\}$. It is immediate to see that, since the execution of ALBA^e on $\varphi \leq \psi$ simulates the execution of ALBA on $s \leq t$, and since the latter execution does not "see" the side conditions introduced by ALBA^e, the execution of ALBA^e so defined is safe. Let us show that if the system generated by ALBA from $s \leq t$ is in Ackermann shape, then so is the corresponding ALBA^e system. Firstly, we observe that if the first system is in Ackermann shape for a given p, then all the strictly Ω-smaller variables have already been solved for. Moreover, all the occurrences of p which agree with ε are in display. Consequently, on the ALBA^e side, all the corresponding occurrences are in display. Moreover, there cannot be more critical occurrences of p in the system generated by $\varphi \leq \psi$. Indeed, such occurrences could only pertain to side conditions. However, we can show by induction on Ω that no critical variable occurrences can belong to side conditions. Indeed, if p is Ω-minimal, then when displaying for critical occurrences of p, the minimal valuation cannot contain any variable.

If an adjunction rule such as

$$\frac{\pi(\varphi) \leq \psi}{\pi(\bot) \leq \psi \quad \varphi \leq \blacksquare_\pi \psi}$$

has been applied in the process of displaying such a critical occurrence, then p occurs in φ, $\blacksquare_\pi \psi$ is a subformula of the minimal valuation, and hence is pure. The other adjunction rules can be treated similarly. And so the side condition $\pi(\bot) \leq \psi$ cannot contain any proposition variables. The induction case is similar.

If an approximation rule such as

$$\frac{\mathbf{i} \leq \pi(\psi)}{[\mathbf{i} \leq \pi(\bot)] \quad ⅋ \quad [\mathbf{j} \leq \psi \quad \mathbf{i} \leq \Diamond_\pi(\mathbf{j})]}$$

has been applied in the process of displaying such a critical occurrence, then p occurs in ψ, and the generated side condition is pure altogether. The other approximation rules can be treated similarly. The induction case is similar.

□

6.6 Relativized canonicity via ALBA^e

In this section, we use ALBA^e to obtain the relativized canonicity of meta-inductive DLE-inequalities.

6.6.1. Definition. Let K be a class of DLEs which is closed under taking canonical extensions, and let $\varphi \leq \psi$ be a DLE-inequality. We say that $\varphi \leq \psi$ is *canonical relative to* K if the intersection of K and the class of DLEs defined by $\varphi \leq \psi$ is closed under taking canonical extensions.

Specifically, we aim at proving the following theorem:

6.6.2. Theorem. *Let $\pi(p), \sigma(p)$ be positive unary terms and $\lambda(p)$ and $\rho(p)$ be negative unary terms in a given DLE-language. Let $\mathbb{A}$ be a DLE such that*

$$\mathbb{A} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q) \quad \mathbb{A} \models \sigma(p) \wedge \sigma(q) \leq \sigma(p \wedge q)$$

$$\mathbb{A} \models \rho(p \vee q) \leq \rho(p) \wedge \rho(q) \quad \mathbb{A} \models \lambda(p) \vee \lambda(q) \leq \lambda(p \wedge q).$$

Let $\varphi \leq \psi$ be a meta-inductive DLE-inequality with respect to Φ, where $\Phi : \{\diamondsuit, \boxdot, \lhd, \rhd\} \to \{\pi, \sigma, \lambda, \rho\}$. Then

$$\mathbb{A} \models \varphi \leq \psi \;\Rightarrow\; \mathbb{A}^\delta \models \varphi \leq \psi.$$

Proof. The strategy follows the usual U-shaped argument illustrated below:

$$\begin{array}{lcl} \mathbb{A} \models \varphi \leq \psi & & \mathbb{A}^\delta \models \varphi \leq \psi \\ \Updownarrow & & \\ \mathbb{A}^\delta \models_A \varphi \leq \psi & & \Updownarrow \\ \Updownarrow & & \\ \mathbb{A}^\delta \models_A \mathsf{ALBA}^e(\varphi \leq \psi), & \Leftrightarrow & \mathbb{A}^\delta \models \mathsf{ALBA}^e(\varphi \leq \psi), \end{array}$$

Since $\varphi \leq \psi$ is meta-inductive, by Theorem 6.5.4, we can assume without loss of generality that there exists a safe and successful execution of ALBA^e. For the proof, we need to argue that under the assumption of the theorem, all the rules of ALBA^e are sound on $\mathbb{A}^\delta$, both under arbitrary and under admissible assignments. The soundness of the approximation and adjunction rules for the new connectives has been discussed in Section 6.4, and the argument is entirely similar for arbitrary and for admissible valuations. The soundness of the Ackermann rule under admissible assignments follows from Propositions B.2.5, together with Proposition B.2.3 and Lemma B.2.4. □

6.6.3. Example. By the theorem above, the inequality $\Diamond\Box\Diamond\Box p \leq \Box\Diamond\Box\Diamond p$, which is meta-inductive w.r.t. $\pi(p) = \Diamond\Box\Diamond(p)$ (cf. Example 6.5.2), is canonical relative to the class of DLEs defined by the inequality $\Diamond\Box\Diamond(p \vee q) \leq \Diamond\Box\Diamond p \vee \Diamond\Box\Diamond q$.

6.7 Examples

In this section, we illustrate the execution of ALBA^e on a few examples.

6.7.1. Example. Consider the inequality $\pi(p \vee q) \leq \pi(p) \vee \pi(q)$. The first approximation rule now yields:

$$\{ \ \mathbf{j}_0 \leq \pi(p \vee q), \quad \pi(p) \vee \pi(q) \leq \mathbf{m}_0 \ \} \, ;$$

by applying the splitting rule, this system is rewritten into:

$$\{ \ \mathbf{j}_0 \leq \pi(p \vee q), \quad \pi(p) \leq \mathbf{m}_0, \quad \pi(q) \leq \mathbf{m}_0 \ \} \, ,$$

by applying the adjunction rule for π, this system is rewritten into:

$$\{ \ \mathbf{j}_0 \leq \pi(p \vee q), \quad p \leq \blacksquare_\pi(\mathbf{m}_0) \quad \pi(\bot) \leq \mathbf{m}_0, \quad q \leq \blacksquare_\pi(\mathbf{m}_0) \ \} \, ,$$

to which the left Ackermann rule can be applied to eliminate p:

$$\{ \ \mathbf{j}_0 \leq \pi(\blacksquare_\pi(\mathbf{m}_0) \vee q), \quad \pi(\bot) \leq \mathbf{m}_0, \quad q \leq \blacksquare_\pi(\mathbf{m}_0) \ \} \, ,$$

by applying left Ackermann rule, we can further eliminate q:

$$\{\ \mathbf{j}_0 \leq \pi(\blacksquare_\pi(\mathbf{m}_0) \vee \blacksquare_\pi(\mathbf{m}_0)),\quad \pi(\bot) \leq \mathbf{m}_0\ \}.$$

By applying the formula-rewriting rule, the system above can be equivalently reformulated as the following quasi-inequality:

$$\forall \mathbf{j}_0 \forall \mathbf{m}_0 (\mathbf{j}_0 \leq \pi(\bot) \vee \Diamond_\pi(\blacksquare_\pi(\mathbf{m}_0))\ \ \&\ \ \pi(\bot) \leq \mathbf{m}_0\ )\ \Rightarrow\ \mathbf{j}_0 \leq \mathbf{m}_0)$$

which in its turn can be rewritten as follows:

$$\forall \mathbf{m}_0 (\pi(\bot) \leq \mathbf{m}_0\ \Rightarrow \forall \mathbf{j}_0 (\mathbf{j}_0 \leq \pi(\bot) \vee \Diamond_\pi(\blacksquare_\pi(\mathbf{m}_0))\ \Rightarrow\ \mathbf{j}_0 \leq \mathbf{m}_0))$$

which can be rewritten as follows:

$$\forall \mathbf{m}_0 (\pi(\bot) \leq \mathbf{m}_0\ \Rightarrow \pi(\bot) \vee \Diamond_\pi(\blacksquare_\pi(\mathbf{m}_0)) \leq \mathbf{m}_0)$$

and hence as follows:

$$\forall \mathbf{m}_0 (\pi(\bot) \leq \mathbf{m}_0\ \Rightarrow \pi(\bot) \leq \mathbf{m}_0\ \ \&\ \ \Diamond_\pi(\blacksquare_\pi(\mathbf{m}_0)) \leq \mathbf{m}_0).$$

Since the adjunction between $\Diamond_\pi$ and $\blacksquare_\pi$ implies that $\Diamond_\pi(\blacksquare_\pi(\mathbf{m}_0)) \leq \mathbf{m}_0$ is a tautology, it is easy to see that the quasi-inequality above is equivalent to $\top$. This is of course unsurprising, given that the additional ALBA^e rules rely on the validity of the inequality in input.

6.7.2. Example. In this example we illustrate a safe execution of ALBA^e on the meta-inductive formula $\pi(\sigma(p)) \leq \sigma(\pi(p))$ corresponding to the Geach axiom $\diamonddot\, \boxdot p \leq \boxdot\, \diamonddot\, p$.

Consider the inequality $\pi(\sigma(p)) \leq \sigma(\pi(p))$. The first approximation rule now yields:

$$\{\ \mathbf{j}_0 \leq \pi(\sigma(p)),\quad \sigma(\pi(p)) \leq \mathbf{m}_0\ \};$$

by the approximation rule for π, the system is written into:

$$\{\ \mathbf{j}_0 \leq \pi(\bot),\quad \sigma(\pi(p)) \leq \mathbf{m}_0\ \}, \left\{\begin{array}{ll} \mathbf{j}_0 \leq \Diamond_\pi \mathbf{j}_1, & \sigma(\pi(p)) \leq \mathbf{m}_0 \\ \mathbf{j}_1 \leq \sigma(p) & \end{array}\right\}$$

by applying the adjunction rule for σ, this system is rewritten into:

$$\{\ \mathbf{j}_0 \leq \pi(\bot),\quad \sigma(\pi(p)) \leq \mathbf{m}_0\ \}, \left\{\begin{array}{ll} \mathbf{j}_0 \leq \Diamond_\pi \mathbf{j}_1, & \sigma(\pi(p)) \leq \mathbf{m}_0 \\ j \leq \sigma(\top), & \blacklozenge_\sigma \mathbf{j}_1 \leq p \end{array}\right\}$$

by applying the monotonicity rule for p, this system is rewritten into:

$$\{\ \mathbf{j}_0 \leq \pi(\bot),\quad \sigma(\pi(\bot)) \leq \mathbf{m}_0\ \}, \left\{\begin{array}{ll} \mathbf{j}_0 \leq \Diamond_\pi \mathbf{j}_1, & \sigma(\pi(p)) \leq \mathbf{m}_0 \\ \mathbf{j}_1 \leq \sigma(\top), & \blacklozenge_\sigma \mathbf{j}_1 \leq p \end{array}\right\}$$

by applying the right-handed Ackermann rule for p, this system is rewritten into the following system of pure inequalities:

$$\left\{\ \mathbf{j}_0 \leq \pi(\bot),\quad \sigma(\pi(\bot)) \leq \mathbf{m}_0\ \right\}, \left\{\begin{array}{ll} \mathbf{j}_0 \leq \Diamond_\pi \mathbf{j}_1, & \sigma(\pi(\blacklozenge_\sigma \mathbf{j}_1)) \leq \mathbf{m}_0 \\ \mathbf{j}_1 \leq \sigma(\top) & \end{array}\right\}$$

In parallel to this execution, we show the execution of ALBA, to which the safe execution of ALBA^e corresponds.

Consider the inequality $\Diamonddot \boxdot p \leq \boxdot \Diamonddot p$. The first approximation rule now yields:

$$\left\{\ \mathbf{j}_0 \leq \Diamonddot \boxdot p,\quad \boxdot \Diamonddot p \leq \mathbf{m}_0\ \right\};$$

by the approximation rule for $\Diamonddot$, the system is written into:

$$\left\{\begin{array}{ll} \mathbf{j}_0 \leq \Diamonddot \mathbf{j}_1, & \boxdot \Diamonddot p \leq \mathbf{m}_0 \\ \mathbf{j}_1 \leq \boxdot p & \end{array}\right\}$$

by applying the adjunction rule for $\boxdot$, this system is rewritten into:

$$\left\{\begin{array}{ll} \mathbf{j}_0 \leq \Diamonddot \mathbf{j}_1, & \boxdot \Diamonddot p \leq \mathbf{m}_0 \\ \blacklozenge \mathbf{j}_1 \leq p & \end{array}\right\}$$

by applying the right-handed Ackermann rule for p, this system is rewritten into the following system of pure inequalities:

$$\left\{\ \mathbf{j}_0 \leq \Diamonddot \mathbf{j}_1,\quad \boxdot \Diamonddot \blacklozenge \mathbf{j}_1 \leq \mathbf{m}_0\ \right\}$$

6.8 Conclusions

In this chapter, we developed and applied ALBA to achieve two different but closely related results. We derived the canonicity of additivity obtained in [154] via pseudo-correspondence as an application of an ALBA-reduction. The key to this result is having expanded the basic language which ALBA manipulates with additional modal operators and their adjoints. With a similar expansion, we obtained a relativized canonicity result for the class of meta-inductive inequalities, which is, by definition, parametric in given term functions $\pi, \sigma, \lambda, \rho$. Clearly, relativized canonicity (cf. Definition 6.6.1) boils down to canonicity if K is the class of all DLEs, which embeds the canonicity via pseudo-correspondence result as a special case of the relativized canonicity result.

Together with the notion of relativized canonicity, we can consider the notion of correspondence relativized to a given class K. A natural question to ask is whether successful runs of ALBA^e generate pure quasi-inequalities which, under the standard translation, are relativized correspondents of the input formula/inequality w.r.t. the class K defined by the following inequalities:

$$\pi(p \vee q) \leq \pi(p) \vee \pi(q) \qquad \sigma(p) \wedge \sigma(q) \leq \sigma(p \wedge q)$$
$$\lambda(p \wedge q) \leq \lambda(p) \vee \lambda(q) \qquad \rho(p) \wedge \rho(q) \leq \rho(p \vee q)$$

Unfortunately, we can answer the question in the negative. For the correspondents effectively calculated by ALBA^e to be true correspondents within K, i.e. relativized correspondents w.r.t. this class, the rules of ALBA^e would have to be sound on all perfect DLEs in K. Now, as we mentioned in Remark 6.4.3, certain rules of ALBA^e are sound only on perfect DLEs which are canonical extensions. Indeed, there are perfect DLEs on which $\pi(p) := \Diamond\Box(p)$ is additive but not completely additive[5]. In these lattices, the identity $\pi(p) = \Diamond_\pi(p) \vee \pi(\bot)$ does not hold, and hence the ALBA^e rule based on it is not sound.

However, if we restrict ourselves to the case of finite DLEs, the correspondents effectively calculated by ALBA^e are true correspondents within the finite slice of K, of which they define elementary subclasses.

In this chapter, the canonicity result in [154] has been slightly generalized so as to apply to non-unary term functions which are positive w.r.t. some order-type ε. The axioms which are proved to be canonical state the additivity of those term functions seen as maps from ε-powers of DLEs. It remains an open question whether a similar result can be proven for non-unary maps and axioms stating their coordinatewise additivity.

[5]To see this, the following considerations are sufficient: for every perfect DLE $\mathbb{B}$, π^B is completely additive iff $\mathbb{B} \models \pi(p) = \Diamond_\pi(p) \vee \pi(\bot)$ iff $\mathbb{B} \models C(\pi)$. If for any perfect DLE the additivity of π^B implies its complete additivity, then we could add $\mathbb{B} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q)$ to the chain of equivalences mentioned above. Hence we would have shown that $\mathbb{B} \models \pi(p \vee q) \leq \pi(p) \vee \pi(q)$ iff $\mathbb{B} \models C(\pi)$ for any perfect DLE $\mathbb{B}$, i.e. the additivity of π would have a first-order correspondent, contradicting the well known fact that Fine's formula is canonical but not elementary.

Chapter 7

Subordinations, closed relations, and compact Hausdorff spaces

In this chapter, which is based on [70], we use correspondence like arguments on topological spaces to develop an alternative duality for de Vries algebras. By the celebrated Stone duality [142], the category of Boolean algebras and Boolean homomorphisms is dually equivalent to the category of Stone spaces (compact Hausdorff zero-dimensional spaces) and continuous maps. De Vries [57] generalized Stone duality to the category of compact Hausdorff spaces and continuous maps. Objects of the dual category are complete Boolean algebras $\mathbb{B}$ with a binary relation $\prec$ (called by de Vries a compingent relation) satisfying certain conditions that resemble the definition of a proximity on a set [124].

Another generalization of Stone duality central to modal logic is the Jónsson-Tarski duality [98] between the categories of modal algebras and modal spaces. The dual of a modal algebra $(\mathbb{B}, \Box)$ is the modal space (X, R), where X is the Stone dual of $\mathbb{B}$ (the space of ultrafilters of $\mathbb{B}$), while R is the dual of $\Box$ (see, e.g., [41, 107, 30]). Unlike the modal case, in de Vries duality we do not split the dual space of $(\mathbb{B}, \prec)$ in two components. Instead, we work with the space of "$\prec$-closed" filters which are maximal with this property.

The aim of this chapter is to develop a "modal-like" duality for de Vries algebras by splitting the dual space of a de Vries algebra $(\mathbb{B}, \prec)$ in two parts: the Stone dual of $\mathbb{B}$ and the dual of $\prec$. If X is the de Vries dual of $(\mathbb{B}, \prec)$, then the Stone dual Y of $\mathbb{B}$ is the Gleason cover of X [21]. We show that the irreducible map $\pi : Y \to X$ gives rise to what we call an irreducible equivalence relation R on Y, which is the dual of $\prec$. It follows that compact Hausdorff spaces are in 1-1 correspondence with pairs (Y, R), where Y is an extremally disconnected compact Hausdorff space and R is an irreducible equivalence relation on Y. We call such pairs Gleason spaces, and introduce the category of Gleason spaces, where morphisms are relations rather than functions, and composition is not relation composition. We prove that the category of Gleason spaces is equivalent to the category of compact Hausdorff spaces and continuous maps, and is dually equivalent to the category of de Vries algebras and de Vries morphisms, thus providing an alternate "modal-like" duality for de Vries algebras.

For this, we first introduce a general concept of a subordination $\prec$ on a Boolean algebra $\mathbb{B}$. Examples of subordinations (that satisfy additional conditions) are (i) modal operators $\Box$, (ii) de Vries' compingent relations, (iii) lattice subordinations of [22], etc. We show that subordinations on a Boolean algebra $\mathbb{B}$ dually correspond to closed relations on the Stone space X of $\mathbb{B}$.

We note that a subordination can be seen as a generalization of the modal operator $\Box$ (see Section 2). If we generalize the modal operator $\Diamond$ the same way, then we arrive at the well-known concept of a precontact relation and a precontact algebra [58, 60]. Since subordinations and precontact relations are definable from each other, the representation of precontact relations can be obtained from the representation of subordinations and vice versa. The representation of precontact algebras is given in [58] (see also [60, 6, 7]), but since there are no proofs given in [58], we include all the proofs here. In addition, we also provide duality for the corresponding morphisms, thus establishing a full categorical duality for the categories of interest.

The chapter is organized as follows. In Section 2, we briefly recall de Vries duality. In Section 3, we introduce the concept of a subordination on a Boolean algebra, show that subordinations are in 1-1 correspondence with precontact relations, and give a number of useful examples of subordinations. In Section 4, we prove that subordinations on a Boolean algebra $\mathbb{B}$ are in 1-1 correspondence with closed relations on the Stone space of $\mathbb{B}$, and develop a full categorical duality for the category of Boolean algebras with subordinations, thus generalizing [58]. In Section 5, we show that on objects the duality of Section 3 can be derived from the generalized Jónsson-Tarski duality. In Section 6, we prove that modally definable subordinations are dually described by means of Esakia relations. As a corollary, we derive the well-known duality between the categories of modal algebras and modal spaces. In Section 7, we characterize those subordinations whose dual relations are reflexive, transitive, and/or symmetric, thus obtaining results similar to [60, 58]. In Section 8, we show that a subordination is a lattice subordination iff its dual relation is a Priestley quasi-order. The duality result of [22] follows as a corollary. Finally, in Section 9 we introduce irreducible equivalence relations, Gleason spaces, and give a "modal-like" alternative to de Vries duality.

7.1 Preliminaries

In this section, we briefly recall the de Vries duality for compact Hausdorff spaces.

7.1.1. Definition. ([57, 21]) A *de Vries algebra* is a pair $(\mathbb{A}, \prec)$ consisting of a complete Boolean algebra $\mathbb{A}$ and a binary relation $\prec$ on $\mathbb{A}$ satisfying the following

(S1) $0 \prec 0$ and $1 \prec 1$;

(S2) $a \prec b, c$ implies $a \prec b \wedge c$;

(S3) $a, b \prec c$ implies $a \vee b \prec c$;

(S4) $a \leq b \prec c \leq d$ implies $a \prec d$.

(S5) $a \prec b$ implies $a \leq b$;

(S6) $a \prec b$ implies $\neg b \prec \neg a$;

(S7) $a \prec b$ implies there is $c \in B$ with $a \prec c \prec b$;

(S8) $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

7.1.2. Definition. ([57, 21]) A map $h : \mathbb{A} \to \mathbb{B}$ between two de Vries algebras is a *de Vries morphism* if it satisfies the following conditions:

(M1) $h(0) = 0$.

(M2) $h(a \wedge b) = h(a) \wedge h(b)$.

(M3) $a \prec b$ implies $\neg h(\neg a) \prec h(b)$.

(M4) $h(a) = \bigvee\{h(b) : b \prec a\}$.

For de Vries morphisms $h : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{B} \to \mathbb{C}$, their composition is given by

$$(g * h)(a) = \bigvee\{g(h(b)) : b \prec a\}.$$

Let **DeV** be the category of de Vries algebras and de Vries morphisms, and **KHaus** be the category of compact Hausdorff spaces and continuous maps.

Recall that a subset U of a topological space X is *regular open* if $U = \mathsf{Int}(\mathsf{Cl}(U))$, where $\mathsf{Int}(.)$ and $\mathsf{Cl}(.)$ are topological interior and closure operators, respectively. The set of regular open sets of a compact Hausdorff space X denoted by $\mathsf{RO}(X)$ form a complete Boolean algebra. For $U, V \in \mathsf{RO}(X)$ define $U \prec V$ if $\mathsf{Cl}(U) \subseteq V$. We define a contravariant functor $\Phi : \mathbf{KHaus} \to \mathbf{DeV}$ as $\Phi(X) = (\mathsf{RO}(X), \prec)$. Then $\Phi(X)$ is a de Vries algebra. For a continuous map $f : X \to Y$, let $\Phi(f) : \mathsf{RO}(Y) \to \mathsf{RO}(X)$ be given by $\Phi(f)(U) = \mathsf{Int}(\mathsf{Cl}(f^{-1}(U)))$ for each $U \in \mathsf{RO}(Y)$. Then $\Phi(f)$ is a de Vries morphism.

For a de Vries algebra $(\mathbb{A}, \prec)$ and $\mathbb{B} \subset \mathbb{A}$, let

$$\begin{aligned} \uparrow\!\!\!\!\uparrow B &= \{a \in \mathbb{A} : \exists b \in \mathbb{B} \text{ with } b \prec a\} \\ \downarrow\!\!\!\!\downarrow B &= \{a \in \mathbb{A} : \exists b \in \mathbb{B} \text{ with } a \prec b\}. \end{aligned}$$

A *filter* F of a de Vries algebra $\mathbb{A}$ is *round* if $F = \uparrow\!\!\!\!\uparrow F$. The maximal round filters are called *ends*. We define a contravariant functor $\Psi : \mathbf{Dev} \to \mathbf{KHaus}$ as follows. For a de Vries algebra $\mathbb{A}$, let $\Psi(\mathbb{A})$ be the space of ends of A topologized by the basis of sets $\{\varepsilon(a) : a \in \mathbb{A}\}$, where $\varepsilon(a) = \{E \in \Psi(\mathbb{A}) : a \in E\}$ for each $a \in \mathbb{A}$. For a de Vries morphism $h : \mathbb{A} \to \mathbb{B}$, let $\Psi(h) : \Psi(\mathbb{B}) \to \Psi(\mathbb{A})$ be given by $\Psi(h)(E) =^{\uparrow} h^{-1}(E)$ for each $E \in \Psi(\mathbb{B})$. Then $\Psi(h)$ is continuous.

7.1.3. Theorem (de Vries duality [57]). *The functors Φ and Ψ defined above provide a dual equivalence between the categories* **DeV** *and* **KHaus**.

7.2 Subordinations on Boolean algebras

In this section, we introduce the concept of a subordination on a Boolean algebra. We show that subordinations are in 1-1 correspondence with precontact relations, and that modal operators, de Vries' compingent relations, and lattice subordinations of [22] are all examples of subordinations.

7.2.1. Definition. A *subordination* on a Boolean algebra $\mathbb{B}$ is a binary relation $\prec$ satisfying:

(S1) $0 \prec 0$ and $1 \prec 1$;

(S2) $a \prec b, c$ implies $a \prec b \wedge c$;

(S3) $a, b \prec c$ implies $a \vee b \prec c$;

(S4) $a \leq b \prec c \leq d$ implies $a \prec d$.

7.2.2. Remark. It is an easy consequence of the axioms that $0 \prec a \prec 1$ for each $a \in \mathbb{B}$. In fact, (S1) can equivalently be stated this way.

7.2.3. Example. We recall [60, 58] that a *proximity* or *precontact* on a Boolean algebra $\mathbb{B}$ is a binary relation δ satisfying

(P1) $a\ \delta\ b \ \Rightarrow\ a, b \neq 0.$

(P2) $a\ \delta\ b \vee c \ \Leftrightarrow\ a\ \delta\ b$ or $a\ \delta\ c.$

(P3) $a \vee b\ \delta\ c \ \Leftrightarrow\ a\ \delta\ c$ or $b\ \delta\ c.$

Let $\prec$ be a subordination on $\mathbb{B}$. Define a binary relation $\delta_\prec$ by $a\delta_\prec b$ iff $a \not\prec \neg b$. It is routine to verify that $\delta_\prec$ is a precontact relation on $\mathbb{B}$. Conversely, if δ is a precontact relation on $\mathbb{B}$, then define $\prec_\delta$ by $a \prec_\delta b$ iff $a \not\delta \neg b$. Then it is easy to see that $\prec_\delta$ is a subordination on $\mathbb{B}$. Moreover, $a \prec b$ iff $a \prec_{\delta_\prec} b$, and $a\delta b$ iff $a\delta_{\prec_\delta} b$. Thus, subordinations are in 1-1 correspondence with precontact relations on $\mathbb{B}$.

We recall that an alternative definition of a modal operator on a Boolean algebra $\mathbb{B}$ is a unary function $\Box : B \to B$ that preserves finite meets (including 1), and a modal algebra is a pair $(\mathbb{B}, \Box)$, where $\mathbb{B}$ is a Boolean algebra and $\Box$ is a modal operator on $\mathbb{B}$. We show that modal operators are in 1-1 correspondence with special subordinations.

7.2.4. Example. Let $\mathbb{B}$ be a Boolean algebra and let $\Box$ be a modal operator on $\mathbb{B}$. Set $a \prec_\Box b$ provided $a \leq \Box b$. Since $\Box 1 = 1$, it is clear that $\prec_\Box$ satisfies (S1). As $\Box(b \wedge c) = \Box b \wedge \Box c$, we also have that $\prec_\Box$ satisfies (S2). That $\prec_\Box$ satisfies (S3) is obvious, and since $\Box$ is order-preserving, $\prec_\Box$ satisfies (S4). Therefore, $\prec_\Box$ is a subordination on $\mathbb{B}$. Note that $\prec_\Box$ is a special subordination on $\mathbb{B}$ that in addition satisfies the following condition: for each $a \in \mathbb{B}$, the element $\Box a$ is the largest element of the set $\{x \in \mathbb{B} : x \prec_\Box a\}$.

7.2.5. Definition. Let $\mathbb{B}$ be a Boolean algebra and let $\prec$ be a subordination on $\mathbb{B}$. We call $\prec$ *modally definable* provided the set $\{x \in \mathbb{B} : x \prec a\}$ has a largest element for each $a \in \mathbb{B}$.

In Example 7.2.4 we saw that if $\Box$ is a modal operator, then $\prec_\Box$ is a modally definable subordination. The converse is also true.

7.2.6. Example. Let $\mathbb{B}$ be a Boolean algebra and let $\prec$ be a modally definable subordination on $\mathbb{B}$. Define $\Box_\prec : B \to B$ by

$$\Box_\prec a \;=\; \text{the largest element of } \{x \in \mathbb{B} : x \prec a\}.$$

By (S1), $\Box_\prec 1 = 1$. In addition, by (S4), $\Box_\prec(a \wedge b) \leq \Box_\prec a \wedge \Box_\prec b$, and by (S2) and (S4), $\Box_\prec a \wedge \Box_\prec b \leq \Box_\prec(a \wedge b)$. Therefore, $\Box_\prec$ is a modal operator on $\mathbb{B}$. Moreover, $\Box_{\prec_\Box} a = \Box a$ and $a \prec_{\Box_\prec} b$ iff $a \prec b$. Thus, modal operators on $\mathbb{B}$ are in 1-1 correspondence with modally definable subordinations on $\mathbb{B}$.

Other examples of subordinations are quasi-modal operators of [38], lattice subordinations of [22] and compingent relations of [57].

7.2.7. Definition. ([22]) A subordination $\prec$ on a Boolean algebra $\mathbb{B}$ is a *lattice subordination* if in addition $\prec$ satisfies

$$a \prec b \textit{ implies that there exists } c \in \mathbb{B} \text{ with } c \prec c \text{ and } a \leq c \leq b.$$

7.2.8. Definition. ([57]) A subordination $\prec$ on a Boolean algebra $\mathbb{B}$ is a *compingent relation* if in addition it satisfies:

(S5) $a \prec b$ implies $a \leq b$;

(S6) $a \prec b$ implies $\neg b \prec \neg a$;

(S7) $a \prec b$ implies there is $c \in \mathbb{B}$ with $a \prec c \prec b$;

(S8) $a \neq 0$ implies there is $b \neq 0$ with $b \prec a$.

We let **Sub** be the category whose objects are pairs $(\mathbb{B}, \prec)$, where $\mathbb{B}$ is a Boolean algebra and $\prec$ is a subordination on $\mathbb{B}$, and whose morphisms are Boolean homomorphisms h satisfying $a \prec b$ implies $h(a) \prec h(b)$.

7.3 Subordinations and closed relations

In this section, we show that subordinations on a Boolean algebra $\mathbb{B}$ can be dually described by means of closed relations on the Stone space of $\mathbb{B}$, and work out a full categorical duality between the category of Boolean algebras with subordinations and the category of Stone spaces with closed relations. These results generalize the results of [58].

7.3.1. Definition. Let X be a topological space and let R be a binary relation on X. We call R a *closed relation* provided R is a closed set in the product topology on $X \times X$.

The next lemma generalizes [26, Prop. 2.3], where a characterization of closed quasi-orders (reflexive and transitive relations) is given. In fact, the proofs of (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are the same as in [26], so we only sketch them. For the rest of the implications, we provide all details.

7.3.2. Lemma. *Let X be a compact Hausdorff space and let R be a binary relation on X. The following conditions are equivalent.*

1. *R is a closed relation.*
2. *For each closed subset F of X, both $R[F]$ and $R^{-1}[F]$ are closed.*
3. *If A is an arbitrary subset of X, then $\overline{R[A]} \subseteq R[\overline{A}]$ and $\overline{R^{-1}[A]} \subseteq R^{-1}[\overline{A}]$.*
4. *If $(x, y) \notin R$, then there is an open neighbourhood U of x and an open neighbourhood V of y such that $R[U] \cap V = \varnothing$.*

Proof. (1) $\Rightarrow$ (2): Suppose that F is a closed subset of X. As X is compact Hausdorff, the projections $\pi_1, \pi_2 : X \times X \to X$ are closed maps. Since R is a closed relation, $R[F] = \pi_2((F \times X) \cap R)$, and $R^{-1}[F] = \pi_1((X \times F) \cap R)$, both $R[F]$ and $R^{-1}[F]$ are closed subsets of X.

(2) $\Rightarrow$ (3): Let A be an arbitrary subset of X. Since $A \subseteq \overline{A}$, we have $R[A] \subseteq R[\overline{A}]$. As $\overline{A}$ is closed, by (2), $R[\overline{A}]$ is also closed. Therefore, $\overline{R[A]} \subseteq R[\overline{A}]$. A similar argument gives $\overline{R^{-1}[A]} \subseteq R^{-1}[\overline{A}]$.

(3) $\Rightarrow$ (4): Let $(x, y) \notin R$. Then $y \notin R[x]$. Since X is Hausdorff, $\{x\}$ is closed. By (3), $\overline{R[x]} \subseteq R[x]$, so $R[x]$ is also closed. As X is compact Hausdorff, and hence regular, there exist disjoint open sets W, V such that $R[x] \subseteq W$ and $y \in V$. Set $U = X - R^{-1}[X - W]$. Since $X - W$ is closed, by (3), $\overline{R^{-1}[X - W]} \subseteq R^{-1}[X - W]$. Therefore, $R^{-1}[X - W]$ is closed, hence $U = X - R^{-1}[X - W]$ is open. Moreover, $R[x] \subseteq W$ implies $x \in U$. If $v \in R[U] \cap V$, then there is $u \in U$ with uRv. This yields $v \in R[u] \subseteq W$, so $W \cap V \neq \varnothing$. The obtained contradiction proves that $R[U] \cap V = \varnothing$. Thus, U is an open neighbourhood of x, V is an open neighbourhood of y, and $R[U] \cap V = \varnothing$.

(4) $\Rightarrow$ (1): Let $(x, y) \notin R$. By (4), there is an open neighbourhood U of x and an open neighbourhood V of y such that $R[U] \cap V = \varnothing$. Therefore, there is an open neighbourhood $U \times V$ of (x, y) such that $(U \times V) \cap R = \varnothing$. Thus, R is a closed subset of $X \times X$. $\square$

For $i = 1, 2$, let R_i be a relation on X_i. Following [25], we call a map $f : X_1 \to X_2$ *stable* provided xR_1y implies $f(x)R_2f(y)$. The following is straightforward.

7.3.3. Lemma. *The following are equivalent:*

1. $f : X_1 \to X_2$ *is stable.*
2. $f(R_1[x]) \subseteq R_2[f(x)]$ *for each* $x \in X_1$.
3. $R_1^{-1}[f^{-1}(y)] \subseteq f^{-1}(R_2^{-1}[y])$ *for each* $y \in X_2$.

Proof. Easy. $\square$

We recall that a *Stone space* is a compact, Hausdorff, zero-dimensional space. The celebrated Stone duality yields that the category of Boolean algebras and Boolean homomorphisms is dually equivalent to the category of Stone spaces and continuous maps (cf. 2.5.1). We next extend Stone duality to the category **Sub**.

Let **StR** be the category whose objects are pairs (X, R), where X is a Stone space and R is a closed relation on X, and whose morphisms are continuous stable morphisms. We will prove that **Sub** is dually equivalent to **StR**.

For a Boolean algebra $\mathbb{B}$, let X be the set of ultrafilters of $\mathbb{B}$. For $a \in \mathbb{B}$, set $\varphi(a) = \{x \in X : a \in x\}$, and topologize X by letting $\{\varphi(a) : a \in \mathbb{B}\}$ be a basis for the topology. The resulting space is called the Stone space of $\mathbb{B}$ and is denoted $\mathbb{B}_*$.

7.3.4. Definition. For $(\mathbb{B}, \prec) \in$ **Sub**, let $(\mathbb{B}, \prec)_* = (X, R)$, where X is the Stone space of $\mathbb{B}$ and xRy iff $\twoheaduparrow x \subseteq y$.

7.3.5. Lemma. *If* $(\mathbb{B}, \prec) \in$ **Sub**, *then* $(\mathbb{B}, \prec)_* \in$ **StR**.

Proof. It is sufficient to prove that R is a closed relation on X. Let $(x, y) \notin R$. Then $\Uparrow x \not\subseteq y$. Therefore, there are $a \in x$ and $b \notin y$ with $a \prec b$.
Claim. $a \prec b$ implies $R[\varphi(a)] \subseteq \varphi(b)$.
Proof.[Proof of Claim] Let $v \in R[\varphi(a)]$. Then there is $u \in \varphi(a)$ with uRv. Therefore, $a \in u$ and $\Uparrow u \subseteq v$. Since $a \prec b$, we have $b \in v$, so $v \in \varphi(b)$. Thus, $R[\varphi(a)] \subseteq \varphi(b)$. □

We set $U = \varphi(a)$ and $V = X - \varphi(b)$. Then U is an open neighbourhood of x, V is an open neighbourhood of y, and $R[U] \cap V = \varnothing$. Thus, by Lemma 7.3.2, R is a closed relation on X, which completes the proof. □

7.3.6. Definition. For $i = 1, 2$, let $(\mathbb{B}_i, \prec_i) \in \mathbf{Sub}$ and let $(X_i, R_i) = (\mathbb{B}_i, \prec_i)_*$. For a morphism $h : \mathbb{B}_1 \to \mathbb{B}_2$ in **Sub**, let $h_* : X_2 \to X_1$ be given by $h_*(x) = h^{-1}(x)$.

7.3.7. Lemma. *If h is a morphism in* **Sub***, then h_* is a morphism in* **StR***.*

Proof. By Stone duality, h_* is a well-defined continuous map. Suppose $x, y \in X_2$ with xR_2y. Then $\Uparrow_2 x \subseteq y$. Let $b \in \Uparrow_1 h^{-1}(x)$. So there is $a \in h^{-1}(x)$ with $a \prec_1 b$. Since h is a morphism in **Sub**, we have $h(a) \prec_2 h(b)$. Therefore, $h(b) \in \Uparrow_2 x$. This implies $h(b) \in y$. Thus, $b \in h^{-1}(y)$, yielding $\Uparrow_1 h^{-1}(x) \subseteq h^{-1}(y)$. Consequently, h_* is a stable continuous map, hence a morphism in **StR**. □

7.3.8. Definition. Define $(-)_* : \mathbf{Sub} \to \mathbf{StR}$ as follows. If $(\mathbb{B}, \prec) \in \mathbf{Sub}$, then $(B, \prec)_* = (X, R)$, and if h is a morphism in **Sub**, then $h_* = h^{-1}$. Applying Lemmas 7.3.5 and 7.3.7 it is straightforward to verify that $(-)_*$ is a well-defined contravariant functor.

For a topological space X, let $\mathsf{Clop}(X)$ be the set of clopen subsets of X. Then it is well known and easy to see that $\mathsf{Clop}(X)$ is a Boolean algebra with respect to the set-theoretic operations of union, intersection, and complement.

7.3.9. Definition. For $(X, R) \in \mathbf{StR}$, let $(X, R)^* = (\mathsf{Clop}(X), \prec)$, where $U \prec V$ iff $R[U] \subseteq V$.

7.3.10. Lemma. *If $(X, R) \in \mathbf{StR}$, then $(X, R)^* \in \mathbf{Sub}$.*

Proof. Since $R[\varnothing] = \varnothing$, it is clear that $\prec$ satisfies (S1). That $\prec$ satisfies (S2) is obvious. From $R[U \cup V] = R[U] \cup R[V]$ it follows that $\prec$ satisfies (S3). Finally, as $U \subseteq V$ implies $R[U] \subseteq R[V]$, we obtain that $\prec$ satisfies (S4). Thus, R is a subordination on $\mathsf{Clop}(X)$, and hence $(X, R)^* \in \mathbf{Sub}$. □

7.3.11. Definition. For $i = 1, 2$, let $(X_i, R_i) \in \mathbf{StR}$ and let $(\mathbb{B}_i, \prec_i) = (X_i, R_i)^*$. For a morphism $f : X_1 \to X_2$ in $\mathbf{StR}$, let $f^* : \mathsf{Clop}(X_2) \to \mathsf{Clop}(X_1)$ be given by $f^*(U) = f^{-1}(U)$.

7.3.12. Lemma. *If f is a morphism in* $\mathbf{StR}$*, then f^* is a morphism in* $\mathbf{Sub}$.

Proof. It follows from Stone duality that f^* is a Boolean homomorphism. Let $U, V \in \mathsf{Clop}(X_2)$ with $U \prec_2 V$. Then $R_2[U] \subseteq V$. This implies $f^{-1}(R_2[U]) \subseteq f^{-1}(V)$. Since f is a stable map, by Lemma 7.3.3, $R_1[f^{-1}(U)] \subseteq f^{-1}(R_2[U])$. Therefore, $R_1[f^{-1}(U)] \subseteq f^{-1}(V)$. Thus, $f^{-1}(U) \prec f^{-1}(V)$, and hence f^* is a morphism in $\mathbf{Sub}$. □

7.3.13. Definition. Define $(-)^* : \mathbf{StR} \to \mathbf{Sub}$ as follows. If $(X, R) \in \mathbf{StR}$, then $(X, R)^* = (\mathsf{Clop}(X), \prec)$, and if f is a morphism in $\mathbf{StR}$, then $f^* = f^{-1}$. Applying Lemmas 7.3.10 and 7.3.12 it is straightforward to see that $(-)^*$ is a contravariant functor.

7.3.14. Lemma. *Let $(\mathbb{B}, \prec) \in$* $\mathbf{Sub}$ *and let $\varphi : \mathbb{B} \to (\mathbb{B}_*)^*$ be the Stone map. Then $a \prec b$ iff $\varphi(a) \prec \varphi(b)$.*

Proof. Let $a, b \in \mathbb{B}$ with $a \prec b$. By the Claim in the proof of Lemma 7.3.5, this implies $R[\varphi(a)] \subseteq \varphi(b)$, so $\varphi(a) \prec \varphi(b)$. Next suppose that $a \not\prec b$. Then $b \notin \Uparrow a$. Since $\prec$ is a subordination, it is easy to see that $\Uparrow a$ is a filter. Therefore, by the ultrafilter theorem, there is an ultrafilter x such that $\Uparrow a \subseteq x$ and $b \notin x$.
Claim. $\Uparrow a \subseteq x$ implies that there is an ultrafilter y such that $a \in y$ and $\Uparrow y \subseteq x$.
Proof.[Proof of Claim] Let $F = \uparrow a$ and $I = B - x$. Then F is a filter containing a and I is an ideal. We show that $\Uparrow F \cap I = \varnothing$. If $c \in \Uparrow F \cap I$, then $c \in I$ and there is $d \in F$ with $d \prec c$. Therefore, $a \leq d \prec c$ and $c \notin x$. Thus, $a \prec c$, so $c \in \Uparrow a$ and $c \notin x$. This yields $\Uparrow a \not\subseteq x$, a contradiction. Consequently, the set Z consisting of the filters G satisfying $a \in G$ and $\Uparrow G \subseteq x$ is nonempty because $F \in Z$. It is easy to see that $(Z, \subseteq)$ is an inductive set, hence by Zorn's lemma, Z has a maximal element, say y. We show that y is an ultrafilter. Suppose $c, \neg c \notin y$. Let F_1 be the filter generated by $\{c\} \cup y$ and F_2 be the filter generated by $\{\neg c\} \cup y$. Since F_1 and F_2 properly contain y, they do not belong to Z, so $\Uparrow F_1, \Uparrow F_2 \not\subseteq x$. This gives $d_1, d_2 \in y$ and $e \notin x$ such that $c \wedge d_1, \neg c \wedge d_2 \prec e$. By (S3) and distributivity, $(c \vee \neg c) \wedge (c \vee d_2) \wedge (d_1 \vee \neg c) \wedge (d_1 \vee d_2) \prec e$. But $(c \vee \neg c) \wedge (c \vee d_2) \wedge (d_1 \vee \neg c) \wedge (d_1 \vee d_2) \in y$, so $e \in \Uparrow y \subseteq x$. The obtained contradiction proves that y is an ultrafilter. Since $y \in Z$, we have $a \in y$ and $\Uparrow y \subseteq x$, which completes the proof of the claim. □

It follows from the Claim that there is $y \in \mathbb{B}_*$ such that $y \in \varphi(a)$ and yRx. Therefore, $x \in R[\varphi(a)]$. On the other hand, $x \notin \varphi(b)$. Thus, $R[\varphi(a)] \not\subseteq \varphi(b)$, yielding $\varphi(a) \not\prec \varphi(b)$. $\square$

For a Stone space X, define $\psi : X \to (X^*)_*$ by $\psi(x) = \{U \in \mathsf{Clop}(X) : x \in U\}$. It follows from Stone duality that ψ is a homeomorphism.

7.3.15. Lemma. *Let $(X, R) \in$ **StR** and let $\psi : X \to (X^*)_*$ be given as above. Then xRy iff $\psi(x)R\psi(y)$.*

Proof. First suppose that xRy. To see that $\psi(x)R\psi(y)$ we must show that ${\uparrow}\psi(x) \subseteq \psi(y)$. Let $V \in {\uparrow}\psi(x)$. Then there is $U \in \psi(x)$ with $U \prec V$. Therefore, $x \in U$ and $R[U] \subseteq V$. Thus, $y \in V$, so ${\uparrow}\psi(x) \subseteq \psi(y)$, and hence $\psi(x)R\psi(y)$.

Conversely, suppose that $(x, y) \notin R$. Since X has a basis of clopens and R is a closed relation, by Lemma 7.3.2, there exist a clopen neighbourhood U of x and a clopen neighbou rhood W of y such that $R[U] \cap W = \varnothing$. Set $V = X - W$. Then $U \in \psi(x)$, $V \notin \psi(y)$, and $R[U] \subseteq V$. Therefore, $U \prec V$, so $V \in {\uparrow}\psi(x)$, but $V \notin \psi(y)$. Thus, $(\psi(x), \psi(y)) \notin R$. $\square$

7.3.16. Theorem. *The categories* **Sub** *and* **StR** *are dually equivalent.*

Proof. By Definition 7.3.8, $(-)_* :$ **Sub** $\to$ **StR** is a well-defined contravariant functor, and by Definition 7.3.13, $(-)^* :$ **StR** $\to$ **Sub** is a well-defined contravariant functor. By Stone duality and Lemmas 7.3.14 and 7.3.15, each $(\mathbb{B}, \prec) \in$ **Sub** is isomorphic in **Sub** to $((B, \prec)_*)^*$ and each $(X, R) \in$ **StR** is isomorphic in **StR** to $((X, R)^*)_*$. That these isomorphisms are natural is easy to see. Thus, **Sub** is dually equivalent to **StR**. $\square$

7.3.17. Remark. As follows from Example 7.2.3, there is a 1-1 correspondence between subordinations and precontact relations on a Boolean algebra $\mathbb{B}$. Therefore, each precontact algebra $(\mathbb{B}, \delta)$ can be represented as $(\mathsf{Clop}(X), \delta_R)$, where (X, R) is the dual of $(\mathbb{B}, \prec_\delta)$ and $U\delta_R V$ iff $R[U] \cap V \neq \varnothing$. This yields the representation theorem of [58, Thm. 3]. In fact, this representation theorem can be generalized to a full categorical duality. Let **PCon** be the category of precontact algebras and Boolean homomorphisms h satisfying $h(a)\ \delta\ h(b)$ implies $a\ \delta\ b$. Then an obvious generalization of Example 7.2.3 gives that the categories **Sub** and **PCon** are isomorphic. Thus, by Theorem 7.3.16, **PCon** is dually equivalent to **StR**.

7.4 Subordinations, strict implications, and Jónsson-Tarski duality

In this section, we show that on objects the duality of the previous section can also be derived from the generalized Jónsson-Tarski duality.

7.4.1. Definition. Let $\mathbb{B}$ be a Boolean algebra and let $\mathbf{2}$ be the two-element Boolean algebra. We call a map $\rightarrow: B \times B \rightarrow \mathbf{2}$ a *strict implication* if it satisfies

(I1) $0 \rightarrow a = a \rightarrow 1 = 1$.

(I2) $(a \vee b) \rightarrow c = (a \rightarrow c) \wedge (b \rightarrow c)$.

(I3) $a \rightarrow (b \wedge c) = (a \rightarrow b) \wedge (a \rightarrow c)$.

7.4.2. Example. Let $\prec$ be a subordination on a Boolean algebra $\mathbb{B}$. Define $\rightarrow_{\prec}: \mathbb{B} \times B \rightarrow \mathbf{2}$ by

$$a \rightarrow_{\prec} b = \begin{cases} 1 & \text{if } a \prec b, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $\rightarrow_{\prec}$ is a strict implication. Conversely, if $\rightarrow: B \times B \rightarrow \mathbf{2}$ is a strict implication, then define $\prec_{\rightarrow} \subseteq B \times B$ by

$$a \prec_{\rightarrow} b \text{ iff } a \rightarrow b = 1.$$

It is straightforward to see that $\prec_{\rightarrow}$ is a subordination on $\mathbb{B}$. Moreover, $a \prec b$ iff $a \prec_{\rightarrow_{\prec}} b$ and $a \rightarrow b = a \rightarrow_{\prec_{\rightarrow}} b$. Thus, there is a 1-1 correspondence between subordinations and strict implications on $\mathbb{B}$.

This observation opens the door for obtaining the duality for subordinations from Jónsson-Tarski duality [98]. Let A, B, C be Boolean algebras and X, Y, Z be their Stone spaces, respectively. Suppose that $f : A \times B \rightarrow C$ is a map. Following the terminology of [141], we call f a *meet-hemiantimorphism in the first coordinate* provided

- $f(0, b) = 1$,
- $f(a \vee b, c) = f(a, c) \wedge f(b, c)$;

and a *meet-hemimorphism in the second coordinate* provided

- $f(a, 1) = 1$,
- $f(a, b \wedge c) = f(a, c) \wedge f(b, c)$.

By the generalized Jónsson-Tarski duality [79, 141], such maps are dually described by special ternary relations $S \subseteq X \times Y \times Z$. For $z \in Z$, let

$$S^{-1}[z] := \{(x, y) \in X \times Y : (x, y, z) \in S\},$$

and for $U \in \mathsf{Clop}(X)$ and $V \in \mathsf{Clop}(Y)$, let

$$\Box_S(U, V) := \{z \in Z : (\forall x \in X)(\forall y \in Y)\,[(x, y, z) \in S \Rightarrow x \notin U \text{ or } y \in V]\}.$$

7.4.3. Definition. We call $S \subseteq X \times Y \times Z$ a *JT-relation (Jónsson-Tarski relation)* provided

(JT1) $S^{-1}[z]$ is closed for each $z \in Z$,

(JT2) $\Box_S(U, V)$ is clopen for each $U \in \mathsf{Clop}(X)$ and $V \in \mathsf{Clop}(Y)$.

By the generalized Jónsson-Tarksi duality [79, 141], the dual ternary relation $S \subseteq X \times Y \times Z$ of $f : A \times B \to C$ is given by

$$(x, y, z) \in S \text{ iff } (\forall a \in A)(\forall b \in \mathbb{B})(f(a, b) \in z \text{ implies } a \notin x \text{ or } b \in y); \tag{7.1}$$

and the dual map $f : \mathsf{Clop}(X) \times \mathsf{Clop}(Y) \to \mathsf{Clop}(Z)$ of $S \subseteq X \times Y \times Z$ is given by

$$f(U, V) = \Box_S(U, V). \tag{7.2}$$

Now let $\to$ be a strict implication on a Boolean algebra $\mathbb{B}$. By Definition 7.4.1, $\to$ is a meet-hemiantimorphism in the first coordinate and a meet-hemimorphism in the second coordinate. Let X be the Stone space of $\mathbb{B}$. The Stone space of $\mathbf{2}$ is the singleton discrete space $\{z\}$, where $z = \{1\}$ is the only ultrafilter of $\mathbf{2}$. Therefore, the dual ternary relation $S \subseteq X \times X \times \{z\}$ of $\to$ is given by

$$(x, y, z) \in S \text{ iff } (\forall a, b \in \mathbb{B})(a \to b = 1 \text{ implies } a \notin x \text{ or } b \in y).$$

The ternary relation S gives rise to the binary relation$R \subseteq X \times X$ by setting

$$xRy \text{ iff } (x, y, 1) \in S.$$

If $\prec$ is the subordination corresponding to the strict implication $\to$, then $a \prec b$ iff $a \to b = 1$. Therefore, the binary relation R is given by

$$xRy \text{ iff } (\forall a, b \in \mathbb{B})(a \prec b \text{ implies } a \notin x \text{ or } b \in y).$$

7.4.4. Proposition. *Let $\prec$ be a subordination on a Boolean algebra $\mathbb{B}$, and let (X, R) be the dual of $(\mathbb{B}, \prec)$. Then ${\Uparrow}x \subseteq y$ iff $(\forall a, b \in \mathbb{B})(a \prec b$ implies $a \notin x$ or $b \in y)$.*

Proof. First suppose that ${\Uparrow}x \subseteq y$. Let $a \prec b$ and $a \in x$. Then $b \in {\Uparrow}x$, so $b \in y$. Conversely, suppose $(\forall a, b \in \mathbb{B})(a \prec b$ implies $a \notin x$ or $b \in y)$. If $b \in {\Uparrow}x$, then there is $a \in x$ with $a \prec b$. Therefore, $y \in \mathbb{B}$, and hence ${\Uparrow}x \subseteq y$. $\Box$

Applying Proposition 7.4.4 then yields

$$xRy \text{ iff } {\Uparrow}x \subseteq y.$$

Consequently, the dual binary relation R of a subordination $\prec$ can be described from the dual ternary relation S of the corresponding strict implication. In fact, if $S \subseteq X \times X \times \{z\}$ is a JT-relation, then (JT2) is redundant, while (JT1) means that R is a closed relation.

The converse is also true. Given a closed relation R on a Stone space X, define the ternary relation $S \subseteq X \times X \times \{z\}$ by

$$(x, y, z) \in S \text{ iff } xRy.$$

Since R is a closed relation, S satisfies (JT1), and S satisfies (JT2) trivially, hence S is a JT-relation. Let $\rightarrow\colon \mathsf{Clop}(X) \times \mathsf{Clop}(Y) \to \mathbf{2}$ be the corresponding strict implication. Then

$$U \to V = \begin{cases} 1 & \text{if } (\forall x \in X)(\forall y \in Y)(xRy \Rightarrow x \notin U \text{ or } y \in V) \\ 0 & \text{otherwise.} \end{cases}$$

7.4.5. Proposition. *Let X be a Stone space, R be a closed relation on X, and $U, V \in \mathsf{Clop}(X)$. Then $R[U] \subseteq V$ iff $(\forall x, y \in X)(xRy$ implies $x \notin U$ or $y \in V)$.*

Proof. First suppose that $R[U] \subseteq V$, xRy, and $x \in U$. Then $y \in R[U]$, so $y \in V$. Conversely, suppose that $(\forall x, y \in X)(xRy$ implies $x \notin U$ or $y \in V)$. If $y \in R[U]$, then there is $x \in U$ with xRy. Therefore, $y \in V$, and hence $R[U] \subseteq V$. $\square$

If $\prec$ is the subordination corresponding to $\rightarrow$, then it follows from Proposition 7.4.5 that $U \prec V$ iff $R[U] \subseteq V$ iff $U \to V = 1$. This shows that on objects our duality for subordinations is equivalent to a special case of the generalized Jónsson-Tarski duality.

7.5 Modally definable subordinations and Esakia relations

In this section, we show that modally definable subordinations dually correspond to Esakia relations, and derive the well-known duality between the categories of modal algebras and modal spaces from the duality of Section 3.

7.5.1. Definition. Let X be a Stone space. We call a binary relation R on X an *Esakia relation* provided $R[x]$ is closed for each $x \in X$ and $U \in \mathsf{Clop}(X)$ implies $R^{-1}[U] \in \mathsf{Clop}(X)$.

7.5.2. Remark.

1. Let $\mathcal{V}(X)$ be the Vietoris space (cf. 8.1.2) of X. It is well known (see, e.g., [64]) that R is an Esakia relation iff the map $\rho_R : X \to \mathcal{V}(X)$ given by $\rho(x) = R[x]$ is a well-defined continuous map. Because of this, Esakia relations are also called *continuous relations*.

2. It is easy to see that Esakia relations are exactly the inverses of binary JT-relations with the same source and target (see, e.g., [79]). Inverses of binary JT-relations with not necessarily the same source and target were first studied by Halmos [92].

It is a standard argument that each Esakia relation is closed, but there exist closed relations that are not Esakia relations. In fact, for a closed relation R on a Stone space X, the following are equivalent:

1. R is an Esakia relation.
2. $U \in \mathsf{Clop}(X)$ implies $R^{-1}[U] \in \mathsf{Clop}(X)$.
3. U open implies $R^{-1}[U]$ is open.

Therefore, Esakia relations form a proper subclass of closed relations, hence correspond to special subordinations. We show that they correspond to modally definable subordinations. Our proof is a generalization of [22, Lem. 5.6].

7.5.3. Lemma.

1. *Suppose that* $(\mathbb{B}, \prec) \in \mathbf{Sub}$ *and* $(X, R) = (B, \prec)_*$. *If* $\prec$ *is modally definable, then* R *is an Esakia relation.*
2. *Suppose that* R *is an Esakia relation on a Stone space* X *and* $(\mathbb{B}, \prec) = (X, R)^*$. *Then* $\prec$ *is modally definable.*

Proof. (1) Suppose that $\prec$ is modally definable and $\Box_\prec$ is the largest element of $\{b \in \mathbb{B} : b \prec a\}$.
Claim. $\varphi(\Box_\prec a) = X - R^{-1}[X - \varphi(a)]$.
Proof.[Proof of Claim] We have $x \in X - R^{-1}[X - \varphi(a)]$ iff $R[x] \subseteq \varphi(a)$. This is equivalent to $(\forall y \in X)(\twoheaduparrow x \subseteq y \Rightarrow a \in y)$. Since $\twoheaduparrow x$ is a filter, by the ultrafilter theorem, it is the intersection of the ultrafilters containing it. Therefore, the last condition is equivalent to $a \in \twoheaduparrow x$. Because $\Box_\prec$ is the largest element of $\{b \in \mathbb{B} : b \prec a\}$, this is equivalent to $\Box_\prec a \in x$, which means that $x \in \varphi(\Box_\prec a)$. Thus, $\varphi(\Box_\prec a) = X - R^{-1}[X - \varphi(a)]$. $\Box$

Now, let $U \in \mathsf{Clop}(X)$. Then $X - U \in \mathsf{Clop}(X)$, so there is $a \in \mathbb{B}$ with $\varphi(a) = X - U$. Therefore, $\varphi(\Box_\prec a) = X - R^{-1}[X - \varphi(a)] = X - R^{-1}[U]$. This yields $X - R^{-1}[U] \in \mathsf{Clop}(X)$, so $R^{-1}[U] \in \mathsf{Clop}(X)$. Since R is also a closed relation, we conclude that R is an Esakia relation.

(2) Let $U \in \mathsf{Clop}(X)$. We show that $X - R^{-1}[X - U]$ is the largest element of $\{V \in \mathsf{Clop}(X) : V \prec U\}$. Let $y \in R[X - R^{-1}[X - U]]$. Then there is $x \in X - R^{-1}[X - U]$ with xRy. From $x \in X - R^{-1}[X - U]$ it follows that $R[x] \subseteq U$. Therefore, $y \in U$, yielding $X - R^{-1}[X - U] \prec U$. Suppose that

$V \in \mathsf{Clop}(X)$ with $V \prec U$. Then $R[V] \subseteq U$, so $V \subseteq X - R^{-1}[X - U]$. Thus, $X - R^{-1}[X - U]$ is the largest element of $\{V \in \mathsf{Clop}(X) : V \prec U\}$, and hence $\prec$ is modally definable. $\square$

We recall that a *modal space* is a pair (X, R), where X is a Stone space and R is an Esakia relation on X. Modal spaces are also known as *descriptive frames*. They are fundamental objects in the study of modal logic as they serve as dual spaces of modal algebras (see, e.g., [41, 107, 30]).

Let $\mathbf{MS^{st}}$ be the category whose objects are modal spaces and whose morphisms are continuous stable morphisms. Let also **MSub** be the full subcategory of **Sub** consisting of the objects $(\mathbb{B}, \prec)$ of **Sub** in which $\prec$ is modally definable. It is an immediate consequence of Theorem 7.3.16 and Lemma 7.5.3 that **MSub** is dually equivalent to $\mathbf{MS^{st}}$.

But modal logicians are more interested in p-morphism rather than stable morphisms since they dually correspond to modal algebra homomorphisms. We recall that a modal homomorphism is a Boolean homomorphism $h : \mathbb{B}_1 \to \mathbb{B}_2$ such that $h(\Box_1 a) = \Box_2 h(a)$. We also recall that a p-morphism is a stable morphism $f : X_1 \to X_2$ such that $f(x) R_2 y$ implies the existence of $z \in X_1$ with $x R_1 z$ and $f(z) = y$ (cf. Definition 2.5.2. Let **MA** be the category whose objects are modal algebras and whose morphisms are modal homomorphisms, and let **MS** be the category whose objects are modal spaces and whose morphisms are continuous p-morphisms. (Note that **MS** is not a full subcategory of $\mathbf{MS^{st}}$.) It is a standard result in modal logic that **MA** is dually equivalent to **MS**. We next show how to obtain this dual equivalence from our results.

Let $h : \mathbb{B}_1 \to \mathbb{B}_2$ be a morphism in **MSub**. For $a \in \mathbb{B}_1$, let $\Box_1 a$ be the largest element of $\{x \in \mathbb{B}_1 : x \prec_1 a\}$, and for $b \in \mathbb{B}_2$, let $\Box_2 b$ be the largest element of $\{y \in \mathbb{B}_2 : y \prec_2 b\}$. Since $\Box_1 a \prec_1 a$, we have $h(\Box_1 a) \prec_2 h(a)$. Therefore, $h(\Box_1 a) \leq_2 \Box_2 h(a)$. Conversely, suppose that h is a Boolean homomorphism satisfying $h(\Box_1 a) \leq_2 \Box_2 h(a)$ for each $a \in \mathbb{B}_1$. Let $a, b \in \mathbb{B}_1$ with $a \prec_1 b$. Then $a \leq_1 \Box_1 b$. Therefore, $h(a) \leq_2 h(\Box_1 b) \leq_2 \Box_2 h(b)$. Thus, $h(a) \prec_2 h(b)$, and h is a morphism in **MSub**.

We call a morphism h in **MSub** a *modal homomorphism* if $h(\Box_1 a) = \Box_2 h(a)$. Let $\mathbf{MSub^m}$ be the category whose objects are the objects of **MSub** and whose morphisms are modal homomorphisms. Then $\mathbf{MSub^m}$ is a non-full subcategory of **MSub**. From Examples 7.2.4 and 7.2.6 and the discussion above it is evident that $\mathbf{MSub^m}$ is isomorphic to **MA**.

We show that $\mathbf{MSub^m}$ is dually equivalent to **MS**. For this, taking into account the dual equivalence of **MSub** and $\mathbf{MS^{st}}$, it is sufficient to see that if h is a morphism in $\mathbf{MSub^m}$, then h_* is a morphism in **MS**, and that if f is a morphism in **MS**, then f^* is a morphism in $\mathbf{MSub^m}$. This is proved in the next lemma, which generalizes [22, Lem. 5.7].

7.5.4. LEMMA.

1. *Let* $(\mathbb{B}_1, \prec_1), (B_2, \prec_2) \in \mathbf{MSub^m}$ *and* $h : \mathbb{B}_1 \to B_2$ *be a morphism in* $\mathbf{MSub^m}$. *Then* h_* *is a morphism in* $\mathbf{MS}$.

2. *Let* $(X_1, R_1), (X_2, R_2) \in \mathbf{MS}$ *and* $f : X_1 \to X_2$ *be a morphism in* $\mathbf{MS}$. *Then* f^* *is a morphism in* $\mathbf{MSub^m}$.

Proof. (1) From the dual equivalence of $\mathbf{MSub}$ and $\mathbf{MS^{st}}$ we know that h_* is continuous and stable. Suppose that $h_*(x)R_1y$. Then $\Uparrow_1 h^{-1}(x) \subseteq y$. Let F be the filter generated by $\Uparrow_2 x \cup h(y)$ and let I be the ideal generated by $h(\mathbb{B}_1 - y)$. If $F \cap I \neq \varnothing$, then there exist $a \in \Uparrow_2 x$, $b \in y$, and $c \notin y$ such that $a \wedge_2 h(b) \leq_2 h(c)$. Therefore, $a \leq_2 h(b \to_1 c)$. From $a \in \Uparrow_2 x$ it follows that there is $d \in x$ with $d \prec_2 a$. So $d \leq_2 \Box_2 a$. But $a \leq_2 h(b \to_1 c)$ implies $\Box_2 a \leq_2 \Box_2 h(b \to_1 c) = h(\Box_1(b \to_1 c))$. This yields $\Box_1(b \to_1 c) \in h^{-1}(x)$, so $b \to_1 c \in \Uparrow_1 h^{-1}(x) \subseteq y$, which is a contradiction since $b \in y$ and $c \notin y$. Thus, $F \cap I = \varnothing$, and by the ultrafilter theorem, there is an ultrafilter z containing F and missing I. From $\Uparrow_2 x \subseteq z$ it follows that xR_2z, and from $h(y) \subseteq z$ and $h(B_1 - y) \cap z = \varnothing$ it follows that $h^{-1}(z) = y$. Consequently, there is z such that xR_2z and $h_*(z) = y$, yielding that h_* is a morphism in $\mathbf{MS}$.

(2) From the dual equivalence of $\mathbf{MSub}$ and $\mathbf{MS^{st}}$ we know that f^* is a Boolean homomorphism satisfying $U \prec_2 V$ implies $f^*(U) \prec_1 f^*(V)$ for each $U, V \in \mathsf{Clop}(X_2)$. Therefore, $f^*(\Box_2 U) \leq_1 \Box_1 f^*(U)$ for each $U \in \mathsf{Clop}(X_2)$. Suppose that $x \in \Box_1 f^*(U)$. Then $R_1[x] \subseteq f^{-1}(U)$, so $f(R_1[x]) \subseteq U$. Since f is a bounded morphism, $f(R_1[x]) = R_2[f(x)]$. Therefore, $R_2[f(x)] \subseteq U$, yielding $f(x) \in \Box_2 U$. Thus, $x \in f^{-1}(\Box_2 U)$. This implies that $f^*(\Box_2 U) = \Box_1 f^*(U)$ for each $U \in \mathsf{Clop}(X_2)$, hence f^* is a morphism in $\mathbf{MSub^m}$. $\Box$

As a consequence, we obtain that $\mathbf{MSub^m}$ is dually equivalent to $\mathbf{MS}$, and since $\mathbf{MSub^m}$ is isomorphic to $\mathbf{MA}$, as a corollary, we obtain the well-known dual equivalence of $\mathbf{MA}$ and $\mathbf{MS}$. Below we summarize the results of this section.

7.5.5. Theorem.

1. $\mathbf{MSub}$ *is dually equivalent to* $\mathbf{MS^{st}}$.

2. $\mathbf{MSub^m}$ *is isomorphic to* $\mathbf{MA}$.

3. $\mathbf{MSub^m}$ *is dually equivalent to* $\mathbf{MS}$, *hence* $\mathbf{MA}$ *is dually equivalent to* $\mathbf{MS}$.

7.6 Further duality results

In modal logic, modal algebras corresponding to reflexive, transitive, and/or symmetric modal spaces play an important role. In this section, we characterize those $(\mathbb{B}, \prec) \in \mathbf{Sub}$ which correspond to $(X, R) \in \mathbf{StR}$ with R reflexive, transitive, and/or symmetric. Since there is a 1-1 correspondence between subordinations and precontact relations, these results are similar to the ones given in [60, 58], but our proofs are different.

7.6.1. Lemma. *Let* $(\mathbb{B}, \prec) \in$ **Sub** *and let* (X, R) *be the dual of* $(\mathbb{B}, \prec)$.

1. *R is reflexive iff $\prec$ satisfies (S5).*
2. *R is symmetric iff $\prec$ satisfies (S6).*
3. *R is transitive iff $\prec$ satisfies (S7).*

Proof. (1) First suppose that R is reflexive. Let $a, b \in \mathbb{B}$ with $a \prec b$. By Lemma 7.3.14, $\varphi(a) \prec \varphi(b)$, so $R[\varphi(a)] \subseteq \varphi(b)$. Since R is reflexive, $\varphi(a) \subseteq R[\varphi(a)]$. Therefore, $R[\varphi(a)] \subseteq \varphi(b)$ implies $\varphi(a) \subseteq \varphi(b)$. Thus, $a \leq b$, and hence $\prec$ satisfies (S5). Next suppose that $\prec$ satisfies (S5). Let $x \in X$ and $a \in {\twoheaduparrow} x$. Then there is $b \in x$ with $b \prec a$. This, by (S5), yields $b \leq a$, so $a \in x$. Therefore, ${\twoheaduparrow} x \subseteq x$, which means that xRx. Thus, R is reflexive.

(2) Suppose that R is symmetric. Let $a, b \in \mathbb{B}$ with $a \prec b$. By Lemma 7.3.14, $R[\varphi(a)] \subseteq \varphi(b)$. We show that $R[X - \varphi(b)] \subseteq X - \varphi(a)$. Let $x \in R[X - \varphi(b)]$. Then there is $y \notin \varphi(b)$ with yRx. Since R is symmetric, xRy. If $x \in \varphi(a)$, then $y \in R[\varphi(a)]$, so $y \in \varphi(b)$, a contradiction. Therefore, $x \notin \varphi(a)$, and hence $R[X - \varphi(b)] \subseteq X - \varphi(a)$. This implies $R[\varphi(\neg b)] \subseteq \varphi(\neg a)$. Applying Lemma 7.3.14 again yields $\neg b \prec \neg a$. Thus, $\prec$ satisfies (S6). Conversely, suppose that $\prec$ satisfies (S6). Let $x, y \in X$ with xRy. Then ${\twoheaduparrow} x \subseteq y$. Suppose $a \in {\twoheaduparrow} y$. So there is $b \in y$ with $b \prec a$. By (S6), $\neg a \prec \neg b$. Since ${\twoheaduparrow} x \subseteq y$ and $\neg b \notin y$, we see that $\neg a \notin x$. Therefore, as x is an ultrafilter, $a \in x$, yielding ${\twoheaduparrow} y \subseteq x$. Thus, yRx, and hence R is symmetric.

(3) Suppose that R is transitive. Let $a, b \in \mathbb{B}$ with $a \prec b$. Then $R[\varphi(a)] \subseteq \varphi(b)$. Therefore, $\varphi(a) \subseteq X - R^{-1}[X - \varphi(b)]$. Denoting $X - R^{-1}[X - \varphi(b)]$ by $\Box_R \varphi(b)$, we obtain $\varphi(a) \subseteq \Box_R \varphi(b)$. Since R is transitive, $\Box_R \varphi(b) \subseteq \Box_R \Box_R \varphi(b)$. This implies $\varphi(a) \subseteq \Box_R \Box_R \varphi(b)$, so $R[\varphi(a)] \subseteq \Box_R \varphi(b)$. Because R is a closed relation, $R[\varphi(a)]$ is closed and $\Box_R \varphi(b)$ is open. Thus, as X is a Stone space, there is clopen U with $R[\varphi(a)] \subseteq U \subseteq \Box_R \varphi(b)$. But $U = \varphi(c)$ for some $c \in \mathbb{B}$. The first inclusion gives $\varphi(a) \prec \varphi(c)$ and the second yields $\varphi(c) \prec \varphi(b)$. Consequently, there is $c \in \mathbb{B}$ with $a \prec c \prec b$, and $\prec$ satisfies (S7). Conversely, suppose that $\prec$ satisfies (S7). Let $x, y, z \in X$ with xRy and yRz. Then ${\twoheaduparrow} x \subseteq y$ and ${\twoheaduparrow} y \subseteq z$. Suppose $a \in {\twoheaduparrow} x$. Then there is $b \in x$ with $b \prec a$. By (S7), there is $c \in \mathbb{B}$ with $b \prec c \prec a$. From $b \prec c$ and $b \in x$, we have $c \in {\twoheaduparrow} x$, hence $c \in y$. But then $c \prec a$ yields $a \in {\twoheaduparrow} y$, so $a \in z$. Thus, xRz, and hence R is transitive. $\Box$

7.6.2. Remark. Let $(\mathbb{B}, \prec) \in$ **Sub** and let (X, R) be the dual of $(\mathbb{B}, \prec)$. Lemma 7.6.1 shows that axioms (S5), (S6), and (S7) correspond to elementary conditions on

R. Developing a general theory which characterizes the class of axioms for subordinations corresponding to elementary conditions on R is closely related to the field of Sahlqvist correspondence theory. A Sahlqvist correspondence for logics corresponding to precontact algebras is developed in [5].

7.6.3. Definition.

1. Let **SubK4** be the full subcategory of **Sub** consisting of the $(\mathbb{B}, \prec) \in \mathbf{Sub}$ that satisfy (S7).

2. Let **SubS4** be the full subcategory of **Sub** consisting of the $(\mathbb{B}, \prec) \in \mathbf{Sub}$ that satisfy (S5) and (S7).

3. Let **SubS5** be the full subcategory of **Sub** consisting of the $(\mathbb{B}, \prec) \in \mathbf{Sub}$ that satisfy (S5), (S6), and (S7).

Clearly **SubS5** is a full subcategory of **SubS4**, and **SubS4** is a full subcategory of **SubK4**.

7.6.4. Definition.

1. Let $\mathbf{StR^{tr}}$ be the full subcategory of **StR** consisting of the $(X, R) \in \mathbf{StR}$, where R is transitive.

2. Let $\mathbf{StR^{qo}}$ be the full subcategory of **StR** consisting of the $(X, R) \in \mathbf{StR}$, where R is a quasi-order (that is, R is reflexive and transitive).

3. Let $\mathbf{StR^{eq}}$ be the full subcategory of **StR** consisting of the $(X, R) \in \mathbf{StR}$, where R is an equivalence relation.

Clearly $\mathbf{StR^{eq}}$ is a full subcategory of $\mathbf{StR^{qo}}$, and $\mathbf{StR^{qo}}$ is a full subcategory of $\mathbf{StR^{tr}}$. The next theorem is an immediate consequence of Theorem 7.3.16 and Lemma 7.6.1.

7.6.5. Theorem.

1. **SubK4** *is dually equivalent to* $\mathbf{StR^{tr}}$.

2. **SubS4** *is dually equivalent to* $\mathbf{StR^{qo}}$.

3. **SubS5** *is dually equivalent to* $\mathbf{StR^{eq}}$.

7.6.6. Remark. We recall (see, e.g., [58]) that a precontact algebra $(\mathbb{B}, \delta)$ is a *contact algebra* if it satisfies the following two axioms:

(P4) $a \neq 0$ implies $a\ \delta\ a$.

(P5) $a\ \delta\ b$ implies $b\ \delta\ a$.

Let **Con** be the full subcategory of **PCon** consisting of contact algebras. By Example 7.2.3, it is straightforward to see that (P4) is the δ-analogue of axiom (S5), while (P5) is the δ-analogue of axiom (S6). Therefore, **Con** is isomorphic to the full subcategory of **Sub** whose objects satisfy axioms (S5) and (S6). Thus, by Theorem 7.3.16 and Lemma 7.6.1, **Con** is dually equivalent to the full subcategory of **StR** consisting of such $(X, R) \in$ **StR**, where R is reflexive and symmetric.

7.6.7. Remark. We recall that a modal algebra $(\mathbb{B}, \Box)$ is a **K4**-*algebra* if $\Box a \leq \Box\Box a$ for each $a \in \mathbb{B}$; a **K4**-algebra is an **S4**-*algebra* if $\Box a \leq a$ for each $a \in \mathbb{B}$; and an **S4**-algebra is an **S5**-*algebra* if $a \leq \Box\Diamond a$ for each $a \in \mathbb{B}$. Let **K4**, **S4**, and **S5** denote the categories of **K4**-algebras, **S4**-algebras, and **S5**-algebras, respectively.

We also let **TRS** be the category of transitive modal spaces, **QOS** be the category of quasi-ordered modal spaces, and **EQS** be the category of modal spaces, where the relation is an equivalence relation. Then it is a well-known fact in modal logic that **K4** is dually equivalent to **TRS**, **S4** is dually equivalent to **QOS**, and **S5** is dually equivalent to **EQS**. These results can be obtained as corollaries of our results as follows.

Let $\mathbf{SubK4^m}$, $\mathbf{SubS4^m}$, and $\mathbf{SubS5^m}$ be the subcategories of **SubK4**, **SubS4**, and **SubS5**, respectively, where morphisms are modal morphisms. It is then clear that $\mathbf{SubK4^m}$ is isomorphic to **K4**, $\mathbf{SubS4^m}$ is isomorphic to **S4**, and $\mathbf{SubS5^m}$ is isomorphic to **S5**. It is also obvious that $\mathbf{SubK4^m}$ is dually equivalent to **TRS**, $\mathbf{SubS4^m}$ is dually equivalent to **QOS**, and $\mathbf{SubS5^m}$ is dually equivalent to **EQS**. The duality results for **K4**, **S4**, and **S5** follow.

7.7 Lattice subordinations and the Priestley separation axiom

An interesting class of subordinations is that of lattice subordinations of [22]. In this section, we show that a subordination $\prec$ on a Boolean algebra $\mathbb{B}$ is a lattice subordination iff in the dual space (X, R) of $(\mathbb{B}, \prec)$, the relation R is a Priestley quasi-order. The duality result of [22, Cor. 5.3] follows as a corollary.

We recall that a lattice subordination is a subordination $\prec$ that in addition satisfies

(S9) $a \prec b \Rightarrow (\exists c \in \mathbb{B})(c \prec c\ \&\ a \leq c \leq b)$.

By [22, Lem. 2.2], a lattice subordination satisfies (S5) and (S7). In addition, since c is reflexive, in the above condition, $a \leq c \leq b$ can be replaced by $a \prec c \prec b$. Therefore, a lattice subordination is a subordination that satisfies (S5) and a stronger form of (S7), where it is required that the existing c is reflexive.

If $\prec$ is a lattice subordination on $\mathbb{B}$, then as follows from the previous section, in the dual space (X, R), we have that R is a quasi-order. But more is true. Let (X, R) be a quasi-ordered set. We call a subset U of X an *R-upset* provided $x \in U$ and xRy imply $y \in U$. Similarly U is an *R-downset* if $x \in U$ and yRx imply $y \in U$. We recall (see, e.g., [134, 26]) that a quasi-order R on a compact Hausdorff space X satisfies the *Priestley separation axiom* if $(x, y) \notin R$ implies that there is a clopen R-upset U such that $x \in U$ and $y \notin U$. If R satisfies the Priestley separation axiom, then we call R a *Priestley quasi-order*. Each Priestley quasi-order is closed, but the converse is not true in general [143, 26]. A *quasi-ordered Priestley space* is a pair (X, R), where X is a Stone space and R is a Priestley quasi-order on X. As was proved in [22, Cor. 5.3], lattice subordinations dually correspond to Priestley quasi-orders. To see how to derive this result from our results, we will use freely the following well-known fact about quasi-ordered Priestley spaces:

If A, B are disjoint closed subsets of a quasi-ordered Priestley space (X, R), with A an R-upset and $\mathbb{B}$ an R-downset, then there is a clopen R-upset U containing A and disjoint from $\mathbb{B}$.

7.7.1. LEMMA. *Let $\prec$ be a subordination on $\mathbb{B}$ and let (X, R) be the dual of $(\mathbb{B}, \prec)$. Then R is a Priestley quasi-order iff $\prec$ satisfies (S9).*

Proof. First suppose that R is a Priestley quasi-order. Let $a, b \in \mathbb{B}$ with $a \prec b$. By Lemma 7.3.14, $R[\varphi(a)] \subseteq \varphi(b)$. Therefore, $R[\varphi(a)] \cap (X - \varphi(b)) = \varnothing$. Since $R[\varphi(a)]$ is an R-upset, this yields $R[\varphi(a)] \cap R^{-1}[X - \varphi(b)] = \varnothing$. As $R[\varphi(a)]$ and $R^{-1}[X - \varphi(b)]$ are disjoint closed sets with $R[\varphi(a)]$ an R-upset and $R^{-1}[X - \varphi(b)]$ an R-downset, there is a clopen R-upset U containing $R[\varphi(a)]$ and disjoint from $R^{-1}[X - \varphi(b)]$. But $U = \varphi(c)$ for some $c \in \mathbb{B}$. Since U is an R-upset, $R[\varphi(c)] \subseteq \varphi(c)$, so $c \prec c$. As $\varphi(a) \subseteq R[\varphi(a)] \subseteq \varphi(c)$, we have $a \leq c$. Finally, since $\varphi(c)$ is disjoint from $R^{-1}[X - \varphi(b)]$, we also have $\varphi(c) \cap (X - \varphi(b)) = \varnothing$, so $\varphi(c) \subseteq \varphi(b)$, and hence $c \leq b$. Thus, $\prec$ satisfies (S9).

Next suppose that $\prec$ satisfies (S9). Then $\prec$ satisfies (S5) and (S7), hence R is a quasi-order. Let $x, y \in X$ with $(x, y) \notin R$. Then ${\Uparrow}x \not\subseteq y$. Therefore, there are $a, b \in \mathbb{B}$ with $a \in x$, $a \prec b$, and $b \notin y$. By (S9), there is $c \in \mathbb{B}$ with $c \prec c$ and $a \leq c \leq b$. From $c \prec c$ it follows that $R[\varphi(c)] \subseteq \varphi(c)$, so $\varphi(c)$ is a clopen R-upset of X. Since $a \in x$ and $a \leq c$, we have $c \in x$, so $x \in \varphi(c)$. As $c \leq b$ and $b \notin y$, we also have $c \notin y$, hence $y \notin \varphi(c)$. Thus, there is a clopen R-upset $\varphi(c)$ such that $x \in \varphi(c)$ and $y \notin \varphi(c)$, yielding that R is a Priestley quasi-order. $\square$

Let **LSub** be the full subcategory of **Sub** consisting of the $(\mathbb{B}, \prec) \in$ **Sub**, where $\prec$ is a lattice subordination. Let also **QPS** be the full subcategory of **StR** consisting of quasi-ordered Priestley spaces. It is an immediate consequence of our results that the dual equivalence of **Sub** and **StR** restricts to a dual equivalence of **LSub** and **QPS**. Thus, we arrive at the following result of [22, Cor. 5.3].

7.7.2. Theorem. **LSub** *is dually equivalent to* **QPS**.

7.8 Irreducible equivalence relations, compact Hausdorff spaces, and de Vries duality

In this final section we introduce irreducible equivalence relations, Gleason spaces, and provide a "modal-like" alternative to de Vries duality. We recall [57] that a *compingent algebra* is a pair $(\mathbb{B}, \prec)$, where $\mathbb{B}$ is a Boolean algebra and $\prec$ is a binary relation on $\mathbb{B}$ satisfying (S1)–(S8). In other words, a compingent algebra is an object of **SubS5** that in addition satisfies (S8). It follows from our duality results that the dual of $(\mathbb{B}, \prec) \in$ **SubS5** is a pair (X, R), where X is a Stone space and R is a closed equivalence relation on X. Since X is compact Hausdorff and R is a closed equivalence relation on X, the factor-space X/R is also compact Hausdorff. In order to give the dual description of (S8), we recall that a continuous map $f : X \to Y$ between compact Hausdorff spaces is *irreducible* provided the f-image of each proper closed subset of X is a proper subset of Y.

7.8.1. Definition. *We call a closed equivalence relation R on a compact Hausdorff space X* irreducible *if the factor-map $\pi : X \to X/R$ is irreducible.*

7.8.2. Remark. Clearly a closed equivalence relation R on a compact Hausdorff space X is irreducible iff for each proper closed subset F of X, we have $R[F]$ is a proper subset of X. If X is a Stone space, then an immediate application of Esakia's lemma ([64, 24]) yields that we can restrict the condition to proper clopen subsets of X.

7.8.3. Lemma. *Let $(\mathbb{B}, \prec) \in$ **SubS5** and let (X, R) be the dual of $(\mathbb{B}, \prec)$. Then the closed equivalence relation R is irreducible iff $\prec$ satisfies (S8).*

Proof. First suppose that R is irreducible. Let $a \in \mathbb{B}$ with $a \neq 0$. Then $\varphi(a) \neq \varnothing$, so $X - \varphi(a)$ is a proper closed subset of X. Since R is irreducible, $R[X - \varphi(a)]$ is a proper subset of X. Therefore, $X - R[X - \varphi(a)] \neq \varnothing$, and as $R[X - \varphi(a)]$ is closed, $X - R[X - \varphi(a)]$ is open. As X is a Stone space, there is a nonempty clopen subset U of X contained in $X - R[X - \varphi(a)]$. But $U = \varphi(b)$ for some $b \in \mathbb{B}$. Since $U \neq \varnothing$, we have $b \neq 0$. As $\varphi(b) \subseteq X - R[X - \varphi(a)]$ and R is an equivalence relation, $R[\varphi(b)] \subseteq \varphi(a)$. Thus, there is $b \neq 0$ with $b \prec a$, and so $\prec$ satisfies (S8).

Next suppose that $\prec$ satisfies (S8). Let F be a proper closed subset of X. Then $X - F$ is a nonempty open subset of X. Since X is a Stone space, there is a nonempty clopen set contained in $X - F$. Therefore, there is $a \in \mathbb{B} - \{0\}$ with $\varphi(a) \subseteq X - F$. By (S8), there is $b \in \mathbb{B} - \{0\}$ with $b \prec a$. Thus, $R[\varphi(b)] \subseteq \varphi(a)$. As R is an equivalence relation, this yields $\varphi(b) \subseteq X - R[X - \varphi(a)] \subseteq X - R[F]$. So $R[F] \subseteq X - \varphi(b)$. Since $b \neq 0$, we see that $X - \varphi(b)$ is a proper subset of X, hence $R[F]$ is a proper subset of X. Consequently, R is irreducible. $\square$

Let **Com** be the full subcategory of **SubS5** consisting of compingent algebras; that is, **Com** consists of the objects of **SubS5** that in addition satisfy (S8). Let also $\mathbf{StR^{ieq}}$ be the full subcategory of $\mathbf{StR^{eq}}$ consisting of the pairs (X, R), where R is an irreducible equivalence relation on a Stone space X. The above results yield:

7.8.4. THEOREM. **Com** *is dually equivalent to* $\mathbf{StR^{ieq}}$.

We recall that a space X is *extremally disconnected* if the closure of every open set is open. We call an extremally disconnected Stone space an *ED-space.* (Equivalently, ED-spaces are extremally disconnected compact Hausdorff spaces.) It is well known that a Boolean algebra $\mathbb{B}$ is complete iff its Stone space X is an ED-space. Since a de Vries algebra is a complete compingent algebra, the duals of de Vries algebras are pairs (X, R), where X is an ED-space and R is an irreducible equivalence relation on X.

7.8.5. DEFINITION. We call a pair (X, R) a *Gleason space* if X is an ED-space and R is an irreducible equivalence relation on X.

Our choice of the name is motivated by the fact that Gleason spaces arise naturally by taking Gleason covers [78] of compact Hausdorff spaces. Indeed, suppose X is compact Hausdorff and (Y, π) is the Gleason cover of X, with $\pi : Y \to X$ the canonical irreducible map. Then Y is an ED-space. Define R on Y by xRy iff $\pi(x) = \pi(y)$. Since π is an irreducible map, it is easy to see that R is an irreducible equivalence relation on Y, hence (Y, R) is a Gleason space. In fact, each Gleason space arises this way because if (Y, R) is a Gleason space, then as R is a closed equivalence relation, the factor-space $X := Y/R$ is compact Hausdorff. Moreover, since R is irreducible, the factor-map $\pi : Y \to X$ is an irreducible map, yielding that (Y, π) is (homeomorphic to) the Gleason cover of X [78]. Thus, we have a convenient 1-1 correspondence between compact Hausdorff spaces and Gleason spaces, and both dually correspond to de Vries algebras.

It is an easy consequence of (M1) and (M3) in Definition 7.1.2 that a de Vries morphism h also satisfies $h(1) = 1$. Therefore, each de Vries morphism is a meet-hemimorphism [92]. Let X be the Stone space of A and Y be the Stone space of $\mathbb{B}$. As follows from [92], meet-hemimorphisms $h : A \to B$ are dually characterized by relations $r \subseteq Y \times X$ satisfying $r[y]$ is closed for each $y \in Y$ and $r^{-1}[U]$ is clopen for each clopen $U \subseteq X$. In [92] such relations are called *Boolean relations.*

7.8.6. REMARK.

1. In [92] Halmos worked with join-hemimorphisms, which generalize the modal operator $\Diamond$, while meet-hemimorphisms generalize the modal operator $\Box$.

2. Boolean relations are exactly the inverses of binary JT-relations, and if $X = Y$, then Boolean relations are Esakia relations (see Remark 7.5.2(2)).

We recall that the dual correspondence between $h : A \to B$ and $r \subseteq Y \times X$ is obtained as follows. Given $h : A \to B$, define $r \subseteq Y \times X$ by setting

$$(y, x) \in r \text{ iff } (\forall a \in A)(h(a) \in y \Rightarrow a \in x).$$

Conversely, given $r : Y \times X$, define $h : \mathsf{Clop}(X) \to \mathsf{Clop}(Y)$ by setting

$$h(U) = Y - r^{-1}[X - U].$$

In order to simplify notation, instead of $(y, x) \in r$, we will often write yrx. We also set

$$\Box_r U := Y - r^{-1}[X - U].$$

Thus, $h(U) = \Box_r U$.

7.8.7. Definition. Suppose $r \subseteq Y \times X$.

1. We say that r is *cofinal* provided $(\forall y \in Y)(\exists x \in X)(yrx)$.

2. We say that r satisfies the *forth condition* provided

$$(\forall y, y' \in Y)(\forall x, x' \in X)(yRy' \;\&\; yrx \;\&\; y'rx' \Rightarrow xRx').$$

$$\begin{tikzcd} y' \arrow[r, "r"] & x' \\ y \arrow[u, "R"] \arrow[r, "r"'] & x \arrow[u, dashed, "R"'] \end{tikzcd}$$

3. We say that r satisfies the *de Vries condition* provided

$$(\forall U \in \mathsf{Clop}(X))(r^{-1}(U) = \mathrm{int}(r^{-1}R^{-1}[U])).$$

7.8.8. Lemma. *Let $(A, \prec)$ and $(\mathbb{B}, \prec)$ be de Vris algebras, (X, R) be the dual of $(A, \prec)$, and (Y, R) be the dual of $(\mathbb{B}, \prec)$. Suppose $h : A \to B$ is a meet-hemimorphism and $r \subseteq Y \times X$ is its dual.*

1. *h satisfies (M1) iff r is cofinal.*

2. *h satisfies (M3) iff r satisfies the forth condition.*

3. *h satisfies (M4) iff r satisfies the de Vries condition.*

Proof. (1) We have $h(0) = 0$ iff $\Box_r(\varnothing) = \varnothing$, which happens iff $r^{-1}[X] = Y$. This in turn is equivalent to $(\forall y \in Y)(\exists x \in X)(yrx)$. Thus, h satisfies (M1) iff r is cofinal.

(2) First suppose that h satisfies (M3). Let $y, y' \in Y$ and $x, x' \in X$ with yRy', yrx, and $y'rx'$. To see that xRx' we must show that ${\uparrow\!\!\uparrow} x \subseteq x'$. Let $b \in {\uparrow\!\!\uparrow} x$. Then there is $a \in x$ with $a \prec b$. By (M3), $\neg h(\neg a) \prec h(b)$. Since $a \in x$, we have $\neg a \notin x$. As yrx, this yields $h(\neg a) \notin y$. Because y is an ultrafilter, $\neg h(\neg a) \in y$. Therefore, $h(b) \in {\uparrow\!\!\uparrow} y$. Since yRy', this gives $h(b) \in y'$. Thus, by $y'rx'$, we obtain $b \in x'$, so xRx'. Consequently, r satisfies the forth condition.

Next suppose that r satisfies the forth condition. Let $a, b \in A$ with $a \prec b$. Then $R[\varphi(a)] \subseteq \varphi(b)$. We have $\varphi(\neg h(\neg a)) = r^{-1}[\varphi(a)]$ and $\varphi(h(b)) = \Box_r \varphi(b)$. Therefore, to see that $\neg h(\neg a) \prec h(b)$, it is sufficient to show that $R[r^{-1}[\varphi(a)]] \subseteq \Box_r \varphi(b)$. Let $y' \in R[r^{-1}[\varphi(a)]]$. Then there are $x \in \varphi(a)$ and $y \in Y$ with yRy' and yrx. Suppose $x' \in X$ with $y'rx'$. So yRy', yrx, and $y'rx'$, which by the forth condition gives xRx'. Therefore, $x' \in R[\varphi(a)]$, yielding $x' \in \varphi(b)$. Thus, $y' \in \Box_r \varphi(b)$. Consequently, $R[r^{-1}[\varphi(a)]] \subseteq \Box_r \varphi(b)$, and hence h satisfies (M3).

(3) We recall that if $S \subseteq A$, then $\varphi(\bigvee S) = \overline{\bigcup\{\varphi(s) : s \in S\}}$. Therefore, for each $a \in A$, we have $\varphi(h(a)) = \Box_r \varphi(a)$ and

$$\begin{aligned} \varphi(\bigvee\{h(b) : b \prec a\}) &= \overline{\bigcup\{\Box_r \varphi(b) : R[\varphi(b)] \subseteq \varphi(a)\}} \\ &= \overline{\bigcup\{\Box_r \varphi(b) : \varphi(b) \subseteq \Box_R \varphi(a)\}} \\ &= \overline{\Box_r \Box_R \varphi(a)}. \end{aligned}$$

Thus, h satisfies (M4) iff $\Box_r \varphi(a) = \overline{\Box_r \Box_R \varphi(a)}$ for each $a \in A$. This is equivalent to $Y - r^{-1}[U] = Y - \mathrm{int}(r^{-1}R^{-1}[U])$ for each $U \in \mathsf{Clop}(U)$. This in turn is equivalent to $r^{-1}[U] = \mathrm{int}(r^{-1}R^{-1}[U])$ for each $U \in \mathsf{Clop}(U)$, yielding that h satisfies (M4) iff r satisfies the de Vries condition. $\Box$

7.8.9. Definition. Let (Y, R) and (X, R) be Gleason spaces. We call a relation $r \subseteq Y \times X$ a *de Vries relation* provided r is a cofinal Boolean relation satisfying the forth and de Vries conditions.

As follows from Lemma 7.8.8, de Vries relations dually correspond to de Vries morphisms. As with de Vries morphisms, because of the de Vries condition, the composition of two de Vries relations may not be a de Vries relation. Thus, for two de Vries relations $r_1 \subseteq X_1 \times X_2$ and $r_2 \times X_2 \times X_3$, we define $r_2 * r_1 \subseteq X_1 \times X_3$ as follows. Let $h_1 : \mathsf{Clop}(X_2) \to \mathsf{Clop}(X_1)$ be the dual of r_1 and $h_2 : \mathsf{Clop}(X_3) \to \mathsf{Clop}(X_2)$ be the dual of r_2. Let $h_3 = h_1 * h_2$ be the composition of h_1 and h_2 in the category **DeV** of de Vries algebras. Then $h_3 : \mathsf{Clop}(X_3) \to \mathsf{Clop}(X_1)$ is a de Vries morphism. Let $r_3 \subseteq X_1 \times X_3$ be the dual of h_3, and set $r_3 = r_2 * r_1$. With this composition, Gleason spaces and de Vries relations form a category we denote by **Gle**. We also let **KHaus** denote the category of compact Hausdorff spaces and continuous maps. The following is an immediate consequence of the above observations.

7.8.10. THEOREM. **Gle** *is dually equivalent to* **DeV***, hence* **Gle** *is equivalent to* **KHaus***.*

Thus, **Gle** is another dual category to **DeV**. This provides an alternative more "modal-like" duality to de Vries duality.

7.8.11. REMARK. The functor Φ : **Gle** $\to$ **KHaus** establishing an equivalence of **Gle** and **KHaus** can be constructed directly, without first passing to **DeV**. For $(X, R) \in$ **Gle**, let $\Phi(X, R) = X/R$. Clearly $X/R \in$ **KHaus**. For $r : Y \times X$ a morphism in **Gle**, let $\Phi(r) = f$, where $f : Y/R \to X/R$ is defined as follows. Let $\pi : X \to X/R$ be the quotient map. Since r is cofinal, for each $y \in Y$ there is $x \in X$ with yrx. We set $f(\pi(y)) = \pi(x)$, where yrx. Since r satisfies the forth condition, f is well defined, and as r is a Boolean relation, f is continuous. Thus, f is a morphism in **KHaus**. From this it is easy to see that Φ is a functor. We already saw that there is a 1-1 correspondence between Gleason spaces and compact Hausdorff spaces. The functor Φ is full because for each continuous function $f : Y \to X$ between compact Hausdorff spaces, $f = \Phi(r)$, where r is the de Vries relation corresponding to the de Vries dual of f. Finally, the functor is faithful because among the cofinal Boolean relations r that satisfy the forth condition and yield the same continuous function $f : Y \to X$ in **KHaus**, there is the largest one, which satisfies the de Vries condition. Consequently, by [121, Thm. IV.4.1], Φ : **Gle** $\to$ **KHaus** is an equivalence.

7.9 Conclusion

In this chapter, we discussed duality for Boolean algebras with subordination relations. We showed that the category of Boolean algebras with subordinations and subordination preserving Boolean homomorphisms is dually equivalent to the category of Stone spaces with closed relations and continuous stable maps. As a particular instance of our duality we obtained a new duality for de Vries algebras (that is, Boolean algebras with compingent relations). We also showed how to derive the de Vries duality between de Vries algebras and compact Hausdorff spaces from our duality.

Next step in this research is to develop logical calculi for the dualities discussed in this chapter. Balbiani et al. [6, 7] already have a number of interesting results in this direction. We aim to continue logical investigations of compact Hausdorff spaces via Boolean algebras with relations in a forthcoming in a future work. Another interesting direction would be to develop a *topological correspondence theory* which characterizes the class of axioms for subordinations corresponding to elementary conditions on the dual relation R on the topological space.

Chapter 8

Sahlqvist preservation for topological fixed-point logic

In this chapter, which is based on [29], we study correspondence and canonicity for modal fixed-point logic beyond the zero-dimensional setting of Stone spaces. By *topological fixed-point logic*, we mean a family of fixed-point logics that admit topological interpretations, and where the fixed-point operators are evaluated with respect to these topological interpretations. We concentrate on a variant of topological fixed-point logic whose models are modal compact Hausdorff spaces (MKH-spaces for short). These spaces were introduced in [24] as a generalization of modal spaces (descriptive frames), which are central order-topological structures appearing in modal logic. In [24], duality and various properties of MKH-spaces were studied for positive modal languages without any fixed-point operators. [27] studied topological fixed-point logic based on descriptive μ-frames. This is a restricted class of modal spaces (descriptive frames) that admits a topological interpretation of fixed-point operators. In this chapter, we investigate topological semantics of fixed-point operators (we consider only the least fixed-point operator) similar to the ones discussed in [27], but in the framework of MKH-spaces of [24]. This way the methods of [27] are extended to a wider class of models and the language of [24] is expanded by incorporating (topological) fixed-point operators.

Modal spaces, which are dual to Boolean algebra with operators, also admit a coalgebraic representation. The Vietoris space of closed sets of a Stone space [156], is a standard construction in topology. The construction naturally extends to an endofunctor on a Stone space. It turns out that the category of modal spaces and continuous p-morphisms, is isomorphic to the category of coalgebras for the Vietoris functor on the category of Stone spaces and continuous maps [1, 112]. The Vietoris functor, however, can be defined in the more general setting of compact Hausdorff spaces. An MKH-space is defined as a concrete realization of the Vietoris functor on a compact Hausdorff space. In particular, an MKH-space is a tuple (W, R) where W is a compact Hausdorff space and R

is a continuous relation on W, meaning the corresponding map from W to its Vietoris space is continuous. An example of an MKH-space is the interval $[0,1]$ with the binary relation $\leq$. It is well known that $[0,1]$ is compact and Hausdorff, but not zero-dimensional. In [24] modal compact regular frames and modal de Vries algebras were introduced as the algebraic structures dual to MKH-spaces, and a Sahlqvist preservation and correspondence result for the positive modal language was proved.

In this chapter, we advance the study of MKH-spaces by extending the positive modal language of [24] with fixed-point operators. We introduce and compare the different semantics of positive modal language extended with a least fixed-point operator over MKH-spaces. In modal spaces formulas are evaluated as clopen (both closed and open) sets. Note that clopen subsets, in general, do not form a complete lattice. Thus, there may exist fixed-point formulas that cannot be interpreted on a modal space as an intersection of *clopen pre-fixed points*. To overcome this, *descriptive mu-frames* (modal mu-spaces) were introduced in [3] as those descriptive frames that admit a topological interpretation of the least fixed-point operator. The main motivation to study this semantics is that every axiomatic system of modal mu-calculus is complete with respect to descriptive mu-frames [3]. Moreover, powerful Sahlqvist correspondence and completeness results hold for mu-calculus over descriptive mu-frames [27]. Unlike descriptive frames, every least fixed-point formula can be interpreted in an MKH-space as the interior of the intersection of *open pre-fixed points*. This makes MKH-spaces a natural candidate to study the topological semantics of fixed-point operators.

The key contributions of this chapter is a Sahlqvist preservation theorem for topological fixed-point logic over MKH-spaces. We define a Sahlqvist sequent in our language. By preservation, we mean the following: a Sahlqvist sequent in the language of positive modal logic with a least fixed-point operator is valid under arbitrary open assignments if, and only if, it is valid under arbitrary set-theoretic assignments. Since we are no longer in the setting of zero-dimensional spaces, the Sahlqvist preservation result in [27] fails for the clopen semantics for the fixed-point operator. We overcome this by introducing an alternative topological semantics where the pre-fixed point of a map f is defined as an open set U such that $f(\overline{U}) \subseteq U$, where $\overline{U}$ is the topological closure of a set U. We call such sets *topological pre-fixed points*. This alternative interpretation of fixed-point operators is different from the classical fixed-point operators, and thus enhances their expressivity. The alternative interpretation can be used for other (topological) interpretations of fixed-point operators (e.g., in logics for spatial reasoning).

The fixed-point is then computed as an intersection of all topological pre-fixed points. For this new semantics and *shallow modal formulas* we prove an analogue of Esakia's lemma, from which our preservation result follows immediately. We show that the new semantics has a nice algebraic counterpart when restricted to shallow modal formulas. Finally, we also show that the Sahlqvist sequent in our language has a frame correspondent in $\mathsf{FO} + \mathsf{LFP}$, which is the first-order language extended with fixed-point operators with topological interpretations. We also provide a few examples of Sahlqvist sequence and their corresponding $\mathsf{FO} + \mathsf{LFP}$-formulas on MKH-spaces.

Finally, we note on an unfortunate overlap of terminology in modal logic and point-free topology: the meaning of the term "frame" in modal logic differs from its meaning in point-free topology. By now both terms are well established in the modal logic and point-free topology literature. We follow these standard terminology hoping that it will not generate confusion. In particular, in Section 4 of the chapter we use the term "frame" in the context of point-free topology and in Section 6 we refer to "frame conditions" which have a standard meaning in the modal logic literature.

The chapter is organized as follows: in Section 2, we introduce preliminary definitions on Vietoris construction and MKH-spaces. In Section 3 we introduce and compare different semantics of the least fixed-point operator over MKH-spaces. In Section 4 we look into the algebraic semantics for our language. In Section 5 we show the Esakia's lemma and Sahlqvist preservation theorem. In Section 6, we prove a correspondence theorem for Sahlqvist sequents followed by examples in Section 7. We conclude and present directions for future research in Section 8.

8.1 Preliminaries

In this section, we recall a few preliminary definitions and results.

8.1.1. Definition. Let $\mathbf{C}$ be a category and let $\mathcal{T} : \mathbf{C} \to \mathbf{C}$ be an endofunctor. A *$\mathcal{T}$-coalgebra* is a pair (X, σ), where $\sigma : X \to \mathcal{T}X$ is a morphism in $\mathbf{C}$. A morphism between two coalgebras (X, σ) and (X', σ') is a morphism f in $\mathbf{C}$ such that the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\sigma"'] & X' \arrow[d, "\sigma'"] \\ \mathcal{T}X \arrow[r, "\mathcal{T}f"'] & \mathcal{T}X' \end{tikzcd}$$

8.1.2. Definition. For a topological space W and $U \subseteq W$ an open set, consider the sets

$$\Box U = \{F \subseteq W : F \text{ is closed and } F \subseteq U\}$$
$$\Diamond U = \{F \subseteq W : F \text{ is closed and } F \cap U \neq \varnothing\}.$$

Then the *Vietoris space* $\mathcal{V}(W)$ of W is defined to have the closed sets of W as its points, and the collection of all sets $\Box U, \Diamond U$, where $U \subseteq W$ is open, as a subbasis for its topology.

It is a standard result in topology that if W is a Stone space, then so is $\mathcal{V}(W)$ (see, eg., [62], p. 380). Let **Stone** be the category of Stone spaces and continuous maps. The Vietoris construction $\mathcal{V}$ extends to a functor $\mathcal{V}$: **Stone** $\to$ **Stone**, which sends a Stone space W to $\mathcal{V}(W)$ and a continuous map $f : W \to Y$ to $\mathcal{V}(f)$ where $\mathcal{V}(f)(F) = f[F]$ for all closed sets $F \subseteq W$. In considering $\mathcal{V}$-coalgebras, note that if R is a relation on W, then $\rho_R : W \to \mathcal{P}(W)$ given by $\rho_R(w) = R[w]$ is a well-defined continuous map from W to $\mathcal{V}(W)$ iff (W, R) is a modal space. This leads to the following theorem.

8.1.3. Theorem ([1, 112, 64]). **MS** *is isomorphic to the category of* $\mathcal{V}$*-coalgebras on* **Stone***.*

It is known that the Vietoris functor can be defined in the more general setting of compact Hausdorff spaces (see, e.g., [62], p. 244). The category of compact Hausdorff spaces and continuous maps is denoted by **KHaus**. The Vietoris construction yields a functor $\mathcal{V}$: **KHaus** $\to$ **KHaus** where a continuous map $f : W \to Y$ is taken to $\mathcal{V}(f)$ with $\mathcal{V}(f)(F) = f[F]$ for all closed sets $F \subseteq W$. It is natural to consider coalgebras for this functor. We first define the notion of a continuous relation on a compact Hausdorff space.

8.1.4. Definition. A relation R on a compact Hausdorff space W is *point closed*, if the relational image $R[w]$ is a closed set for each $w \in W$. Further, R is *continuous* if it is point closed and the map $\rho_R : W \to \mathcal{V}(W)$, taking a point w to $R[w]$ is a continuous map from the space W to its Vietoris space $\mathcal{V}(W)$. In other words, R is continuous if (X, ρ_R) is a Vietoris coalgebra.

8.1.5. Proposition. *([24]) A relation R on a compact Hausdorff space W is continuous iff R satisfies the following conditions:*

1. *$R[w]$ is closed for each $w \in W$.*
2. *$R^{-1}[F]$ is closed for each closed $F \subseteq W$.*
3. *$R^{-1}[U]$ is open for each open $U \subseteq W$.*

8.1.6. Definition. A *modal compact Hausdorff space* or an MKH-space is a tuple (W, R) such that W is a compact Hausdorff space and R is a continuous relation on W. Let **MKHaus** be the category of MKH-spaces and continuous p-morphisms.

8.1.7. Theorem ([24]). **MKHaus** *is isomorphic to the category of* $\mathcal{V}$*-coalgebras on* **KHaus***.*

8.2 Topological fixed-point semantics

In this section, we discuss various semantics for the modal mu-calculus on modal compact Hausdorff spaces. We restrict our language to positive modal logic. Given a set Prop of countably infinite propositional variables, the *modal mu-formulas* in our language are inductively defined by the following rule

$$\varphi := \bot \mid \top \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Diamond\varphi \mid \Box\varphi \mid \mu x\varphi$$

where $p, x \in \mathsf{Prop}$. Note that we have only the least fixed-point operator in our language. An occurrence of x in φ is said to be bound if it is in the scope of a μx, and free, otherwise. We interpret formulas in our language over MKH-spaces. Given an MKH-space (W, R), let $\mathcal{F} \subseteq \mathcal{P}(W)$ be such that $(\mathcal{F}, \subseteq)$ is a sublattice[1] of $(\mathcal{P}(W), \subseteq)$. That is, $\varnothing, W \in \mathcal{F}$ and if $U, V \in \mathcal{F}$, then $U \cap V \in \mathcal{F}$ and $U \cup V \in \mathcal{F}$. We denote the (infinite) meets and joins in $\mathcal{F}$ by $\bigwedge^{\mathcal{F}}$ and $\bigvee^{\mathcal{F}}$, respectively. If $(\mathcal{F}, \subseteq)$ is complete, then infinite meets and joins always exist. As we will see below, $\bigwedge^{\mathcal{F}}$ and $\bigvee^{\mathcal{F}}$ may differ from set-theoretic intersection and union. An *assignment* h is a map from the set of propositional variables Prop to $\mathcal{F}$. For each modal mu-formula φ, we denote by $[\![\varphi]\!]_h^{\mathcal{F}}$, the set of points satisfying φ under assignment h. Given $S \subseteq W$, let $\langle R\rangle(S) = R^{-1}[S]$ and $[R](S) = W \backslash (R^{-1}[W \backslash S])$. We define the semantics of a modal mu-formula φ, by induction on the complexity of formulas as follows:

$$\begin{aligned}
[\![\bot]\!]_h^{\mathcal{F}} &= \varnothing, \\
[\![\top]\!]_h^{\mathcal{F}} &= W, \\
[\![p]\!]_h^{\mathcal{F}} &= h(p), \\
[\![\varphi \wedge \psi]\!]_h^{\mathcal{F}} &= [\![\varphi]\!]_h^{\mathcal{F}} \cap [\![\psi]\!]_h^{\mathcal{F}}, \\
[\![\varphi \vee \psi]\!]_h^{\mathcal{F}} &= [\![\varphi]\!]_h^{\mathcal{F}} \cup [\![\psi]\!]_h^{\mathcal{F}}, \\
[\![\Diamond\varphi]\!]_h^{\mathcal{F}} &= \langle R\rangle([\![\varphi]\!]_h^{\mathcal{F}}), \\
[\![\Box\varphi]\!]_h^{\mathcal{F}} &= [R]([\![\varphi]\!]_h^{\mathcal{F}}),
\end{aligned}$$

where $p \in \mathsf{Prop}$.

Let $\varphi(x, p_1, \ldots, p_n)$ be a modal mu-formula. The semantics of φ is defined for all assignments h using the definition above. For a fixed assignment h, φ and h give rise to a map $f_{\varphi,h} : \mathcal{F} \to \mathcal{F}$ defined by $f_{\varphi,h}(U) = [\![\varphi]\!]_{h_x^U}^{\mathcal{F}}$, where $U \in \mathcal{F}$, $h_x^U(x) = U$ and $h_x^U(y) = h(y)$ for each propositional variable $y \neq x$. Since we have restricted our language to positive modal formulas, $f_{\varphi,h}$ is a monotone map with respect to the inclusion order. Assume that $(\mathcal{F}, \subseteq)$ is a complete lattice. Therefore, by the Knaster-Tarski theorem (cf. Theorem 5.1.1), $f_{\varphi,h}$ has a

[1] Note that this requirement is not essential (see Remark 8.2.2), but it always holds in the examples that we consider in this chapter. So we find it convenient to make this restriction.

least fixed-point. We define $[\![\mu x\varphi]\!]_h^{\mathcal{F}}$ to be the least fixed-point of $f_{\varphi,h}$, which, is computed as follows

$$[\![\mu x\varphi]\!]_h^{\mathcal{F}} = \bigwedge^{\mathcal{F}}\{U \in \mathcal{F} : [\![\varphi]\!]_{h_x^U}^{\mathcal{F}} \subseteq U\}.$$

A set $U \in \mathcal{F}$ such that $[\![\varphi]\!]_{h_x^U}^{\mathcal{F}} \subseteq U$ is called a *pre-fixed point*.

Note that the powerset $(\mathcal{P}(W), \subseteq)$ is a complete lattice where meets and joins are set-theoretic intersections and unions. Therefore, if $\mathcal{F} = \mathcal{P}(W)$, then

$$[\![\mu x\varphi]\!]_h^{\mathcal{P}(W)} = \bigcap\{U \in \mathcal{P}(W) : [\![\varphi]\!]_{h_x^U}^{\mathcal{P}(W)} \subseteq U\}.$$

In the complete lattice $(\mathsf{Cl}(W), \subseteq)$ of closed sets of a topological space, infinite meets are intersections and infinite joins are the closure of the union. Thus, if $\mathcal{F} = \mathsf{Cl}(W)$, then

$$[\![\mu x\varphi]\!]_h^{\mathsf{Cl}(W)} = \bigcap\{U \in \mathsf{Cl}(W) : [\![\varphi]\!]_{h_x^U}^{\mathsf{Cl}(W)} \subseteq U\}.$$

Finally, in the complete lattice $(\mathsf{Op}(W), \subseteq)$ of open sets of a topological space infinite meets are the interior of the intersection and joins are unions. Thus, if $\mathcal{F} = \mathsf{Op}(W)$, then

$$[\![\mu x\varphi]\!]_h^{\mathsf{Op}(W)} = \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : [\![\varphi]\!]_{h_x^U}^{\mathsf{Op}(W)} \subseteq U\}\right),$$

where Int is the interior operator.

If $\mathcal{F} = \mathcal{P}(W)$, then $[\![.]\!]_h^{\mathcal{F}}$ is called *classical* or *set-theoretic semantics*. If $\mathcal{F} = \mathsf{Cl}(W)$, then $[\![.]\!]_h^{\mathcal{F}}$ is called *closed semantics*, and if $\mathcal{F} = \mathsf{Op}(W)$, then $[\![.]\!]_h^{\mathcal{F}}$ is called *open semantics*. The assignment h is called a *set-theoretic assignment* if $h(p) \in \mathcal{P}(W)$, *closed* if $h(p) \in \mathsf{Cl}(W)$, and *open* if $h(p) \in \mathsf{Op}(W)$, for each $p \in \mathsf{Prop}$.

The following example illustrates how to compute modal mu-formulas in MKH-spaces.

8.2.1. Example. Consider the interval $[0, 1] \subseteq \mathbb{R}$ with the subspace topology. It is an example of a compact Hausdorff space which is not totally disconnected. The only clopen sets are $[0, 1]$ and $\varnothing$. Consider the relation $\leq$ on this space which gives, $\leq [a] = [a, 1]$, which shows that $\leq$ is point closed. Also, for an open set $U \subseteq [0, 1]$ with supremum b we have $\langle\leq\rangle U = [0, b)$, which is open in the subspace topology. Checking that $\langle\leq\rangle$ of a closed set is closed is similar. Therefore, the relation $\leq$ satisfies the conditions of the Proposition 8.1.5, which shows $([0, 1], \leq)$ is an MKH-space. Moreover, it is not a modal space.

Consider a modal mu-formula, $\mu x(p \vee \Diamond x)$ with the open assignment of p given by $h(p) = (\frac{1}{3}, \frac{2}{3})$. The valuation for the formula is given by

$$[\![\mu x(\, p \vee \Diamond x)]\!]_h^{\mathsf{Op}(W)} = \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : h(p) \cup \langle\leq\rangle U \subseteq U\}\right).$$

As noted above, for an open set $U \subseteq [0,1]$ with supremum b we have $\langle\leq\rangle U = [0,b)$. The only open sets U which satisfy $h(p) \cup \langle\leq\rangle U \subseteq U$, are the ones which are of the form $[0,b)$ and contain $h(p)$. The interior of the intersection of all such sets will be the set $[0,\frac{2}{3})$, which is the least fixed-point of the formula.

8.2.2. Remark. The requirement that $(\mathcal{F},\subseteq)$ is a complete lattice is not necessary for interpreting fixed-point operators. It is sufficient to demand that the meet of the sets of type $\{U \in \mathcal{F} : [\![\varphi]\!]^{\mathcal{F}}_{h^U_x} \subseteq U\}$, for each φ and h, exist in $\mathcal{F}$. The lattice $(\mathcal{F},\subseteq)$ may not be complete, but such meets may still exist in $\mathcal{F}$. For example, for a modal space (W,R) the lattice $(\mathsf{Clop}(W),\subseteq)$ of its clopen sets may not be complete. *Descriptive mu-frames* are those modal spaces where meets of the sets of type $\{U \in \mathcal{F} : [\![\varphi]\!]^{\mathcal{F}}_{h^U_x} \subseteq U\}$, for each φ and h, are clopen, see e.g., [3] and [27]. They provide completeness for any axiomatic system of modal mu-calculus. Moreover, a version of Sahlqvist theorem holds for descriptive mu-frames [27]. We view MKH-spaces as generalizations of descriptive mu-frames. Similarly the results in this chapter generalize the results of [27] to the case of MKH-spaces.

8.2.3. Remark. Also, note that the regular open (closed) sets of a topological space form a complete Boolean algebra [62]. These sets provide important topological structures for interpreting modal mu-formulas. Note that these Boolean algebras are not sublattices of the powerset Boolean algebra, see e.g., [62]. As already noted in the footnote in the previous page, the demand that $(\mathcal{F},\subseteq)$ is a sublattice of the powerset, is made only for convenience and could be easily dropped in order to accommodate interesting examples such as regular open (closed) sets. Since we do not consider regular open and closed sets in this chapter, we are going to keep this restriction.

The key property of MKH-spaces is that modal operators $\Box$ and $\Diamond$ can be interpreted on open sets. The next theorem shows that modal mu-formulas can also be interpreted on open sets of an MKH-space.

8.2.4. Theorem. *The open semantics of modal mu-formulas is well-defined, that is, if h is an open assignment, then $[\![\varphi]\!]^{\mathsf{Op(W)}}_h$ is an open set, for any modal mu-formula φ.*

Proof. The proof is by induction on the complexity of φ. In the base case, when $\varphi = \top, \bot$ or $p \in \mathsf{Prop}$, $[\![\varphi]\!]^{\mathsf{Op}(W)}_h$ is an open set, since $\varnothing$, W are open sets and $h(p)$ is an open assignment. For the induction step if $\varphi = \varphi_1 \vee \varphi_2$ or $\varphi_1 \wedge \varphi_2$, $[\![\varphi]\!]^{\mathsf{Op}(W)}_h$ is also open since finite intersection and union of open sets is open. If $\varphi = \Diamond\psi$, $[\![\Diamond\psi]\!]^{\mathsf{Op}(W)}_h = \langle R\rangle([\![\psi]\!]^{\mathsf{Op}(W)}_h)$, which is open by Proposition 8.1.5, as $[\![\psi]\!]^{\mathsf{Op}(W)}_h$ is open by induction hypothesis. The case when $\varphi = \Box\psi$ is similar and uses the fact that $[R]U$ is open for an open U. Finally, if $\varphi = \mu x\psi$, since we define the semantics of $\mu x\psi$ to be equal to the interior of an intersection of open sets, $[\![\varphi]\!]^{\mathsf{Op}(W)}_h$

will be an open set. □

In order to simplify the notation, instead of $[\![\varphi(p_1,\dots,p_n)]\!]^{\mathcal{F}}_h$ with $h(p_i) = U_i, 1 \leq i \leq n$, we will sometimes simply write $\varphi(U_1,\dots,U_n)^{\mathcal{F}}$ or just $\varphi(U_1,\dots,U_n)$ if it is clear from the context. We now show that the semantics for $\mu x\varphi$ defined above, gives the least fixed-point of φ.

8.2.5. Lemma. *Let (W,R) be an MKH-space, $\mathcal{F} \subseteq \mathcal{P}(W)$ a complete lattice and h such an assignment that $[\![\varphi]\!]^{\mathcal{F}}_h \in \mathcal{F}$ for each modal mu-formula φ. Then the valuation function is monotone, that is, for $U, V \in \mathcal{F}$ such that $U \subseteq V$, for every modal mu-formula, $[\![\varphi]\!]^{\mathcal{F}}_{h^U_x} \subseteq [\![\varphi]\!]^{\mathcal{F}}_{h^V_x}$.*

Proof. The above lemma can be proved by induction on the complexity of the formula φ. The basic modal cases are well known. For the case when $\varphi = \mu y\psi$, we want to show that for $U \subseteq V$ we have $[\![\mu y\psi]\!]^{\mathcal{F}}_{h^U_x} \subseteq [\![\mu y\psi]\!]^{\mathcal{F}}_{h^V_x}$. By induction hypothesis, we have $[\![\psi]\!]^{\mathcal{F}}_{h^U_x} \subseteq [\![\psi]\!]^{\mathcal{F}}_{h^V_x}$. This means that for each $C \in \mathcal{F}$, if $[\![\psi]\!]^{\mathcal{F}}_{h^V_x} \subseteq C$, then $[\![\psi]\!]^{\mathcal{F}}_{h^U_x} \subseteq C$. Therefore, $\bigwedge\{C : [\![\psi]\!]^{\mathcal{F}}_{f^C_y} \subseteq C\} \subseteq \bigwedge\{C : [\![\psi]\!]^{\mathcal{F}}_{g^C_y} \subseteq C\}$, where $f = h^U_x$ and $g = h^V_x$. Hence, $[\![\mu x\psi]\!]^{\mathcal{F}}_{h^U_x} \subseteq [\![\mu x\psi]\!]^{\mathcal{F}}_{h^V_x}$, and $[\![\varphi]\!]^{\mathcal{F}}_{h^U_x} \subseteq [\![\varphi]\!]^{\mathcal{F}}_{h^V_x}$. □

8.2.6. Theorem. *For a modal mu-formula φ, the map given by $(U \mapsto [\![\varphi]\!]^{\mathsf{Op}(W)}_{h^U_x})$, where h is an open assignment, has the least fixed-point $[\![\mu x\varphi]\!]^{\mathsf{Op}(W)}_h$.*

Proof. We know that $\mathsf{Op}(W)$ is a complete lattice, and $[\![\varphi]\!]^{\mathsf{Op}(W)}_{h^U_x}$ is monotone as shown in the previous lemma. Therefore, from the Knaster-Tarski theorem, it follows that $[\![\mu x\varphi]\!]^{\mathsf{Op}(W)}_h$ is its least fixed-point.

□

8.2.1 Open fixed-point semantics

In this section, we focus on the open semantics for the least fixed-point operator. We first prove the following theorem which shows that if we restrict ourselves to open assignments, the interpretation of any modal mu-formula under the set-theoretic semantics is the same as in the open semantics.

8.2.7. Theorem. *Let (W,R) be an MKH-space and h be an open assignment. Then, for each for each modal mu-formula φ, $[\![\varphi]\!]^{\mathcal{P}(W)}_h = [\![\varphi]\!]^{\mathsf{Op}(W)}_h$.*

Proof. We prove the lemma by induction on the complexity of formulas. The cases $\varphi = \top$ or $\bot$, $\varphi = \varphi_1 \wedge \varphi_2$ or $\varphi_1 \vee \varphi_2$, $\varphi = \Diamond\psi$ or $\Box\psi$ are obvious. Now assume $\varphi = \mu x\psi$ and suppose the result holds for ψ. We let $f_{\psi,h}$ and $g_{\psi,h}$ be a map such that $f_{\psi,h}(U) = [\![\psi]\!]^{\mathsf{Op(W)}}_{h^U_x}$ and $g_{\psi,h}(U) = [\![\psi]\!]^{\mathcal{P}(W)}_{h^U_x}$.

We have seen earlier that the least fixed-point can also be computed as the limit of the following increasing sequence of sets,

$$\varnothing \subseteq f_{\psi,h}(\varnothing) \subseteq f^2_{\psi,h}(\varnothing) \subseteq \dots f^{\alpha}_{\psi,h}(\varnothing) \dots$$

$$\varnothing \subseteq g_{\psi,h}(\varnothing) \subseteq g^2_{\psi,h}(\varnothing) \subseteq \dots g^{\alpha}_{\psi,h}(\varnothing) \dots$$

By the induction hypothesis $f_{\psi,h}(U) = g_{\psi,h}(U)$, for each $U \in \mathsf{Op}(\mathsf{W})$. So $f^n_{\psi,h}(\varnothing) = g^n_{\psi,h}(\varnothing)$, for each $n \in \omega$. As h is an open assignment, each $f^n_{\psi,h}(\varnothing)$ is an open set. So their join is just the union. Thus, $f^{\omega}_{\psi,h}(\varnothing) = \bigcup_{n\in\omega} f^n_{\psi,h}(\varnothing) = \bigcup_{n\in\omega} g^n_{\psi,h}(\varnothing) = g^{\omega}_{\psi,h}(\varnothing)$. Continuing this process transfinitely we obtain that for each ordinal α we have $f^{\alpha}_{\psi,h}(\varnothing) = g^{\alpha}_{\psi,h}(\varnothing)$. This implies that $[\![\mu x\psi]\!]^{\mathsf{Op(W)}}_h = [\![\mu x\psi]\!]^{\mathcal{P}(W)}_h$.

□

Note that the above theorem holds only when h is open. In the following we will be dealing with assignments that in general are not open. For such assignments, the above theorem may not hold as Example 8.2.8 below shows.

8.2.8. Example. Consider the interval $I = [0,1] \subseteq \mathbb{R}$ with the subspace topology. Note that this is an MKH-space. We compute the fixed-point of the modal mu-formula $\varphi = \mu x(p \vee x)$ on this interval with an assignment $h(p) = [\frac{1}{2}, \frac{2}{3})$ which is not open. It is easy to see that the least fixed-point of φ, with the set-theoretic semantics is $[\frac{1}{2}, \frac{2}{3})$. In case of open semantics, the least fixed-point is the interval $(\frac{1}{2}, \frac{2}{3})$. So, this example shows that, if the assignment is not open, then the least fixed-point of a modal mu-formula may not be the same in set-theoretic and open semantics.

We now show that the semantics of the least fixed-point operator simplifies in the case of open semantics and open assignments. To this end, we define a new semantics $||\varphi||^{\mathcal{F}}_h$, where $\mathcal{F} \subseteq \mathcal{P}(W)$ is complete. It agrees with $[\![\cdot]\!]$ on all clauses except for the one for the fixed-point operator which we define as follows

$$||\mu x\varphi||^{\mathcal{F}}_h = \bigcap\{U \in \mathcal{F} : ||\varphi||^{\mathcal{F}}_{h^U_x} \subseteq U\}.$$

8.2.9. Lemma. *Let (W,R) be an MKH-space, $\mathcal{F} \subseteq \mathcal{P}(W)$ a complete lattice, and h be an assignment such that $||\varphi||^{\mathcal{F}}_h \in \mathcal{F}$ for every modal mu-formula φ. Then the valuation function is monotone, that is, for $U,V \in \mathcal{F}$ such that $U \subseteq V$, for every modal mu-formula φ, $||\varphi||^{\mathcal{F}}_{h^U_x} \subseteq ||\varphi||^{\mathcal{F}}_{h^V_x}$.*

Proof. Similar to the proof of Lemma 8.2.5. □

8.2.10. THEOREM. *Let (W, R) be an MKH-space and $\mathcal{F} \subseteq \mathcal{P}(W)$ be a complete sublattice*[2]*. If h is an arbitrary assignment such that $[\![\varphi]\!]_h^{\mathcal{F}} \in \mathcal{F}$ for each modal mu formula φ, then $[\![\varphi]\!]_h^{\mathcal{F}} = ||\varphi||_h^{\mathcal{F}}$ for each modal mu-formula φ.*

Proof. We prove the theorem by induction on the complexity of φ. Suppose $\varphi = \mu x\psi$. By induction hypothesis, $[\![\psi]\!]_h^{\mathcal{F}} = ||\psi||_h^{\mathcal{F}}$ for any assignment h such that $[\![\varphi]\!]_h^{\mathcal{F}} \in \mathcal{F}$. From definition, $[\![\mu x\psi]\!]_h^{\mathcal{F}} = \bigwedge^{\mathcal{F}}\{U \in \mathcal{F} : [\![\psi]\!]_{h_x^U}^{\mathcal{F}} \subseteq U\}$. Since $[\![\mu x\psi]\!]_h^{\mathcal{F}} \subseteq U$ for each pre-fixed point $U \in \mathcal{F}$ and $[\![\psi]\!]_{h_x^U}^{\mathcal{F}} = ||\psi||_{h_x^U}^{\mathcal{F}}$,

$$[\![\mu x\psi]\!]_h^{\mathcal{F}} \subseteq \bigcap\{U \in \mathcal{F} : ||\psi||_{h_x^U}^{\mathcal{F}} \subseteq U\} = ||\mu x\psi||_h^{\mathcal{F}}.$$

For the converse inclusion, let U be a pre-fixed point of ψ with respect to the semantics $||\cdot||$. By induction hypothesis, U is a pre-fixed point of ψ with respect to the semantics $[\![\cdot]\!]$. Using Lemma 8.2.5, the induction hypothesis and $[\![\mu x\psi]\!]_h^{\mathcal{F}} \subseteq U$, it follows that $[\![\psi]\!]_{h_x^{[\![\mu x\psi]\!]}}^{\mathcal{F}} \subseteq [\![\psi]\!]_{h_x^U}^{\mathcal{F}}$, and $||\psi||_{h_x^U}^{\mathcal{F}} \subseteq U$. But this implies that $[\![\psi]\!]_{h_x^{[\![\mu x\psi]\!]}}^{\mathcal{F}} \subseteq \bigwedge^{\mathcal{F}}\{U \in \mathcal{F} : [\![\psi]\!]_{h_x^U}^{\mathcal{F}} \subseteq U\} = [\![\mu x\psi]\!]_h^{\mathcal{F}}$. Since $||\psi||_{h_x^{[\![\mu x\psi]\!]}}^{\mathcal{F}} = [\![\psi]\!]_{h_x^{[\![\mu x\psi]\!]}}^{\mathcal{F}}$, it follows that $||\psi||_{h_x^{[\![\mu x\psi]\!]}}^{\mathcal{F}}$ is a pre-fixed point. Moreover, since $[\![\psi]\!]_{h_x^{[\![\mu x\psi]\!]}}^{\mathcal{F}} \in \mathcal{F}$, by our induction hypothesis, $||\psi||_{h_x^{[\![\mu x\psi]\!]}}^{\mathcal{F}}$ belongs to $\mathcal{F}$. Hence, $\bigcap\{U \in \mathcal{F} : ||\psi||_{h_x^U}^{\mathcal{F}} \subseteq U\} \subseteq [\![\mu x\psi]\!]_h^{\mathcal{F}}$. This finishes the proof of the theorem.

□

8.2.11. COROLLARY. *Let (W, R) be an MKH-space. If h is an open assignment, then*

$$[\![\mu x\varphi]\!]_h^{\mathsf{Op}(W)} = \bigcap\{U \in \mathsf{Op}(W) : [\![\varphi]\!]_{h_x^U}^{\mathsf{Op}(W)} \subseteq U\}.$$

Proof. The result follows directly form Theorems 8.2.4 and 8.2.10. □

By Theorem 8.2.7, the open semantics for open assignments coincides with the classical semantics. However, in this chapter, we are more interested in the topological semantics of fixed-point operators. Moreover, we aim at proving an analogue of the Sahlqvist theorem of [27]. For this purpose, it is essential to prove an analogue of Esakia's lemma. As we will show in Section 5.1 Esakia's lemma fails for the open semantics considered above. We remedy this by introducing a new topological semantics for fixed-point operators. For this we will first need to recall from [24] the algebraic semantics and duality for MKH-spaces.

8.3 Algebraic semantics

A duality between compact Hausdorff spaces and compact regular frames was established by Isbell [95] (see also [96]). In [24], Isbell duality was extended to

[2]By a complete sublattice, we mean a sublattice which is complete.

a duality between modal compact Hausdorff spaces and modal compact regular frames. We briefly recall this duality and later show that the duality extends to the language of positive modal mu-calculus.

8.3.1. Definition. A *frame* L is a complete lattice that satisfies $a \wedge \bigvee S = \bigwedge\{a \wedge s \mid s \in S\}$, where $S \subseteq L$. It is *compact* if whenever $\bigvee S = 1$, there is a finite subset $T \subseteq S$ with $\bigvee T = 1$. A map $f : L \to M$ between frames is a *frame homomorphism* if it preserves finite meets and arbitrary joins.

Suppose L is a frame. For each $a \in L$ there is a largest element of L whose meet with a is zero, called the *pseudocomplement* of a and written $\neg a$. For $a, b \in L$ we say a is well inside b and write $a \prec b$ if $\neg a \vee b = 1$. We say L is *regular* if $a = \bigvee\{b \mid b \prec a\}$ for each $a \in L$.

Given a topological space X, the collection $\mathsf{Op}(X)$ of all open sets of X is a frame. For a continuous map $f : X \to Y$ between spaces, define $\Omega f = f^{-1} : \mathsf{Op}(Y) \to \mathsf{Op}(X)$. It can be checked that Ω is a contravariant functor from the category of topological spaces to the category of frames. Given a frame L, a filter $F \subseteq L$ is called *complete* if $\bigvee A \in F$ implies that there is $a \in A$ such that $a \in F$. The set $\mathfrak{p}L$ of complete filters forms a topological space with the basis $\alpha(a) = \{x \in \mathfrak{p}L \mid a \in x\}$ where $a \in L$.

For a frame homomorphism $h : L \to M$, the map $\mathfrak{p}h : \mathfrak{p}M \to \mathfrak{p}L$ sending a $x \in \mathfrak{p}M$ to $h^{-1}(x)$ is well defined and continuous. $\mathfrak{p}$ is a contravariant functor between the category of frames and the category of topological spaces. The functors Ω and $\mathfrak{p}$ give dual equivalence when we restrict them to appropriate subcategories.

8.3.2. Theorem (Isbell duality [95, 96]). *The functors Ω and $\mathfrak{p}$ provide a dual equivalence between the category* **KHaus** *of compact Hausdorff spaces and continuous maps and the category* **KRFrm** *of compact regular frames and frame homomorphisms.*

8.3.3. Definition. A *modal compact regular frame* (abbreviated: *MKR-frame*) is a triple $L = (L, \Box, \Diamond)$ where L is a compact regular frame, and $\Box, \Diamond$ are unary operations on L satisfying the following conditions.

1. $\Box$ preserves finite meets, so $\Box 1 = 1$ and $\Box(a \wedge b) = \Box a \wedge \Box b$.

2. $\Diamond$ preserves finite joins, so $\Diamond 0 = 0$ and $\Diamond(a \vee b) = \Diamond a \vee \Diamond b$.

3. $\Box(a \vee b) \leq \Box a \vee \Diamond b$ and $\Box a \wedge \Diamond b \leq \Diamond(a \wedge b)$.

4. $\Box, \Diamond$ preserve directed joins, so $\Diamond \bigvee S = \bigvee\{\Diamond s \mid s \in S\}, \Box \bigvee S = \bigvee\{\Box s \mid s \in S\}$ for any up-directed S.

For MKR-frames $L = (L, \Box, \Diamond)$ and $M = (M, \Box, \Diamond)$, an *MKR-morphism* from L to M is a frame homomorphism $h : L \to M$ that satisfies $h(\Box a) = \Box h(a)$ and $h(\Diamond a) = \Diamond h(a)$ for each $a \in L$. Let **MKRFrm** be the category whose objects are MKR-frames and whose morphisms are MKR-morphisms.

8.3.4. Definition. ([24]) For $\mathcal{M} = (W, R)$ an MKH-space, $\Omega\mathcal{M} = (\mathsf{Op}(W), [R], \langle R\rangle)$. For a continuous p-morphism $f : W \to V$ between MKH-spaces (W, R) and (W', R') define $\Omega f : \mathsf{Op}(W') \to \mathsf{Op}(W)$ by $\Omega f = f^{-1}$.

8.3.5. Definition. For $\mathcal{L} = (L, \Box, \Diamond)$ an MKR-frame, $\mathfrak{p}\mathcal{L} = (W, R)$ where $W = \mathfrak{p}L$ and R is a relation on W defined by PRQ iff $a \in Q$ implies $\Diamond a \in P$ for all $a \in L$ (alternatively, by $\Box a \in P$ implies $a \in Q$). For a modal frame homomorphism $h : L \to M$, between MKR-frames $\mathcal{L} = (L, \Box, \Diamond)$ and $\mathcal{M} = (M, \Box, \Diamond)$ we define $\mathfrak{p}h : \mathfrak{p}M \to \mathfrak{p}L$ as $(\mathfrak{p}h) = h^{-1}$.

8.3.6. Theorem ([24]). *The functors Ω and $\mathfrak{p}$ defined above, provide a dual equivalence between* **MKHaus** *and* **MKRFrm**.

The positive modal mu-formulas in our language can be interpreted over a modal compact regular frame $\mathcal{L} = (L, \Box, \Diamond)$. An algebra assignment h is a map from propositional variables to L. The semantics of propositional connectives are given in a standard way. The formulas $\Box\varphi$ and $\Diamond\varphi$ are interpreted using $\Box$ and $\Diamond$ in $\mathcal{L}$. Let h_x^a denote the map which agrees with h on all variables except for x and which maps x to a. The semantics of $\mu x\varphi$ is given by

$$[\mu x\varphi]_h^L = \bigwedge\{a \in L : [\varphi]_{h_x^a} \leq a\}.$$

Using the Knaster-Tarski theorem, it is easy to see that $[\mu x\varphi]_h$ is the least fixed-point of the map given by $(a \mapsto [\varphi]_{h_x^a})$.

The next theorem shows that computing a modal mu-formula φ in (W, R) or algebraically in its dual frame yields the same result.

8.3.7. Theorem. *Let (W, R) be an MKH-space and $(\mathsf{Op}(W), \Box, \Diamond)$ be the dual MKR-frame. For each modal mu-formula φ and open assignment h, we have $[\varphi]_h^{\mathsf{Op}(W)} = [\![\varphi]\!]_h^{\mathsf{Op}(W)}$.*

Proof. The proof is by induction on the complexity of φ. For the propositional and modal cases we refer to the modal Isbell duality in [24, Prop. 3.10]. If $\varphi = \mu x\psi(x, p_1, \ldots, p_n)$, by induction hypothesis $[\psi]_h^L = [\![\psi]\!]_h^{\mathsf{Op}(W)}$. Let $\mathbf{U} = \{U \in \mathsf{Op}(W) : [\![\psi]\!]_{h_x^U} \subseteq U\}$. The result now follows from the fact that $\bigwedge \mathbf{U} = \mathrm{Int}(\bigcap \mathbf{U})$, which is true because in $\mathsf{Op}(W)$ the meet is the interior of the intersection. $\Box$

We now introduce an alternative semantics for $\mu x\varphi$ as follows.

8.3.8. Definition. Let $(L, \Box, \Diamond)$ be an MKR-frame and h an assignment. For each propositional and modal connective, the alternative semantics $[.]_h^{L'}$ is eaual to the standard semantics $= [.]_h$. For a least fixed-point formula, we define

$$[\mu x \varphi]_h^{L'} = \bigwedge \{a \in L : \exists b \in L \text{ s.t. } a \prec b \text{ and } [\varphi]_{h_x^b}^{L'} \leq a\}.$$

We will now define its topological counter-part.

8.3.9. Definition. Let (W, R) be an MKH-space and h an open assignment. For each modal formula φ we let $[\![\varphi]\!]_h^{\mathsf{Op}(W)'} = [\![\varphi]\!]_h^{\mathsf{Op}(W)}$ and we let

$$[\![\mu x \varphi]\!]_h^{\mathsf{Op}(W)'} = \operatorname{Int} \bigcap \{U \in \mathsf{Op}(W) : \exists V \in \mathsf{Op}(W) \text{ s.t. } \overline{U} \subseteq V \text{ and } [\![\varphi]\!]_{h_x^V}^{\mathsf{Op}(W)'} \subseteq U\}.$$

The next theorem shows that the two new interpretations of the fixed-point operator coincide for MKH-spaces.

8.3.10. Theorem. *Let (W, R) be an MKH-space and $(\mathsf{Op}(W), \Box, \Diamond)$ be the dual MKR-frame. For any modal mu-formula φ we have $[\varphi]_h^{\mathsf{Op}(W)'} = [\![\varphi]\!]_h^{\mathsf{Op}(W)'}$.*

Proof. We prove the theorem by induction on the complexity of φ. We only consider the case $\varphi = \mu x \psi$. First note that in the frame $\mathsf{Op}(W)$ for $U, V \in \mathsf{Op}(W)$ we have $U \prec V$ iff $\overline{U} \subseteq V$. The rest of the proof follows from duality and the fact that meets in $\mathsf{Op}(W)$ are the interior of the intersection. $\Box$

We will use this new algebraic interpretation of the fixed-point operator in the next section. In particular, we will give yet another (topological) interpretation of the fixed-point operator. But we will show that in some important cases the topological and algebraic interpretations of the fixed-point operator coincide.

8.4 Sahlqvist preservation

In this section, we define Sahlqvist sequents in our language and prove a preservation result for these sequents using Esakia's lemma. We begin by introducing an alternative topological semantics for the fixed-point operator.

8.4.1 An alternative fixed-point semantics

In case of classical modal logic, Esakia's lemma shows that in modal spaces the valuation of a positive formula φ on a closed set is equal to the intersection of valuations of φ on clopen sets containing this closed set [64], [138]. This was extended in [27] to positive modal mu-formulas and descriptive mu-frames. An analogue of Esakia's lemma for MKH-spaces and positive modal formulas was proved in [24]. In the case of MKH-spaces clopen sets are replaced by open sets. First, we show that an analogue of Esakia's lemma does not hold for the open semantics defined in Section 3. This motivates the introduction of a new topological semantics for fixed-point operators for which a fixed-point analogue of Esakia's lemma will be shown in Section 8.4.2.

8.4.1. Example. Consider an MKH-space $([0,1],\leq)$ and a modal mu-formula $\mu x(p\vee x)$, such that $h(p)=[\frac{1}{2},1]$. The least fixed-point of the formula is computed as the interior of the intersection of those open sets U, for which $h(p)\cup\ U\subseteq U$, or $[\frac{1}{2},1]\ \cup\ U\subseteq U$. This is equal to the interior of $[\frac{1}{2},1]$, which is $(\frac{1}{2},1]$.

Let $\mathcal{A}=\{U\in\mathsf{Op}(W):[\frac{1}{2},1]\subseteq U\}$. Then $[\frac{1}{2},1]=\bigcap\mathcal{A}$. Let $\varphi=\mu x(p\vee x)$. If Esakia's lemma were true, we would have

$$[\![\mu x(p\vee x)]\!]^{\mathsf{Op}(W)}_{h_p^{[\frac{1}{2},1]}}=\bigcap\{[\![\mu x(p\vee x)]\!]^{\mathsf{Op}(W)}_{h_p^U}:U\in\mathcal{A}\}.$$

It is easy to check that with $h'(p)=A\in\mathcal{A}$, the least fixed-point of the formula $\mu x(p\vee x)$ is equal to A itself. The intersection of all the least fixedpoints, or A's in this case, is the closed set $[\frac{1}{2},1]$. So, we have

$$(1/2,1]=[\![\mu x(p\vee x)]\!]^{\mathsf{Op}(W)}_{h_p^{[\frac{1}{2},1]}}\supsetneq\bigcap\{[\![\mu x(p\vee x)]\!]^{\mathsf{Op}(W)}_{h_p^U}:U\in\mathcal{A}\}=[1/2,1]\,.$$

Therefore, Esakia's lemma fails for modal mu-formulas for open semantics.

We remedy this by introducing an alternative semantics for the least fixed-point operator. For an important class of modal mu-formulas this semantics will coincide with the semantics introduced in the previous section. We first introduce an alternative notion of a pre-fixed-point of a modal formula φ.

8.4.2. Definition. Let (W,R) be an MKH-space and h be an open assignment. The Boolean and modal operators for the *topological semantics* $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_h$ are interpreted in the same way as in the case of open semantics. Finally, for a formula φ with free variable x, we set

$$[\![\mu x\varphi]\!]^{\overline{\mathsf{Op}(W)}}_h=\mathrm{Int}\left(\bigcap\{U\in\mathsf{Op}(W):[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h_x^{\overline{U}}}\subseteq U\}\right),$$

where $\overline{U}$ is the closure of U.

The difference between topological and the open semantics is that the pre-fixed points in the topological semantics are taken with respect to the closure of a set. Sets U such that $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h_x^{\overline{U}}}\subseteq U$ will be called *topological pre-fixed points.*

8.4.3. Example. Consider the interval $I=[0,1]$ with the usual metric topology. We compute the valuation of fixed-point operator according to the topological semantics defined above. Consider a modal mu-formula $\mu x(p\vee x)$ and an open assignment $h(p) = (\frac{1}{3}, \frac{2}{3})$. As we saw in Example 8.2.1, $[\![\mu x(p \vee x)]\!]_h^{\mathsf{Op}(I)} = (1/3, 2/3)$. For the new semantics we have

$$[\![\mu x(p \vee x)]\!]_h^{\overline{\mathsf{Op}(I)}} = \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(I) : ((1/3, 2/3)) \cup \overline{U} \subseteq U\}\right).$$

It can be checked that the only open $U \subseteq [0, 1]$ which satisfies $(\frac{1}{3}, \frac{2}{3}) \cup \overline{U} \subseteq U$, is $U = [0, 1]$. Now this is a pre-fixed point but not the least fixed-point, in the sense that it is not the least open pre-fixed point. We have seen earlier in the Example 8.2.1 that the set $(\frac{1}{3}, \frac{2}{3})$ is the least open pre-fixed point for the formula $\mu x(p \vee x)$.

The following lemma shows that the topological semantics $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ is well-defined.

8.4.4. Lemma. *Let (W, R) be an MKH-space and h an open assignment. Then for each modal mu formula φ $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ is an open set.*

Proof. We want to show that if we restrict ourselves to open assignments, then the open semantics $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ is an open set. It is easy to see this for the cases when φ is a modal formula, since the valuation function is the same as in the case of usual semantics. In the case when $\varphi = \mu x\psi$, $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ is still open since we define it to be the interior of intersection of sets U such that $[\![\psi]\!]_{h_x^{\overline{U}}}^{\overline{\mathsf{Op}(W)}} \subseteq U$. □

The following lemma connects the topological semantics with the algebraic semantics discussed in the previous section.

8.4.5. Lemma. *For an MKH-space (W, R), if $F_1, \ldots, F_n$ are closed sets and $\varphi(x_1, \ldots, x_n)$ is a modal formula*[3]*, then $[\![\varphi]\!]_{h_{x_1,\ldots,x_n}^{F_1,\ldots,F_n}}^{\overline{\mathsf{Op}(W)}}$ is a closed set.*

Proof. The above lemma can be proved by induction on the complexity of φ. For the base case when $\varphi = p, \bot$ or $\top$, the lemma follows trivially. If $\varphi = \varphi_1 \vee \varphi_2$ or $\varphi_1 \wedge \varphi_2$, the lemma holds since finite union and intersection of closed sets is closed. If $\varphi = \Box\psi$ or $\varphi = \Diamond\psi$, the lemma is true because of the conditions on R in Proposition 8.1.5. □

8.4.6. Theorem. *For an MKH-space (W, R), $U \subseteq W$ is a topological pre-fixed of a modal formula $\varphi(x)$ as defined above iff there exists an open V such that $\overline{U} \subseteq V$ and $[\![\varphi]\!]_{h_x^V}^{\overline{\mathsf{Op}(W)}} \subseteq U$.*

[3] φ does not have any fixed-point operator

Proof. Note that φ is a modal formula and does not contain any fixed-point operators. The direction from right to left is easy. If there is an open V such that $\overline{U} \subseteq V$, by monotonicity, $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{U}}_x} \subseteq [\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{V}_x} \subseteq U$. For the converse direction, since $\varphi(x)$ is a modal formula, Esakia's lemma for positive modal formulas and MKH-spaces ([24, Lemma 7.8]) holds for it. So, we have

$$[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{U}}_x} = \bigcap \{ [\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{V'}}_x} : \overline{U} \subseteq V' \text{ \& } V \in \mathsf{Op}(W)\}.$$

Then, $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{U}}_x} \subseteq U$ implies that $\bigcap \{ [\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{V'}}_x} : \overline{U} \subseteq V' \text{ \& } V \in \mathsf{Op}(W)\} \subseteq U$. As φ is a modal formula, $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{V'}}_x}$ is a closed set using Lemma 8.4.5. Therefore, by compactness of W, there is an open V with $\overline{U} \subseteq V$ such that $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{V}}_x} \subseteq U$. But then $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{V}_x} \subseteq [\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{V}}_x} \subseteq U$. So, we found V with $\overline{U} \subseteq V$ such that $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{V}_x} \subseteq U$. $\square$

We restrict the syntax of modal mu-formulas so that we only have a modal formula in the scope of a fixed-point connective.

8.4.7. Definition. A *shallow modal mu-formula* is modal mu-formula such that only a modal formula (without fixed-point operators) can occur in the scope of the least fixed-point operator.

8.4.8. Example. A simple example of a shallow modal mu-formula is $\mu x(\Diamond p \vee x)$. We cannot have the formula $\mu x \mu y(\Diamond p \vee x) \wedge (\Box p \vee y)$ in our language since the nesting of fixed-point operators is not allowed by the syntax, but we can have $\mu x(\Diamond p \vee x) \wedge \ \mu y(\Box p \vee y)$. To see more concrete cases, one can check that the computational tree logic (CTL), linear temporal logic (LTL) and propositional dynamic logic (PDL) have shallow fixed-point connectives. For example, the iteration diamond $\langle \alpha^* \rangle$ of the PDL can be expressed as the least fixed-point of the modal formula $p \vee \langle \alpha \rangle x$, that is, $\mu x\, (p \vee \langle \alpha \rangle x)$. We note, however, that both PDL and CTL do allow for nesting of operators, even if each operator is "shallow".

The following theorem connects the topological semantics with the algebraic semantics discussed in the previous section.

8.4.9. Theorem. *Let (W, R) be an MKH-space. Then for each shallow modal mu-formula φ, and an open assignment h, we have $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h} = [\![\varphi]\!]^{\mathsf{Op}(W)'}$.*

Proof. We prove the lemma by induction on the complexity of the formula φ. The only case that needs to be checked is $\varphi = \mu x \psi$, where ψ is a modal formula.

But then by Theorem 8.4.6, and the definitions of $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ and $[\![\varphi]\!]^{\mathsf{Op}(W)'}$, we immediately obtain that $[\![\mu x\psi]\!]_h^{\overline{\mathsf{Op}(W)}} = [\![\mu x\psi]\!]^{\mathsf{Op}(W)'}$.

□

8.4.10. THEOREM. *Let (W, R) be an MKH-space and h be an open assignment. Then for each modal μ-formula $\varphi(x)$, $[\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ is monotone. That is, for $U \subseteq V$, s.t. $U, V \in \mathsf{Op}(W)$*

$$U \subseteq V \ \ \textit{implies} \ \ [\![\varphi]\!]_{h_x^U}^{\overline{\mathsf{Op}(W)}} \subseteq [\![\varphi]\!]_{h_x^V}^{\overline{\mathsf{Op}(W)}}$$

Proof. We prove the lemma by induction on the complexity of φ and show the induction step only for the case when $\varphi = \mu y\psi(y, x)$. By induction hypothesis, the lemma holds for ψ, that is, for all $U, V \subseteq W$ and $C \in \mathsf{Op}(W)$, we have

$$\begin{aligned}
U \subseteq V \Rightarrow \ & [\![\psi]\!]_{h_{y,x}^{\overline{C},U}}^{\overline{\mathsf{Op}(W)}} \subseteq [\![\psi]\!]_{h_{y,x}^{\overline{C},V}}^{\overline{\mathsf{Op}(W)}} \\
\Rightarrow \ & \text{If } [\![\psi]\!]_{h_{y,x}^{\overline{C},V}}^{\overline{\mathsf{Op}(W)}} \subseteq C, \ \text{ then } \ [\![\psi]\!]_{h_{y,x}^{\overline{C},U}}^{\overline{\mathsf{Op}(W)}} \subseteq C \\
\Rightarrow \ & \{C : [\![\psi]\!]_{h_{y,x}^{\overline{C},V}}^{\overline{\mathsf{Op}(W)}} \subseteq C\} \subseteq \{C : [\![\psi]\!]_{h_{y,x}^{\overline{C},U}}^{\overline{\mathsf{Op}(W)}} \subseteq C\} \\
\Rightarrow \ & \bigcap\{C : [\![\psi]\!]_{h_{y,x}^{\overline{C},U}}^{\overline{\mathsf{Op}(W)}} \subseteq C\} \subseteq \bigcap\{C : [\![\psi]\!]_{h_{y,x}^{\overline{C},V}}^{\overline{\mathsf{Op}(W)}} \subseteq C\} \\
\Rightarrow \ & \mathrm{Int}\left(\bigcap\{C : [\![\psi]\!]_{h_{y,x}^{\overline{C},U}}^{\overline{\mathsf{Op}(W)}} \subseteq C\}\right) \subseteq \mathrm{Int}\left(\bigcap\{C : [\![\psi]\!]_{h_{y,x}^{\overline{C},V}}^{\overline{\mathsf{Op}(W)}} \subseteq C\}\right) \\
\Rightarrow \ & [\![\mu y\psi]\!]_{h_x^U}^{\overline{\mathsf{Op}(W)}} \subseteq [\![\mu y\psi]\!]_{h_x^V}^{\overline{\mathsf{Op}(W)}}.
\end{aligned}$$

□

We have already seen in the Example 8.4.3 that the alternative semantics of the formula $\mu x\varphi$ does not give the least fixed-point of φ. In the following lemma, we show that if h is an open assignment, then $[\![\mu x\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$, gives a pre-fixed point of φ. This is similar to [27], where the semantics of the least fixed-point operator is the standard semantics, which is not necessarily the least fixed-point.

8.4.11. THEOREM. *The topological semantics for the fixed-point operator $[\![\mu x\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ under an open assignment h, gives a pre-fixed point of the formula φ.*

Proof. In order to show that $[\![\mu x\varphi(x, p_1, \dots, p_n)\}]\!]_h^{\overline{\mathsf{Op}(W)}}$ is a pre-fixed point, we need to show that $[\![\varphi]\!]_{h_x^S}^{\overline{\mathsf{Op}(W)}} \subseteq S$, where $S = \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : [\![\varphi]\!]_{h_x^{\overline{U}}}^{\overline{\mathsf{Op}(W)}} \subseteq U\}\right)$. Let $\mathbf{U} = \{U \in \mathsf{Op}(W) : [\![\varphi]\!]_{h_x^{\overline{U}}}^{\overline{\mathsf{Op}(W)}} \subseteq U\}$. Since $S \subseteq U \subseteq \overline{U}$, for all $U \in \mathbf{U}$, we have

$[\![\varphi]\!]_{h_x^S}^{\overline{\mathsf{Op}(W)}} \subseteq [\![\varphi]\!]_{h_x^{\overline{U}}}^{\overline{\mathsf{Op}W)}} \subseteq U$. So $[\![\varphi]\!]_{h_x^S}^{\overline{\mathsf{Op}(W)}} \subseteq \bigcap\{U \in \mathsf{Op}(W) : [\![\varphi]\!]_{h_x^{\overline{U}}}^{\overline{\mathsf{Op}(W)}} \subseteq U\}$. By Lemma 8.4.4, $[\![\varphi]\!]_{h_x^S}^{\overline{\mathsf{Op}(W)}}$ is open. So $[\![\varphi]\!]_{h_x^S}^{\overline{\mathsf{Op}(W)}} \subseteq \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : [\![\varphi]\!]_{h_x^{\overline{U}}}^{\overline{\mathsf{Op}(W)}}\right) = S$. Therefore, S is a pre-fixed point. □

8.4.2 Esakia's lemma

In this section, we work with only shallow modal mu-formulas. We prove an Esakia's lemma for MKH-spaces which will be used later to prove a Sahlqvist theorem for the shallow modal fixed-point formulas. Let W be any set. Recall that a set $\mathbb{F} \subseteq \mathcal{P}(W)$ is *downward directed* if for each $F, F' \in \mathbb{F}$, there exists $F'' \in \mathbb{F}$ such that $F'' \subseteq F \cap F'$.

8.4.12. Lemma (Esakia's lemma). *Let (W, R) be an MKH-space. Let F, $F_1, \ldots, F_n \subseteq W$ be closed sets and let $\mathcal{A} \subseteq \mathsf{Op}(W)$ be a downward directed family of open sets such that $\bigcap \mathcal{A} = F$. Then, for each positive shallow modal μ-formula $\varphi(x, x_1, \ldots, x_n)$, we have*

$$[\![\varphi]\!]_{h_{x,\vec{x}}^{F,\vec{F}}}^{\overline{\mathsf{Op}(W)}} = \bigcap\{[\![\varphi]\!]_{h_{x,\vec{x}}^{C,\vec{F}}}^{\overline{\mathsf{Op}(W)}} : C \in \mathcal{A}\}$$

where, $\vec{F} = (F_1, \ldots F_n)$ and $\vec{x} = (x_1, \ldots, x_n)$.

Proof. Throughout this proof, we adopt the following simplified notation: we use $\varphi(F, \vec{F})$ instead of $[\![\varphi]\!]_{h_{x,\vec{x}}^{F,\vec{F}}}^{\overline{\mathsf{Op}(W)}}$.

First, note that $\varphi(F, \vec{F}) = \bigcap\{\varphi(C, \vec{F}) : C \in \mathcal{A}\}$ follows from $\varphi(F, \vec{F}) = \bigcap\{\varphi(\overline{C}, \vec{F}) : C \in \mathcal{A}\}$, where $\overline{C}$ is the closure of C, as a result of the following claim.

Claim. $\varphi(F, \vec{F}) = \bigcap\{\varphi(\overline{C}, \vec{F}) : C \in \mathcal{A}\}$, implies $\varphi(F, \vec{F}) = \bigcap\{\varphi(C, \vec{F}) : C \in \mathcal{A}\}$.

Proof.[Proof of Claim] From Lemma 8.4.10, we have that φ is monotone. So, if $F \subseteq C \subseteq \overline{C}$, then $\varphi(F, \vec{F}) \subseteq \varphi(C, \vec{F}) \subseteq \varphi(\overline{C}, \vec{F})$, which implies $\varphi(F, \vec{F}) \subseteq \bigcap\{\varphi(C, \vec{F}) : C \in \mathcal{A}\} \subseteq \bigcap\{\varphi(\overline{C}, \vec{F}) : C \in \mathcal{A}\}$. Therefore, if we show $\varphi(F, \vec{F}) = \bigcap\{\varphi(\overline{C}, \vec{F}) : C \in \mathcal{A}\}$, we get $\varphi(F, \vec{F}) = \bigcap\{\varphi(C, \vec{F}) : C \in \mathcal{A}\}$ □

We show $\varphi(F, \vec{F}) = \bigcap\{\varphi(\overline{C}, \vec{F}) : C \in \mathcal{A}\}$ by induction on the complexity of φ. For the cases not involving the fixed-point operator, we refer to the proof of [24, Lemma 7.8]. For the case when $\varphi = \mu x\psi(x, y, \vec{x})$, we need to show

$$\mu x\psi(x, F, \vec{F}) = \bigcap\{\mu x\psi(x, \overline{C}, \vec{F}) : C \in \mathcal{A}\}$$

For each $C \in \mathcal{A}$, we have $F \subseteq C \subseteq \overline{C}$, which implies $\mu x\psi(x, F, \vec{F}) \subseteq \mu x\psi(x, \overline{C}, \vec{F})$ using Lemma 8.4.10. Therefore, $\mu x\psi(x, F, \vec{F}) \subseteq \bigcap\{\mu x\psi(x, \overline{C}, \vec{F}) : C \in \mathcal{A}\}$.

For the other direction, suppose $w \in \bigcap\{\mu x\psi(x, \overline{C}, \vec{F}) : C \in \mathcal{A}\}$. This implies that $w \in \mu x\psi(x, \overline{C}, \vec{F})$, for each $C \in \mathcal{A}$. As a result,

$$w \in \operatorname{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : \psi(\overline{U}, \overline{C}, \vec{F}) \subseteq U\}\right),$$

using the definition of the alternative semantics for the least fixed-point operator. Therefore, there exists a neighborhood U_w of w such that $U_w \subseteq \bigcap\{U \in \mathsf{Op}(W) : \psi(\overline{U}, \overline{C}, \vec{F}) \subseteq U\}$. So, for each $C \in \mathcal{A}$, and each $V \in \mathsf{Op}(W)$ with $\psi(\overline{V}, \overline{C}, \vec{F}) \subseteq V$ we have $U_w \subseteq V$.

Assume $U \in \mathsf{Op}(W)$ is such that $\psi(\overline{U}, F, \vec{F}) \subseteq U$. By the induction hypothesis, $\psi(\overline{U}, F, \vec{F}) = \bigcap\{\psi(\overline{U}, \overline{C}, \vec{F}) : C \in \mathcal{A}\}$. Hence, $\bigcap\{\psi(\overline{U}, \overline{C}, \vec{F}) : C \in \mathcal{A}\} \subseteq U$. By Lemma 8.4.5, each $\psi(\overline{U}, \overline{C}, \vec{F})$ is a closed set. Therefore, as U is open, by compactness, there exist finitely many $C_1, \ldots, C_k \in \mathcal{A}$ such that $\bigcap_{i=1}^{k} \psi(\overline{U}, \overline{C_i}, \vec{F}) \subseteq U$. As $\mathcal{A}$ is downward directed, there exists a $C \in \mathcal{A}$ such that $C \subseteq \bigcap_{i=1}^{k} C_i$ which implies $\overline{C} \subseteq \overline{\bigcap_{i=1}^{k} C_i} \subseteq \bigcap_{i=1}^{k} \overline{C_i}$.

Finally, by Lemma 8.4.10, $\psi(\overline{U}, \overline{C}, \vec{F}) \subseteq U$ which implies $U_w \subseteq U$. Therefore, it follows that $w \in \mu x\psi(x, F, \vec{F})$.

□

8.4.13. Corollary. *Let (W, R) be an MKH-space, $\vec{F} = (F_1, \ldots, F_n)$, $\vec{G} = G_1, \ldots, G_k \subseteq W$ be closed sets and $\varphi(\vec{x}, \vec{y})$ be a modal mu-formula, where $\vec{x} = (x_1, \ldots, x_n)$ and $\vec{y} = (y_1, \ldots, y_k)$. Then,*

1. $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{F_1,\ldots,F_n,\vec{G}}_{x_1,\ldots,x_n,\vec{y}}} = \bigcap\{[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{C_1,\ldots,C_n,\vec{G}}_{x_1,\ldots,x_n,\vec{y}}} : F_i \subseteq C_i \in \mathsf{Op}(W), 1 \leq i \leq n\}.$

2. $[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{F_1,\ldots,F_n,\vec{G}}_{x_1,\ldots,x_n,\vec{y}}} = \bigcap\{[\![\varphi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{C_1,\ldots,C_n,\vec{G}}_{x_1,\ldots,x_n,\vec{y}}} : F_i \subseteq C_i \in A_i, 1 \leq i \leq n\}$, *where $A_i \subseteq \mathsf{Op}(W)$ is downward directed and $\bigcap A_i = F_i$, for each $1 \leq i \leq n$.*

Proof. The result follows from Lemma 8.4.12 by a trivial induction. □

8.4.14. Remark. From the proof of the Esakia's Lemma, one can see why do we need to restrict our syntax to shallow modal mu-formulas. In order to use the compactness property to get a finite intersection, from an infinite intersection, we need the set $S = [\![\psi]\!]^{\overline{\mathsf{Op}(W)}}_{h^{\overline{U},\overline{C},\vec{F}}_{x,y,\vec{x}}}$ to be closed. If ψ contains fixed-points, S may not necessarily be a closed set.

To see this let ψ be the formula $\mu x(p \vee x)$. We consider the space $\mathbb{N}$ of natural numbers with the discrete topology. The Alexandroff one-point compactification $\alpha\mathbb{N}$ of $\mathbb{N}$ is a compact Hausdorff (and also zero-dimensional) space. This space is obtained by adding ∞ to $\mathbb{N}$. A set U is open in $\alpha\mathbb{N}$ if $U \subseteq \mathbb{N}$ or $U = V \cup \{\infty\}$ for a cofinite subset $V \subseteq \mathbb{N}$. Let $h(p) = \{n \in \mathbb{N} : n \text{ is even}\} \cup \{\infty\}$ be a closed valuation. Then it is easy to check that the evaluation of the formula $\mu x(p \vee x)$ under the alternative semantics is equal to the set $\mathrm{Int}(\{n \in \mathbb{N} : n \text{ is even}\} \cup \{\infty\}) = \{n \in \mathbb{N} : n \text{ is even}\}$. Obviously this is open but not a closed set. This justifies why we work with shallow modal mu-formulas ensuring that ψ does not have any fixed-point operators and S is a closed set as a result of Lemma 8.4.5. The above example underlines once again the non-standard nature of this semantics. Note that in the standard semantics the evaluation of the formula $\mu x(p \vee x)$ is equal to the evaluation of the atom p.

8.4.3 Sahlqvist formulas

In this section, we define a Sahlqvist formula and Sahlqvist sequent in our language. We then prove a version of Sahlqvist preservation result using the Esakia's lemma proved in the previous section for shallow modal fixed-point logic. In fact, with an analogue of the Esakia's lemma at hand the proof follows the standard patter of a proof of Sahlqvist theorem via topological frame see e.g., [138], [89], [27], [75], [24]. Thus, we will only underline the main steps. The details can be found in any of the above reference.

8.4.15. Definition. Let (W, R) be an MKH-space and h an assignment. For each formulas φ and ψ we say that $\varphi \vdash$ is true in W under h if $\overline{[\![\varphi]\!]_h^{\mathsf{Op}(W)}} \subseteq \overline{[\![\psi]\!]_h^{\mathsf{Op}(W)}}$. We say that $\varphi \vdash \psi$ is *topologically valid* in (W, R) and write $W \models \varphi \vdash \psi$ if $\overline{[\![\varphi]\!]_h^{\mathsf{Op}(W)}} \subseteq \overline{[\![\psi]\!]_h^{\mathsf{Op}(W)}}$ for each open assignment h. We say that $\varphi \vdash \psi$ is *valid* in (W, R) and write $\mathcal{M}W \models \varphi \vdash \psi$ if $\overline{[\![\varphi]\!]_h^{\mathsf{Op}(W)}} \subseteq \overline{[\![\psi]\!]_h^{\mathsf{Op}(W)}}$ for each assignment h.

8.4.16. Definition. Define $\Box^0 p = p$ and $\Box^{n+1} p = \Box^n p$. Recall that a boxed atom is a formula of the form $\Box^n \bot$, $\Box^n \top$, or $\Box^n p$ for some propositional variable p and $n \geq 0$. A *Sahlqvist antecedent* is obtained from boxed atoms by applying $\wedge$ and $\Diamond$.

8.4.17. Definition. A sequent $\varphi \vdash \psi$ is called a *Sahlqvist sequent* if φ is a Sahlqvist antecedent and ψ is a shallow modal mu-formula in our language.

8.4.18. Theorem (Sahlqvist preservation). *Let (W, R) be an MKH-space and $\varphi \vdash \psi$ be a Sahlqvist sequent. Then the following are equivalent.*

1. *$W \models \varphi \vdash \psi$.*
2. *$\mathcal{M}W \models \varphi \vdash \psi$.*

Proof.(Sketch) Obviously, (2) implies (1). Now suppose $\mathcal{M}W \not\models \varphi \vdash \psi$. Then there exists a set-theoretic assignment f and a point $w \in W$ such that $w \in [\![\varphi]\!]_f^{\overline{\mathsf{Op}(W)}}$ and $w \notin [\![\psi]\!]_f^{\overline{\mathsf{Op}(W)}}$. But since φ is Sahlqvist, there is a minimal closed assignment g such that $w \in [\![\varphi]\!]_g^{\overline{\mathsf{Op}(W)}}$ and $w \notin [\![\psi]\!]_g^{\overline{\mathsf{Op}(W)}}$. By the Esakia lemma there exists an open assignment h such that $w \notin [\![\psi]\!]_h^{\overline{\mathsf{Op}(W)}}$. Finally, by monotonicity, $x \in [\![\psi]\!]_h^{\overline{\mathsf{Op}(W)}}$. Thus, $W \not\models \varphi \vdash \psi$. $\square$

8.5 Sahlqvist correspondence

The aim of this section is to show that every Sahlqvist sequent is equivalent to a frame condition, which can be expressed in a first-order language with a least fixed-point operator ($\mathsf{FO} + \mathsf{LFP}$). The language $\mathsf{FO} + \mathsf{LFP}$ [61] has a countably infinite set of variables, a binary relation symbol R, and a unary predicate P, for each propositional variable $p \in \mathsf{Prop}$. A formula χ in $\mathsf{FO} + \mathsf{LFP}$ is said to be an $\mathsf{FO} + \mathsf{LFP}$-*frame condition* if it does not contain free variables or predicate symbols.

Let $\mathcal{M} = (W, R)$ be an MKH-space and h be an open assignment. We interpret formulas in $\mathsf{FO} + \mathsf{LFP}$ over (W, R), such that $P^{\mathcal{M}} = h(p) \in \mathsf{Op}(W)$ for every $p \in \mathsf{Prop}$. Let g be a first-order assignment of variables. The satisfaction of a $\mathsf{FO} + \mathsf{LFP}$ formula ξ, denoted by $(\mathcal{M}, h, g) \models \xi$, is defined in a standard way using induction on ξ. For a $\mathsf{FO} + \mathsf{LFP}$ formula $\xi(v, X)$, where v is a first-order variable and X is a unary predicate, let $h_x^{\overline{U}}$ denote the assignment of the variable x to the set $\overline{U}$ and g_u^w denote the first-order assignment mapping variable v to $w \in W$. Let $F(U) = \{w \in W : (\mathcal{M}, h_x^{\overline{U}}, g_v^w) \models \xi(v, X)\}$. The semantics of $(\mu(X, v)\xi(v, X)\varphi))(u)$, can be defined as follows

$$(\mathcal{M}, h, g) \models (\mu(X, v)\xi(v, X))(u) \text{ iff } g(u) \in \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : F(\overline{U}) \subseteq U\}\right)$$

8.5.1. Definition. Let u, v be first-order variables. The standard translation of a modal mu-formula into the language $\mathsf{FO} + \mathsf{LFP}$ is inductively defined as follows

- $ST_u(\bot) = \bot$,
- $ST_u(\top) = \top$,
- $ST_u(p) = P(u)$, where $p \in \mathsf{Prop}$,
- $ST_u(\varphi \wedge \psi) = ST_u(\varphi) \wedge ST_u(\psi)$,
- $ST_u(\varphi \vee \psi) = ST_u(\varphi) \vee ST_u(\psi)$,
- $ST_u(\Diamond\varphi) = \exists v(R(u, v) \wedge ST_v(\varphi)$,

- $ST_u(\Box\varphi) = \forall v(R(u,v) \to ST_v(\varphi))$,
- $ST_u(\mu x\varphi) = (\mu(X,v)ST_v(\varphi))(u)$,
- $ST_u(\varphi \vdash \psi) = ST_u(\varphi) \to ST_u(\psi)$.

8.5.2. Proposition. *Let $\mathcal{M} = (W,R)$ be an MKH-space, h be an open assignment and φ be a modal mu-formula. For each $w \in W$ and a first-order assignment g_u^w mapping variable v to w, we have,*

1. $w \in [\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}}$ *iff* $(\mathcal{M}, h, g_u^w) \models ST_u(\varphi)$.
2. $\forall h \left(w \in [\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}} \right)$ *iff* $(\mathcal{M}, g_u^w) \models \forall P_1 \ldots \forall P_n\, ST_u(\varphi)$.
3. $\forall h \, \forall w \left(w \in [\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}} \right)$ *iff* $\mathcal{M} \models \forall P_1 \ldots \forall P_n \forall u\; ST_u(\varphi)$.

Proof. The Proposition easily follows from an induction on the complexity of φ.
$\Box$

8.5.3. Theorem. *Let (W,R) be an MKH-space and $\varphi \vdash \psi$ be a Sahlqvist sequent. Then there is a frame condition $\chi(\varphi,\psi)$ in $\mathsf{FO} + \mathsf{LFP}$ such that*

$$(W,R) \models \chi(\varphi,\psi) \textit{ iff } \varphi \vdash \psi \textit{ is valid in } (W,R).$$

Proof. We give an algorithm to effectively compute the first order frame correspondent $\chi(\varphi,\psi)$ of $\varphi \vdash \psi$.

Step 1 Since $\varphi \vdash \psi$ is valid in (W,R), $\forall w \in W$,

$$w \in [\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}} \Rightarrow w \in [\![\psi]\!]_h^{\overline{\mathsf{Op}(W)}}$$

Fix $w \in W$. Let $p_1, \ldots, p_n \in \mathsf{Prop}$ be the set of propositional variables occurring in φ. We compute the minimal assignment $h_0(p_i)$, $1 \leq i \leq n$ for each propositional variables as follows: let $\beta_1, \ldots, \beta_{m_i}$ be the boxed atoms in φ which contain p_i, with $\beta_j = \Box^{d_j} p_i, 1 \leq j \leq m_i$ and $d_j \geq 0$. Let $R^0[w] = \{w\}$ and $R^n[w] = \{w' \in W : \exists w_1, \ldots, w_n \text{ s.t. } wRw_1R \ldots Rw_n \text{ and } w_n = w'\}$ for $n \geq 1$. The minimal valuation for p_i is equal to $h_0(p_i) = R^{d_1}[w] \cup \ldots \cup R^{d_{m_i}}[w]$.

Step 2 Let h_0 be the minimal assignment computed in Step 1. The syntactic shape of the Sahlqvist formula ensures that we have the following equivalence.

Claim. If φ is a Sahlqvist antecedent, then

$$\forall h \left(w \in [\![\varphi]\!]_h^{\overline{\mathsf{Op}(W)}} \Rightarrow w \in [\![\psi]\!]_h^{\overline{\mathsf{Op}(W)}} \right) \quad \text{iff} \quad \forall h_0 \left(w \in [\![\varphi]\!]_{h_0}^{\overline{\mathsf{Op}(W)}} \Rightarrow w \in [\![\psi]\!]_{h_0}^{\overline{\mathsf{Op}(W)}} \right) \tag{8.1}$$

Proof.[Proof of Claim] The direction from left to right is clear. We prove the converse by contraposition. Suppose there exists an arbitrary assignment h such that $w \in \llbracket\varphi\rrbracket_h^{\overline{\mathsf{Op}(W)}}$ and $w \notin \llbracket\psi\rrbracket_h^{\overline{\mathsf{Op}(W)}}$. We show that there exists a minimal valuation h_0 such that $w \in \llbracket\varphi\rrbracket_{h_0}^{\overline{\mathsf{Op}(W)}}$ and $w \notin \llbracket\psi\rrbracket_{h_0}^{\overline{\mathsf{Op}(W)}}$, using an induction on the complexity of φ.

The base case with $\varphi = \bot$ is trivial. If $\varphi = \Box^n p$, it is easy to check that $w \in \llbracket\Box^n p\rrbracket_h^{\overline{\mathsf{Op}(W)}}$ if, and only if, $w \in \llbracket\Box^n p\rrbracket_{h_0}^{\overline{\mathsf{Op}(W)}}$, where $h_0(p) = R^n[w]$ is the minimal valuation computed in Step 1. Since ψ is a positive formula and $h_0(p) \subseteq h(p)$, it follows that $w \notin \llbracket\psi\rrbracket_{h_0}^{\overline{\mathsf{Op}(W)}}$. If $\varphi = \varphi_1 \wedge \varphi_2$, by induction hypothesis, there exist minimal valuations $g_0(p) \subseteq h(p)$ and $k_0(p) \subseteq h(p)$ for φ_1 and φ_2 respectively. Let $h_0(p) = g_0(p) \cup k_0(p)$, which implies $h_0(p) \subseteq h(p)$. Hence, $w \notin \llbracket\psi\rrbracket_{h_0}^{\overline{\mathsf{Op}(W)}}$ If $\varphi = \Diamond\varphi_1$, the minimal valuation h_0 such that $w \in \llbracket\varphi\rrbracket_{h_0}^{\overline{\mathsf{Op}(W)}}$ and $w \notin \llbracket\psi\rrbracket_{h_0}^{\overline{\mathsf{Op}(W)}}$, is the same as the minimal valuation for φ_1. $\Box$

Step 3 We showed in Step 2 that a Sahlqvist sequent is valid under an arbitrary assignment if and only if it is valid under a minimal assignment. As it is shown below, the minimal assignment h_0 computed in Step 1 is first-order definable. Hence, it ensures that the frame condition corresponding to a Sahlqvist sequent is in $\mathsf{FO} + \mathsf{LFP}$.

Let $\chi'(\varphi, \psi) = \forall P_1 \ldots \forall P_n \forall u\ ST_u(\varphi \vdash \psi)$. Suppose $h_0(p_i) = R^{d_1}[w] \cup \ldots \cup R^{d_{m_i}}[w]$ for $p_i \in \mathsf{Prop}$. The $\mathsf{FO} + \mathsf{LFP}$ condition $\chi(\varphi, \psi)$ is obtained from $\chi'(\varphi, \psi)$ by replacing $\forall P_i$ with $\forall z_i$, where z_i is a fresh first order variable, and each atomic formula of the form $P_i(v)$ with an $\mathsf{FO} + \mathsf{LFP}$ formula $\theta_i = \exists v_0, \ldots, v_n[z_i = v \wedge \bigwedge_{j=0}^{n-1} v_j R v_{j+1} \wedge v_n = v]$, which says 'there exists an R-path from z_i to v in n steps'.

Claim. The $\mathsf{FO} + \mathsf{LFP}$ sentence $\chi(\varphi, \psi)$ is the frame condition for $\varphi \vdash \psi$.

Proof.[Proof of Claim] The minimal valuation for all the propositional variables in φ computed above are first-order definable. Hence, it follows using Proposition 8.5.2.3 that $\chi(\varphi, \psi)$ is an $\mathsf{FO} + \mathsf{LFP}$ frame condition. $\Box$

The proof of the theorem follows from the claim. $\Box$

8.5.4. Example. Consider the sequent $\Diamond p \vdash \Box\Diamond^* p$, where $\Diamond^* p = \mu x(p \vee \Diamond x)$. The standard translation of the sequent is given as follows

$$\begin{aligned} ST_u(\Diamond p \vdash \Box\Diamond^* p) = &\exists v_1(R(u, v_1) \wedge P(v_1)) \to \forall v_2(R(u, v_2) \to \\ &\mu(X, v_3)(P(v_3) \vee \exists v_4(R(v_3, v_4) \wedge X(v_4)))(v_2)). \end{aligned}$$

The propositional variable p does not occur in scope of any box in the antecedent. Hence, the minimal valuation for p is $h_0(p) = \{w\}$. According to the algorithm in Theorem 8.5.3, the $\mathsf{FO} + \mathsf{LFP}$ frame condition $\chi(\Diamond p, \Box\Diamond^* p)$ is

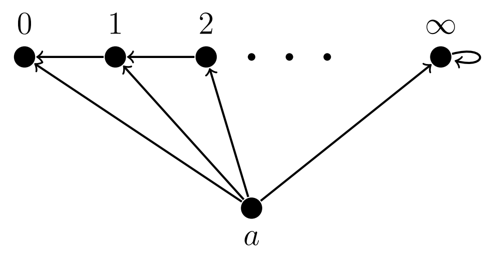

Figure 8.1: Alexandroff compactification of $\mathbb{N}$ with an isolated point

obtained by replacing all occurrences of $P(v_i)$ with $z_i = v_i$, where z_i is a new variable

$$\begin{aligned}\chi(\Diamond p, \Box\Diamond^* p) = &\forall z_1 \forall z_2 \forall u \exists v_1 \forall v_2 (R(u, v_1) \wedge (z_1 = v_1)) \to (R(u, v_2) \to \\ &\mu(X, v_3)((z_2 = v_3) \vee \exists v_4(R(v_3, v_4) \wedge X(v_4)))(v_2)).\end{aligned}$$

8.5.5. Example. Consider the sequent $\Diamond\Box\bot \vdash \Box\Diamond^*\Box\bot$, where $\Diamond^*\Box\bot = \mu x(\Box\bot \vee \Diamond x)$. Since there are no propositional variables in the sequent, its first order correspondence is obtained from its standard translation by quantifying over the free variable as follows

$$\begin{aligned}\chi(\Diamond\Box\bot,\ \Box\Diamond^*\Box\bot) = &\forall u \exists v_1 (R(u, v_1) \wedge \forall v_2 (R(v_1, v_2) \to \bot)) \to \forall v_3 (R(u, v_3) \to \\ &\mu(X, v_4)(\forall v_5 (R(v_4, v_5) \to \bot) \vee \exists v_6 (R(v_4, v_6) \wedge X(v_6)))(v_3)).\end{aligned}$$

This simplifies to

$$\begin{aligned}\chi(\Diamond\Box\bot,\ \Box\Diamond^*\Box\bot) = &\forall u \exists v_1 (R(u, v_1) \wedge \forall v_2 (\neg R(v_1, v_2))) \to \forall v_3 (R(u, v_3) \\ &\to \mu(X, v_4)(\forall v_5 (\neg R(v_4, v_5)) \vee \exists v_6 (R(v_4, v_6) \wedge X(v_6)))(v_3)).\end{aligned} \tag{8.2}$$

We now give a semantic interpretation of the sequent. Consider the space of $\mathbb{N}$ of natural numbers with the discrete topology. The Alexandroff one-point compactification of this space obtained by adding ∞ is a compact and Hausdorff space. The open sets of the compact space are open sets in the discrete topology of the original space and cofinite sets including infinity. We further add an isolated point a to the space after compactification, as seen in Figure 8.1. Let $W = \mathbb{N} \cup \{\infty, a\}$ with the topology described above. The relation $R = \{(n, n-1) : n \in \mathbb{N} \text{ and } n \geq 1\} \cup \{\infty, \infty\} \cup \{a, 0\} \cup \{a, \infty\}$ on W makes (W, R) an MKH-space.

The antecedent $\Diamond\Box\bot$ of the sequent is valid at points a and 1. The classical semantics of the formula $\Diamond^*\Box\bot$ in the consequent is given as

$$[\![\Diamond^*\Box\bot]\!]^{\mathsf{Op}(W)} = \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : \big(\{0\} \cup R^{-1}(U)\big) \subseteq U\}\right)$$

For any open $U = \{0, 1, \ldots, k\} \cup \{a\}$, where $k \in \mathbb{N}$, $R^{-1}(U) = \{0, \ldots, k, k + 1\} \cup \{a\}$. Hence, the open sets U which satisfy the condition $(\{0\} \cup R^{-1}(U)) \subseteq U$ are $\{0, a\} \cup \mathbb{N}$ and $\{0, a, \infty\} \cup \mathbb{N}$. As a result, $[\![\Diamond^*\Box\bot]\!]^{\mathsf{Op}(W)} = \{0, a\} \cup \mathbb{N}$.

In our closure semantics the semantics of $\Diamond^*\Box\bot$ is

$$[\![\Diamond^*\Box\bot]\!]^{\overline{\mathsf{Op}(W)}} = \mathrm{Int}\left(\bigcap\{U \in \mathsf{Op}(W) : \left(\{0\} \cup R^{-1}(\overline{U})\right) \subseteq U\}\right)$$

The closure of the open set $\{0, a\} \cup \mathbb{N}$ is $\{0, a, \infty\} \cup \mathbb{N}$. Therefore, it does not satisfy the condition $\left(\{0\} \cup R^{-1}(\overline{U})\right) \subseteq U$. The only open set which satisfies the condition is $U = \mathbb{N} \cup \{a, \infty\}$. Hence, $\Diamond^*\Box\bot$ is valid everywhere in (W, R). As a result, the sequent $\Diamond\Box\bot \vdash \Box\Diamond^*\Box\bot$ is valid. Therefore, it follows from Theorem 8.5.3 that the $\mathsf{FO} + \mathsf{LFP}$ frame condition $\chi(\Diamond\Box\bot \vdash \Box\Diamond^*\Box\bot)$ obtained above is valid on (W, R).

8.6 Conclusion

In this chapter, we studied different topological semantics of the least fixed-point operator on MKH-spaces. We showed that for an open assignment, set-theoretic and open semantics coincide. We gave an interpretation of the least fixed-point operator on compact regular frames and showed that the duality between compact Hausdorff spaces and compact regular locales extends to the language with the least fixed-point operator. For Sahlqvist preservation, we introduced a new topological semantics for the least fixed-point operator as the intersection of topological pre-fixed-points. In the new semantics, we proved that Esakia'a lemma holds for the class of shallow fixed-point formulas which do not have any nesting of fixed-point operators. As a consequence of Esakia's lemma, we obtained our main preservation result which states that a Sahlqvist sequent in our language is valid under open assignments on an MKH-space if and only if it is valid under arbitrary assignments. We also showed that a Sahlqvist sequent is valid in an MKH-space if and only if the condition expressible in $\mathsf{FO} + \mathsf{LFP}$ corresponding to the sequent is valid on the space. Finally, using examples we illustrated that the alternative topological semantics for the least fixed-point operator is different from the usual semantics over MKH-spaces.

One criticism of the semantics considered in the chapter might be that it is specially tailored for proving Esakias lemma and obtaining the Sahlqvist preservation result this way. Although this might be a valid criticism, we note that the fixed-point operators considered in the chapter are new and topological in nature. These operators often differ from the classical fixed-point operators and thus enrich the realm and expressivity of the existing fixed-point operators. We also believe that this point of view opens up a wider perspective for other (topological) interpretations of fixed-point operators (e.g., via regular open or closed sets, convex sets, polygons, rectangles, etc.).

We conclude with a few open problems and future directions that can be explored. An interesting problem is whether our results hold for the greatest fixed-point operator and formulas with mixed fixed-point operators. Also regular open sets play an important role in the semantics of spatial logics, and are suitable for modal mu-calculus with negation. Therefore, the fixed-point semantics for regular open sets is an interesting and, for now, unexplored area that deserves attention.

The completeness of Kozen's axiomatization [105] over MKH-spaces is another open problem. In [3] Kozen's axiomatization was shown to be complete with respect to descriptive mu-frames, or equivalently with respect to modal mu-algebras. In our case, the algebraic structures which provide the semantics are compact regular frames. These structures have infinitary operations, while our language has connectives of finite arity. This leads to a major question on what should be the logical counterpart of these structures. Does this have to be an infinitary logic or the infinitary operations of compact regular frames can be encoded in a finitary logic?

Another possible direction is to explore the expressivity results for our language with a fixed-point operator over compact Hausdorff spaces (see eg., [36]). It would be interesting to find examples of standard topological properties which can be expressed with the alternative fixed-point semantics and e.g., to find an analogue of the Goldblatt-Thomasson theorem [30, Section 3.8].

Appendix A

Success of ALBA on inductive and recursive inequalities

A.1 ALBA^r succeeds on inductive inequalities

In the present subsection, we discuss the success of ALBA^r in both the DLR and the HAR setting simultaneously. We will use the symbol $\mathcal{L}$ to refer generically to either of DLR or HAR. The treatment of the present subsection is similar to that of [50, Section 10], hence, in what follows, we expand only on details which are specific to the regular setting. Let us start with some auxiliary definitions and lemmas. Unlike the corresponding definitions in [50, Section 10], the definitions below are given in terms of the positive classification (cf. [46, Section 6.2]).

A.1.1. Definition. Given an order type $\varepsilon \in \{1, \partial\}^n$, a signed generation tree $*\varphi$ of a term $\varphi(p_1, \ldots, p_n) \in \mathcal{L}^+$ is called *ε-conservative* if all connectives occurring on ε-critical branches of $*\varphi$ are from the base language $\mathcal{L}$.

The next definition extends the notion of inductive terms and inequalities to $\mathcal{L}^+_{term}$ and $\mathcal{L}^+$, essentially by keeping Definition 4.4.7 intact and simply forbidding connectives belonging properly to the extended language on critical branches. Nevertheless, since this definition will be used extensively, we write it out in full here.

A.1.2. Definition. [(Ω, ε)-inductive $\mathcal{L}^+$-terms and inequalities] For any order type ε and any irreflexive and transitive relation Ω on $p_1, \ldots p_n$, the generation tree $*s$ ($* \in \{-, +\}$) of a $\mathcal{L}^+$-term $s(p_1, \ldots p_n)$ is *(Ω, ε)-inductive* if

1. it is ε-conservative, and
2. for each $1 \leq i \leq n$, on every ε-critical branch with leaf labelled p_i is good (cf. Definition 4.4.3), and moreover, every binary node $\star(\alpha \circ \beta)$ in P_1 for $\star \in \{+, -\}$, such that the critical branch lies in β satisfies the following conditions:

(a) $\varepsilon^\partial(\star\alpha)$ (resp. $\varepsilon(\star\alpha)$) if $\circ$ is positive (resp. negative) in the first coordinate, and

(b) $p_j <_\Omega p_i$ for every p_j occurring in α.

An $\mathcal{L}^+$-inequality $s \leq t$ is (Ω, ε)-*inductive* if the trees $+s$ and $-t$ are both (Ω, ε)-inductive.

Notice that in the DLR setting, $\circ$ in item (a) above can only be $+\vee$ and $-\wedge$, and hence the options in brackets are ruled out.

The next definition makes use of auxiliary notions in Definition 4.4.3.

A.1.3. Definition. [Definite (Ω, ε)-inductive $\mathcal{L}^+$-terms and inequalities] For any $\mathcal{L}^+$-term $s(p_1, \ldots p_n)$ such that the signed generation tree $*s$ $(* \in \{-, +\})$ is (Ω, ε)-inductive, $*s$ is *definite* (Ω, ε)-*inductive* if, in addition, there are no occurences of $+\vee$ or $-\wedge$ nodes in the segment P_2 of any ε-critical branch of $*s$. The term s is *definite* (Ω, ε)-*left inductive* (resp. *definite* (Ω, ε)-*right inductive*) if $+s$ (resp. $-s$) is definite (Ω, ε)-inductive. An inequality $s \leq t \in \mathcal{L}_{term}$ is *definite* (Ω, ε)-*inductive* if the trees $+s$ and $-t$ are both definite (Ω, ε)-inductive.

The definition of definite inductive inequalities is meant to capture the syntactic shape of inductive inequalities after preprocessing (see Lemma A.1.4 below). During preprocessing, all occurences of $+\vee$ and $-\wedge$ in the segment P_2 of every critical branch can be surfaced and then eliminated via exhaustive applications of the splitting rule.

The proof of the following lemma is analogous to that of [50, Lemma 10.4].

A.1.4. Lemma. *Let $\{s_i \leq t_i\}$ be the set of inequalities obtained by preprocessing an (Ω, ε)-inductive $\mathcal{L}$-inequality $s \leq t$. Then each $s_i \leq t_i$ is a definite (Ω, ε)-inductive inequality.*

A.1.5. Definition. [Definite good shape] An inequality $s \leq t \in \mathcal{L}^+$ is in *definite* (Ω, ε)-*good shape* if either of the following conditions hold:

1. s is pure, $+t$ is definite (Ω, ε)-inductive, and moreover, if $+t$ contains a skeleton node on an ε-critical branch, then s is a nominal.[1]

2. t is pure, $-s$ is definite (Ω, ε)-inductive and moreover, if $-s$ contains a skeleton node on an ε-critical branch, then t is a co-nominal.

Clearly, if an inequality $s \leq t$ is definite (Ω, ε)-inductive, then the two inequalities obtained by applying the first approximation rule to it are in definite (Ω, ε)-good shape. Next, we would like to prove a 'good-shape lemma' for definite inductive

[1] Note that the sides have been swapped around: We require that the *righthand* side of the inequality must be *left* inductive. This is so because the first approximation rule swaps the sides of inequalities.

inequalities. In particular, we would like to show that the application of the reduction rules does not spoil good shape. Actually the application the following rules might spoil good shape:

$$\frac{x \wedge y \leq z}{\overline{x \leq y \rightarrow z}} \qquad \frac{z \leq y \vee x}{\overline{z - y \leq x}}$$

This happens e.g. when z is pure and y is not. A solution to this is provided by allowing only applications of the rules above which are *restricted* to the cases in which the term y that switches sides is pure.

A.1.6. Lemma. *If $s \leq t$ is in definite (Ω, ε)-good shape, then any inequality $s' \leq t'$ obtained from $s \leq t$, by either the application of a splitting rule, of an approximation rule, or of a residuation rule for a unary connective from-top-to-bottom, or of the application of a residuation rule restricted as indicated above, is again in definite (Ω, ε)-good shape. Moreover, any side condition introduced by an application of an adjunction rule is pure on both sides.*

Proof. The proof of the lemma above is analogous to that of [50, Lemma 10.6]. We only discuss the rules and the additional statement specific to the regular setting. Consider for instance the case in which s is definite inductive, t is pure and the root of s is f for $\eta_f = 1$. Then the following adjunction rule is applicable:

$$\frac{f(s') \leq t}{f(\bot) \leq t \quad s' \leq \blacksquare_f t}$$

Then, both inequalities in the conclusion are in definite (Ω, ε)-good shape. Indeed, the side-condition is all pure, as required by the second part of the statement, and $s' \leq \blacksquare_f t$ is in definite good shape, because otherwise, the inequality $f(s') \leq t$ would not be, contrary to the assumptions. The remaining cases are analogous and are omitted. □

A.1.7. Definition. [(Ω, ε)-Ackermann form] A set of $\mathcal{L}^+$-inequalities $\{s_i \leq t_i\}_{i \in I}$ is in *reduced* 1*-Ackermann form with respect to a variable* p if, for every $i \in I$, either

1. s_i is pure and $t_i = p$, or
2. s_i is positive in p and t_i is negative in p.

Similarly, the set $\{s_i \leq t_i\}_{i \in I}$ is in *reduced* ∂*-Ackermann form* with respect to a variable p if, for every $i \in I$, either

1. $s_i = p$ and t_i is pure, or

2. s_i is negative in p and t_i is positive in p.

For any irreflexive, transitive ordering Ω on $p_1, \ldots, p_n$ and any order-type $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, a set $\{s_i \leq t_i\}_{i \in I}$ of inequalities is in *reduced* (Ω, ε)*-Ackermann form* if it is in reduced ε_k-Ackermann form w.r.t. every Ω-minimal variable p_k.

A.1.8. PROPOSITION. *Any finite set* $\{s_i \leq t_i\}_{i \in I}$ *of inequalities which are in definite* (Ω, ε)*-good shape can be transformed into a set* $\{s'_i \leq t'_i\}_{i \in I'}$ *which is in reduced* (Ω, ε)*-Ackermann form, through the exhaustive and safe application, only to non-pure inequalities, of the* $\wedge$*-splitting,* $\vee$*-splitting, approximation, and unary residuation rules top-to-bottom, as well as the restricted application of the binary residuation rules top-to-bottom.*

Proof. The proof is analogous to that of [50, Proposition 10.10], and makes use of the counterparts, in the $\mathcal{L}$-setting, of [50, Lemma 10.8, Corollary 10.9], which are here omitted, since their statement and proof reproduce the mentioned ones verbatim. Notice that by Lemma A.1.6 the side conditions do not contain proposition variables, which guarantees that if the rules are applied to display critical variable occurrences, then they are applied safely. □

A.1.9. THEOREM. *For each inductive inequality, there exists a safe and successful execution of* ALBA^r *on it.*

Proof. Let $s_0 \leq t_0$ be an (Ω, ε)-inductive inequality. By Lemma A.1.4, preprocessing $s_0 \leq t_0$ will yield a finite set $\{s_i \leq t_i\}_{i \in I}$ of definite (Ω, ε)-inductive inequalities. The execution of the algorithm now branches and proceeds separately on each of these inequalities. Each $s \leq t \in \{s_i \leq t_i\}_{i \in I}$ is replaced with $\{\mathbf{i} \leq s, t \leq \mathbf{m}\}$. Notice that $\mathbf{i} \leq s$ and $t \leq \mathbf{m}$ are in definite (Ω, ε)-good shape. Hence, by Proposition A.1.8, the system $\{\mathbf{i} \leq s, t \leq \mathbf{m}\}$ can be transformed, through the *safe* application of the rules of the algorithm, into a set of inequalities in reduced (Ω, ε)-Ackermann form. To this set, the Ackermann-rule can then be applied to eliminate all Ω-minimal propositional variables.
The Ackermann rule, applied to a set of inequalities in reduced (Ω, ε)-Ackermann form, replaces propositional variables with pure terms, therefore the resulting set of inequalities is in definite (Ω', ε)-good shape, where Ω' is the restriction of Ω to the non Ω-minimal variables. Indeed, the application of an Ackermann rule turns all ε-critical branches corresponding to Ω-minimal variables into non-critical branches, and leaves the critical branches corresponding to the other variables unaffected.

Now another cycle of reduction rules, applied safely, will lead to a new set of inequalities in reduced (Ω', ε)-Ackermann form, from which an application of the Ackermann rule will eliminate all the Ω'-minimal variables, and so on. Since the number of variables in $s_0 \leq t_0$ is finite, after a finite number of cycles the algorithm will output a set of pure inequalities. □

A.2 Success on recursive μ-inequalities

The aim of the present section is to show that the enhanced version of ALBA is successful on ε-recursive inequalities.

A.3 Preprocess, first approximation and approximation

Indeed, let $\eta \leq \beta$ be an ε-recursive inequality. We proceed as in ALBA and preprocess this inequality by applying splitting and $(\top)$ and $(\bot)$ exhaustively. This might produce multiple inequalities, on each of which we proceed separately. On each such inequality, denoted again $\eta \leq \beta$, we proceed to first approximation, which yields the following quasi-inequality:

$$\forall \overline{p} \forall \mathbf{i} \forall \mathbf{m}[(\mathbf{i} \leq \eta \ \& \ \beta \leq \mathbf{m}) \Rightarrow \mathbf{i} \leq \mathbf{m}]. \tag{A.1}$$

Because its consequent is always pure, we only concentrate on its antecedent. Since the outer skeleton of β and η is built exactly as the outer part of an inductive modal formula, the ordinary approximation rules can be applied so as to surface the inner skeleton. So we can equivalently rewrite $\mathbf{i} \leq \eta \ \& \ \beta \leq \mathbf{m}$ as the conjunction of a set of inequalities which, whenever they contain critical variables in the scope of fixed points occurring as skeleton nodes, are of the form

$$\mathbf{i} \leq \mu X.\psi'(\overline{p}) \qquad \text{and} \qquad \nu X.\varphi'(\overline{p}) \leq \mathbf{m}, \tag{A.2}$$

where $\mu X.\psi'(\overline{p})$ and $\nu X.\varphi'(\overline{p})$ are sentences. (For the critical branches which do not contain such fixed points, we further proceed by exhaustively applying the approximation rules as in ALBA).

A.3.1. PROPOSITION. *1. The inequality $\mathbf{i} \leq \mu X.\psi'$ in (A.2) is of the form $\mathbf{i} \leq \mu X.\psi(\overline{\varphi}/\overline{y}, \overline{\gamma}/\overline{z})$, where $\mu X.\psi(\overline{y}, X, \overline{z})$ is an $(\overline{y}, \overline{z})$-$IF^{\Diamond}_{\delta}$-formula for some order-type δ over $\overline{y}$;*

2. the inequality $\nu X.\varphi' \leq \mathbf{m}$ in (A.2) is of the form $\nu X.\varphi(\overline{\psi}/\overline{y}, X, \overline{\gamma}/\overline{z}) \leq \mathbf{m}$, where $\nu X.\varphi(\overline{y}, X, \overline{z})$ is an $(\overline{y}, \overline{z})$-$IF^{\Box}_{\delta}$-formula for some order-type δ over $\overline{y}$.

Proof. Notice that preprocessing, first approximation and ordinary approximation rules do not involve fixed points. Hence a proof very similar to that given for [50, Lemmas 10.4 and 10.6] proves that $+\mu X.\psi'$ and $-\nu X.\varphi'$ are non trivially ε-recursive. Hence the statement immediately follows from Lemma A.3.2 below. $\square$

A.3.2. Lemma. *1. If $+\psi'$ is non-trivially ε-recursive, and the P_3-paths of all critical branches are of length 0, then ψ' is of the form $\psi(\overline{\varphi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$, where $\psi(\overline{x}, \overline{z})$ is an $(\overline{x}, \overline{z})$-$IF^{\Diamond}_{\delta}$-formula for $\overline{x} = \overline{y} \oplus \overline{X}$, some order-type δ over $\overline{x}$, and $\overline{\varphi}$ and $\overline{\gamma}$ $\mathcal{L}$-sentences.*

2. If $-\varphi'$ is non-trivially ε-recursive, and the P_3-paths of all critical branches are of length 0, then φ' is of the form $\varphi(\overline{\psi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$, where $\varphi(\overline{x}, \overline{z})$ is an $(\overline{x}, \overline{z})$-$IF^{\Box}_{\delta}$-formula for $\overline{x} = \overline{y} \oplus \overline{X}$, some order-type δ over $\overline{x}$, and $\overline{\psi}$ and $\overline{\gamma}$ $\mathcal{L}$-sentences.

Proof. Let us define the *skeleton depth* of an ε-recursive generation tree $*\xi$, with $* \in \{+, -\}$, to be the maximum length of the P_2 paths in $*\xi$ leading to critical variables. The proof proceeds by simultaneous induction on the skeleton depths of $+\psi'$ and $-\varphi'$.

If the depth of $+\psi'$ is 0, then the critical branches will consist only of PIA nodes, i.e., $+\psi'$ is non-trivially ε-PIA, and by Definition 5.3.1.1 ψ' is a sentence; hence we let $\psi = x_1$ which is $\mathrm{IF}^{\Diamond}_{\delta}$ with $\delta = (1)$, and $\varphi_1 = \psi'$. Analogously for the base case of $-\varphi'$.

As for the induction step, let us suppose that the depth of $+\psi'$ is $k+1$, and that the statement is true for generation trees satisfying the assumptions and of depth not greater than k. We proceed by cases, depending on the form of $+\psi'$.

If $+\psi'$ is of the form $+\mu X.\psi'_1$, then by the induction hypothesis applied to $+\psi'_1$, we have that ψ'_1 is of the form $\psi_1(\overline{\varphi}/\overline{y}, \overline{X}'\overline{\gamma}/\overline{z})$ where $\overline{X}' = \overline{X} \oplus X$ and $\psi_1(\overline{x}', \overline{z})$ is an $(\overline{x}', \overline{z})$-$\mathrm{IF}^{\Diamond}_{\delta}$-formula for $\overline{x}' = \overline{y} \oplus \overline{X}' = \overline{x} \oplus X$ and some order-type $\delta' = \delta \oplus 1$ over $\overline{x} \oplus X$, and the $\overline{\varphi}$ and $\overline{\gamma}$ are sentences. Hence we let $\psi = \mu X.\psi_1(\overline{x}, X, \overline{z})$ which is an $(\overline{x}, \overline{z})$-$\mathrm{IF}^{\Diamond}_{\delta}$-formula.

If $+\psi'$ is of the form $+(\psi'_1 \vee \psi'_2)$, then, by the induction hypothesis applied to $+\psi'_1$ and $+\psi'_2$, we have that ψ'_1 and ψ'_2 are of the form $\psi_1(\overline{\varphi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$ and $\psi_2(\overline{\varphi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$, respectively, satisfying the statement for some order-types δ over $\overline{x} = \overline{y} \oplus \overline{X}$. We let $\psi = \psi_1(\overline{x}, \overline{z}) \vee \psi_2(\overline{x}, \overline{z})$, which is $\mathrm{IF}^{\Diamond}_{\delta}$. Hence ψ' is of the form $\psi = \psi_1(\overline{\varphi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z}) \vee \psi_2(\overline{\varphi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$, as required.

If $+\psi'$ is of the form $+(\chi - \varphi')$ with χ a sentence and $\varepsilon^{\partial}(+\chi)$, then $-\varphi'$ is ε-recursive, and hence, by the induction hypothesis, φ' is of the form $\varphi(\overline{\psi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$, where $\varphi(\overline{x}, \overline{z})$ is an $(\overline{x}, \overline{z})$-$\mathrm{IF}^{\Box}_{\delta^{\partial}}$-formula for some order-type δ over $\overline{x} = \overline{y} \oplus \overline{X}$, and the $\overline{\psi}$ and $\overline{\gamma}$ are sentences. Then we let $\psi = z - \varphi_1(\overline{x}, \overline{z})$, which is $(\overline{x}, \overline{z} \oplus z)$-$\mathrm{IF}^{\Diamond}_{\delta}$, where z is a fresh variable. Hence ψ' is of the form $(z - \varphi(\overline{\psi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z}_1))[\chi/z]$ as required.

If $+\psi'$ is of the form $+(\psi' - \chi)$ with χ a sentence and $\varepsilon(+\chi)$, then $+\psi'$ is ε-recursive, and hence by the induction hypothesis ψ' is of the form $\psi(\overline{\varphi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$ where $\psi(\overline{x}, \overline{z})$ is an $(\overline{x}, \overline{z})$-$\mathrm{IF}^{\Diamond}_{\delta}$-formula for some order-type δ over $\overline{x} = \overline{y} \oplus \overline{X}$, and the $\overline{\psi}$ and $\overline{\gamma}$ are sentences. Then we let $\psi = \psi(\overline{x}, \overline{z}) - z$ which is $(\overline{x}, \overline{z} \oplus z)$-$\mathrm{IF}^{\Diamond}_{\delta}$, where z is a fresh variable. Hence ψ' is of the form $(\psi(\overline{\psi}/\overline{y}, \overline{\gamma}/\overline{z}) - z)[\chi/z]$ as required.

The remaining cases are left to the reader. □

A.3.3. Remark. Actually, Lemma A.3.2 can be strengthened to the following:

1. Let ψ' be such that $+\psi'$ is non-trivially ε-recursive, and the P_3-paths of all critical branches are of length 0. Then ψ' is of the form $\psi(\overline{\varphi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$ where $\psi(\overline{x}, \overline{z})$ is an $(\overline{x}, \overline{z})$-IF$^{\Diamond}_{\delta}$-formula, for $\overline{x} = \overline{y} \oplus \overline{X}$ and some order-type δ over $\overline{x}$, and the $\overline{\varphi}$ and $\overline{\gamma}$ are sentences. Moreover, if $\overline{y} = (y_1, \ldots, y_n)$ then, for each $1 \leq i \leq n$, $+\varphi_i$ is ε-PIA if $\delta_i = 1$ and $-\varphi_i$ is ε-PIA if $\delta_i = \partial$. Finally $\varepsilon^{\partial}(+\psi(\overline{x}, \overline{\gamma}/\overline{z}))$.

2. Let φ' be such that $-\varphi'$ is non-trivially ε-recursive, and the P_3-paths of all critical branches are of length 0. Then φ' is of the form $\varphi(\overline{\psi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$ where $\varphi(\overline{x}, \overline{z})$ is an $(\overline{x}, \overline{z})$-IF$^{\Box}_{\delta}$-formula, for $\overline{x} = \overline{y} \oplus \overline{X}$ and some order-type δ over $\overline{x}$, and the $\overline{\varphi}$ and $\overline{\gamma}$ are sentences. Moreover, if $\overline{y} = (y_1, \ldots, y_n)$ then, for each $1 \leq i \leq n$, $-\psi_i$ is ε-PIA if $\delta_i = 1$ and $+\psi_i$ is ε-PIA if $\delta_i = \partial$. Finally $\varepsilon^{\partial}(-\varphi(\overline{x}, \overline{\gamma}/\overline{z}))$.

Hence, Proposition A.3.1 can be strengthened in an analogous way.

The proof of the enhanced Lemma A.3.2 is essentially a refined version of the induction in the original proof. The base case as it stands already verifies this strengthening. In particular, for $+\psi'$ we have $\psi(\overline{x}, \overline{\gamma}/\overline{z}) = x_1$ and $\varepsilon^{\partial}(x_1)$.

We illustrate the rest of the induction by considering the case when $+\psi'$ is of the form $+(\chi - \varphi')$ with χ a sentence and $\varepsilon^{\partial}(+\chi)$. Then $-\varphi'$ is non-trivially ε-recursive, and hence, by the strengthened induction hypothesis, φ' is of the form $\varphi(\overline{\psi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$, where $\varphi(\overline{x}, \overline{z})$ is an $(\overline{x}, \overline{z})$-IF$^{\Box}_{\delta^{\partial}}$-formula for some order-type δ over $\overline{x} = \overline{y} \oplus \overline{X}$, and the $\overline{\psi}$ and $\overline{\gamma}$ are sentences. Moreover, $\overline{y} = (y_1, \ldots, y_n)$ and for every $1 \leq i \leq n$, the generation tree $-\psi_i$ is non-trivially ε-PIA if $(\delta^{\partial})_i = 1$ (i.e., $\delta_i = \partial$) and $+\psi_i$ is non-trivially ε-PIA if $\delta^{\partial}_i = \partial$ (i.e., $\delta_i = 1$). Then we let $\psi = z - \varphi(\overline{x}, \overline{z})$, which is $(\overline{x}, \overline{z} \oplus z)$-IF$^{\Diamond}_{\delta}$, where z is a fresh variable. Moreover, for $1 \leq i \leq n$ we let $\varphi_i = \psi_i$. Hence ψ' is of the form $(z - \varphi(\overline{\psi}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z}))[\chi/z]$, with $\psi_1 \ldots \psi_n$ playing the role of $\varphi_1 \ldots \varphi_n$. Finally, $\varepsilon^{\partial}(\chi - \varphi(\overline{x}, \overline{\gamma}/\overline{z}))$, since $\varepsilon^{\partial}(+\chi)$, and the induction hypothesis implies that $\varepsilon(+\varphi(\overline{x}, \overline{\gamma}/\overline{z}))$.

Proposition A.3.1 and Lemma 5.4.3 together say that the approximation rules (μ^{δ}-A) and (ν^{δ}-A) can be applied to the inequalities (A.2), respectively.[2] In addition to this, by the enhancement of Proposition A.3.1 discussed in remark A.3.3, we can assume w.l.o.g. that any inequality sitting in the antecedents of the quasi-inequalities produced by these rule applications and containing a critical branch is of the form

$$\mathbf{j} \leq \varphi \quad \text{or} \quad \psi \leq \mathbf{n}, \tag{A.3}$$

[2]Note that applying one of these approximation rules within the antecedent of a quasi-inequality may split that quasi-inequality into the conjunction of several quasi-inequalities, on each of which we proceed separately.

where φ and ψ are sentences (see Definition 5.3.1.1), and moreover $+\varphi$ and $-\psi$ are non-trivially ε-PIA. Lemma A.4.1 in the next subsection, together with the fact that φ and ψ are sentences, ensures that the appropriate adjunction rules (IF_R^σ) and (IF_L^σ) are respectively applicable to these inequalities.

A.4 Application of adjunction rules

If ε and δ are order-types over $\overline{p}$ and $\overline{x}$ respectively, and $\overline{p}$ is not longer than $\overline{x}$ and has length n, then we abuse terminology and say that δ restricts to ε if $\varepsilon_i = \delta_i$ for each $1 \leq i \leq n$.

A.4.1. LEMMA. *Let ε be an order-type over $\overline{p}$.*

1. *Let $\varphi'(\overline{p}, \overline{X})$ be such that $+\varphi'$ is non-trivially ε-PIA. Then $\varphi'(\overline{p}, \overline{X})$ is of the form $\varphi(\overline{p}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$ where $\varphi(\overline{x}, \overline{z})$ is a normal $(\overline{x}, \overline{z})$-$\mathrm{IF}_\delta^\Box$-formula with $\overline{x} = \overline{y} \oplus \overline{X}$ and δ is an order-type over $\overline{x}$ which restricts to ε over $\overline{y}$. Moreover, $\varepsilon^\partial(\gamma) \prec +\varphi'$ for each $\gamma \in \overline{\gamma}$. Finally each $z \in \overline{z}$ occurs at most once in $\varphi(\overline{x}, \overline{z})$.*

2. *Let $\psi'(\overline{p}, \overline{X})$ be such that $-\psi'$ is non-trivially ε-PIA. Then $\psi'(\overline{p}, \overline{X})$ is of the form $\psi(\overline{p}/\overline{y}, \overline{X}, \overline{\gamma}/\overline{z})$ where $\psi(\overline{x}, \overline{z})$ is a normal $(\overline{x}, \overline{z})$-$\mathrm{IF}_\delta^\Diamond$-formula with $\overline{x} = \overline{y} \oplus \overline{X}$ and δ is an order-type over $\overline{x}$ which restricts to ε over $\overline{y}$. Moreover, $\varepsilon^\partial(\gamma) \prec -\psi'$ for each $\gamma \in \overline{\gamma}$. Finally each $z \in \overline{z}$ occurs at most once in $\psi(\overline{x}, \overline{z})$.*

Proof. It is sufficient to show that the formulas $\varphi(\overline{x}, \overline{z})$ and $\psi(\overline{x}, \overline{z})$ in the statement of the lemma are $(\overline{x}, \overline{z})$-$\mathrm{IF}_\delta^\Box$- and $(\overline{x}, \overline{z})$-$\mathrm{IF}_\delta^\Diamond$-formulas, respectively, since normality will then follow from Proposition 5.4.8.

Let us define the *PIA depth* of a non-trivial ε-PIA generation tree $*\xi$, with $* \in \{+, -\}$, to be the maximum length of its critical branches. The proof proceeds by simultaneous induction on the PIA depths of $+\varphi'$ and $-\psi'$.

If the depth of $+\varphi'$ is 0, then $\varphi' = p_1$ such that $\varepsilon_1 = 1$, so we let $\varphi = y_1$ which is $\mathrm{IF}_\delta^\Box$ with $\delta = (1)$.

Analogously for the base case of $-\psi'$.

As for the induction step, let us suppose that the depth of $+\varphi'$ is $k + 1$ and that the statement is true for generation trees satisfying the assumptions and of depth not greater than k. We proceed by cases depending on the form of $+\varphi'$.

If $+\varphi'$ is of the form $+\nu X.\varphi_1'(\overline{p}, \overline{X}')$ for $\overline{X}' = \overline{X} \oplus X$, then by the induction hypothesis applied to $+\varphi_1'$, we have that $\varphi_1'(\overline{p}, \overline{X}')$ is of the form $\varphi_1(\overline{p}/\overline{y}, \overline{X}', \overline{\gamma}/\overline{z})$, where $\varphi_1(\overline{x}', \overline{z})$ is an $(\overline{x}', \overline{z})$-$\mathrm{IF}_{\delta'}^\Box$ for $\overline{x}' = \overline{y} \oplus \overline{X}' = \overline{x} \oplus X$ with $\delta' = \delta \oplus 1$ where δ is an order-type over $\overline{x}$ which restricts to ε over $\overline{y}$, and each $z \in \overline{z}$ occurs at most once. Moreover, $\varepsilon^\partial(\gamma) \prec +\varphi_1'$ for each $\gamma \in \overline{\gamma}$. Hence we let $\varphi = \nu X.\varphi_1(\overline{x} \oplus X, \overline{z})$, which is an $(\overline{x}, \overline{z})$-$\mathrm{IF}_\delta^\Box$-formula, in which each $z \in \overline{z}$ occurs at most once.

If $+\varphi'$ is of the form $+(\varphi_1'(\overline{p},\overline{X}) \wedge \varphi_2'(\overline{p},\overline{X}))$, then by the induction hypothesis applied to $+\varphi_1'$ and $+\varphi_2'$ we have that φ_j' is of the form $\varphi_j(\overline{\varphi}/\overline{y},\overline{X},\overline{\gamma}/\overline{z})$ which satisfies the statement for some order-type δ over $\overline{x} = \overline{y} \oplus \overline{X}$ which restricts to ε over $\overline{y}$. We let $\varphi = \varphi_1(\overline{x},\overline{z}) \wedge \varphi_2(\overline{xz})$, which is $\mathrm{IF}_\delta^\Box$. Hence φ' is of the form $\varphi_1(\overline{p}/\overline{y},\overline{X},\overline{\gamma}/\overline{z}) \wedge \varphi_2(\overline{p}/\overline{y},\overline{X},\overline{\gamma}/\overline{z})$, as required.

If $+\varphi'$ is of the form $+(\psi'(\overline{p},\overline{X}) \to \chi)$ with $\varepsilon^\partial(+\chi)$, then $-\psi'$ is non-trivially ε-PIA, and hence by the induction hypothesis ψ' is of the form $\psi(\overline{p}/\overline{y},\overline{X},\overline{\gamma}/\overline{z})$ where $\psi(\overline{x},\overline{z})$ is an $(\overline{x},\overline{z})$-$\mathrm{IF}_{\delta^\partial}^\Diamond$-formula for $\overline{x} = \overline{y} \oplus \overline{X}$ and δ an order-type over $\overline{x}$ which restricts to ε over $\overline{y}$, in which each $z \in \overline{z}$ occurs at most once. Moreover, $\varepsilon^\partial(\gamma) \prec -\psi'$ for each $\gamma \in \overline{\gamma}$. Then we let $\varphi = \psi(\overline{x},\overline{z}) \to z$, where z is a fresh variable, which is $(\overline{x},\overline{z}\oplus z)$-$\mathrm{IF}_\delta^\Box$. Hence φ' is of the form $(\psi(\overline{p}/\overline{y},\overline{X},\overline{\gamma}/\overline{z}) \to z)[\chi/z]$ as required. Finally, $\varepsilon^\partial(\gamma) \prec -\psi' \prec +\varphi'$ for each $\gamma \in \overline{\gamma}$, and also $\varepsilon^\partial(+\chi)$ implies that $\varepsilon^\partial(\chi) \prec +\varphi'$.

The remaining cases are left to the reader. $\Box$

Analyzing lemma A.4.1 above we note that it guarantees us the following:

1. To every inequality $\mathbf{j} \leq \varphi$ with $+\varphi$ non-trivially ε-PIA, of a suitable instance of the (IF_R^σ) rule can be applied.

2. By a 'suitable' instance we mean one given in terms of a normal $(\overline{y},\overline{z})$-$\mathrm{IF}_\delta^\Box$-formula with $\delta = \varepsilon$ and in which all and only the ε-critical variable occurrences in $+\varphi$ are substituted for y-positions.

3. Consequently, all non-critical variable occurrences are in the $\overline{\gamma}$.

An analogous order-dual list of facts holds for inequalities $\psi \leq \mathbf{m}$ with $-\psi$ non-trivially ε-PIA.

A.5 The ε-Ackermann shape

Applying suitable instances of the rules (IF_R^σ) and (IF_L^σ) to the inequalities $\mathbf{j} \leq \varphi$ and $\psi \leq \mathbf{m}$ in (A.3), respectively, yields, for each $1 \leq i \leq n$,

$$\mathsf{LA}_i^\delta(\varphi)[\mathbf{i}/u,\overline{\gamma}/\overline{z}] \leq^{\varepsilon_i} p_i \quad \text{and} \quad p_i \leq^{\varepsilon_i} \mathsf{RA}_i^\delta(\psi)[\mathbf{m}/u,\overline{\gamma}/\overline{z}] \qquad \text{(A.4)}$$

respectively. Lemma 5.5.3 implies that the polarity of the $\overline{\gamma}$ remains invariant under such applications. Indeed, we know that $\varepsilon^\partial(\gamma) \prec +\varphi$ for each $\gamma \in \overline{\gamma}$. For each $1 \leq i \leq n$, if $\delta_i = 1$, then Lemma 5.5.3 implies that γ occurs in $\mathsf{LA}_i^\delta(\varphi)[\mathbf{i}/u,\overline{\gamma}/\overline{z}]$ with the opposite polarity to what it had in φ. Moreover, the rule yields $\mathsf{LA}_i^\delta(\varphi)[\mathbf{i}/u,\overline{\gamma}/\overline{z}] \leq^1 p_i$, hence $\varepsilon^\partial(\gamma) \prec -\mathsf{LA}_i^\delta(\varphi)[\mathbf{i}/u,\overline{\gamma}/\overline{z}]$. If $\delta_i = \partial$, then Lemma 5.5.3 implies that γ occurs in $\mathsf{LA}_i^\delta(\varphi)[\mathbf{i}/u,\overline{\gamma}/\overline{z}]$ with the same polarity to what it had in φ. Moreover, the rule yields $\mathsf{LA}_i^\delta(\varphi)[\mathbf{i}/u,\overline{\gamma}/\overline{z}] \leq^\partial p_i$, i.e, $p_i \leq$

$\mathsf{LA}_i^\delta(\varphi)[\mathbf{i}/u,\overline{\gamma}/\overline{z}]$, hence $\varepsilon^\partial(\gamma) \prec +\mathsf{LA}_i^\delta(\varphi)[\mathbf{i}/u,\overline{\gamma}/\overline{z}]$. The application of (IF_L^σ) to $\psi \leq \mathbf{m}$ is analyzed order-dually.

Since the only occurrences of $\overline{p}$ in $\mathsf{LA}_i^\delta(\varphi)[\mathbf{i}/u,\overline{\gamma}/\overline{z}]$ and in $\mathsf{RA}_i^\delta(\psi)[\mathbf{m}/u,\overline{\gamma}/\overline{z}]$ are of course the ones sitting in the $\overline{\gamma}$, the polarity invariance discussed above implies that the displayed inequalities in (A.4) are of the form

$$\alpha(\overline{p}) \leq p_i \quad \text{or} \quad p_j \leq \alpha'(\overline{p}), \tag{A.5}$$

with $\varepsilon_i = 1$, $\varepsilon_j = \partial$, $\varepsilon^\partial(-\alpha(\overline{p}))$, and $\varepsilon^\partial(+\alpha'(\overline{p}))$.

A.5.1. Definition. Given an order-type ε over $\overline{p}$, a set of $\mathcal{L}^+$-inequalities is in *ε-Ackermann shape* if each inequality in the set is of one of the following forms:

1. $\alpha(\overline{p}) \leq p_i$ with $\varepsilon_i = 1$ and $\varepsilon^\partial(-\alpha(\overline{p}))$;
2. $p_j \leq \alpha'(\overline{p})$ if $\varepsilon_j = \partial$ and $\varepsilon^\partial(+\alpha'(\overline{p}))$;
3. $\gamma \leq \gamma'$ with $\varepsilon^\partial(-\gamma)$ and $\varepsilon^\partial(+\gamma')$.

Taking stock of the reduction process up to this point, we see that the obtained system is now in ε-Ackermann shape. Indeed, through the application of approximation rules, the ε-critical occurrences in the antecedent of (A.1) have been ripped out, were made to sit in inequalities of the form (A.3), and then displayed in inequalities as in (A.5), which are clearly in one of the required shape of Definition A.5.1.1 or A.5.1.2. Moreover, the approximation rules produced, besides the inequalities of the form (A.3), also inequalities of the form $\mathbf{i} \leq \psi(\overline{x},\gamma/\overline{z})$ and $\varphi(\overline{x},\gamma/\overline{z}) \leq \mathbf{m}$, which, by remark A.3.3, are of the form prescribed in Definition A.5.1.3.

Exactly in the same way as was shown in [50], applying the Ackermann rule to a set of inequalities in ε-Ackermann shape produces a set of inequalities which is again in ε-Ackermann shape. Hence all occurrences of propositional variables may be eliminated by repeated applications of the Ackermann rule. Thus we shown that this procedure applied to an ε-recursive inequality in input produces a set of pure quasi-inequalities in $\mathcal{L}^+$, each of which can be equivalently translated into FO+LFP.

Appendix B

Topological Ackermann Lemmas

B.1 Intersection lemmas

B.1.1. Definition. 1. A DLE^{++} formula is *syntactically closed* if all occurrences of nominals, $\overline{f}^{(i)}$ for $\varepsilon_f(i) = \partial$, $\underline{g}^{(i)}$ for $\varepsilon_g(i) = 1$, $\blacktriangleleft_\lambda, \blacklozenge_\sigma$ are positive, and all occurrences of co-nominals, $\overline{f}^{(i)}$ for $\varepsilon_f(i) = 1$, $\underline{g}^{(i)}$ for $\varepsilon_g(i) = \partial$, $\blacktriangleright_\rho, \blacksquare_\pi$ are negative;

2. A DLE^{++} formula is *syntactically open* if all occurrences of nominals, $\overline{f}^{(i)}$ for $\varepsilon_f(i) = \partial$, $\underline{g}^{(i)}$ for $\varepsilon_g(i) = 1$, $\blacktriangleleft_\lambda, \blacklozenge_\sigma$ are negative, and all occurrences of co-nominals, $\overline{f}^{(i)}$ for $\varepsilon_f(i) = 1$, $\underline{g}^{(i)}$ for $\varepsilon_g(i) = \partial$, $\blacktriangleright_\rho, \blacksquare_\pi$ are positive.

Recall that $\blacksquare_\pi, \blacklozenge_\sigma, \blacktriangleleft_\lambda, \blacktriangleright_\rho \colon \mathbb{A}^\delta \to \mathbb{A}^\delta$ respectively denote the adjoints of the maps $\Diamond_\pi, \Box_\sigma, \lhd_\lambda, \rhd_\rho$. Then Lemma 6.3.9 immediately implies the following facts, which will be needed for the soundness of the topological Ackermann rule:

B.1.2. Lemma. *1. If $o \in O(\mathbb{A}^\delta)$ and $\pi(\bot) \leq o$, then $\blacksquare_\pi(o) = \bigvee\{a \mid a \leq \blacksquare_\pi(o)\} \in O(\mathbb{A}^\delta)$.*

2. If $k \in K(\mathbb{A}^\delta)$ and $\sigma(\top) \geq k$, then $\blacklozenge_\sigma(k) = \bigwedge\{a \mid a \geq \blacklozenge_\sigma(k)\} \in K(\mathbb{A}^\delta)$.

3. If $o \in O(\mathbb{A}^\delta)$ and $\lambda(\top) \leq o$, then $\blacktriangleleft_\lambda (o) = \bigwedge\{a \mid a \geq \blacktriangleleft_\lambda (o)\} \in K(\mathbb{A}^\delta)$.

4. If $k \in K(\mathbb{A}^\delta)$ and $\rho(\bot) \geq k$, then $\blacktriangleright_\rho (k) = \bigvee\{a \mid a \leq \blacktriangleright_\rho (k)\} \in O(\mathbb{A}^\delta)$.

In the remainder of the chapter, we work under the assumption that the values of all parameters (propositional variables, nominals and conominals) occurring in the term functions mentioned in the statements of propositions and lemmas are given by admissible assignments.

B.1.3. Lemma. *Let $\varphi(p)$ be syntactically closed, $\psi(p)$ be syntactically open, $c \in K(A^\delta)$ and $o \in O(A^\delta)$.*

1. *If $\varphi(p)$ is positive in p, $\psi(p)$ is negative in p, and*

   (a) $\pi(\bot) \leq \psi'^{\mathbb{A}^\delta}(c)$ *for any subformula* $\blacksquare_\pi \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   (b) $\sigma(\top) \geq \psi'^{\mathbb{A}^\delta}(c)$ *for any subformula* $\blacklozenge_\sigma \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   (c) $\lambda(\top) \leq \psi'^{\mathbb{A}^\delta}(c)$ *for any subformula* $\blacktriangleleft_\lambda \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   (d) $\rho(\bot) \geq \psi'^{\mathbb{A}^\delta}(c)$ *for any subformula* $\blacktriangleright_\rho \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   then $\varphi(c) \in K(\mathbb{A}^\delta)$ *and* $\psi(c) \in O(\mathbb{A}^\delta)$ *for each* $c \in K(\mathbb{A}^\delta)$.

2. *If $\varphi(p)$ is negative in p, $\psi(p)$ is positive in p, and*

   (a) $\pi(\bot) \leq \psi'^{\mathbb{A}^\delta}(o)$ *for any subformula* $\blacksquare_\pi \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   (b) $\sigma(\top) \geq \psi'^{\mathbb{A}^\delta}(o)$ *for any subformula* $\blacklozenge_\sigma \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   (c) $\lambda(\top) \leq \psi'^{\mathbb{A}^\delta}(o)$ *for any subformula* $\blacktriangleleft_\lambda \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   (d) $\rho(\bot) \geq \psi'^{\mathbb{A}^\delta}(o)$ *for any subformula* $\blacktriangleright_\rho \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   then $\varphi(o) \in K(\mathbb{A}^\delta)$ *and* $\psi(o) \in O(\mathbb{A}^\delta)$ *for each* $o \in O(\mathbb{A}^\delta)$.

Proof. The proof proceeds by simultaneous induction on φ and ψ. It is easy to see that φ cannot be $\mathbf{m}$, and the outermost connective of φ cannot be $\overline{f}^{(i)}$ with $\varepsilon_f(i) = 1$, or $\underline{g}^{(j)}$ with $\varepsilon_g(j) = \partial$, or $\blacksquare_\pi, \blacktriangleright_\rho, \rightarrow$. Similarly, ψ cannot be $\mathbf{i}$, and the outermost connective of ψ cannot be $\underline{g}^{(j)}$ with $\varepsilon_g(j) = 1$, or $\overline{f}^{(i)}$ with $\varepsilon_f(i) = \partial$, or $\blacklozenge_\sigma, \blacktriangleleft_\lambda, -$.

The basic cases, that is, $\varphi = \bot, \top, p, q, \mathbf{i}$ and $\psi = \bot, \top, p, q, \mathbf{m}$ are straightforward.

Assume that $\varphi(p) = \blacklozenge_\sigma \varphi'(p)$. Since $\varphi(p)$ is positive in p, the subformula $\varphi'(p)$ is syntactically closed and positive in p, and assumptions 1(a)-1(d) hold also for $\varphi'(p)$. Hence, by inductive hypothesis, $\varphi'(c) \in K(\mathbb{A}^\delta)$ for any $c \in K(\mathbb{A}^\delta)$. In particular, assumption 1(b) implies that $\sigma(\top) \geq \varphi'(c)$. Hence, by Lemma B.1.2, $\blacklozenge_\sigma \varphi'(c) \in K(\mathbb{A}^\delta)$, as required. The case in which $\varphi(p)$ is negative in p is argued order-dually.

The cases in which $\varphi(p) = \blacklozenge_\sigma \varphi'(p), \blacktriangleleft_\lambda \varphi'(p), \blacktriangleright_\rho \varphi'(p)$ are similar to the one above.

The cases of the remaining connectives are treated as in [50, Lemma 11.9] and the corresponding proofs are omitted.

$\square$

B.1.4. Lemma (Intersection lemma). *Let $\varphi(p)$ be syntactically closed, $\psi(p)$ be syntactically open, $\mathcal{C} \subseteq K(A^\delta)$ be downward-directed, $\mathcal{O} \subseteq O(A^\delta)$ be upward-directed. Then*

1. *if $\varphi(p)$ is positive in p, $\psi(p)$ is negative in p, and*

   (a) $\pi(\bot) \leq \psi'^{\mathbb{A}^\delta}(\bigwedge \mathcal{C})$ *for any subformula* $\blacksquare_\pi \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   (b) $\sigma(\top) \geq \psi'^{\mathbb{A}^\delta}(\bigwedge \mathcal{C})$ *for any subformula* $\blacklozenge_\sigma \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   (c) $\lambda(\top) \leq \psi'^{\mathbb{A}^\delta}(\bigwedge \mathcal{C})$ *for any subformula* $\blacktriangleleft_\lambda \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   (d) $\rho(\bot) \geq \psi'^{\mathbb{A}^\delta}(\bigwedge \mathcal{C})$ *for any subformula* $\blacktriangleright_\rho \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   then

   (a) $\varphi^{\mathbb{A}^\delta}(\bigwedge \mathcal{C}) = \bigwedge\{\varphi^{\mathbb{A}^\delta}(c) : c \in \mathcal{C}'\}$ *for some down-directed subcollection* $\mathcal{C}' \subseteq \mathcal{C}$ *such that* $\varphi^{\mathbb{A}^\delta}(c) \in K(\mathbb{A}^\delta)$ *for each* $c \in \mathcal{C}'$.

   (b) $\psi^{\mathbb{A}^\delta}(\bigwedge \mathcal{C}) = \bigvee\{\psi^{\mathbb{A}^\delta}(c) : c \in \mathcal{C}'\}$ *for some down-directed subcollection* $\mathcal{C}' \subseteq \mathcal{C}$ *such that* $\psi^{\mathbb{A}^\delta}(c) \in O(\mathbb{A}^\delta)$ *for each* $c \in \mathcal{C}'$.

2. *If $\varphi(p)$ is negative in p, $\psi(p)$ is positive in p, and*

   (a) $\pi(\bot) \leq \psi'^{\mathbb{A}^\delta}(\bigvee \mathcal{O})$ *for any subformula* $\blacksquare_\pi \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   (b) $\sigma(\top) \geq \psi'^{\mathbb{A}^\delta}(\bigvee \mathcal{O})$ *for any subformula* $\blacklozenge_\sigma \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   (c) $\lambda(\top) \leq \psi'^{\mathbb{A}^\delta}(\bigvee \mathcal{O})$ *for any subformula* $\blacktriangleleft_\lambda \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   (d) $\rho(\bot) \geq \psi'^{\mathbb{A}^\delta}(\bigvee \mathcal{O})$ *for any subformula* $\blacktriangleright_\rho \psi'(p)$ *of* $\varphi(p)$ *and of* $\psi(p)$,

   then

   (a) $\varphi^{\mathbb{A}^\delta}(\bigvee \mathcal{O}) = \bigwedge\{\varphi^{\mathbb{A}^\delta}(o) : o \in \mathcal{O}'\}$ *for some up-directed subcollection* $\mathcal{O}' \subseteq \mathcal{O}$ *such that* $\varphi^{\mathbb{A}^\delta}(o) \in K(\mathbb{A}^\delta)$ *for each* $o \in \mathcal{O}'$.

   (b) $\varphi^{\mathbb{A}^\delta}(\bigvee \mathcal{O}) = \bigvee\{\varphi^{\mathbb{A}^\delta}(o) : o \in \mathcal{O}'\}$ *for some up-directed subcollection* $\mathcal{O}' \subseteq \mathcal{O}$ *such that* $\varphi^{\mathbb{A}^\delta}(o) \in O(\mathbb{A}^\delta)$ *for each* $o \in \mathcal{O}'$.

Proof. The proof proceeds by simultaneous induction on φ and ψ. It is easy to see that φ cannot be $\mathbf{m}$, and the outermost connective of φ cannot be $\overline{f}^{(i)}$ with $\varepsilon_f(i) = 1$, or $\underline{g}^{(j)}$ with $\varepsilon_g(j) = \partial$, or $\blacksquare_\pi, \blacktriangleright_\rho, \rightarrow$. Similarly, ψ cannot be $\mathbf{i}$, and the outermost connective of ψ cannot be $\underline{g}^{(j)}$ with $\varepsilon_g(j) = 1$, or $\overline{f}^{(i)}$ with $\varepsilon_f(i) = \partial$, or $\blacklozenge_\sigma, \blacktriangleleft_\lambda, -$.

The basic cases in which $\varphi = \bot, \top, p, q, i$ and $\psi = \bot, \top, p, q, m$ are straightforward.

Assume that $\varphi(p) = \blacklozenge_\sigma \varphi'(p)$. Since $\varphi(p)$ is positive in p, the subformula $\varphi'(p)$ is syntactically closed and positive in p, and assumptions 1(a)-1(d) hold also for $\varphi'(p)$. Hence, by inductive hypothesis, $\varphi'(\bigwedge \mathcal{C}) = \bigwedge\{\varphi'(c) \mid c \in \mathcal{C}''\}$ for some down-directed subcollection $\mathcal{C}'' \subseteq \mathcal{C}$ such that $\varphi'(c) \in K(\mathbb{A}^\delta)$ for each $c \in \mathcal{C}''$. In particular, assumption 1(b) implies that $\sigma(\top) \geq \varphi'(\bigwedge \mathcal{C}) = \bigwedge\{\varphi'(c) \mid c \in \mathcal{C}''\}$. Notice that $\varphi'(p)$ being positive in p and $\mathcal{C}''$ being down-directed imply that

$\{\varphi'(c) \mid c \in \mathcal{C}''\}$ is down-directed. Hence, by Proposition 6.3.10 applied to $\{\varphi'(c) \mid c \in \mathcal{C}''\}$, we get that $\blacklozenge_\sigma \varphi'(\bigwedge \mathcal{C}) = \blacklozenge_\sigma(\bigwedge\{\varphi'(c) \mid c \in \mathcal{C}''\}) = \bigwedge\{\blacklozenge_\sigma \varphi'(c) \mid c \in \mathcal{C}''\}$. Moreover, there exists some down-directed subcollection $\mathcal{C}' \subseteq \mathcal{C}''$ such that $\blacklozenge_\sigma \varphi'(c) \in K(\mathbb{A}^\delta)$ for each $c \in \mathcal{C}'$ and $\bigwedge\{\blacklozenge_\sigma \varphi'(c) \mid c \in \mathcal{C}'\} = \bigwedge\{\blacklozenge_\sigma \varphi'(c) \mid c \in \mathcal{C}''\}$. This gives us $\blacklozenge_\sigma \varphi'(\bigwedge \mathcal{C}) = \bigwedge\{\blacklozenge_\sigma \varphi'(c) \mid c \in \mathcal{C}'\}$ as required. The case in which $\varphi(p)$ is negative in p is argued order-dually.

The cases in which $\varphi(p) = \blacklozenge_\sigma \varphi'(p), \blacktriangleleft_\lambda \varphi'(p), \blacktriangleright_\rho \varphi'(p)$ are similar to the one above.

The cases of the remaining connectives are treated as in [50, Lemma 11.10] and the corresponding proofs are omitted.

□

B.2 Topological Ackermann Lemma

B.2.1. Definition. A system S of DLE^{++} inequalities is *topologically adequate* when the following conditions hold:

1. if $\varphi \leq \blacksquare_\pi \psi$ is in S, then $\pi(\bot) \leq \psi$ is in S, and
2. if $\blacklozenge_\sigma \varphi \leq \psi$ is in S, then $\sigma(\top) \geq \varphi$ is in S, and
3. if $\varphi \leq \blacktriangleright_\rho \psi$ is in S, then $\rho(\bot) \geq \psi$ is in S, and
4. if $\blacktriangleleft_\lambda \varphi \leq \psi$ is in S, then $\lambda(\top) \leq \varphi$ is in S.

B.2.2. Definition. A system S of DLE^{++} inequalities is *compact-appropriate* if the left-hand side of each inequality in S is syntactically closed and the right-hand side of each inequality in S is syntactically open (cf. Definition B.1.1).

B.2.3. Proposition. *Topological adequacy is an invariant of safe executions of ALBA^e.*

Proof. Preprocessing vacuously preserves the topological adequacy of any input inequality. The topological adequacy is vacuously satisfied up to the first application of an adjunction rule introducing any of $\blacksquare_\pi, \blacklozenge_\sigma, \blacktriangleleft_\lambda, \blacktriangleright_\rho$. Each such application introduces two inequalities, one of which contains the new black connective, and the other one exactly is the side condition required by the definition of topological adequacy for the first inequality to be non-offending. Moreover, at any later stage, safe executions of ALBA do not modify the side conditions, unless for substituting minimal valuations. This, together with the fact that ALBA^e does not contain any rules which allow to manipulate any of $\blacksquare_\pi, \blacklozenge_\sigma, \blacktriangleleft_\lambda, \blacktriangleright_\rho$, guarantees the preservation of topological adequacy. Indeed, if e.g. $\pi(\bot) \leq \psi$ and

$\varphi \leq \blacksquare_\pi \psi$ are both in a topologically adequate quasi-inequality, then the variables occurring in ψ in both inequalities have the same polarity, and in a safe execution, the only way in which they could be modified is if they both receive the same minimal valuations under applications of Ackermann rules. Hence, after such an application, they would respectively be transformed into $\pi(\bot) \leq \psi'$ and $\varphi' \leq \blacksquare_\pi \psi'$ for the *same* ψ'. Thus, the topological adequacy of the quasi-inequality is preserved. □

B.2.4. Lemma. *Compact-appropriateness is an invariant of ALBA^e executions.*

Proof. Entirely analogous to the proof of [50, Lemma 9.5]. □

B.2.5. Proposition (Right-handed Topological Ackermann Lemma). *Let S be a topologically adequate system of DLE inequalities which is the union of the following disjoint subsets:*

- *S_1 consists only of inequalities in which p does not occur;*
- *S_2 consists of inequalities of the type $\alpha \leq p$, where α is syntactically closed and p does not occur in α;*
- *S_3 consists of inequalities of the type $\beta(p) \leq \gamma(p)$ where $\beta(p)$ is syntactically closed and positive in p, and $\gamma(p)$ be syntactically open and negative in p,*

Then the following are equivalent:

1. *$\beta^{\mathbb{A}^\delta}(\bigvee \alpha^{\mathbb{A}^\delta}) \leq \gamma^{\mathbb{A}^\delta}(\bigvee \alpha^{\mathbb{A}^\delta})$ for all inequalities in S_3, where $\bigvee \alpha$ abbreviates $\bigvee\{\alpha \mid \alpha \leq p \in S_2\}$;*
2. *There exists $a_0 \in \mathbb{A}$ such that $\bigvee \alpha^{\mathbb{A}^\delta} \leq a_0$ and $\beta^{\mathbb{A}^\delta}(a_0) \leq \gamma^{\mathbb{A}^\delta}(a_0)$ for all inequalities in S_3.*

Proof.

($\Leftarrow$) By the monotonicity of $\beta_i(p)$ and antitonicity of $\gamma_i(p)$ in p for $1 \leq i \leq n$, together with $\alpha^{\mathbb{A}^\delta} \leq a_0$ we have that $\beta_i^{\mathbb{A}^\delta}(\alpha^{\mathbb{A}^\delta}) \leq \beta_i^{\mathbb{A}^\delta}(a_0) \leq \gamma_i^{\mathbb{A}^\delta}(a_0) \leq \gamma_i^{\mathbb{A}^\delta}(\alpha^{\mathbb{A}^\delta})$.

($\Rightarrow$) Since the quasi-inequality is topologically adequate, by Lemma B.1.4.1, $\alpha^{\mathbb{A}^\delta} \in K(A^\delta)$.

Hence, $\alpha^{\mathbb{A}^\delta} = \bigwedge\{a \in \mathbb{A} : \alpha^{\mathbb{A}^\delta} \leq a\}$, making it the meet of a downward-directed set of clopen elements. Therefore, we can rewrite each inequality in S_3 as

$$\beta^{\mathbb{A}^\delta}(\bigwedge\{a \in \mathbb{A} : \alpha^{\mathbb{A}^\delta} \leq a\}) \leq \gamma^{\mathbb{A}^\delta}(\bigwedge\{a \in \mathbb{A} : \alpha^{\mathbb{A}^\delta} \leq a\}).$$

Since β is syntactically closed and positive in p, γ is syntactically open and negative in p, again by topological adequacy, we can apply Lemma B.1.4 and get that

$$\bigwedge\{\beta^{\mathbb{A}^\delta}(a) : a \in \mathcal{A}_1\} \leq \bigvee\{\gamma_i^{\mathbb{A}^\delta}(b) : b \in \mathcal{A}_2\}$$

for some $\mathcal{A}_1, \mathcal{A}_2 \subseteq \{a \in \mathbb{A} : \alpha^{\mathbb{A}^\delta} \leq a\}$ such that $\beta^{\mathbb{A}^\delta}(a) \in K(\mathbb{A}^\delta)$ for each $a \in \mathcal{A}_1$, and $\gamma^{\mathbb{A}^\delta}(b) \in O(\mathbb{A}^\delta)$ for each $b \in \mathcal{A}_2$. By compactness,

$$\bigwedge\{\beta_i^{\mathbb{A}^\delta}(a) : a \in \mathcal{A}_1'\} \leq \bigvee\{\gamma_i^{\mathbb{A}^\delta}(b) : b \in \mathcal{A}_2'\}$$

for some finite subsets $\mathcal{A}_1' \subseteq \mathcal{A}_1$ and $\mathcal{A}_2' \subseteq \mathcal{A}_2$. Then let $a^* = \bigwedge\{\bigwedge \mathcal{A}_1' \wedge \bigwedge \mathcal{A}_2' \mid \beta \leq \gamma \in S_3\}$. Clearly, $a^* \in \mathbb{A}$, and $\alpha^{\mathbb{A}^\delta} \leq a^*$. By the monotonicity of $\beta(p)$ and the antitonicity of $\gamma(p)$ in p for each $\beta \leq \gamma$ in S_3, we have $\beta^{\mathbb{A}^\delta}(a^*) \leq \beta^{\mathbb{A}^\delta}(a)$ and $\gamma_i^{\mathbb{A}^\delta}(b) \leq \gamma_i^{\mathbb{A}^\delta}(a^*)$ for all $a \in \mathcal{A}_1'$ and all $b \in \mathcal{A}_2'$. Therefore,

$$\beta_i^{\mathbb{A}^\delta}(a^*) \leq \bigwedge\{\beta_i^{\mathbb{A}^\delta}(a) : a \in \mathcal{A}_1'\} \leq \bigvee\{\gamma_i^{\mathbb{A}^\delta}(b) : b \in \mathcal{A}_2'\} \leq \gamma_i^{\mathbb{A}^\delta}(a^*)$$

for each $\beta \leq \gamma$ in S_3. □

B.2.6. Proposition (Left-handed Topological Ackermann Lemma).

Let S be a topologically adequate system of DLE inequalities which is the union of the following disjoint subsets:

- *S_1 consists only of inequalities in which p does not occur;*
- *S_2 consists of inequalities of the type $p \leq \alpha$, where α is syntactically open and p does not occur in α;*
- *S_3 consists of inequalities of the type $\beta(p) \leq \gamma(p)$ where $\beta(p)$ is syntactically closed and negative in p, and $\gamma(p)$ be syntactically open and positive in p,*

Then the following are equivalent:

1. *$\beta^{\mathbb{A}^\delta}(\bigwedge \alpha^{\mathbb{A}^\delta}) \leq \gamma^{\mathbb{A}^\delta}(\bigwedge \alpha^{\mathbb{A}^\delta})$ for all inequalities in S_3, where $\bigwedge \alpha$ abbreviates $\bigwedge\{\alpha \mid p \leq \alpha \in S_2\}$;*
2. *There exists $a_0 \in \mathbb{A}$ such that $a_0 \leq \bigwedge \alpha^{\mathbb{A}^\delta}$ and $\beta^{\mathbb{A}^\delta}(a_0) \leq \gamma^{\mathbb{A}^\delta}(a_0)$ for all inequalities in S_3.*

Proof. The proof is similar to the proof of the right-handed Ackermann lemma and is omitted. □

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Samenvatting

In dit proefschrift bestuderen we correspondentie en canoniciteit in de niet-klassieke logica, middels het gebruik van algebraïsche en orde-topologische methoden. Correspondentietheorie beoogt te beantwoorden op welke wijze modale, eerste- en tweede-orde talen met elkaar interacteren en overlappen in hun gezamenlijke semantische omgeving. Dit proefschrift betreft Sahlqvist correspondentietheorie, een lijn van onderzoek in correspondentietheorie die oorspronkelijk ontwikkeld is voor klassieke modale logica, en een systematische vertaling geeft tussen klassieke modale logica en eerste-orde logica.

Modale talen die geïnterpreteerd worden over relationele structuren, zijn expressieve fragmenten van tweede-orde logica. Het beroemde Sahlqvist-van-Benthem algoritme, de hoeksteen van de correspondentietheorie, stelt echter dat voor iedere formule in een grote syntactische klasse van modale formules genaamd Sahlqvist formules, de correspondent in feite een eerste-orde zin is. Deze correspondent kan bovendien efficiënt berekend worden. Canoniciteit is nauw verwant aan correspondentie, en garandeert dat logica's die geaxiomatiseerd zijn door deze formules semantisch volledig zijn met betrekking tot eerste-orde definieerbare klasses van relationele structuren.

Het eerste gedeelte van dit proefschrift richt zich op algebraïsche methoden voor correspondentie en canoniciteit. In Hoofdstuk 3 presenteren we een algebraïsche aanpak voor Sahlqvist-type correspondentie resultaten door de klassieke Sahlqvist correspondentie stelling te bewijzen voor modale logica in de algebraïsche setting van complexe algebra's. In de algebraische setting kunnen reductie strategieën om tweede-orde variabelen te elimineren volledig in theoretische termen geformuleerd worden. De orde-theoretische condities die de toepasbaarheid van deze strategieën garanderen, leiden ook tot een positieve karakterizering van Sahlqvist en inductieve formules voor verschillende signaturen. Conradie en Palmigiano [50] ontwikkelden een Ackermann Lemma Based Algorithm (ALBA) voor distributieve modale logica, gebasseerd op een algebraïsche analyse van de correspondentietheorie. In Hoofdstuk 4 en 5, respectievelijk, breiden we

het ALBA algoritme uit naar de reguliere modale logica (modale logica met niet-normale modaliteiten) en de intuïtionistische modale mu-calculus. Ook geven we een syntactische definitie van de klasse van inductieve ongelijkheden in deze talen en laten we zien dat het ALBA algoritme werkt op deze ongelijk-heden. In Hoofdstuk 6 ontwikkelen we een versie van ALBA voor distributieve tralie uitbreidingen (DLEs), waarmee we de canoniciteit bewijzen van zekere syntactisch gedefinieerde klasses van DLE-ongelijkheden (de meta-inductieve ongelijkheden) met betrekking tot de structuren waarin de formules die de additiviteit van bepaalde gegeven termen beweren geldig zijn.

Het tweede gedeelte van het proefschrift richt zich op orde-topologische methoden. In Hoofdstuk 7 introduceren we het concept van een subordinatie van een Boolese algebra en ontwikkelen we een categoriale dualiteit tussen Boolese algebra's met een subordinatie en Stone ruimtes met een gesloten relatie. We breiden deze dualiteit uit om te laten zien dat de categorie van de Vries algebra's dual is aan de categorie van Gleason ruimtes: extreem onsamenhangende ruimtes met een gesloten irreducibele equivalentierelatie. Hiermee geven we een alternatieve Jónsson-Tarski stijl dualiteit voor de de Vries dualiteit tussen compacte Hausdorff ruimtes en de Vries algebra's, wat een mogelijkheid biedt voor het ontwikkelen van een topologische correspondentie theorie en een logische calculus voor compacte Hausdorff ruimtes . In Hoofdstuk 8 bewijzen we een Sahlqvist correspondentie en canoniciteit stelling voor topologische dekpuntlogica in het geval van modaal compacte Hausdorff ruimtes. Dit is een generalisatie van het Sambin-Vaccaro bewijs voor de canoniciteit van de taal van positieve modale mu-calculus geïnterpreteerd over modale compacte Hausdorf ruimtes.

Abstract

In this thesis, we study correspondence and canonicity for non-classical logic using algebraic and order-topological methods. Correspondence theory is aimed at answering the question of how precisely modal, first-order, second-order languages interact and overlap in their shared semantic environment. The line of research in correspondence theory which concerns the present thesis is Sahlqvist correspondence theory — which was originally developed for classical modal logic, and provides a systematic translation between classical modal logic and first-order logic.

Modal languages are expressive fragments of second-order logic when interpreted over relational structures. However, the celebrated *Sahlqvist-van Benthem theorem*, which is the cornerstone of the correspondence theory, states that for every formula in a large, syntactically defined class of modal formulas called *Sahlqvist formulas*, the correspondent is, in fact, a first-order sentence. Moreover, this correspondent can be computed effectively. Canonicity is closely related to correspondence, and ensures that logics axiomatized by these formulas are complete with respect to relational semantics. Thus, correspondence and canonicity together establish that Sahlqvist logics are semantically complete with respect to first-order definable classes of relational structures.

The first part of the thesis focuses on algebraic methods for correspondence and canonicity. In chapter 3, we introduce the algebraic approach to Sahlqvist-type correspondence results by proving the classical Sahlqvist correspondence theorem for basic modal logic in the algebraic setting of complex algebras of frames. In the algebraic setting, the reduction strategies for the elimination of the second order variables can be formulated entirely in order-theoretic terms. The order-theoretic conditions that guarantee the applicability of these strategies also lead to a positive characterization of Sahlqvist and inductive formulas across different signatures. Conradie and Palmigiano [50] develop an Ackermann Lemma Based Algorithm (ALBA) for distributive modal logic based on an algebraic analysis of the correspondence theory. We extend the algorithm ALBA to regular modal logic

(modal logic with non-normal modalities) and intuitionistic modal mu-calculus in Chapters 4 and 5, respectively. Moreover, we syntactically define the class of inductive inequalities in these languages, and show that the algorithm succeeds on them. In Chapter 6, we develop a version of ALBA for distributive lattice expansions (DLEs), using which we prove the canonicity of certain syntactically defined classes of DLE-inequalities (called the *meta-inductive inequalities*), relative to the structures in which the formulas asserting the additivity of some given terms are valid.

The second part focuses on order-topological methods. In Chapter 7, we introduce the concept of a subordination on a Boolean algebra, and develop a full categorical duality between Boolean algebras with a subordination and Stone spaces with a closed relation. We further extend this duality to show that the category of de Vries algebras is dual to the category of Gleason spaces, which are extremely disconnected spaces with a closed irreducible equivalence relation. This provides an alternative Jnsson-Tarski style duality to de Vries duality between de compact Hausdorff spaces and de Vries algebras. It also offers a possibility for developing a topological correspondence theory, and a logical calculus for modal compact Hausdorff spaces. In Chapter 8, we prove a Sahlqvist correspondence and canonicity theorem for topological fixed-point logic on compact Hausdorff spaces. This generalizes the Sambin-Vaccaro proof of canonicity for the language of positive modal mu-calculus interpreted over modal compact Hausdorff spaces.

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