Correspondence of phase transition points and singularities of thermodynamic geometry of black holes

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Abstract We explore a formulation of the thermodynamic geometry of black holes and prove that the divergent points of the specific heat correspond exactly to the singularities of the thermodynamic curvature. We investigate this correspondence for different types of black holes. This formulation can also be applied to an arbitrary thermodynamic system.

1 Introduction

In 1971, Hawking [1] stated that the area, A, of the event horizon of a black hole can never decrease in physical processes. It was later noted by Bekenstein [2] that this result was analogous to the statement of the ordinary second law of thermodynamics, namely that the total entropy, S, of a closed system never decreases. Bekenstein proposed that the entropy of a black hole is proportional to its area. The correspondence between the thermodynamics of black holes and the well-known first and second laws of thermodynamics was further studied in [3]. However, it is still a challenging problem to find the statistical origin of black-hole thermodynamics.

The geometric formulation of thermodynamics is a useful tool in the study of some aspects of physical systems. For instance, Weinhold [4] introduced in the equilibrium space a Riemannian metric defined in terms of the second derivatives of the internal energy with respect to the entropy and other extensive variables of a thermodynamic system. Moreover, in 1979, Ruppeiner [5] introduced a Riemannian metric structure in thermodynamic fluctuation theory, and related it to the second derivatives of the entropy. The Ruppeiner metric is based on the thermodynamic state space geometry. For the second-order phase transitions, the Ruppeiner scalar curvature (R) is expected to diverge at the critical point [6–9]. Over the last decade, thermodynamic geometry and some of

its new formulations have also been applied to black holes [10–15]. Another geometric formulation of thermodynamics was proposed by Quvedo [16]. Quvedo's method incorporates Legendre invariance in a natural way, and it allows us to derive Legendre invariant metrics in the space of equilibrium states. However, this method not only contains certain ambiguities but also fails to explain the correspondence between phase transitions and singularities of the scalar curvature for the phantom Reissner–Nordstrom–AdS black hole [17–19].

In this paper, we explore a formulation for the thermodynamic geometry of black holes, and we prove that the divergent points of the specific heat correspond exactly to the singularities of the thermodynamic geometry. The outline of this paper is as follows. In Sect. 2, certain analytical techniques are used to prove that singularities of the specific heat and the scalar curvature occur on identical points. In Sect. 3, we study the thermodynamic geometry of the phantom Reissner–Nordstrom–AdS (anti-RN-(A)dS) black hole. We also study another phantom solution, which is called a black plane [20], and compare Quvedo's method with our proposed formulation. In Sect. 4, we study the Kerr Newman black holes [21] and show that the phase transitions exactly occur on identical points of the curvature singularities. Finally, in Sect. 5, we discuss our results.

2 Specific heat and thermodynamic geometry singularities

The first law of thermodynamics for RN black holes [22], characterized by their mass M and charge Q, can be written as follows:

$$\mathrm{d}M = T\mathrm{d}S + \Phi\mathrm{d}Q \tag{1}$$

where Φ is the potential difference between the horizon and infinity and T is the Hawking temperature:

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$$T = \left(\frac{\partial M}{\partial S}\right)_Q \tag{2}$$

$$\Phi = \left(\frac{\partial M}{\partial Q}\right)_S \tag{3}$$

The heat capacity at constant potential is given by

$$C_\Phi = T\left(\frac{\partial S}{\partial T}\right)_\Phi = \frac{1}{T\left(\frac{\partial T}{\partial S}\right)_\Phi} \tag{4}$$

Weinhold introduced a geometric formulation of thermodynamics for the first time [23]. A few years later, Ruppeiner [5] developed another geometric formulation for thermodynamics and statistical mechanics. The Weinhold metric is defined as the second derivative of the internal energy with respect to the entropy and other extensive parameters. We have

$$g_{ij}^W = \frac{\partial^2 M(X^K)}{\partial X^i \partial X^J}; \quad X^i = (S, N^a) \tag{5}$$

where S is the entropy and N^a determines all other extensive variables of the system. The Ruppeiner metric is defined as the second derivative of the entropy of the system with respect to the internal energy and other extensive variables, and it is given by

$$g_{ij}^R = \frac{\partial^2 S(X^K)}{\partial X^i \partial X^J}; \quad X^i = (U, N^a) \tag{6}$$

where U is the internal energy and N^a determines all other extensive variables of the system. The line elements in the Weinhold and Ruppeiner geometries [24] are conformally related:

$$\mathrm{d}S_R^2 = \frac{\mathrm{d}S_W^2}{T}. \tag{7}$$

The Ruppeiner metric of a black hole can be obtained by the following relations:

$$g_{SS} = \frac{1}{T}\left(\frac{\partial^2 M}{\partial S^2}\right) = \frac{1}{T}\frac{\partial}{\partial S}\left(\frac{\partial M}{\partial S}\right)_Q = \frac{1}{T}\left(\frac{\partial T}{\partial S}\right)_Q \tag{8}$$

$$g_{SQ} = \frac{1}{T}\left(\frac{\partial^2 M}{\partial Q \partial S}\right) = \frac{1}{T}\frac{\partial}{\partial Q}\left(\frac{\partial M}{\partial S}\right)_Q$$
$$= \frac{1}{T}\left(\frac{\partial T}{\partial Q}\right)_S = \frac{1}{T}\left(\frac{\partial \Phi}{\partial S}\right)_Q = g_{QS} \tag{9}$$

$$g_{QQ} = \frac{1}{T}\left(\frac{\partial^2 M}{\partial Q^2}\right) = \frac{1}{T}\frac{\partial}{\partial Q}\left(\frac{\partial M}{\partial Q}\right)_S = \frac{1}{T}\left(\frac{\partial \Phi}{\partial Q}\right)_S \tag{10}$$

where M, S, and Q are mass, entropy, and charge of the black hole, respectively. In this section, we investigate the phase transition points of the heat capacity C_Φ at a constant electric potential and show that they exactly correspond to the singularities of the scalar curvature $R(S, Q)$. Using the first law, the following Maxwell relation can be obtained:

$$\left(\frac{\partial T}{\partial Q}\right)_S = \left(\frac{\partial \Phi}{\partial S}\right)_Q \tag{11}$$

We define $\overline{M}$ as a new conjugate potential of $M(S, Q)$ in order to determine another useful Maxwell equation. $\overline{M}$ is related to $M(S, Q)$ by the following Legendre transformation:

$$\overline{M}(S, \Phi) = M(S, Q) - \Phi Q \tag{12}$$

For this new function, the first law of thermodynamics will be

$$\mathrm{d}\overline{M} = T\mathrm{d}S - Q\mathrm{d}\Phi \tag{13}$$

As a result, we obtain another Maxwell relation:

$$\left(\frac{\partial T}{\partial \Phi}\right)_S = -\left(\frac{\partial Q}{\partial S}\right)_\Phi \tag{14}$$

Moreover, the metric elements for this conjugate potential are defined as in the following equations:

$$\overline{g}_{SS} = \frac{1}{T}\left(\frac{\partial^2 \overline{M}}{\partial^2 S}\right) = \frac{1}{T}\frac{\partial}{\partial S}\left(\frac{\partial \overline{M}}{\partial S}\right)_\Phi = \frac{1}{T}\left(\frac{\partial T}{\partial S}\right)_\Phi \tag{15}$$

$$\overline{g}_{S\Phi} = \frac{1}{T}\left(\frac{\partial^2 \overline{M}}{\partial \Phi \partial S}\right) = \frac{1}{T}\frac{\partial}{\partial \Phi}\left(\frac{\partial \overline{M}}{\partial S}\right)_\Phi = \frac{1}{T}\left(\frac{\partial T}{\partial \Phi}\right)_S$$
$$= -\frac{1}{T}\left(\frac{\partial Q}{\partial S}\right)_\Phi = \overline{g}_{\Phi S} \tag{16}$$

$$\overline{g}_{\Phi\Phi} = \frac{1}{T}\left(\frac{\partial^2 \overline{M}}{\partial \Phi^2}\right) = \frac{1}{T}\frac{\partial}{\partial \Phi}\left(\frac{\partial \overline{M}}{\partial \Phi}\right)_S = -\frac{1}{T}\left(\frac{\partial Q}{\partial \Phi}\right)_S \tag{17}$$

The last part of the metric elements in (8–10) are written by using Maxwell's equations (11) and (14). For calculating the scalar curvature for two dimensions associated with the Ruppeiner metric, we use the following relation:

$$R = \frac{\begin{vmatrix} g_{SS} & g_{QQ} & g_{SQ} \\ g_{SS,S} & g_{QQ,S} & g_{SQ,S} \\ g_{SS,Q} & g_{QQ,Q} & g_{SQ,Q} \end{vmatrix}}{-2\begin{vmatrix} g_{SS} & g_{SQ} \\ g_{SQ} & g_{QQ} \end{vmatrix}^2} \tag{18}$$

The inverse of the heat capacity C_Φ can be written as

$$C_\Phi^{-1} = \frac{1}{T}\left(\frac{\partial T}{\partial S}\right)_\Phi = -\frac{1}{T}\left(\frac{\partial T}{\partial \Phi}\right)_S\left(\frac{\partial \Phi}{\partial S}\right)_T. \tag{19}$$

We will prove that the square root of the denominator of $R(S, Q)$ is proportional to the inverse of C_Φ:

$$(g_{SS}g_{QQ} - (g_{SQ})^2) \propto -\frac{1}{T}\left(\frac{\partial T}{\partial \Phi}\right)_S\left(\frac{\partial \Phi}{\partial S}\right)_T \tag{20}$$

The above equation means that the phase transition points correspond to the singularities of $R(S, Q)$. Substituting the metric elements (8–10) in the left-hand side of (20) and using (11) yield

$$-\frac{1}{T^2}\left(\frac{\partial T}{\partial Q}\right)_S\left[\left(\frac{\partial Q}{\partial S}\right)_T\left(\frac{\partial \Phi}{\partial Q}\right)_S+\left(\frac{\partial \Phi}{\partial S}\right)_Q\right] \qquad (21)$$

Equation (21) can be simplified by using the following equation:

$$d\Phi=\left(\frac{\partial \Phi}{\partial S}\right)_Q dS+\left(\frac{\partial \Phi}{\partial Q}\right)_S dQ \qquad (22)$$

or

$$\left(\frac{\partial \Phi}{\partial S}\right)_T=\left(\frac{\partial \Phi}{\partial S}\right)_Q+\left(\frac{\partial \Phi}{\partial Q}\right)_S\left(\frac{\partial Q}{\partial S}\right)_T \qquad (23)$$

Considering T as a function of Q and S, we have

$$dT=\left(\frac{\partial T}{\partial Q}\right)_S dQ+\left(\frac{\partial T}{\partial S}\right)_Q dS \qquad (24)$$

Therefore, we get another useful equation:

$$\left(\frac{\partial T}{\partial Q}\right)_S=\frac{\left(\frac{\partial T}{\partial \Phi}\right)_S}{\left(\frac{\partial Q}{\partial \Phi}\right)_S} \qquad (25)$$

By substituting Eqs. (23) and (25) in (21), the left-hand side of Eq. (20) can be simplified as

$$-\frac{1}{T^2}\left(\frac{\left(\frac{\partial T}{\partial \Phi}\right)_S}{\left(\frac{\partial Q}{\partial \Phi}\right)_S}\right)\left(\frac{\partial \Phi}{\partial S}\right)_T=\frac{\left(\frac{\partial \Phi}{\partial Q}\right)_S}{T}C_\Phi^{-1} \qquad (26)$$

We conclude that for a finite value of $(\partial \Phi/\partial Q)_S$ and $T\neq 0$, we have

$$\begin{vmatrix} g_{SS} & g_{SQ} \\ g_{SQ} & g_{QQ} \end{vmatrix}=\frac{\left(\frac{\partial \Phi}{\partial Q}\right)_S}{T}C_\Phi^{-1} \qquad (27)$$

This means that the singularities of $R(S, Q)$ correspond exactly to the phase transition points. A similar calculation for conjugate potential yields

$$\begin{vmatrix} \overline{g}_{SS} & \overline{g}_{S\Phi} \\ \overline{g}_{S\Phi} & \overline{g}_{\Phi\Phi} \end{vmatrix}=-\frac{\left(\frac{\partial Q}{\partial \Phi}\right)_S}{T}C_Q^{-1} \qquad (28)$$

This means that the singularities of $\overline{R}(S, \Phi)$ are the same as the transition points of C_Q, which identify the Davies curve [25]. Moreover, the curvature singularity of the free-energy metric is also located at the Davies curve of C_Q.

The Helmholtz free-energy is related to $M(S, Q)$ by the following Legendre transformation:

$$\overline{\overline{M}}(Q, T)=M(S, Q)-TS \qquad (29)$$

For this new function, the first law of thermodynamics can be expressed as

$$d\overline{\overline{M}}(Q, T)=-SdT+\Phi dQ \qquad (30)$$

The metric elements for the Helmholtz free-energy can be defined by the following equations:

$$\overline{\overline{g}}_{TT}=\frac{1}{T}\left(\frac{\partial^2 \overline{\overline{M}}}{\partial T^2}\right)=-\frac{1}{T}\left(\frac{\partial S}{\partial T}\right)_Q \qquad (31)$$

$$\overline{\overline{g}}_{TQ}=\frac{1}{T}\left(\frac{\partial^2 \overline{\overline{M}}}{\partial T\partial Q}\right)=-\frac{1}{T}\left(\frac{\partial S}{\partial Q}\right)_S$$
$$=\frac{1}{T}\left(\frac{\partial \Phi}{\partial T}\right)_Q=\overline{\overline{g}}_{QT} \qquad (32)$$

$$\overline{\overline{g}}_{TT}=\frac{1}{T}\left(\frac{\partial^2 \overline{\overline{M}}}{\partial Q^2}\right)=\frac{1}{T}\left(\frac{\partial \Phi}{\partial Q}\right)_T \qquad (33)$$

We can also write the metric elements of the Helmholtz free energy in the same coordinates of the conjugate potential $\overline{M}(S, \Phi)$ by using a transformation matrix. This matrix changes coordinates from (T, Q) to (S, Φ). The transformation matrix can be written in the form

$$N=\begin{pmatrix} \left(\frac{\partial T}{\partial S}\right)_\Phi & \left(\frac{\partial T}{\partial \Phi}\right)_S \\ \left(\frac{\partial Q}{\partial S}\right)_\Phi & \left(\frac{\partial Q}{\partial \Phi}\right)_S \end{pmatrix} \qquad (34)$$

Using the transformation matrix, we can show that the metric elements of the $\overline{\overline{M}}(T, Q)$ in the new coordinates (S, Φ) are the same as the metric elements of the $\overline{M}(S, \Phi)$:

$$\overline{g}'_{ij}=N_{ik}^T\,\overline{\overline{g}}_{kl}\,N_{lj} \qquad (35)$$

where N^T is the transpose of N. Therefore, the singularity points of the scalar curvature for both Helmholtz free-energy function and conjugate potential occur exactly at the same phase transition points of C_Q (28). In other words, the line element of the free energy $\overline{\overline{M}}(Q, T)$ is associated with the line element of the conjugate potential $\overline{M}(S, \Phi)$ [24]. We have

$$ds^2(\overline{\overline{M}})=-ds^2(\overline{M}) \qquad (36)$$

Furthermore, we are able to define the metric element of $\overline{\overline{M}}$ in terms of the metric elements of $\overline{M}$ by the conformal transformation

$$\overline{\overline{g}}_{ij}=-\overline{g}_{ij}=-\frac{1}{T}\left(\frac{\partial^2 \overline{M}}{\partial X^i\partial X^j}\right);\quad X^i=(S, \Phi) \qquad (37)$$

The above calculations confirm our (trivial) expectation that the curvature is independent of any specific coordinate choice of the thermodynamic quantities. We can also prove that for black holes with three parameters, the phase transition points of the heat capacity $C_{\Phi,\Omega}$ at a constant electric potential and angular velocity correspond exactly to the singularities of the scalar curvature $R(S, Q, J)$. The scalar curvature is proportional to the inverse of the square determinant of the metric:

$$R(S, Q, J) \propto \begin{vmatrix} g_{SS} & g_{SQ} & g_{SJ} \\ g_{QS} & g_{QQ} & g_{QJ} \\ g_{JS} & g_{JQ} & g_{JJ} \end{vmatrix}^{-2} \tag{38}$$

After a long but straightforward calculation, the inverse of the heat capacity $C_{\Phi,\Omega}$ can be written as follows (see the Appendix):

$$\left[\left(\frac{\partial \Omega}{\partial J} \right)_{S,Q} \left(\frac{\partial \phi}{\partial Q} \right)_{S,J} - \left(\frac{\partial \Omega}{\partial Q} \right)_{S,J} \left(\frac{\partial \phi}{\partial J} \right)_{S,Q} \right]$$
$$\times \frac{(C_{\Phi,\Omega})^{-1}}{T^2} = \begin{vmatrix} g_{SS} & g_{SQ} & g_{SJ} \\ g_{QS} & g_{QQ} & g_{QJ} \\ g_{JS} & g_{JQ} & g_{JJ} \end{vmatrix} \tag{39}$$

Therefore, we conclude that the singularities of $R(S, Q, J)$ correspond to the phase transition points of $C_{\Phi,\Omega}$ and, also, that those of $\overline{R}(S, \Phi, \Omega)$ correspond to the phase transitions of $C_{Q,J}$. We have

$$\left[\left(\frac{\partial Q}{\partial \Phi} \right)_{S,\Omega} \left(\frac{\partial J}{\partial \Omega} \right)_{S,\Phi} - \left(\frac{\partial J}{\partial \Phi} \right)_{S,\Omega} \left(\frac{\partial Q}{\partial \Omega} \right)_{S,\Phi} \right]$$
$$\times \frac{(C_{Q,J})^{-1}}{T^2} = \begin{vmatrix} \overline{g}_{SS} & \overline{g}_{S\Phi} & \overline{g}_{S\Omega} \\ \overline{g}_{\Phi S} & \overline{g}_{\Phi\Phi} & \overline{g}_{\Phi\Omega} \\ \overline{g}_{\Omega S} & \overline{g}_{\Omega\Phi} & \overline{g}_{\Omega\Omega} \end{vmatrix} \tag{40}$$

We may generalize the above-mentioned relations to a large class of black holes with an arbitrary number of parameters. The first law of thermodynamics for black holes with $(n+1)$ parameters can be written as follows:

$$dM = T\,dS + \sum_{i=1}^{n} \Phi_i Q_i \tag{41}$$

It is clear that the energy M is a function of $n+1$ extensive variables (S, Q_i). In addition, we can consider $(n+1)$ pairs of intensive/extensive variables (T, S) and (Φ_i, Q_i). The Ruppenier metric for black holes with $(n+1)$ extensive variables can also be written

$$g_{ij}^{R} = \frac{1}{T} \left(\frac{\partial^2 M}{\partial X^i \partial X^j} \right); \quad X^i = (S, Q_1, Q_2, \ldots, Q_n) \tag{42}$$

In general, we contend that the transition points of the heat capacity $C_{\Phi_1,\ldots,\Phi_n}$ are the same as the singularity points of the scalar curvature $R(S, Q_1, \ldots, Q_n)$. The scalar curvature $R(S, Q_1, \ldots, Q_n)$ is proportional to the inverse of the square determinant of the metric:

$$R(S, Q_1, \ldots, Q_n) \propto \begin{vmatrix} g_{SS} & g_{SQ_1} & \cdots & g_{SQ_n} \\ g_{Q_1 S} & g_{Q_1 Q_1} & \cdots & g_{Q_1 Q_n} \\ \vdots & \vdots & \vdots & \vdots \\ g_{Q_n S} & g_{Q_n Q_1} & \cdots & g_{Q_n Q_n} \end{vmatrix}^{-2} \tag{43}$$

Using Eq. (43), it is expected that the inverse of the heat capacity $C_{\Phi_1,\ldots,\Phi_n}$ can be obtained by

$$\left[\frac{(C_{\Phi_1,\Phi_2,\ldots,\Phi_n})^{-1}}{T^n} \left(\frac{\partial(\Phi_1, \Phi_2, \ldots, \Phi_n)}{\partial(Q_1, Q_2, \ldots, Q_n)_S} \right) \right]$$
$$= \begin{vmatrix} g_{SS} & g_{SQ_1} & \cdots & g_{SQ_n} \\ g_{Q_1 S} & g_{Q_1 Q_1} & \cdots & g_{Q_1 Q_n} \\ \vdots & \vdots & \vdots & \vdots \\ g_{Q_n S} & g_{Q_n Q_1} & \cdots & g_{Q_n Q_n} \end{vmatrix} \tag{44}$$

where

$$\frac{\partial(\Phi_1, \Phi_2, \ldots, \Phi_n)}{\partial(Q_1, Q_2, \ldots, Q_n)_S} = \begin{vmatrix} \left(\frac{\partial \Phi_1}{\partial Q_1}\right) & \left(\frac{\partial \Phi_1}{\partial Q_2}\right) & \cdots & \left(\frac{\partial \Phi_1}{\partial Q_n}\right) \\ \left(\frac{\partial \Phi_2}{\partial Q_1}\right) & \left(\frac{\partial \Phi_2}{\partial Q_2}\right) & \cdots & \left(\frac{\partial \Phi_2}{\partial Q_n}\right) \\ \vdots & \vdots & \vdots & \vdots \\ \left(\frac{\partial \Phi_n}{\partial Q_1}\right) & \left(\frac{\partial \Phi_n}{\partial Q_2}\right) & \cdots & \left(\frac{\partial \Phi_n}{\partial Q_n}\right) \end{vmatrix} \tag{45}$$

Thus, the singularities of $R(S, Q_1, \ldots, Q_n)$ correspond to the phase transition points of $C_{\Phi_1,\ldots,\Phi_n}$. Although we have proved Eqs. (39) and (40), we do not know a rigorous proof of Eq. (44) at this time. A general proof for Eq. (44) remains as an open problem. We hope to solve this problem in the near future. On the other hand, the conjugate potential $\overline{M}(S, \Phi_1, \ldots, \Phi_n)$ can be obtained from $M(S, Q_1, \ldots, Q_n)$ by the Legendre transformation

$$\overline{M}(S, \Phi_1, \ldots, \Phi_n) = M(S, Q_1, \ldots, Q_n) - \sum_{i=1}^{n} \Phi_i Q_i \tag{46}$$

by defining the metric elements for the conjugate potential $\overline{M}(S, \Phi_1, \ldots, \Phi_n)$ as

$$\overline{g}_{ij} = \frac{1}{T} \left(\frac{\partial^2 \overline{M}}{\partial X^i \partial X^j} \right); \quad X^i = (S, \Phi_1, \Phi_2, \ldots, \Phi_n) \tag{47}$$

We can assert that the singularity points of $\overline{R}(S, \Phi_1, \Phi_2, \ldots, \Phi_n)$ correspond to the phase transitions of $C_{Q_1, Q_2, \ldots, Q_n}$:

$$\frac{(C_{Q_1, Q_2, \ldots, Q_n})^{-1}}{T^n} \left(\frac{\partial(Q_1, Q_2, \ldots, Q_n)}{\partial(\Phi_1, \Phi_2, \ldots, \Phi_n)_S} \right)$$
$$= \begin{vmatrix} \overline{g}_{SS} & \overline{g}_{S\Phi_1} & \cdots & \overline{g}_{S\Phi_n} \\ \overline{g}_{\Phi_1 S} & \overline{g}_{\Phi_1 \Phi_1} & \cdots & \overline{g}_{\Phi_1 \Phi_n} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{g}_{\Phi_n S} & \overline{g}_{\Phi_n \Phi_1} & \cdots & \overline{g}_{\Phi_n \Phi_n} \end{vmatrix} \tag{48}$$

We have collected the mass (M), the conjugate potential ($\overline{M}$), capacities, scalar curvature functions, and temperature for various black holes and listed them in Tables 1, 2 and 3. It is clear from these tables that both the heat capacity and the scalar curvature diverge at the same point. For Kerr, EMGB, and EYMGB black holes, a factor proportional to temperature appears in the denominator of the Ricci scalar. In order to probe this argument, we start with the metric

Table 1 Thermodynamic variables and scalar curvature functions for Kerr and RN (Reissner–Nordstrom) black holes

Kerr	RN
$M(S,J) = \sqrt{\frac{S}{4\pi} + \frac{J^2\pi}{S}}$	$M(S,Q) = \frac{\sqrt{S\pi}}{2}\left(\frac{1}{\pi} + \frac{Q^2}{S}\right)$
$T(S,J) = \frac{S^2 - 4J^2\pi^2}{4S^{3/2}\sqrt{\pi}\sqrt{S^2 + 4J^2\pi^3}}$	$T(S,Q) = \frac{S - Q^2\pi}{4S^{3/2}\sqrt{\pi}}$
$R(S,J) = -\frac{S(S^2 + 12\pi^2 J^2)}{(S^2 + 4\pi^2 J^2)(4\pi^2 J^2 - S^2)}$	$R(S,Q) = 0$
$C_\Omega(S,J) = 2\frac{(-S^2 + 4J^2\pi^2)S^3}{(S^2 + 4J^2\pi^2)^2}$	$C_\Phi(S,Q) = -2S$
$\overline{M}(S,\Omega) = \sqrt{\frac{S}{4\pi} - \left(\frac{S\Omega}{2\pi}\right)^2}$	$\overline{M}(S,\Phi) = \sqrt{\frac{S}{4\pi}}(\Phi^2 - 1)$
$T(S,\Omega) = \frac{-2S\Omega^2 + \pi}{4\pi\sqrt{S(-S\Omega^2 + \pi)}}$	$T(S,\Phi) = \frac{1-\Phi^2}{4\sqrt{\pi}S}$
$\overline{R}(S,\Omega) = 4\frac{(-S\Omega^2 + \pi)\pi^2(\pi^2 - 2S\Omega^2\pi - 8S^2\Omega^4)}{(\pi^2 - 8S\Omega^2\pi + 4S^2\Omega^4)^2(-2S\Omega^2 + \pi)S}$	$\overline{R}(S,\Phi) = -\frac{-1+\Phi^2}{(-1+3\Phi^2)^2 S}$
$C_J(S,\Omega) = -2\frac{\pi S(-2S\Omega^2 + \pi)}{4S^2\Omega^4 - 8S\Omega^2\pi + \pi^2}$	$C_Q(S,\Phi) = -2\frac{S(\Phi^2 - 1)}{-1+3\Phi^2}$

Table 2 Thermodynamic variables and scalar curvature functions for BTZ (Banados–Teitelboim–Zanelli) and EMGB (Einstein–Maxwell–Gauss–Bonnet) black holes

Λ cosmological constant, k Chern–Simons coupling constant and α Gauss–Bonnet coupling constant

BTZ	EMGB
$M(S,J) = \frac{S^2}{16\pi^2 l^2} + \frac{4\pi^2 J^2}{S^2}$	$M(S,Q) = \pi\alpha + \frac{\pi Q^2}{6\sqrt[3]{S^2}} + \pi^2\sqrt[3]{S^2} - \frac{\pi\Lambda\sqrt[3]{S^4}}{12}$
$T(S,J) = \frac{S^4 - 64\pi^4 J^2 L^2}{8\pi^2 L^2 S^3}$	$T(S,Q) = \frac{\pi(-Q^2 + 3S^{4/3} - \Lambda S^2)}{9S^{5/3}}$
$R(S,J) = 0$	$R(S,Q) = -\frac{\Lambda(-6S^{10/3}Q^2 + \Lambda^2 S^6 - 9S^{14/3} + 3Q^4 S^2 - 4Q^2\Lambda S^4 + 8S^{16/3}\Lambda)}{S(-Q^2 + 3S^{4/3} + \Lambda S^2)^2(-Q^2 + 3S^{4/3} - \Lambda S^2)}$
$C_\Omega(S,J) = S$	$C_\Phi(S,Q) = -3\frac{(-Q^2 + 3S^{4/3} - \Lambda S^2)S}{-Q^2 + 3S^{4/3} + \Lambda S^2}$
$\overline{M}(S,\Omega) = \frac{S^2}{16\pi^2 l^2} - \frac{\Omega^2 S^2}{16\pi^2}$	$\overline{M}(S,\Phi) = \pi\alpha - \frac{3\Phi\sqrt[3]{S^2}}{\pi} + \frac{\pi\sqrt[3]{S^2}}{12} - \frac{\pi\Lambda\sqrt[3]{S^4}}{12}$
$T(S,\Omega) = -\frac{S(-1+\Omega^2 L^2)}{8\pi^2 L^2}$	$T(S,\Phi) = -\frac{9\Phi^2 - 3\pi^2 + \pi^2\Lambda S^{2/3}}{9\pi\sqrt[3]{S}}$
$\overline{R}(S,\Omega) = 2\frac{-1+\Omega^2 L^2}{(1+3\Omega^2 L^2)^2 S}$	$\overline{R}(S,\Phi) = -\frac{A(S,\Phi)}{S^{5/3}(-45\Phi^2 + 3\pi^2 + \pi^2\Lambda S^{2/3})^2(9\Phi^2 - 3\pi^2 + \pi^2\Lambda S^{2/3})}$
$C_J(S,\Omega) = -\frac{S(-1+\Omega^2 L^2)}{1+3\Omega^2 L^2}$	$C_Q(S,\Phi) = 3S\frac{(9\Phi^2 - 3\pi^2 + \pi^2\Lambda S^{2/3})}{-45\Phi^2 + 3\pi^2 + \pi^2\Lambda S^{2/3}}$

Table 3 Thermodynamic variables and scalar curvature functions for EGB (Einstein–Gauss–Bonnet) and EYMGB (Einstein–Yang–Mills–Gauss–Bonnet) black holes

EGB	EYMGB
$M(S,Q) = \sqrt[3]{S^2} + \frac{Q^2}{3\sqrt[3]{S^2}}$	$M(S,Q) = \sqrt[3]{S^2} - \frac{2Q^2\ln(S)}{3}$
$T(S,Q) = \frac{2(3S^{4/3} - Q^2)}{9S^{5/3}}$	$T(S,Q) = \frac{2(S - Q^2\sqrt[3]{S})}{3S^{4/3}}$
$R(S,Q) = 0$	$R(S,Q) = -\frac{B(S,Q)}{6\left(-S+Q^2\sqrt[3]{S}\right)\left(-\ln(S)S + 3\ln(S)Q^2\sqrt[3]{S} + 6Q^2\sqrt[3]{S}\right)^3}$
$C_\Phi(S,Q) = -3S$	$C_\Phi(S,Q) = -\frac{3\ln(S)\left(-S+Q^2\sqrt[3]{S}\right)S}{-\ln(S)S + 6Q^2\sqrt[3]{S} + 3Q^2\ln(S)\sqrt[3]{S}}$
$\overline{M}(S,\Phi) = \sqrt[3]{S^2}\left(1 - \frac{3}{4}\Phi^2\right)$	$\overline{M}(S,\Phi) = \sqrt[3]{S^2} + \frac{3\Phi^2}{8\ln(S)}$
$T(S,\Phi) = -\frac{-4+3\Phi^2}{6\sqrt[3]{S}}$	$T(S,\Phi) = -\frac{-16(\ln(S))^2 S + 9\Phi^2\sqrt[3]{S}}{24S^{4/3}(\ln(S))^2}$
$\overline{R}(S,\Phi) = -8\frac{-4+3\Phi^2}{(-4+15\Phi^2)^2 S}$	$\overline{R}(S,\Phi) - \frac{D(S,\Phi)}{(\ln(S))^2\left(-16(\ln(S))^2 S + 9\Phi^2\sqrt[3]{S}\right)S^{\frac{20}{3}}\left(-16(\ln(S))^2 S^2 + 27\Phi^2 S^{4/3}\right)^3}$
$C_Q(S,\Phi) = -3\frac{(-4+3\Phi^2)S}{-4+15\Phi^2}$	$C_Q(S,\Phi) = -3\frac{S\left(-16(\ln(S))^2 S + 9\Phi^2\sqrt[3]{S}\right)}{-16(\ln(S))^2 S + 27\Phi^2\sqrt[3]{S}}$

$$\overline{g}_{\gamma\beta} = \frac{1}{T}\left(\frac{\partial^2 \overline{M}}{\partial \gamma \partial \beta}\right) = \frac{1}{T} h_{\gamma\beta} \tag{49}$$

to show that

$$\overline{g}_{\gamma\beta,\beta} = \partial_\beta \left(\frac{1}{T} h_{\gamma\beta}\right) = \frac{1}{T^2}\tilde{h}_{\gamma\beta,\beta} \tag{50}$$

where

$$\tilde{h}_{\gamma\beta,\beta} = -h_{\gamma\beta}\partial_\beta T + T h_{\gamma\beta,\beta} \tag{51}$$

By replacing these equations in the numerator of the scalar curvature, we have

$$\begin{vmatrix} \frac{1}{T}h_{\beta\beta} & \frac{1}{T}h_{\gamma\gamma} & \frac{1}{Th_{\gamma\beta}} \\ \frac{1}{T^2}\tilde{h}_{\beta\beta,\beta} & \frac{1}{T^2}\tilde{h}_{\gamma\gamma,\beta} & \frac{1}{T^2}\tilde{h}_{\gamma\beta,\beta} \\ \frac{1}{T^2}\tilde{h}_{\beta\beta,\gamma} & \frac{1}{T^2}\tilde{h}_{\gamma\gamma,\gamma} & \frac{1}{T^2}\tilde{h}_{\gamma\beta,\gamma} \end{vmatrix} \propto \frac{1}{T^5} \tag{52}$$

For the denominator of the scalar curvature we have

$$\begin{vmatrix} \frac{1}{T}h_{\gamma\gamma} & \frac{1}{T}h_{\gamma\beta} \\ \frac{1}{T}h_{\gamma\beta} & \frac{1}{T}h_{\beta\beta} \end{vmatrix}^2 \propto \frac{1}{T^4} \tag{53}$$

Consequently

$$\overline{R} \propto \frac{1}{T} \tag{54}$$

On the other hand, for RN, BTZ, and EGB black holes, two elements of the metric are equal to zero. For RN and EGB black holes, we have $\partial_S(\overline{g}_{S\Phi}) = \partial_\Phi(\overline{g}_{SS}) = 0$ and, for BTZ, $\partial_S(\overline{g}_{S\Omega}) = \partial_\Omega(\overline{g}_{SS}) = 0$. Therefore, the numerator of the scalar curvature is proportional to $\frac{1}{T^3}$ and the scalar curvature is proportional to the temperature ($\overline{R} \propto T$).

3 Thermodynamic geometry of phantom Reissner–Nordestrom–AdS and black plane

Recently, a new solution of Einstein–anti-Maxwell theory with a cosmological constant, called the anti-Reissner–Nordstrom-(A)de Sitter solution, has been investigated [18]. This new solution has led to the following thermodynamic expression for the mass of this black hole:

$$M = \frac{1}{2}(S/\pi)^{3/2}\left(\frac{\pi}{S} - \frac{\Lambda}{3} + \frac{\eta\pi^2 Q^2}{S^2}\right) \tag{55}$$

where Λ is the cosmological constant, which might behave as $\Lambda > 0$ (dS) or $\Lambda < 0$ (AdS). At $\eta = 1$, we have a solution for Reissner–Nordstrom–AdS, while $\eta = -1$, due to the negative energy of the field of spin 1, gives us a solution for anti-Reissner–Nordstrom–AdS (phantom). The Hawking temperature, T, the electric potential, Φ, and C_Q are defined as follows:

$$T = \left(\frac{\partial M}{\partial S}\right) = \frac{-\pi S + \Lambda S^2 + \eta\pi^2 Q^2}{-4(\pi S)^{3/2}} \tag{56}$$

$$\Phi = \left(\frac{\partial M}{\partial Q}\right) = \frac{(S/\pi)^{3/2}\eta\pi^2 Q}{S^2} \tag{57}$$

$$\begin{aligned} C_Q &= T\left(\frac{\partial S}{\partial T}\right)_Q = \frac{T}{\left(\frac{\partial T}{\partial S}\right)_Q} \\ &= \frac{-2S(-\pi S + \Lambda S^2 + \eta\pi^2 Q^2)}{(-\pi S - \Lambda S^2 + 3\eta\pi^2 Q^2)} \end{aligned} \tag{58}$$

In [16], the geometrothermodynamic approach is used to obtain the phase transition points. However, this theory is not able to produce the correct phase transition points. In summary, the geometrothermodynamics of black holes is considered as a $2n + 1$-dimensional thermodynamic phase space, T, with independent coordinates $\Phi, E^a, I^a, a = 1\ldots n$, where Φ represents the thermodynamic potential, and E^a and I^a are the extensive and intensive thermodynamic variables, respectively. If the space T possesses a non-degenerate metric $G_{AB}(Z^C)$, where $Z^c = \Phi, E^a, I^a$, and one form of Gibbs $\Theta = d\Phi - \delta_{ab}I^a E^b$ (in which δ_{ab} is the Kronecker delta), then

$$\begin{aligned} G &= (d\Phi - \delta_{ab}I^a dE^b) + (\delta_{ab}E^a I^b)(\eta_{cd}dE^c dI^d) \\ \eta_{cd} &= diag(-1, 1, \ldots, 1) \end{aligned} \tag{59}$$

The Gibbs form is invariant under Legendre transformations, written as

$$\{\Phi, E^a, I^a\} \rightarrow \{\tilde{\Phi}, \tilde{E}^a, \tilde{I}^a\} \ \Phi = \tilde{\Phi} - \delta_{ab}\tilde{E}^a \tilde{I}^b \tag{60}$$

where

$$E^a = -\tilde{I}^a \ I^a = \tilde{E}^a \tag{61}$$

On the other hand, if somebody considers a n-dimensional subspace E such that $E \subset T$, we will, therefore, obtain $d\Phi = \delta_{ab}I^a dE^b$, which is called the first law of thermodynamics. In the space E, the Quevedo metric is given by

$$g^Q = \left(E^c \frac{\partial \Phi}{\partial E^c}\right)\left(\eta_{ab}\delta^{bc}\frac{\partial^2 \Phi}{\partial E^c \partial E^d}dE^a dE^b\right) \tag{62}$$

Using (62) and (55), the scalar curvature is obtained as follows:

$$\begin{aligned} R(S, Q) &= \frac{A(S, Q)}{(S\pi + \Lambda S^2 - 3\pi^2\eta Q^2)^2} \\ &\times \frac{1}{(-S\pi + \Lambda S^2 - 3\pi^2\eta Q^2)^3} \end{aligned} \tag{63}$$

where the points $S_1 = -\left(\frac{\pi}{2\Lambda}\right)(1 + \sqrt{1 + 12\eta\Lambda Q^2})$. $S_2 = -\left(\frac{\pi}{2\Lambda}\right)(-1 + \sqrt{1 + 12\eta\Lambda Q^2})$ and $S_3 = -\left(\frac{\pi}{2\Lambda}\right)(1 - \sqrt{1 + 12\eta\Lambda Q^2})$ are singularities of $R(S, Q)$. All the other points are negative or have a complex value for the entropy, and have thus been rejected. The points of the phase transition of the specific heat in Eq. (58) are only S_1 and

S_3. However, the extra point, i.e. S_2, does not correspond to a phase transition. Thus, the geometrodynamic method is not able to provide the same result as the analysis by using the heat capacity of RN–AdS and anti-RN–AdS black holes does.

Using a Legendre transformation, we find a conjugate potential for $M(S, Q)$:

$$\overline{M}(S, \Phi) = M(S, Q) - \Phi Q. \tag{64}$$

By solving Q from (57) and substituting it in (64), we have

$$\overline{M}(S, \Phi) = \frac{1}{2}\left(\frac{S}{\pi}\right)^{\frac{3}{2}}\left(\frac{\pi}{S} - \frac{\Lambda}{3} + \frac{\pi\Phi^2}{\eta S}\right) - \frac{\Phi^2 S^2}{(S/\pi)^{3/2}\eta\pi^2} \tag{65}$$

We define the following metric:

$$\overline{g}_{ij} = \frac{1}{T}\left(\frac{\partial^2 \overline{M}}{\partial X^i \partial X^j}\right); \quad X^i = (S, \Phi) \tag{66}$$

Therefore, we will obtain the scalar curvature, R, as a function of S and Φ. Finally, using the relation (57), we rewrite R as a function of S and Q:

$$\bar{R}(S, Q) = \frac{C(S, Q)}{(-S\pi - \Lambda S^2 + 3\pi^2\eta Q^2)^2} \times \frac{1}{(-S\pi + \Lambda S^2 + \pi^2\eta Q^2)} \tag{67}$$

The roots of the first part of the denominator give S_1 and S_3; i.e., the phase transition points. The second part of the denominator is only zero at $T = 0$ or for extremal black holes. Therefore, the curvature diverges exactly at those points where the heat capacity diverges with no other additional roots.

For the *RN–AdS* black hole, the scalar curvature (67) and the specific heat (58) are depicted in Figs. 1 and 2, respectively, as a function of the entropy and for a fixed value of the electric charge: $Q = 0.25$.

The Ruppeiner curvature can also be used to probe the microstructure of a thermodynamic system [26,27]. The scalar curvature is positive in Fig. 1; we, therefore, expect a fermion-like or short range repulsive behavior for the microstructure of *RN–AdS* black holes.

A change of sign for the heat capacity is usually associated with a drastic change in the stability properties of a thermodynamic system; a negative heat capacity represents a region of instability whereas the stable domain is characterized by a positive heat capacity. For an RN–AdS black hole, the unstable region ($C_Q < 0$) is between S_1 and S_3, while we expect stability for $S < S_1$ and $S > S_3$. The scalar curvature and the specific heat for the *phantom Reissner–Nordestrom–AdS* are depicted in Figs. 3 and 4, respectively.
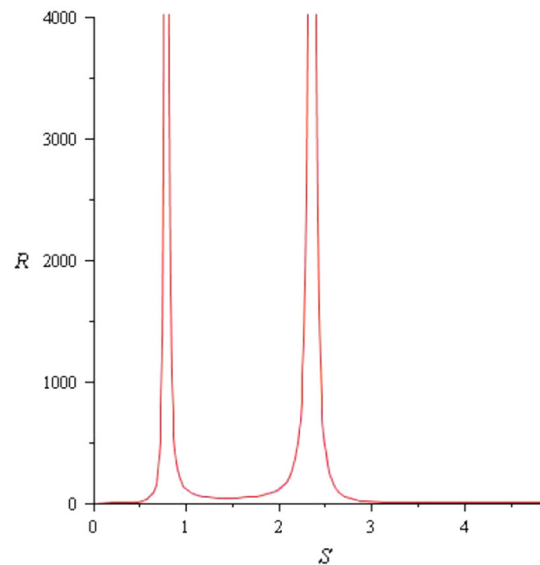

Fig. 1 Graph of the scalar of curvature as a function of entropy, S, in the *RN–AdS* case, for an electric charge $Q = 0.25$ and a cosmological constant $\Lambda = -1$
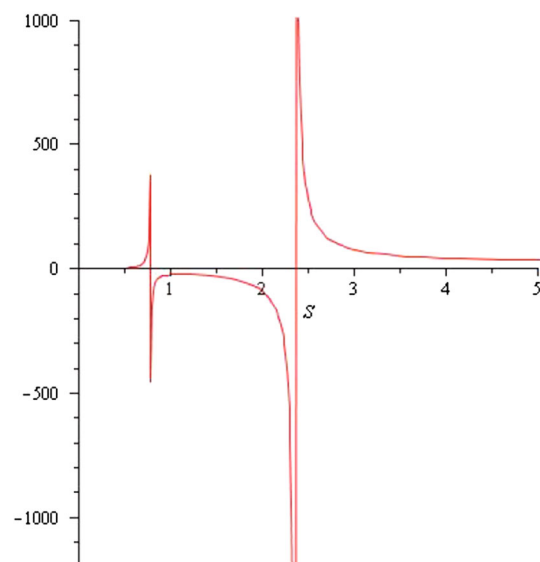

Fig. 2 Graph of the specific heat as a function of entropy, S, in the *RN–AdS* case, for an electric charge $Q = 0.25$ and a cosmological constant $\Lambda = -1$

The four-dimensional black plane is another interesting thermodynamic system with two degrees of freedom [20]. The mass of the black plane is given by

$$M(S, Q) = \frac{\alpha^2 S^2 + \eta\pi^2 Q^2}{\pi\alpha\sqrt{2S}} \tag{68}$$

where Λ is the cosmological constant and $\alpha^2 = \frac{-\Lambda}{3}$. When considering $\eta = 1$, we have a solution for the normal black plane, while $\eta = -1$ gives us a solution for the phantom black plane. Furthermore, the temperature and the electric potential
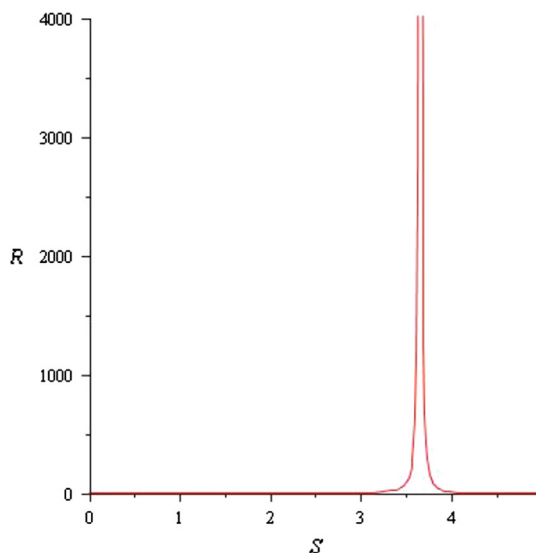
Fig. 3 Graph of the scalar of curvature as a function of the entropy, S, of the *phantom Reissner–Nordestrom–AdS* black hole, for an electric charge $Q = 0.25$ and a cosmological constant $\Lambda = -1$
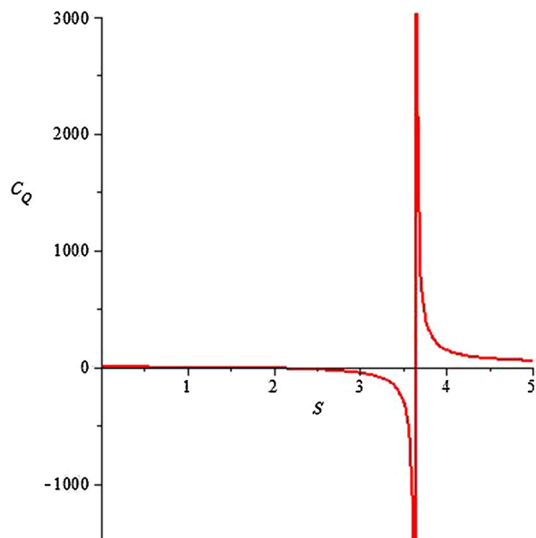

Fig. 4 Graph of the specific heat as a function of the entropy, S, of the *phantom Reissner–Nordestrom–AdS* black hole for an electric charge $Q = 0.25$ and a cosmological constant $\Lambda = -1$

can be written in terms of the entropy and the electric charge:

$$T = \left(\frac{\partial M}{\partial S}\right) = \frac{3\alpha^2 S^2 - \eta\pi^2 Q^2}{\pi\alpha(\sqrt{2S})^3} \tag{69}$$

$$\Phi = \left(\frac{\partial M}{\partial Q}\right) = \frac{2\pi Q}{\alpha\sqrt{2S}} \tag{70}$$

The heat capacity for the black plane can be straightforwardly computed using the fundamental relation (68):

$$C_Q = T\left(\frac{\partial S}{\partial T}\right)_Q = \frac{2S(3\alpha^2 S^2 - \eta\pi^2 Q^2)}{3(\alpha^2 S^2 + \eta\pi^2 Q^2)} \tag{71}$$

Phase transitions are then determined by the roots of the denominator of C_Q; i.e., when the specific heat diverges. Therefore, there exists only one divergent point $S_i = -i\sqrt{\eta}\pi Q/\alpha$. This shows that the normal case ($\eta = 1$) has no phase transition, while the phantom case ($\eta = -1$) possesses a phase transition. For the phantom black plane, we compute the scalar curvature by using our formalism of the thermodynamic geometry. We can consider the following Legendre transformation of $M(S, Q)$:

$$\overline{M}(S, \Phi) = M(S, Q) - \Phi Q \tag{72}$$

Using (70) and replacing Q by $\Phi\alpha\sqrt{\frac{S}{2\pi^2}}$ in (72), we have

$$\overline{M}(S, \Phi) = \frac{\alpha\sqrt{2S}(2S\eta - \Phi^2)}{4\eta\pi} \tag{73}$$

Now, we evaluate the Ricci scalar $R(S, \Phi)$ for the black plane,

$$\overline{R}(S, \Phi) = 4/3 \frac{\Phi^2\left(18\,S\eta + 5\,\Phi^2\right)\eta}{\left(2\,S\eta + \Phi^2\right)^2\left(6\,S\eta - \Phi^2\right)} \tag{74}$$

By replacing relation (70) for Φ, we have

$$\overline{R}(S, Q) = \frac{-2\pi^2 Q^2\alpha^2 S\eta(9S^2\alpha^2 + 5\eta\pi^2 Q^2)}{(S^2\alpha^2 + \eta\pi^2 Q^2)^2(-3S^2\alpha^2 + \eta\pi^2 Q^2)} \tag{75}$$

The roots of the first term in the denominator correspond to the phase transition points. The second factor in the denominator is zero only at the extremal limit ($T = 0$).

4 Thermodynamic geometry of Kerr Newman black hole

Kerr Newman black holes [21] are described by their mass (M), entropy (S), charge (Q), and angular momentum (J). The mass for the Kerr Newman black hole is given by

$$M = \frac{\sqrt{S(4J^2 + S^2 + 2Q^2 S + Q^4)}}{2S} \tag{76}$$

The thermodynamic variables and the first law of thermodynamics are given by

$$T = \left(\frac{\partial M}{\partial S}\right) = \frac{S^2 - 4J^2 - Q^2}{4S^{\frac{3}{2}}\sqrt{S^2 + 4J^2 + Q^2 + 2Q^2 S}}$$
$$= \frac{S^2 - 4J^2 - Q^2}{4S^2 M} \tag{77}$$

$$\Omega = \left(\frac{\partial M}{\partial J}\right) = \frac{2J}{\sqrt{S^2 + 4J^2 + Q^2 + 2Q^2 S}\sqrt{S}}$$
$$= \frac{J}{SM} \tag{78}$$

$$\Phi = \left(\frac{\partial M}{\partial Q}\right) = \frac{4QS + 4Q^3}{4\sqrt{S(4J^2 + S^2 + 2Q^2 S + Q^4)}}$$
$$= \frac{Q(S + Q^2)}{2MS} \tag{79}$$

$$dM = TdS + \Omega dJ + \Phi dQ \tag{80}$$

where T is the Hawking temperature, Ω is the angular velocity, and Φ is the potential deference of the Kerr Newman black holes. The following relations also hold among the seven variables $M, T, S, \Omega, \Phi, Q, J$:

$$2T\left(\frac{1}{S} - \Omega^2\right)^{1/2} = \frac{1}{2S} - \Omega^2 - \frac{Q^2}{S^2} \tag{81}$$

$$\Omega^2 = \frac{1}{S} - \left(\frac{\Phi}{Q}\right)^2 \tag{82}$$

$$S\left(\frac{\Phi}{\Omega}\right) = M + 2TS \tag{83}$$

These relations help us to obtain exact expressions for the heat capacities.

$C_{J,Q}$ is given by

$$C_{J,Q} = \frac{T}{\left(\frac{\partial T}{\partial S}\right)_{J,Q}} = F(S, Q, J)$$
$$\times 1/(48J^4 + 24S^2 J^2 + 32SJ^2 Q^2 + 24J^2 Q^4$$
$$- S^4 + 6S^2 Q^4 + 8SQ^6) \tag{84}$$

The conjugate potential for $M(S, Q, J)$ can be defined as

$$\overline{M}(S, \Omega, \Phi) = M(S, Q, J) - \Omega J - \Phi Q \tag{85}$$

Now, replacing J by (ΩSM) in Eq. (76) and resolving this with respect to M will yield the following relation for M:

$$M(S, \Omega, Q) = \frac{S + Q^2}{2\sqrt{-\Omega^2 S^2 + S}} \tag{86}$$

Using (82) and replacing Q by $\frac{\Phi\sqrt{S}}{\sqrt{1-\Omega^2 S}}$ in (86), we have

$$M(S, \Omega, \Phi) = \frac{(S - S^2 \Omega^2 + S\Phi^2)}{2(1 - S\Omega^2)\sqrt{S - S^2 \Omega^2}} \tag{87}$$

Finally, we obtain the following conjugate potential:

$$\overline{M}(S, \Omega, \Phi) = \frac{S(\Omega^2 S - 1)(\Omega^2 S - 1 + \Phi^2)}{2(1 - S\Omega^2)\sqrt{S - S^2 \Omega^2}} \tag{88}$$

Defining the following metric:

$$\overline{g}_{ij} = \frac{1}{T}\left(\frac{\partial^2 \overline{M}}{\partial X^i \partial X^j}\right); \quad X^i = (S, \Omega, \Phi) \tag{89}$$

We may obtain the resulting scalar curvature in terms of S, Ω, and Φ as follows:

$$\overline{R}(S, \Omega, \Phi) = \frac{E(S, \Omega, \Phi)}{A(S, \Omega, \Phi)^2 B(S, \Omega, \Phi)} \tag{90}$$

We can also rewrite $C_{J,Q}$ as a function of S, Ω, and Φ by replacing the following equations in relation (84):

$$J = \frac{\Omega S}{2}\left(\frac{S}{\sqrt{S - S^2 \Omega^2}} + \frac{\Phi^2}{\sqrt{S - S^2 \Omega^2}\left(S^{-1} - \Omega^2\right)}\right) \tag{91}$$

$$Q = \frac{\Phi\sqrt{S}}{\sqrt{1 - \Omega^2 S}} \tag{92}$$

Thus,

$$C_{J,Q} = \frac{2B(S, \Phi, \Omega)\left(-1 + \Omega^2 S - \phi^2\right)}{A(S, \Phi, \Omega)} \tag{93}$$

where

$$A = 4\Omega^8 S^4 - 16\Omega^6 S^3 + 21\Omega^4 S^2 + 4\Omega^4 \Phi^2 S^2$$
$$- 2\Omega^2 \Phi^2 S - 10\Omega^2 S + 1 - 3\Phi^4 - 2\Phi^2 \tag{94}$$

$$B = 2S(\Omega^4 S^2 - 3\Omega^2 S + 1 - \Phi^2) \tag{95}$$

As a result, the roots of $A(S, \Omega, \Phi)$ correspond to the divergence point of the heat capacity $C_{J,Q}$, while $B(S, \Omega, \Phi)$ is zero only at the extreme points ($T = 0$). For Myers–Perry black holes and a similar calculation, see [28].

5 Conclusion

In this work, we have explored a formulation for the thermodynamic geometry of black holes. This formulation yields a proper expression of the relation between heat capacity and curvature singularities. We also investigated a large class of black holes in all of which the singularity of the specific heat corresponds to that of the scalar curvature. We conclude that our method can be used as a correct and simple formulation for the characterization of the thermodynamic geometry of black holes and other thermodynamic systems.

Appendix

The first law of thermodynamics for a black hole with the three parameters S, Q, and J can be written as follows:

$$dM = TdS + \Phi dQ + \Omega dJ \tag{96}$$

The conjugate potential $\overline{M}(S, \Phi, \Omega)$ can be obtained from $M(S, Q, J)$ by the following Legendre transformation:

$$\overline{M}(S, \Phi, \Omega) = M(S, Q, J) - \Phi Q - \Omega J \tag{97}$$

Therefore, the first law of thermodynamics for this function would be

$$d\overline{M} = T\,dS - Q\,d\Phi - J\,d\Omega \tag{98}$$

We can write Maxwell's relations from Eqs. (96) and (98):

$$\left(\frac{\partial \Phi}{\partial J}\right)_{S,Q} = \left(\frac{\partial \Omega}{\partial Q}\right)_{S,J}; \quad \left(\frac{\partial \Phi}{\partial S}\right)_{J,Q} = \left(\frac{\partial T}{\partial Q}\right)_{J,S}$$

$$\left(\frac{\partial \Omega}{\partial S}\right)_{J,Q} = \left(\frac{\partial T}{\partial J}\right)_{S,Q} \tag{99}$$

$$\left(\frac{\partial Q}{\partial \Omega}\right)_{\Phi,S} = \left(\frac{\partial J}{\partial \Phi}\right)_{S,\Omega}; \quad \left(\frac{\partial Q}{\partial S}\right)_{\Omega,\Phi} = -\left(\frac{\partial T}{\partial \Phi}\right)_{S,\Omega}$$

$$\left(\frac{\partial J}{\partial S}\right)_{\Omega,\Phi} = -\left(\frac{\partial T}{\partial \Omega}\right)_{S,\Phi} \tag{100}$$

In addition, the Ruppeiner metric of the three parameters of the black hole can be expressed by the following relations:

$$g_{SS} = \frac{1}{T}\left(\frac{\partial^2 M}{\partial S^2}\right) = \frac{1}{T}\left(\frac{\partial T}{\partial S}\right)_{J,Q} \tag{101}$$

$$g_{SQ} = g_{QS} = \frac{1}{T}\left(\frac{\partial^2 M}{\partial S \partial Q}\right) = \frac{1}{T}\left(\frac{\partial T}{\partial Q}\right)_{S,J} \tag{102}$$

$$g_{SJ} = g_{JS} = \frac{1}{T}\left(\frac{\partial^2 M}{\partial S \partial J}\right) = \frac{1}{T}\left(\frac{\partial T}{\partial J}\right)_{S,Q} \tag{103}$$

$$g_{QJ} = g_{JQ} = \frac{1}{T}\left(\frac{\partial^2 M}{\partial Q \partial J}\right) = \frac{1}{T}\left(\frac{\partial \Phi}{\partial J}\right)_{S,Q} \tag{104}$$

$$g_{QQ} = \frac{1}{T}\left(\frac{\partial^2 M}{\partial Q^2}\right) = \frac{1}{T}\left(\frac{\partial \Phi}{\partial Q}\right)_{J,S} \tag{105}$$

$$g_{JJ} = \frac{1}{T}\left(\frac{\partial^2 M}{\partial J^2}\right) = \frac{1}{T}\left(\frac{\partial \Omega}{\partial J}\right)_{S,Q} \tag{106}$$

We can expand the right hand side of (39) as follows:

$$g_{SS}(g_{QQ}g_{JJ} - (g_{QJ})^2) - g_{SQ}(g_{SQ}g_{JJ}$$
$$- g_{SJ}g_{QJ}) + g_{SJ}(g_{SQ}g_{QJ} - g_{QQ}g_{SJ}) \tag{107}$$

By replacing the elements of the metric (101)–(106) in (107) and using Maxwell's relations (99) and (100), we obtain the following relation:

$$\begin{vmatrix} g_{SS} & g_{SQ} & g_{SJ} \\ g_{SQ} & g_{QQ} & g_{QJ} \\ g_{SJ} & g_{QJ} & g_{JJ} \end{vmatrix}$$

$$= \frac{(C_{\Phi,\Omega})^{-1}}{T^2}\left(\left(\frac{\partial \Omega}{\partial J}\right)_{S,Q}\left(\frac{\partial \phi}{\partial Q}\right)_{S,J} - \left(\frac{\partial \Omega}{\partial Q}\right)_{S,J}\left(\frac{\partial \phi}{\partial J}\right)_{S,Q}\right)$$

$$= \frac{(C_{\Omega\Phi})^{-1}}{T^2}\left(\frac{\partial(\Omega,\Phi)}{\partial(J,Q)_S}\right) \tag{108}$$

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