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Corrigendum to My Recent Papers on "Representations for Isotropic Functions"

Vol. 36, pp. 166–197, 198–223 (1970)

In the analysis of representations for isotropic functions, I still overlooked a few minor points:

I. In Part 1, Lemma 7, there should be one more list for which C_3 does not necessarily imply C_4. This list was left out in the analysis of Case 3B of the proof of the Congruence Theorem. There should be a case defined as follows:

3Bd. Some, but not all, axial lines in $\ell_3, \ldots, \ell_{p+1}$ lie on ℓ_1, while the remaining axial lines and all oriented lines in $\ell_2, \ldots, \ell_{p+1}$ lie on the same line. In this case, C_4 fails, because two lists $(\ell_1^*, \ell_2^*, \ell_1', \ell_2')$ and $(\hat{\ell}_1^*, \ell_2^*, \ell_1', \ell_2')$ with $\hat{\ell}_1^* = -\ell_1^*$, $\ell_2^* \perp \ell_1^*$, $\ell_1' \parallel \ell_1^*$, and $\ell_2' \parallel \ell_2^*$ obey C_3 but not C_4.

Then the original Case 3Bd should be redesignated as Case 3Be, and the five sub cases, 3Ba–3Be, complete the Case 3B. This correction does not cause any change in the Congruence Theorem or the Equivalence Theorem. But in the Representation Theorem there should be a new list:

11. $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{W}_1, \boldsymbol{W}_2)$. For this list, E_4 is characterized by E_3 and the invariants $\boldsymbol{W}_1 \boldsymbol{v}_1 \cdot \boldsymbol{W}_2 \boldsymbol{v}_2$ and $\boldsymbol{W}_1 \boldsymbol{v}_2 \cdot \boldsymbol{W}_2 \boldsymbol{v}_1$. To see this, let $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{W}_1, \boldsymbol{W}_2)$ and $(\hat{\boldsymbol{v}}_1, \boldsymbol{v}_2, \boldsymbol{W}_1, \boldsymbol{W}_2)$ obey E_3. From the analysis of the new Case 3Bd above, we see that E_4 fails only if

$$\boldsymbol{W}_1 \boldsymbol{v}_1 \cdot \boldsymbol{W}_2 \boldsymbol{v}_2 - \boldsymbol{W}_1 \hat{\boldsymbol{v}}_1 \cdot \boldsymbol{W}_2 \boldsymbol{v}_2 = 4 |\boldsymbol{v}_1| \, |\boldsymbol{v}_2| \, |\boldsymbol{W}_1| \, |\boldsymbol{W}_2| \neq 0.$$

Thus the following item should be added to the table of invariants:

Variables	*Invariants*
$\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{W}_1, \boldsymbol{W}_2$	$\boldsymbol{W}_1 \boldsymbol{v}_1 \cdot \boldsymbol{W}_2 \boldsymbol{v}_2,\ \boldsymbol{W}_1 \boldsymbol{v}_2 \cdot \boldsymbol{W}_2 \boldsymbol{v}_1$

II. In the analysis of Case 3C of the Representation Theorem the case with $\theta = 0 \bmod \pi$ was left out. For that case, we have $\hat{\boldsymbol{v}}_1 = -\boldsymbol{v}_1$, $\boldsymbol{v}_2 \perp \boldsymbol{v}_1$, $\boldsymbol{A} = \alpha \boldsymbol{e}_1 \otimes \boldsymbol{e}_1 + \beta \boldsymbol{e}_2 \otimes \boldsymbol{e}_2 + \gamma \boldsymbol{e}_3 \otimes \boldsymbol{e}_3$, where $\boldsymbol{v}_1, \boldsymbol{v}_2$ do not vanish, α, β, γ are unequal, and $\boldsymbol{v}_1$ is not parallel to $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, or $\boldsymbol{e}_3$. To characterize E_3 for this case, we need the invariant $\boldsymbol{v}_1 \cdot \boldsymbol{A}^2 \boldsymbol{v}_2$ in addition to the invariant $\boldsymbol{v}_1 \cdot \boldsymbol{A} \boldsymbol{v}_2$. To see this, we have equations

$$(\boldsymbol{v}_1 \cdot \boldsymbol{e}_1)(\boldsymbol{v}_2 \cdot \boldsymbol{e}_1) + (\boldsymbol{v}_1 \cdot \boldsymbol{e}_2)(\boldsymbol{v}_2 \cdot \boldsymbol{e}_2) + (\boldsymbol{v}_1 \cdot \boldsymbol{e}_3)(\boldsymbol{v}_2 \cdot \boldsymbol{e}_3) = 0,$$

$$\alpha(\boldsymbol{v}_1 \cdot \boldsymbol{e}_1)(\boldsymbol{v}_2 \cdot \boldsymbol{e}_1) + \beta(\boldsymbol{v}_1 \cdot \boldsymbol{e}_2)(\boldsymbol{v}_2 \cdot \boldsymbol{e}_2) + \gamma(\boldsymbol{v}_1 \cdot \boldsymbol{e}_3)(\boldsymbol{v}_2 \cdot \boldsymbol{e}_3) = 0,$$

$$\alpha^2(\boldsymbol{v}_1 \cdot \boldsymbol{e}_1)(\boldsymbol{v}_2 \cdot \boldsymbol{e}_1) + \beta^2(\boldsymbol{v}_1 \cdot \boldsymbol{e}_2)(\boldsymbol{v}_2 \cdot \boldsymbol{e}_2) + \gamma^2(\boldsymbol{v}_1 \cdot \boldsymbol{e}_3)(\boldsymbol{v}_2 \cdot \boldsymbol{e}_3) = 0.$$

Since the coefficient matrix is non-singular, we obtain

$$(v_1 \cdot e_1)(v_2 \cdot e_1) = (v_1 \cdot e_2)(v_2 \cdot e_2) = (v_1 \cdot e_3)(v_2 \cdot e_3) = 0,$$

which contradicts the condition that v_1 is not parallel to e_1, e_2, or e_3.

Thus the invariant $v_1 \cdot A^2 v_2$ should be added to the table also; *i.e.*,

Variables	*Invariants*
A, v_1, v_2	$v_1 \cdot A v_2,\ v_1 \cdot A^2 v_2$

III. In Part 2, § 4, I stated that (4.19) was read off from (4.18) by replacing one of the variables A_1, A_2, A_3 by $v \otimes v$. In doing so, I left out two terms. (There are six terms in (4.18) but only four terms in (4.19)!) The correct (4.19) is

$$\mathscr{W}_{(A_1,\,A_2,\,v)} = \mathscr{W}_{(A_1,\,A_2)} + \mathscr{W}_{(A_1,\,v)} + \mathscr{W}_{(A_2,\,v)} + \mathscr{W}_{(A_1 v \otimes A_2 v - A_2 v \otimes A_1 v)} + \mathscr{W}_{(A_1 A_2 v \otimes v - v \otimes A_1 A_2 v)} + \mathscr{W}_{(A_2 A_1 v \otimes v - v \otimes A_2 A_1 v)}.$$

Thus the generators $A_1 A_2 v \otimes v - v \otimes A_1 A_2 v$, $A_2 A_1 v \otimes v - v \otimes A_2 A_1 v$ should be added to the table of generators; *i.e.*,

Variables	*Generators*
A_1, A_2, v	$A_1 v \otimes A_2 v - A_2 v \otimes A_1 v,\ A_1 A_2 v \otimes v - v \otimes A_1 A_2 v$ $A_2 A_1 v \otimes v - v \otimes A_2 A_1 v$

IV. For symmetric tensor-valued isotropic functions, I cited incorrectly a result of RIVLIN & ERICKSEN. As they have shown, the function $A_1^2 A_2^2 + A_2^2 A_1^2$ is redundant in the set of generators. Thus that function should be deleted from the table of generators.

V. Again for symmetric tensor-valued isotropic functions, the function WA^2W is redundant in the set of generators. Thus that function should also be removed from the table of generators. In the proof the function WA^2W was used in Case 5 only. For that case the set $(1, A, A^2, W^2, WAW, WA - AW)$ suffices.

The necessity of the invariants $v \cdot A^2 v$, $W_1 v_1 \cdot W_2 v_2$, and $W_1 v_2 \cdot W_2 v_1$ was observed recently by Professor G. F. SMITH[1] who gave also a counter example for the generator $A_1 A_2 v \otimes v - v \otimes A_1 A_2 v$. That example, of course, is contained in the analysis of the list (A_1, A_2, A_3), since that list includes the list $(A_1, A_2, v \otimes v)$ as a special case.

Mr. SHIH-I CHOU reminded me recently about the redundancy of the function $A_1^2 A_2^2 + A_2^2 A_1^2$.

Some Comments on Irreducibility

The method of RIVLIN & ERICKSEN, which was used by Professor SMITH to determine a functional basis for (scalar) isotropic functions, cannot analyze

[1] G. F. SMITH, On Isotropic Functions of Symmetric Tensors, Skew-symmetric Tensors and Vectors. Int. J. Engng Sci. (forthcoming). I am grateful to Professor A. C. ERINGEN for sending me a copy of Professor SMITH's paper prior to its publication.

effectively the irreducibility of its outcome. In fact, irreducibility is not even mentioned in Professor SMITH'S analysis. That method is based on decomposing the domain of the function into convenient subsets. For each subset a list of invariants sufficient for the representation of isotropic functions on that particular subset is determined. Then the union of the lists of all subsets is taken to be a functional basis. Of course, if the decomposition has many subsets, then to exhaust all subsets remains a major problem. On the other hand, if the decomposition has only a few subsets, then to determine a list of invariants for each subset becomes more difficult. In any event, since for each subset only a partial list of the final functional basis is considered, there is no guarantee that that basis is irreducible.

While Professor SMITH did not mention the concept of irreducibility for a functional basis, for representation of vector-valued isotropic functions he made the following remark:

"The representation is said to be irreducible if no smaller set of vectors will suffice to furnish a complete representation."

Unfortunately, Professor SMITH did not explain what he meant by a "smaller set". Surely he could not mean a set with fewer elements (although his examples seem to suggest that this is the intended meaning), for if "irreducible" were used in that sense, then an "irreducible" basis should contain only one function, and an "irreducible" generating set should contain no more than three vector-functions, or six symmetric tensor-functions, or three skew-symmetric tensor-functions! To obtain such a magic basis, we simply take a function which has different constant values on different equivalence classes. Such a function clearly exists, since the equivalence classes are mutually disjoint, so that their cardinality is the same as that of the real numbers. The same argument can be applied to obtain "irreducible" generating sets.

In my work, *irreducible* means *incapable of being reduced.* Thus a basis is irreducible if no proper subset of it remains a basis, and a generating set is irreducible if no proper subset of it remains a generating set. In other words, a basis is irreducible if its elements are not functionally related, and a generating set is irreducible if it is linearly independent with respect to isotropic functions.

Given any basis or generating set, one can reduce it by removing all redundant elements; one does not reduce it by replacing it by a set of fewer number of elements.

For definiteness, let us call a list of order p dependent if E_p is a consequence of E_q for some q less than p. Then my Equivalence Theorem says that all lists of order five or more are dependent. More specifically, all lists of order one or two are independent, all lists of order three with the exception of lists of three vectors are independent, and the following three lists of order four are independent: $(\boldsymbol{A}_1, \boldsymbol{A}_2, \boldsymbol{v}_1, \boldsymbol{v}_2)$, $(\boldsymbol{A}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{W})$, $(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{W}_1, \boldsymbol{W}_2)$. All other lists are dependent.

My Representation Theorem is then based on the following simple fact: The union of irreducible bases of all independent sublists of a given possibly dependent list is an irreducible basis of the given list. Hence to determine an irreducible basis for dependent lists in general, one only needs irreducible bases for independent lists, and these are given by the table of invariants. An irreducible basis found in this manner has the distinguished property that it contains only functions having the least number of variables sufficient for the representation. A similar property is enjoyed by the irreducible generating sets found in my work.

Note by the Editors. Publication of this corrigendum has been delayed at the author's request.

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(Received November 2, 1970)