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Cosine Similarity Measure Based Multi-attribute Decision-making with Trapezoidal Fuzzy Neutrosophic Numbers

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Abstract. The objective of the study is to present cosine similarity measure based multi-attribute decision making under neutrosophic environment. The assesments of alternatives over the attributes are expressed with trapezoidal fuzzy neutrosophic numbers in which the three independent components namely, truth-membership degree (T), indeterminacy-membership degree (I) and falsity-membership degree (F) are expressed by trapezoidal fuzzy numbers. Cosine similarity measure between two trapezoidal fuzzy neutrosophic numbers and its properties are introduced. Expected value of trapezoidal fuzzy neutrosophic number is defined to determine the attribute weight. With these attribute weights, weighted cosine similarity measure between relative positive ideal alternative and each alternative is determined to find out the best alternative in multi-attribute decision-making problem. Finally, a numerical example is provided to illustrate the proposed approach.

Keywords: Neutrosophic set,Single-valued neutrosophic set,Trapezoidal fuzzy neutrosophic number, Expected value, Cosine similarity measure, Multi-attribute decision making

1 Introduction

Multiple attribute decision-making (MADM) is a process of finding the best option from all the feasible alternatives. In classical MADM methods [1, 2, 3, 4], the ratings and the weights of the attributes are described by crisp values. However, under many conditions, crisp data are inadequate to model real-life situations since human judgments including preferences are often vague and cannot be estimated with an exact numerical value. A more realistic approach may lead to use linguistic assessments instead of exact numerical values i.e. the ratings and weights of the criteria in the problem may be presented by means of linguistic variables. These characteristics indicate the applicability of fuzzy set introduced by Zadeh [5], intuitionistic fuzzy set studied by Atanassov [6] and neutrosophic set pioneered by Smarandache [7] in capturing the decision makers' judgement. However, neutrosophic set [8, 9] generlizes the crisp set [10, 11], fuzzy set [5], intuitionistic fuzzy set [6] and other extension of fuzzy sets. Wang et al. [12] introduced the concept of single valued neutrosophic set from practical point of view. The single valued neutrosophic set consists of three independent membership functions, namely, truth-membership function, indeterminacy-membership function, and falsity-membership function. It is capable of dealing with incomplete, indeterminate, and inconsistent information. The concept of single valued neutrosophic set has been studied and applied in different fields including decision making problems [13, 14, 15, 16, 17, 18, 19, 20, 21].

Several similarity measures in neutrosophic environment have been studied by researchers in the literature. Broumi and Smarandache [22] proposed the Hausdorff distance between neutrosophic sets and some similarity measures based on the Hausdorff distance, set theoretic approach, and matching function to determine the similarity degree between neutrosophic sets. Based on the study of Bhattacharya's distance [23], Broumi and Smarandache [24] proposed cosine similarity measure and established that their proposed similarity measure is more efficient and robust than the existing similarity measures. Pramanik and Mondal [25] proposed cosine similarity measure of rough neutrosophic sets and its application in medical diagnosis.

Majumdar and Samanta [26] developed several similarity measures of single valued neutrosophic sets (SVNSs) based on distances, a maching function, memebership grades, and then proposed an entropy measure for a SVNS. Ye and Zhang [27] proposed three new similarity measures between SVNSs based on the minimum and maximum operators and developed a multiple attribute decision making method based on the weighted similarity measure of SVNSs under single valued neutrosophic environment.
Ye [28] defined generalized distance measure between SVNSs and proposed two distance-based similarity

measures of SVNSs. In the same study, Ye [28] presented a clustering algorithm based on the similarity measures of SVNSs to cluster single-valued neutrosophic data.

Ye [29] also presented the Hamming and Euclidean distances between interval neutrosophic sets (INSs) and their similarity measures and applied them to multiple attribute decision–making problems with interval neutrosophic information. Ye [30] developed three vector similarity measure for SNSs, interval valued neutrosophic sets including the Jaccard [31], Dice [32], and cosine similarity measures [33] for SVNS and INSs and applied them to multicriteria decision-making problems with simplified neutrosophic information. Ye [34] further proposed improved cosine similarity measure of SVNSs and applied it to medical diagnosis with single valued neutrosophic information. Recently, Ye [35] proposed trapezoidal fuzzy neutrosophic number weighted arithmetic averaging (TFNNWAA) operator and a trapezoidal fuzzy neutrosophic number weighted geometric averaging (TFNNWGA) operator to aggregate the trapezoidal fuzzy neutrosophic information. Based on the TFNNWAA and TFNNWGA operators and the score and accuracy functions of a trapezoidal fuzzy neutrosophic numbers, Ye [35] proposed multiple attribute decision making in which the evaluated values of alternatives on the attributes are represented by the form of trapezoidal fuzzy neutrosophic numbers. However, cosine similarity based multiattribute decision making with trapezoidal fuzzy neutrosophic information is yet to appear in the literature.

In this paper, we propose a new approach called "Cosine similarity based multi-attribute decision making with trapezoidal fuzzy neutrosophic numbers". The expected interval and the expected value theorem for trapezoidal fuzzy neutrosophic numbers are established. Cosine similarity measure of trapezoidal fuzzy neutrosophic numbers is also established.

The rest of the paper is organized as follows: Section 2 briefly presents some preliminaries regarding neutrosophic set and single-valued neutrosophic set. In Section 3, definitions of trapezoidal fuzzy neutrosophic number and some operational laws are studied. Section 4 is confined to define the cosine similarity measure between two trapezoidal fuzzy neutrosophic numbers and its properties. Section 5 is devoted to present the cosine similarity measure based multi-attribute decision making with trapezoidal fuzzy neutrosophic numbers. Section 6 represents an illustrative example that shows the effectiveness and applicability of the proposed approach. Finally, section 7 presents the direction of future research and concluding remarks.

2 Some Preliminaries

In this section, we review some basic definitions and concepts that are used to develop the paper.

Definition 1 Let $\boldsymbol{X} = (x_1, x_2, ..., x_n)$ and $\boldsymbol{Y} = (y_1, y_2, ..., y_n)$ be two n-dimensional vectors with positive components. The cosine [33] of two vectors $\boldsymbol{X}$ and $\boldsymbol{Y}$ is the inner product of $\boldsymbol{X}$ and $\boldsymbol{Y}$ divided by the products of their lengths and it can be defined as

$$Cos(X,Y) = \frac{X.Y}{\|X\|_2 \|Y\|_2} \tag{1}$$

satisfying the following properties

i. $0 \le Cos(\mathrm{X},\mathrm{Y}) \le 1$;

ii. $Cos(\mathrm{X},\mathrm{Y}) = Cos(Y,X)$;

iii. $Cos\left(X,Y\right) = 1$, if $X = Y$ i.e. $x_i = y_i$ for i = 1, 2, ..., n.

Definition 2 A fuzzy set [5] $\tilde{A}$ in a universe of discourse X is defined by $\tilde{A} = \left\{\left\langle x, \mu_{\tilde{A}}(x)\right\rangle | x \in X\right\}$, where $\mu_{\tilde{A}}(x)$: $X \rightarrow$ [0, 1] is called the membership function of $\tilde{A}$ and $\mu_{\tilde{A}}(x)$ is the degree of membership to which $x \in \tilde{A}$.

Definition 3 A fuzzy set [5] $\tilde{A}$ defined on the universal set of real number R is said to be a fuzzy number if its membership function has the following characteristics.

i. $\tilde{A}$ is convex i.e. for any $x_1, x_2 \in X$ the membership function $\mu_{\tilde{A}}(x)$ satisfies the inequality $\mu_{\tilde{A}}\left(\lambda x_1 + \left(1-\lambda\right)x_2\right) \ge \min\{\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\}$ for $0 \le \lambda \le 1$.

ii. $\tilde{A}$ is normal i.e., if there exists at least one point $x \in X$ such that $\mu_{\tilde{A}}(x) = 1$

iii. $\mu_{\tilde{A}}(x)$ is piecewise continuous.

Definition 4 A trapezoidal fuzzy number [36] $\tilde{A}$ is denoted by (a_1, a_2, a_3, a_4), where, a_1, a_2, a_3, a_4 are real numbers and its membership function $\mu_{\tilde{A}}(x)$ is defined as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} f(\mathrm{x}) = \dfrac{x-a_1}{a_2-a_1} & a_1 \le x \le a_2; \\ 1 & a_2 \le x \le a_3; \\ g(\mathrm{x}) = \dfrac{a_4-x}{a_4-a_3} & a_3 \le x \le a_4; \\ 0 & otherwise. \end{cases}$$

Then, $\mu_{\tilde{A}}(x)$ satisfies the following conditions:

1. $\mu_{\tilde{A}}(x)$ is a continuous mapping from R to closed interval [0,1],
2. $\mu_{\tilde{A}}(x)$ = 0 for every $x \in (-\infty, a_1]$,
3. $\mu_{\tilde{A}}(x)$ is strictly increasing and continuous on $[a_1, a_2]$,
4. $\mu_{\tilde{A}}(x)$ = 1 for every $x \in [a_2, a_3]$,

5. $\mu_{\tilde{A}}(x)$ is strictly decreasing and continuous on [a_3, a_4],
6. $\mu_{\tilde{A}}(x) = 0$ for every x ∈ [a_4, ∞).

The trapezoidal fuzzy number reduces to a triangular fuzzy number if $a_2 = a_3$.

Definition 5 The expected interval and the expected value of fuzzy number [37] $\tilde{A}$ are expressed as follows:

$$\mathrm{EI}(\tilde{A}) = [E(\tilde{A}^L), E(\tilde{A}^U)] \quad (2)$$

$$\mathrm{EV}(\tilde{A}) = \left(E(\tilde{A}^L), E(\tilde{A}^U)\right)/2 \quad (3)$$

where $E(\tilde{A}^L) = a_2 - \int_{a_1}^{a_2} f(x)dx$ and

$$E(\tilde{A}^U) = a_3 + \int_{a_3}^{a_4} g(x)dx.$$

In case of the trapezoidal fuzzy number the expected interval and the expected value of $\tilde{A}$ = (a_1, a_2, a_3, a_4) can be obtained by using the equations (2) and (3) as follows:

$$\mathrm{EI}(\tilde{A}) = \left[\frac{(a_1 + a_2)}{2}, \frac{(a_3 + a_4)}{2}\right] \quad (4)$$

$$\mathrm{EV}(\tilde{A}) = \frac{(a_1 + a_2 + a_3 + a_4)}{4} \quad (5)$$

Definition 6 Cosine similarity measure [33] is defined as the inner product of two vectors divided by the product of their lengths. It is the cosine of the angle between the vector representations of the two fuzzy sets.

Let us assume that A = $\left(\mu_A(x_1), \mu_A(x_2), ..., \mu_A(x_n)\right)$ and B = $\left(\mu_B(x_1), \mu_B(x_2), ..., \mu_B(x_n)\right)$ are two fuzzy sets in the universe of discourse X = {x_1, x_2,..., x_n}. Then the cosine similarity measure of $\mu_A(x_i)$ and $\mu_B(x_i)$ is

$$C_{Fuzz}(\tilde{A}, \tilde{B}) = \frac{\sum_{i=1}^{n}\left(\mu_A(x_i)\,\mu_B(x_i)\right)}{\sqrt{\sum_{i=1}^{n}\left(\mu_A^2(x_i)\right)}\sqrt{\sum_{i=1}^{n}\left(\mu_B^2(x_i)\right)}} \quad (6)$$

It satisfies the following properties:

i) $0 \le C_{Fuzz}(\tilde{A}, \tilde{B}) \le 1$

ii) $C_{Fuzz}(\tilde{A}, \tilde{B}) = C_{Fuzz}(\tilde{B}, \tilde{A})$

iii) $C_{Fuzz}(\tilde{A}, \tilde{B}) = 1$ if $\tilde{A} = \tilde{B}$.

The value of $C_{Fuzz}(\tilde{A}, \tilde{B})$ is considered zero if $\mu_{\tilde{A}}(x) = 0$ and $\mu_{\tilde{B}}(x) = 0$.

Definition 7 Cosine similarity measure of trapezoidal fuzzy numbers [38]

Let $\tilde{A}$ = (a_1, a_2, a_3, a_4) and $\tilde{B}$ = (b_1, b_2, b_3, b_4) be two trapezoidal fuzzy numbers in the set of real numbers ℝ. The four parameters presented in two numbers $\tilde{A}$ and $\tilde{B}$ can be considered as the vector representations of four elements. Thus the cosine similarity measure of $\tilde{A}$ and $\tilde{B}$ can be defined as the extension of the cosine similarity measure of fuzzy sets as follows:

$$C_{TRFN}(\tilde{A}, \tilde{B}) = \frac{\sum_{i=1}^{n} a_i . b_i}{\sqrt{\sum_{i=1}^{n}(a_i)^2}\sqrt{\sum_{i=1}^{n}(b_i)^2}} \quad (7)$$

It satisfies the following properties:

i) $0 \le C_{TRFN}(\tilde{A}, \tilde{B}) \le 1$

ii) $C_{TRFN}(\tilde{A}, \tilde{B}) = C_{TRFN}(\tilde{B}, \tilde{A})$

iii) $C_{TRFN}(\tilde{A}, \tilde{B}) = 1$, if $\tilde{A} = \tilde{B}$ i.e. $a_i = b_i$ for i = 1, 2, 3, 4.

2.1 Some basic concepts of neutrosophic set

Definition 8

Let *X* be a space of points (objects) with generic element x. Then a neutrosophic set [7] A in *X* is characterized by a truth membership function T_A, an indeterminacy membership function I_A and a falsity membership function F_A. The functions T_A, I_A and F_A are real standard or non-standard subsets of $]^-0, 1^+[$ that is $T_A : X \to]^-0, 1^+[$; $I_A : X \to]^-0, 1^+[$; $F_A : X \to]^-0, 1^+[$

$T_A(x)$, $I_A(x)$, and $F_A(x)$ satisfy the relation

i.e. $^-0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$

Definition 9 The complement [7] A^c of a neutrosophic set *A* is defined as follows:

$T_{A^c}(x) = \{1^+\} - T_A(x)$; $I_{A^c}(x) = \{1^+\} - I_A(x)$;

$F_{A^c}(x) = \{1^+\} - F_A(x)$.

Definition 10 A neutrosophic set [7] *A* is contained in other neutrosophic set *B* i.e., $A \subseteq B$ if and only if the following results hold good.

$\inf T_A(x) \le \inf T_B(x)$, $\sup T_A(x) \le \sup T_B(x)$

$\inf I_A(x) \ge \inf I_B(x)$, $\sup I_A(x) \ge \sup I_B(x)$

$\inf F_A(x) \ge \inf F_B(x)$, $\sup F_A(x) \ge \sup F_B(x)$

for all x in *X*.

Definition 11. Let *X* be a universal space of points (objects), with a generic element $x \in X$. A single-valued neutrosophic set [12] $\tilde{\mathcal{N}} \subset X$ is characterized by a true membership function $T_{\tilde{N}}(x)$, a falsity membership function

$F_{\tilde{N}}(x)$ and an indeterminacy membership function $I_{\tilde{N}}(x)$ with $T_{\tilde{N}}(x)$, $I_{\tilde{N}}(x)$, $F_{\tilde{N}}(x) \in [0, 1]$ for all $x \in X$. For a SVNS $\tilde{N}$, the relation

$$0 \le \sup T_{\tilde{N}}(x) + \sup I_{\tilde{N}}(x) + \sup F_{\tilde{N}}(x) \le 3 \quad (8)$$

holds for $\forall x \in X$.

When X is continuous SVNSs, $\tilde{N}$ can be written as follows:

$$\tilde{N} = \int_x \left\langle T_{\tilde{N}}(x), I_{\tilde{N}}(x), F_{\tilde{N}}(x) \right\rangle / x, \ \forall x \in X.$$

and when X is discrete a SVNSs $\tilde{N}$ can be written as follows:

$$\tilde{N} = \sum_{i=1}^{m} \left\langle T_{\tilde{N}}(x), I_{\tilde{N}}(x), F_{\tilde{N}}(x) \right\rangle / x, \ \forall x \in X.$$

$T_{\tilde{N}}(x)$, $I_{\tilde{N}}(x)$, $F_{\tilde{N}}(x) \in [0, 1]$

Definition 12 The complement $\tilde{N}^c$ of a single-valued neutrosophic set [12] is defined as follows:

$T_{\tilde{N}^c}(x) = F_{\tilde{N}}(x)$; $I_{\tilde{N}^c}(x) = 1 - I_{\tilde{N}}(x)$; $F_{\tilde{N}^c}(x) = T_{\tilde{N}}(x)$

Definition 13 A single-valued neutrosophic set [12] $\tilde{N}_A$ is contained in $\tilde{N}_B$ i.e., $\tilde{N}_A \subseteq \tilde{N}_B$, if and only if

$T_{\tilde{N}_A}(x) \le T_{\tilde{N}_B}(x)$; $I_{\tilde{N}_A}(x) \ge I_{\tilde{N}_B}(x)$; $F_{\tilde{N}_A}(x) \ge F_{\tilde{N}_B}(x)$ for $\forall x \in X$.

Definition 14 Two SVNSs [12] $\tilde{N}_A$ and $\tilde{N}_B$ are equal, i.e. $\tilde{N}_A = \tilde{N}_B$, if and only if $\tilde{N}_A \subseteq \tilde{N}_B$ and $\tilde{N}_A \supseteq \tilde{N}_B$.

Definition 15 The union of two SVNSs [12] $\tilde{N}_A$ and $\tilde{N}_B$ is a SVNS $\tilde{N}_C$, denoted as $\tilde{N}_C = \tilde{N}_A \cup \tilde{N}_B$. Its truth membership, indeterminacy-membership and falsity membership functions are related to those of $\tilde{N}_A$ and $\tilde{N}_B$ as follows:

$T_{\tilde{N}_C}(x) = \max(T_{\tilde{N}_A}(x), T_{\tilde{N}_B}(x))$;

$I_{\tilde{N}_C}(x) = \max(I_{\tilde{N}_A}(x), I_{\tilde{N}_B}(x))$;

$F_{\tilde{N}_C}(x) = \min(F_{\tilde{N}_A}(x), F_{\tilde{N}_B}(x))$ $\forall x \in X$.

Definition 16 The intersection of two SVNSs [12] $\tilde{N}_A$ and $\tilde{N}_B$ is denoted as a SVNS $\tilde{N}_C = \tilde{N}_A \cap \tilde{N}_B$, where truth membership, indeterminacy-membership and falsity membership functions are defined as follows:

$T_{\tilde{N}_C}(x) = \min(T_{\tilde{N}_A}(x), T_{\tilde{N}_B}(x))$;

$I_{\tilde{N}_C}(x) = \min(I_{\tilde{N}_A}(x), I_{\tilde{N}_B}(x))$;

$F_{\tilde{N}_C}(x) = \max(F_{\tilde{N}_A}(x), F_{\tilde{N}_B}(x))$ for all x in X.

Definition 17 The addition of two SVNSs [12] $\tilde{N}_A$ and $\tilde{N}_B$ is a SVNS $\tilde{N}_C = \tilde{N}_A \oplus \tilde{N}_B$, whose three membership degrees related to $\tilde{N}_A$ and $\tilde{N}_B$ are defined as follows:

$T_{\tilde{N}_C}(x) = T_{\tilde{N}_A}(x) + T_{\tilde{N}_B}(x) - T_{\tilde{N}_A}(x).T_{\tilde{N}_B}(x)$;

$I_{\tilde{N}_C}(x) = I_{\tilde{N}_A}(x).I_{\tilde{N}_B}(x)$; $F_{\tilde{N}_C}(x) = F_{\tilde{N}_A}(x).F_{\tilde{N}_B}(x)$ $\forall x \in X$.

Definition 18 The multiplication of two SVNSs [12] $\tilde{N}_A$ and $\tilde{N}_B$ is a SVNS $\tilde{N}_C = \tilde{N}_A \otimes \tilde{N}_B$, whose three membership degrees related to $\tilde{N}_A$ and $\tilde{N}_B$ are defined as follows:

$T_{\tilde{N}_C}(x) = T_{\tilde{N}_A}(x).T_{\tilde{N}_B}(x)$;

$I_{\tilde{N}_C}(x) = I_{\tilde{N}_A}(x) + I_{\tilde{N}_B}(x) - I_{\tilde{N}_A}(x).I_{\tilde{N}_B}(x)$;

$F_{\tilde{N}_C}(x) = F_{\tilde{N}_A}(x) + F_{\tilde{N}_B}(x) - F_{\tilde{N}_A}(x).F_{\tilde{N}_B}(x)$ $\forall x \in X$.

3 Trapezoidal Fuzzy Neutrosophic Number

Definition 19 A neutrosophic set $\tilde{N}_A$ in a universe of discourse X is defined in the following form:

$\tilde{N}_A = \left\{ \left\langle T_{\tilde{N}_A}(x), I_{\tilde{N}_A}(x), F_{\tilde{N}_A}(x) \right\rangle \mid x \in X \right\}$ where, truth membership degree $T_{\tilde{N}_A}(x) : X \in [0, 1]$, indeterminacy membership degree $I_{\tilde{N}_A}(x) : X \in [0, 1]$ and falsity membership degree $F_{\tilde{N}_A}(x) : X \in [0, 1]$. Fuzzy neutrosophic number can be defined by extending a discrete set to a contious set.

Let $\tilde{N}_A$ be a fuzzy neutrosophic number in the set of real numbers R. Then its truth membership function can be defined as follows:

$$T_{\tilde{N}_A}(x) = \left\langle \begin{array}{ll} T^L_{\tilde{N}_A}(x) & a_{11} \le x \le a_{21} \\ 1 & a_{21} \le x \le a_{31} \\ T^U_{\tilde{N}_A}(x) & a_{31} \le x \le a_{41} \\ 0 & otherwise \end{array} \right\rangle \quad (9)$$

The indeterminacy membership function can be defined as follows:

$$I_{\tilde{N}_A}(x) = \left\langle \begin{array}{ll} I^L_{\tilde{N}_A}(x) & b_{11} \le x \le b_{21} \\ 0 & b_{21} \le x \le b_{31} \\ I^U_{\tilde{N}_A}(x) & b_{31} \le x \le b_{41} \\ 1 & otherwise \end{array} \right\rangle \quad (10)$$

and the falsity membership function can be defined as follows:

$$F_{\tilde{\mathcal{N}}_A}(x) = \left\langle \begin{array}{ll} F^L_{\tilde{\mathcal{N}}_A}(x) & c_{11} \le x \le c_{21} \\ 0 & c_{21} \le x \le c_{31} \\ F^U_{\tilde{\mathcal{N}}_A}(x) & c_{31} \le x \le c_{41} \\ 1 & otherwise \end{array} \right\rangle \quad (11)$$

where $0 \le \sup T_{\tilde{\mathcal{N}}_A}(x) + \sup I_{\tilde{\mathcal{N}}_A}(x) + \sup F_{\tilde{\mathcal{N}}_A}(x) \le 3$, $\forall x \in X$ and $a_{11}, a_{21}, a_{31}, a_{41}, b_{11}, b_{21}, b_{31}, b_{41}, c_{11}, c_{21}, c_{31}, c_{41} \in \mathbf{R}$ such that $a_{11} \le a_{21} \le a_{31} \le a_{41}$, $b_{11} \le b_{21} \le b_{31} \le b_{41}$ and $c_{11} \le c_{21} \le c_{31} \le c_{41}$. $T^L_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$, $I^U_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$, and $F^U_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$ are continuous monotonic increasing functions and $T^U_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$, $I^L_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$, and $I^L_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$ are continuous monotonic decreasing functions.

Definition 20 (Trapezoidal Fuzzy Neutrosophic Number) A trapezoidal fuzzy neutrosophic number (TrFNN) [35] $\tilde{\mathcal{N}}_A$ is denoted by $\tilde{\mathcal{N}}_A = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$ in a universe of discourse *X*. The parameters satisfy the following relations $a_1 \le a_2 \le a_3 \le a_4$, $b_1 \le b_2 \le b_3 \le b_4$ and $c_1 \le c_2 \le c_3 \le c_4$.

Its truth membership function is defined as follows:

$$T_{\tilde{\mathcal{N}}_A}(x) = \left\langle \begin{array}{ll} \dfrac{x - a_1}{a_2 - a_1} & a_1 \le x \le a_2 \\ 1 & a_2 \le x \le a_3 \\ \dfrac{a_4 - x}{a_4 - a_3} & a_3 \le x \le a_4 \\ 0 & otherwise \end{array} \right\rangle \quad (12)$$

Its indeterminacy membership function is defined as follows:

$$I_{\tilde{\mathcal{N}}_A}(x) = \left\langle \begin{array}{ll} \dfrac{b_2 - x}{b_2 - b_1} & b_1 \le x \le b_2 \\ 0 & b_2 \le x \le b_3 \\ \dfrac{x - b_3}{b_4 - b_3} & b_3 \le x \le b_4 \\ 1 & otherwise \end{array} \right\rangle \quad (13)$$

and its falsity membership function is defined as follows:

$$F_{\tilde{\mathcal{N}}_A}(x) = \left\langle \begin{array}{ll} \dfrac{c_2 - \mathrm{x}}{c_2 - c_1} & c_1 \le x < c_2 \\ 0 & c_2 \le x \le c_3 \\ \dfrac{x - c_3}{c_4 - c_3} & c_3 < x \le c_4 \\ 1 & otherwise \end{array} \right\rangle \quad (14)$$

3.1 Some operational rules of trapezoidal fuzzy neutrosophic numbers.

Let $\tilde{\mathcal{N}}_A = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$ and $\tilde{\mathcal{N}}_B = \langle (e_1, e_2, e_3, e_4), (f_1, f_2, f_3, f_4), (g_1, g_2, g_3, g_4) \rangle$ be two TrFNNs in the set of real numbers **R**. Then the operation rules [35] for $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ are presented as follows:

1. $$\tilde{\mathcal{N}}_A \oplus \tilde{\mathcal{N}}_B = \left\langle \begin{array}{c} \begin{pmatrix} a_1 + e_1 - a_1 e_1, a_2 + e_2 - a_2 e_2, \\ a_3 + e_3 - a_3 e_3,\ a_4 + e_4 - a_4 e_4 \end{pmatrix}, \\ (b_1 f_1,\ b_2 f_2,\ b_3 f_3,\ b_4 f_4), \\ (c_1 g_1,\ c_2 g_2,\ c_3 g_3,\ c_4 g_4) \end{array} \right\rangle \quad (15)$$

2. $$\tilde{\mathcal{N}}_A \otimes \tilde{\mathcal{N}}_B = \left\langle \begin{array}{c} (a_1 e_1,\ a_2 e_2,\ a_3 e_3,\ a_4 e_4), \\ \begin{pmatrix} b_1 + f_1 - b_1 f_1, b_2 + f_2 - b_2 f_2, \\ b_3 + f_3 - b_3 f_3, b_4 + f_4 - b_4 f_4 \end{pmatrix}, \\ \begin{pmatrix} c_1 + g_1 - c_1 g_1, c_2 + g_2 - c_2 g_2, \\ c_3 + g_3 - c_3 g_3, c_4 + g_4 - c_4 g_4 \end{pmatrix} \end{array} \right\rangle \quad (16)$$

3. $$\lambda \tilde{\mathcal{N}}_A = \left\langle \begin{array}{c} \begin{pmatrix} 1 - (1 - a_1)^\lambda,\ 1 - (1 - a_2)^\lambda, \\ 1 - (1 - a_3)^\lambda,\ 1 - (1 - a_4)^\lambda \end{pmatrix}, \\ (b_1^\lambda, b_2^\lambda, b_3^\lambda, b_4^\lambda),\ (c_1^\lambda, c_2^\lambda, c_3^\lambda, c_4^\lambda) \end{array} \right\rangle \quad (17)$$
for $\lambda > 0$;

4. $$\left(\tilde{\mathcal{N}}_A\right)^\lambda = \left\langle \begin{array}{c} (a_1^\lambda, a_2^\lambda, a_3^\lambda, a_4^\lambda) \\ \begin{pmatrix} 1 - (1 - b_1)^\lambda, 1 - (1 - b_2)^\lambda, \\ 1 - (1 - b_3)^\lambda, 1 - (1 - b_4)^\lambda \end{pmatrix} \\ \begin{pmatrix} 1 - (1 - c_1)^\lambda, 1 - (1 - c_2)^\lambda, \\ 1 - (1 - c_3)^\lambda, 1 - (1 - c_4)^\lambda \end{pmatrix}, \end{array} \right\rangle \quad (18)$$
for $\lambda > 0$.

5. $\tilde{\mathcal{N}}_A = \tilde{\mathcal{N}}_B$ if $a_i = e_i$, $b_i = f_i$ and $c_i = g_i$ hold for i = 1, 2, 3, 4 i.e. $(a_1, a_2, a_3, a_4) = (e_1, e_2, e_3, e_4)$, $(b_1, b_2, b_3, b_4) = (f_1, f_2, f_3, f_4)$ and $(c_1, c_2, c_3, c_4) = (g_1, g_2, g_3, g_4)$.

3.2 Expected value of trapezoidal fuzzy neutrosophic number

Let $\tilde{\mathcal{N}}_A = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$ be the TrFNN characterized by three independent membership degrees in the set of real numbers **R** where, $T_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$ be the truth membership degree, $I_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$ be the indeterminacy degree and $F_{\tilde{\mathcal{N}}_A}(x) \in [0, 1]$ be the falsity membership degree such that the following relation holds.

$0 \le \sup T_{\tilde{\mathcal{N}}_A}(x) + \sup I_{\tilde{\mathcal{N}}_A}(x) + \sup F_{\tilde{\mathcal{N}}_A}(x) \le 3.$

It is also assumed that

$T^L_{\tilde{\mathcal{N}}_A}(x) = \dfrac{x - a_1}{a_2 - a_1}$, $T^U_{\tilde{\mathcal{N}}_A}(x) = \dfrac{a_4 - x}{a_4 - a_3}$ are the two sides of $T_{\tilde{\mathcal{N}}_A}(x)$. Similarly, $I^L_{\tilde{\mathcal{N}}_A}(x) = \dfrac{b_2 - x}{b_2 - b_1}$, $I^U_{\tilde{\mathcal{N}}_A}(x) = \dfrac{x - b_3}{b_4 - b_3}$ are the two sides of $I_{\tilde{\mathcal{N}}_A}(x)$ and $F^L_{\tilde{\mathcal{N}}_A}(x) = \dfrac{c_2 - x}{c_2 - c_1}$, $F^U_{\tilde{\mathcal{N}}_A}(x) = \dfrac{x - c_3}{c_4 - c_3}$ are the two sides of $F_{\tilde{\mathcal{N}}_A}(x)$.

Each of three membership degrees of TrFNN can be taken as the three independent components like fuzzy numbers. Thus similar to fuzzy set, the expected interval or expected value of each membership degree can be determined separately.

Definition 21 We define the expected interval and the expected value of truth membership function

$T_{\tilde{\mathcal{N}}_A}(x) = (a_1, a_2, a_3, a_4)$ of TrFNN $\tilde{\mathcal{N}}_A$ as follows:

$$EI(T_{\tilde{\mathcal{N}}_A}(x)) = \left[\frac{(a_1 + a_2)}{2}, \frac{(a_3 + a_4)}{2}\right] \tag{19}$$

$$EV(T_{\tilde{\mathcal{N}}_A}(x)) = \frac{(a_1 + a_2 + a_3 + a_4)}{4} \tag{20}$$

Similarly, we define the expected interval and the expected value of the indeterminacy membership function of TrFNN as follows:

$$EI(I_{\tilde{\mathcal{N}}_A}(x)) = \left[\frac{(b_1 + b_2)}{2}, \frac{(b_3 + b_4)}{2}\right] \tag{21}$$

$$EV(I_{\tilde{\mathcal{N}}_A}(x)) = \frac{(b_1 + b_2 + b_3 + b_4)}{4} \tag{22}$$

We define the expected interval and the expected value of the falsity membership function of TrFNN as follows:

$$EI(F_{\tilde{\mathcal{N}}_A}(x)) = \left[\frac{(c_1 + c_2)}{2}, \frac{(c_3 + c_4)}{2}\right] \tag{23}$$

$$EV(F_{\tilde{\mathcal{N}}_A}(x)) = \frac{(c_1 + c_2 + c_3 + c_4)}{4} \tag{24}$$

Definition 22 (Truth favorite relative expected value of TrFNN)

Let $\tilde{\mathcal{N}}_A = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$ be the TrFNN in the set of real numbers **R.** Suppose $EV(T_{\tilde{\mathcal{N}}_A}(x))$, $EI(I_{\tilde{\mathcal{N}}_A}(x))$ and $EV(F_{\tilde{\mathcal{N}}_A}(x))$ are the expected values of truth membership, indeterminacy membership and falsity membership component of SVNN $\tilde{\mathcal{N}}_A$. If

$$EV^T(\tilde{\mathcal{N}}_A) = \frac{3EV(T_{\tilde{\mathcal{N}}_A}(x))}{EV(T_{\tilde{\mathcal{N}}_A}(x)) + EV(I_{\tilde{\mathcal{N}}_A}(x)) + EV(F_{\tilde{\mathcal{N}}_A}(x))} \tag{25}$$

then we define $EV(\tilde{\mathcal{N}}_A)$ as the truth favorite relative expected value (TFREV) of $\tilde{\mathcal{N}}_A$.

Theorem 1(Expected value theorem)

Let $\tilde{\mathcal{N}}_A = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$ be the TrFNN in the set of real numbers **R,** then the truth favorite relative expected value (TFREV) of $\tilde{\mathcal{N}}_A$ is defined by

$$EV^T(\tilde{\mathcal{N}}_A) = \frac{3\sum_{i=1}^4 a_i}{\left(\sum_{i=1}^4 a_i + \sum_{i=1}^4 b_i + \sum_{i=1}^4 c_i\right)} \tag{26}$$

Proof: Given that $T_{\tilde{\mathcal{N}}_A}(x)$ is the truth membership, $I_{\tilde{\mathcal{N}}_A}(x)$ is the indeterminacy membership and $F_{\tilde{\mathcal{N}}_A}(x)$ is the falsity membership component of TrFNN $\tilde{\mathcal{N}}_A$. Treating each component of $\tilde{\mathcal{N}}_A$ as the trapezoidal fuzzy number, the combined expected value of the $\tilde{\mathcal{N}}_A$ can be obtained by considering the centroid of three expected values of $T_{\tilde{\mathcal{N}}_A}(x)$, $I_{\tilde{\mathcal{N}}_A}(x)$ and $F_{\tilde{\mathcal{N}}_A}(x)$.

Then, the combined expected value of three membership components can be defined by

$$EV(\tilde{\mathcal{N}}_A) = \frac{1}{3}\left(EV(T_{\tilde{\mathcal{N}}_A}(x)) + EV(I_{\tilde{\mathcal{N}}_A}(x)) + EV(F_{\tilde{\mathcal{N}}_A}(x))\right) \tag{27}$$

Combining Eqs. (20), (22), (24), and (27) we obtain

$$EV(\tilde{\mathcal{N}}_A) = \frac{1}{3}\left(\frac{(a_1 + a_2 + a_3 + a_4)}{4} + \frac{(b_1 + b_2 + b_3 + b_4)}{4} + \frac{(c_1 + c_2 + c_3 + c_4)}{4}\right)$$

$$= \frac{\left(\sum_{i=1}^4 a_i + \sum_{i=1}^4 b_i + \sum_{i=1}^4 c_i\right)}{12} \tag{28}$$

Now, the TFREV of $\tilde{\mathcal{N}}_A$ can be determined by

$$EV^T(\tilde{\mathcal{N}}_A) = \frac{EV(T_{\tilde{\mathcal{N}}_A}(x))}{EV(\tilde{\mathcal{N}}_A)}. \tag{29}$$

Using Eqs.(20) (28) and (29), we obtain the desired TFREV of $\tilde{\mathcal{N}}_A$ as follows:

$$EV^T(\tilde{\mathcal{N}}_A) = \frac{3\sum_{i=1}^{4} a_i}{\left(\sum_{i=1}^{4} a_i + \sum_{i=1}^{4} b_i + \sum_{i=1}^{4} c_i\right)} \tag{30}$$

This completes the proof.

Now, if the corresponding elements of three membership degrees of TrFNN $\tilde{\mathcal{N}}_A$ coincide with each other i.e., when

$(a_1, a_2, a_3, a_4) = (b_1, b_2, b_3, b_4) = (c_1, c_2, c_3, c_4)$ then combined expected value of $\tilde{\mathcal{N}}_A$ would be

$$EV(\tilde{\mathcal{N}}_A) = \frac{(a_1 + a_2 + a_3 + a_4)}{4} \tag{31}$$

and TFREV of $\tilde{\mathcal{N}}_A$ would be $EV^T(\tilde{\mathcal{N}}_A) = 1$.

It is to be noted that if $a_2 = a_3, b_2 = b_3$ and $c_2 = c_3$ of a TrFNN $\tilde{\mathcal{N}}_A = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$

then $\tilde{\mathcal{N}}_A$ reduces to triangular fuzzy neutrosophic number (TFNN) $\tilde{\mathcal{N}}_A = \langle (a_1, a_2, a_4), (b_1, b_2, b_4), (c_1, c_2, c_4) \rangle$. Then according to Eq.(28), the expected value of TFNN $\tilde{\mathcal{N}}_{Tri} = \langle (l_1, l_2, l_3), (m_1, m_2, m_3), (n_1, n_2, n_3) \rangle$ can be defined as follows:

$$EV(\tilde{\mathcal{N}}_{Tri}) = \frac{(l_1 + 2l_2 + l_3 + m_1 + 2m_2 + m_3 + n_1 + 2n_2 + n_3)}{12} \tag{32}$$

and TFREV of $\tilde{\mathcal{N}}_{Tri}$ can be defined as follows:

$$EV^T(\tilde{\mathcal{N}}_{Tri}) = \frac{3(l_1 + 2l_2 + l_3)}{(l_1 + 2l_2 + l_3 + m_1 + 2m_2 + m_3 + n_1 + 2n_2 + n_3)} \tag{33}$$

4 Cosine Similarity Measure of Trapezoidal Fuzzy Neutrosophic Numbers

Definition 23

Let $\tilde{\mathcal{N}}_A = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$ and

$\tilde{\mathcal{N}}_B = \langle (e_1, e_2, e_3, e_4), (f_1, f_2, f_3, f_4), (g_1, g_2, g_3, g_4) \rangle$ be two TrFNNs in the set of real numbers **R**. The twelve parameters considered in $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ can be taken as two vector representations with twelve elements. Thus, a cosine similarity measure between $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ can be determined in a similar manner to the cosine similarity measure between two trapezoidal fuzzy numbers. Then,

$$Cos_{TrFNN}(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B) = \frac{\sum_{i=1}^{4} a_i e_i + \sum_{i=1}^{4} b_i f_i + \sum_{i=1}^{4} c_i g_i}{\left[\begin{array}{l}\left(\sqrt{\sum_{i=1}^{4}(a_i)^2 + \sum_{i=1}^{4}(b_i)^2 + \sum_{i=1}^{4}(c_i)^2}\right) \times \\ \left(\sqrt{\sum_{i=1}^{4}(e_i)^2 + \sum_{i=1}^{4}(f_i)^2 + \sum_{i=1}^{4}(g_i)^2}\right)\end{array}\right]}. \tag{34}$$

The cosine similarity measure $Cos_{TrFNN}(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B)$ of $\tilde{\mathcal{N}}_A$ and $\tilde{\mathcal{N}}_B$ satisfies the following properties:

P1. $0 \le Cos_{TrFNN}(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B) \le 1$

P2 $Cos_{TrFNN}(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B) = Cos_{TrFNN}(\tilde{\mathcal{N}}_B, \tilde{\mathcal{N}}_A)$

P3 $Cos_{TrFNN}(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B) = 1$ for $\tilde{\mathcal{N}}_A = \tilde{\mathcal{N}}_B$

i.e., $a_i = e_i$, $b_i = f_i$ and $c_i = g_i$ for i = 1, 2, 3, 4.

Proof: P1 is shown to be true from the basic definition of cosine value.

P2: Symmetry of Eq. (34) validates the property P2.

P3: By putting $a_i = e_i$, $b_i = f_i$ and $c_i = g_i$ for i = 1, 2, 3, 4 in Eq. (34), the denominator and numerator reduce to $\left(\sum_{i=1}^{4}(a_i)^2 + \sum_{i=1}^{4}(b_i)^2 + \sum_{i=1}^{4}(c_i)^2\right)$ and therefore $Cos_{TrFNN}(\tilde{\mathcal{N}}_A, \tilde{\mathcal{N}}_B) = 1$.

5 Cosine Similarity Based Multiple Attribute Decision-Making Problems with Trapezoidal Fuzzy Neutrosophic Numbers

Let $A_1, A_2, ..., A_m$ be a discrete set of *m* alternatives, and $C_1, C_2, ..., C_n$ be the set of *n* attributes for a multi-attribute decision-making problem. The rating d_{ij} provided by the decision maker describes the performance of the alternative A_i against the attribute C_j. Then the assessment values of the alternatives can be presented in the following decision matrix form.

Table 1. Decision matrix of attribute values

$$D = \langle d_{ij} \rangle_{m\times n} = \begin{array}{c} \\ A_1 \\ A_2 \\ . \\ . \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & ... & C_n \end{array} \\ \begin{bmatrix} d_{11} & d_{12} & ... & d_{1n} \\ d_{21} & d_{22} & ... & d_{2n} \\ ... & ... & ... & ... \\ ... & ... & ... & ... \\ d_{m1} & d_{m2} & ... & d_{mn} \end{bmatrix} \end{array} \tag{35}$$

Step 1. Determination of the most important attributes

In a decision making process, a set of criteria or attributes are to be required to evaluate the best alternative. All attributes are not equal important in the decision making situation. Therefore it is important to choose the set of proper attributes based on experts' opinions.

Step 2. Construction of the decision matrix with TrFNNs

Let us assume that the ratings of alternative A_i (i = 1, 2,…, m) with respect to the attribute C_j (j = 1, 2, …, n) are expressed with TrFNNs. The TrFNN based rating values of the m-th alternative over the n-th attribute can be presented in the following decision matrix.

Table 2. Decision matrix with *TrFNNs*

$$D_{\tilde{N}} = \langle \tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{c}_{ij} \rangle_{m\times n} =$$

$$\begin{bmatrix} \langle \tilde{a}_{11},\tilde{b}_{11},\tilde{c}_{11}\rangle & \langle \tilde{a}_{12},\tilde{b}_{12},\tilde{c}_{12}\rangle & \ldots & \langle \tilde{a}_{1n},\tilde{b}_{1n},\tilde{c}_{1n}\rangle \\ \langle \tilde{a}_{21},\tilde{b}_{21},\tilde{c}_{21}\rangle & \langle \tilde{a}_{22},\tilde{b}_{22},\tilde{c}_{22}\rangle & \ldots & \langle \tilde{a}_{2n},\tilde{b}_{2n},\tilde{c}_{2n}\rangle \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \langle \tilde{a}_{m1},\tilde{b}_{m1},\tilde{c}_{m1}\rangle & \langle \tilde{a}_{m2},\tilde{b}_{m2},\tilde{c}_{m2}\rangle & \ldots & \langle \tilde{a}_{mn},\tilde{b}_{mn},\tilde{c}_{mn}\rangle \end{bmatrix} \quad (36)$$

In the decision matrix $D_{\tilde{N}} = \langle \tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{c}_{ij} \rangle_{m\times n}$, $\tilde{a}_{ij}$ denotes the degree that the alternative A_i (i = 1, 2, ..., m) satisfies the attribute C_j (j = 1, 2, ..., n), $\tilde{b}_{ij}$ denotes the degree of indeterminacy of the alternative A_i over the attribute C_j and $\tilde{c}_{ij}$ denotes the degree that the alternative A_i does not satisfy the attribute C_j. These three membership components $\tilde{a}_{ij}$, $\tilde{b}_{ij}$ and $\tilde{c}_{ij}$ are expressed by the trapezoidal fuzzy numbers with the following properties:

1. a. $\tilde{a}_{ij} = \left(a_{ij}^1, a_{ij}^2, a_{ij}^3, a_{ij}^4\right) \in [0, 1]$;
   b. $\tilde{b}_{ij} = \left(b_{ij}^1, b_{ij}^2, b_{ij}^3, b_{ij}^4\right) \in [0, 1]$;
   c. $\tilde{c}_{ij} = \left(c_{ij}^1, c_{ij}^2, c_{ij}^3, c_{ij}^4\right) \in [0, 1]$;

2. $0 \le a_{ij}^4 + b_{ij}^4 + c_{ij}^4 \le 3$ for i = 1, 2, .., m and j = 1, 2, ..., n.

Step 3. Determination of the weights of attributes

The importance of attributes may not be always same to decision maker in decision-making situation. The information available of the attribute weights is often vague or incomplete in the decision making situation. Let $W = \left(w_1, w_2, ..., w_n\right)^T$ be the vaguely expressed weight vector assigned to the different attributes. In this case the weight of the attribute C_j for j = 1, 2, ..., n can be presented by the TrFNNs. Let us assume that $w_j = \left\langle \left(a_{1j}, a_{2j}, a_{3j}, a_{4j}\right), \left(b_{1j}, b_{2j}, b_{3j}, b_{4j}\right), \left(c_{1j}, c_{2j}, c_{3j}, c_{4j}\right) \right\rangle$ be the TrFNN based weight of attribute C_j. The expected value of w_j (j = 1, 2, ..., n) is determined by using the Eq.(30). These values are to be normalized by the following formula to make dimensionless

$$w_i^N = \frac{EV^T\left(w_i\right)}{\sum_{i=1}^{n} EV^T\left(w_i\right)} \text{ for i = 1, 2, ..., n.} \quad (37)$$

Step 4. Determination of the positive ideal neutrosophic solution (PINS) and the relative positive ideal neutrosophic solution (RPINS) for TrFNNs based neutrosophic decision matrix

Definition 24 Let H be the collection of two types of attributes namely benifit type attribute (P) and cost type attribute (L) in the MADM problems.

The positive ideal neutrosophic solution (PINS) $Q_{\tilde{N}}^+ = [q_{\tilde{N}_1}^+, q_{\tilde{N}_2}^+, ..., q_{\tilde{N}_n}^+]$ is the solution of the decision matrix $D_{\tilde{N}} = \langle T_{ij}, I_{ij}, F_{ij} \rangle_{m\times n}$ where, every component of $Q_{\tilde{N}}^+$ has the following form:

$$q_{\tilde{N}j}^+ = \left\langle \begin{matrix} \left(a_j^{1+}, a_j^{2+}, a_j^{3+}, a_j^{4+}\right), \left(b_j^{1+}, b_j^{2+}, b_j^{3+}, b_j^{4+}\right), \\ \left(c_j^{1+}, c_j^{2+}, c_j^{3+}, c_j^{4+}\right) \end{matrix} \right\rangle$$

$$= \left\langle \begin{matrix} \left(\max_i\{a_{ij}^1\}, \max_i\{a_{ij}^2\}, \max_i\{a_{ij}^3\}, \max_i\{a_{ij}^4\}\right), \\ \left(\max_i\{b_{ij}^1\}, \max_i\{b_{ij}^2\}, \max_i\{b_{ij}^3\}, \max_i\{b_{ij}^4\}\right), \\ \left(\max_i\{c_{ij}^1\}, \max_i\{c_{ij}^2\}, \max_i\{c_{ij}^3\}, \max_i\{c_{ij}^4\}\right) \end{matrix} \right\rangle \quad (38)$$

for the benefit type attribute and

$$q_{\tilde{N}j}^+ = \left\langle \begin{matrix} \left(\min_i\{a_{ij}^1\}, \min_i\{a_{ij}^2\}, \min_i\{a_{ij}^3\}, \min_i\{a_{ij}^4\}\right), \\ \left(\min_i\{b_{ij}^1\}, \min_i\{b_{ij}^2\}, \min_i\{b_{ij}^3\}, \min_i\{b_{ij}^4\}\right), \\ \left(\min_i\{c_{ij}^1\}, \min_i\{c_{ij}^2\}, \min_i\{c_{ij}^3\}, \min_i\{c_{ij}^4\}\right) \end{matrix} \right\rangle \quad (39)$$

for the cost type attribute.

Definition 25 The negative ideal neutrosophic solution (PINS) $Q_{\tilde{N}}^- = [q_{\tilde{N}_1}^-, q_{\tilde{N}_2}^-, ..., q_{\tilde{N}_n}^-]$ is the solution of the decision matrix $\mathrm{D}_{\tilde{N}} = \langle \mathrm{T_{ij}}, \mathrm{I_{ij}}, \mathrm{F_{ij}} \rangle_{\mathrm{m\times n}}$ where, every component of $Q_{\tilde{N}}^-$ has the following form:

$$q_{\tilde{N}j}^- = \left\langle \begin{matrix} \left(a_j^{1-}, a_j^{2-}, a_j^{3-}, a_j^{4-}\right), \left(b_j^{1-}, b_j^{2-}, b_j^{3-}, b_j^{4-}\right), \\ \left(c_j^{1-}, c_j^{2-}, c_j^{3-}, c_j^{4-}\right) \end{matrix} \right\rangle$$

$$=\left\langle \begin{array}{l} \left(\min_i\{a_{ij}^1\},\min_i\{a_{ij}^2\},\min_i\{a_{ij}^3\},\min_i\{a_{ij}^4\}\right), \\ \left(\min_i\{b_{ij}^1\},\min_i\{b_{ij}^2\},\min_i\{b_{ij}^3\},\min_i\{b_{ij}^4\}\right), \\ \left(\min_i\{c_{ij}^1\},\min_i\{c_{ij}^2\},\min_i\{c_{ij}^3\},\min_i\{c_{ij}^4\}\right) \end{array} \right\rangle \quad (40)$$

for the benefit type attribute.

$$q_{\tilde{\mathcal{N}}j}^{-}=\left\langle \begin{array}{l} \left(\max_i\{a_{ij}^1\},\max_i\{a_{ij}^2\},\max_i\{a_{ij}^3\},\max_i\{a_{ij}^4\}\right), \\ \left(\max_i\{b_{ij}^1\},\max_i\{b_{ij}^2\},\max_i\{b_{ij}^3\},\max_i\{b_{ij}^4\}\right), \\ \left(\max_i\{c_{ij}^1\},\max_i\{c_{ij}^2\},\max_i\{c_{ij}^3\},\max_i\{c_{ij}^4\}\right) \end{array} \right\rangle \quad (41)$$

for the cost type attribute.

Step 5. Determination of the weighted cosine similarity measure between each alternative and the ideal alternative

Let w_j be the weight of the attribute C_j for j = 1, 2, ..., n. The weighted cosine similarity measure between the alternative A_i for i = 1, 2, ..., m and the positive ideal alternative $Q_{\tilde{\mathcal{N}}}^{+}$ is

$$Cos_{TrFNN}^{W+}\left(Q_{\tilde{\mathcal{N}}}^{+},A_i\right)=$$

$$\sum_{j=1}^{n} w_j \frac{\sum_{s=1}^4 a_j^{s+}a_{ij}^s+\sum_{s=1}^4 b_j^{s+}b_{ij}^s+\sum_{s=1}^4 c_j^{s+}c_{ij}^s}{\left\{\begin{array}{l}\left(\sqrt{\sum_{s=1}^4\left(a_j^{s+}\right)^2+\sum_{s=1}^4\left(b_j^{s+}\right)^2+\sum_{s=1}^4\left(c_j^{s+}\right)^2}\right)\times \\ \left(\sqrt{\sum_{s=1}^4\left(a_{ij}^{s}\right)^2+\sum_{s=1}^4\left(b_{ij}^{s}\right)^2+\sum_{s=1}^4\left(c_{ij}^{s}\right)^2}\right)\end{array}\right\}} \quad (42)$$

Step 6. Ranking the alternatives

The ranking order of all alternatives can be determined by using the weighted cosine similarity measure between the alternative and the positive ideal alternative defined in Eq. (42). The highest value of $Cos_{TrFNN}^{W+}\left(Q_{\tilde{\mathcal{N}}}^{+},A_i\right)$ reflects the most desired alternative for i = 1, 2, ..., n.

6. Illustrative Example

In this section, multi attribute decision making problem under a trapezoidal fuzzy neutrosophic environment is considered to demonstrate the applicability and the effectiveness of the proposed approach. Let us consider the decision-making problem in which a customer intends to buy a tablet from the set of primarily chosen five alternatives *A* = (A_1, A_2, A_3, A_4, A_5). The customer takes into account the following four attributes:

1. features (C_1);
2. hardware(C_2);
3. affordable price (C_3);
4. customer care (C_4).

Assume that the weight vector of the four attributes provided by the decision maker is expressed by the trapezoidal fuzzy neutrosophic numbers

$$W=[w_1,w_2,w_3,w_4]=$$

$$\left\{\begin{array}{l} \left\langle(0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.2,0.3,0.4)\right\rangle, \\ \left\langle(0.2,0.3,0.4,0.5),(0.1,0.2,0.2,0.2),(0.2,0.2,0.3,0.4)\right\rangle, \\ \left\langle(0.6,0.7,0.8,0.9),(0.0,0.1,0.2,0.3),(0.1,0.1,0.2,0.3)\right\rangle, \\ \left\langle(0.4,0.5,0.6,0.7),(0.0,0.1,0.2,0.3),(0.1,0.2,0.3,0.4)\right\rangle \end{array}\right\} \quad (43)$$

Given that the following trapezoidal fuzzy neutrosophic number based decision matrix according to the experts' assessment of the five alternatives with respect to the four attributes:

Table3. Decision matrix with SVNS

$$D_{\tilde{\mathcal{N}}}=\left\langle \tilde{a}_{ij},\tilde{b}_{ij},\tilde{c}_{ij}\right\rangle_{5\times 4}=$$

$$\left[\begin{array}{l} \left\langle(0.5,0.6,0.7,0.8),(0.1,0.1,0.2,0.3),(0.1,0.2,0.2,0.3)\right\rangle \\ \left\langle(0.3,0.4,0.5,0.5),(0.1,0.2,0.2,0.4),(0.1,0.1,0.2,0.3)\right\rangle \\ \left\langle(0.3,0.3,0.3,0.3),(0.2,0.3,0.4,0.4),(0.6,0.7,0.8,0.9)\right\rangle \\ \left\langle(0.7,0.8,0.8,0.9),(0.1,0.2,0.3,0.3),(0.2,0.2,0.2,0.2)\right\rangle \\ \left\langle(0.1,0.2,0.2,0.3),(0.2,0.2,0.3,0.4),(0.6,0.6,0.7,0.8)\right\rangle \end{array}\right.$$

$$\begin{array}{l} \left\langle(0.1,0.1,0.2,0.3),(0.2,0.2,0.3,0.4),(0.4,0.5,0.6,0.7)\right\rangle \\ \left\langle(0.2,0.3,0.4,0.5),(0.1,0.1,0.2,0.3),(0.2,0.2,0.3,0.3)\right\rangle \\ \left\langle(0.1,0.2,0.2,0.3),(0.2,0.3,0.3,0.4),(0.4,0.5,0.6,0.6)\right\rangle \\ \left\langle(0.5,0.6,0.7,0.7),(0.2,0.2,0.2,0.2),(0.1,0.1,0.2,0.2)\right\rangle \\ \left\langle(0.5,0.6,0.6,0.7),(0.1,0.2,0.3,0.4),(0.2,0.2,0.3,0.4)\right\rangle \end{array}$$

$$\begin{bmatrix} \langle(0.3,0.4,0.4,0.5),(0.1,0.2,0.2,0.3),(0.2,0.2,0.3,0.4)\rangle \\ \langle(0.2,0.2,0.2,0.2),(0.1,0.1,0.1,0.1),(0.6,0.7,0.8,0.8)\rangle \\ \langle(0.2,0.3,0.4,0.5),(0.2,0.3,0.3,0.4),(0.3,0.4,0.4,0.5)\rangle \\ \langle(0.3,0.4,0.4,0.5),(0.1,0.2,0.2,0.3),(0.1,0.2,0.3,0.4)\rangle \\ \langle(0.6,0.7,0.8,0.8),(0.2,0.2,0.3,0.3),(0.1,0.1,0.2,0.3)\rangle \end{bmatrix}$$

$$\begin{bmatrix} \langle(0.4,0.5,0.6,0.7),(0.2,0.2,0.3,0.4),(0.1,0.2,0.3,0.4)\rangle \\ \langle(0.4,0.5,0.6,0.6),(0.2,0.2,0.3,0.3),(0.2,0.3,0.4,0.4)\rangle \\ \langle(0.2,0.2,0.3,0.4),(0.3,0.3,0.3,0.3),(0.3,0.4,0.5,0.6)\rangle \\ \langle(0.1,0.2,0.3,0.4),(0.2,0.2,0.3,0.3),(0.5,0.6,0.7,0.8)\rangle \\ \langle(0.2,0.3,0.4,0.4),(0.1,0.2,0.3,0.4),(0.3,0.4,0.4,0.5)\rangle \end{bmatrix} \quad (44)$$

Step 1. Determination of the weight of attributes

The truth favorite relative expected values (TFREVs) of the assessment of four attributes expressed with TrFNNs can be determined by the Eq. (30) as follows: $EV^T(w_1) = 1.737$, $EV^T(w_2) = 1.31$, $EV^T(w_3) = 2.093$ and $EV^T(w_4) = 1.737$. The normalized expected value of the assessment of four attributes is obtained by using the Eq. (37) as $EV^{TN}(w_1) = 0.2525$; $EV^{TN}(w_2) = 0.1907$; $EV^{TN}(w_3) = 0.3042$ and $EV^{TN}(w_4) = 0.2525$.

Step 2. Determination of the relative positive ideal neutrosophic solution (PINS) for the TrFNNs based neutrosophic decision matrix

The positive ideal solution of the decision matrix $D_{\tilde{N}} = \langle \tilde{a}_{ij}, \tilde{b}_{ij}, \tilde{c}_{ij} \rangle_{5\times 4}$ is $Q^+_{\tilde{N}} = [q^+_{\tilde{N}_1}, q^+_{\tilde{N}_2}, q^+_{\tilde{N}_3}, q^+_{\tilde{N}_4}]$ where,

$$q^+_{\tilde{N}_1} = \left\langle \begin{matrix} (0.7,0.8,0.8,0.9),(0.2,0.3,0.4,0.4), \\ (0.6,0.7,0.8,0.9) \end{matrix} \right\rangle \quad (45)$$

$$q^+_{\tilde{N}_2} = \left\langle \begin{matrix} (0.5,0.6,0.7,0.7),(0.2,0.2,0.3,0.4), \\ (0.4,0.5,0.6,0.7) \end{matrix} \right\rangle \quad (46)$$

$$q^+_{\tilde{N}_3} = \left\langle \begin{matrix} (0.6,0.7,0.8,0.8),(0.2,0.3,0.3,0.4), \\ (0.6,0.7,0.8,0.8) \end{matrix} \right\rangle \quad (47)$$

$$q^+_{\tilde{N}_4} = \left\langle \begin{matrix} (0.4,0.5,0.6,0.7),(0.3,0.3,0.3,0.4), \\ (0.5,0.6,0.7,0.8) \end{matrix} \right\rangle \quad (48)$$

Step 3. Calculation of the weighted cosine similarity measure between each alternative and the ideal alternative

The weighted cosine similarity measures between positive ideal alternative and each alternative are determined by using the Eq. (42) and the results are shown in the table 4.

Table 4. Decision results of weighted cosine similarity measures

Alternative (A_i)	Weighted cosine similarity measure
Alternative (A_1)	0.910296
Alternative (A_2)	0.918177
Alternative (A_3)	**0.928833**
Alternative (A_4)	0.915722
Alternative (A_5)	0.904869
Ranking Order	$A_3 \succ A_2 \succ A_4 \succ A_1 \succ A_5$

Step 4. Ranking of the alternatives

According to the values of weighted cosine similarity measure Table 4 shows that A_3 is the best alternative.

6 Conclusion

In this paper, we have presented cosine similarity measure based multiple attribute decision-making with trapezoidal fuzzy neutrosophic numbers. Expected value theorem and cosine similarity measure of trapezoidal fuzzy neutrosophic numbers are developed. The assessments of alternatives and attribute weights provided by the decision maker are considered with the trapezoidal fuzzy neutrosophic numbers. Ranking order of all alternatives is determined using the proposed cosine similarity measure between positive ideal alternative and each of alternatives. Finally, an illustrative example is provided to show the feasibility of the proposed approach and to demonstrate its practicality and effectiveness. However, the authors hope that the proposed approach will be applicable in medical diagnosis, pattern recognition, and other neutrosophic decision making problems.

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