# Cosmological Krylov Complexity

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In this paper, we study the Krylov complexity (K) from the planar/inflationary patch of the de Sitter space using the two mode squeezed state formalism in the presence of an effective field having sound speed  $c_s$ . From our analysis, we obtain the explicit behavior of Krylov complexity (K) and lancoz coefficients  $(b_n)$  with respect to the conformal time scale and scale factor in the presence of effective sound speed  $c_s$ . Since lancoz coefficients  $(b_n)$  grow linearly with integer n, this suggests that universe acts like a chaotic system during this period. We also obtain the corresponding Lyapunov exponent  $\lambda$  in presence of effective sound speed  $c_s$ . We show that the Krylov complexity (K) for this system is equal to average particle numbers suggesting it's relation to the volume. Finally, we give a comparison of Krylov complexity (K) with entanglement entropy (Von-Neumann) where we found that there is a large difference between Krylov complexity (K) and entanglement entropy for large values of squeezing amplitude. This suggests that Krylov complexity (K) can be a significant probe for studying the dynamics of the cosmological system even after the saturation of entanglement entropy.

#### i. Introduction

Recent years have seen a fair amount of applications coming from quantum information theory into high energy physics and cosmology. [1–6]. One such concept is complexity and chaos [7–10]. Complexity characterizes the notion of difficulty of preparing a state or applying a certain unitary operator while chaos quantifies the sensitivity of the system to the initial condition. While in classical mechanics chaos is a very well-defined quantity, it is not so in quantum mechanics. So, one resorts to various kinds of probes and measures. One recent tool that has been proposed to study operator growth and characterize quantum chaos is Krylov/K complexity [11–19].

In this framework, one tries to understand the Heisenberg evolution of some initial Hermitian operators. Depending on the Hamiltonian and initial operator, the evolution can become extremely complicated. Krylov complexity can capture this growth of the operator. While obtaining Krylov complexity, one also has to obtain the so-called Krylov basis using the Lancoz algorithm. Lancoz algorithm also gives us Lancoz coefficients which are conjectured to be maximum for chaotic systems [12]. In recent years, Krylov complexity has been studied extensively from black hole physics to conformal field theories.

The main motivation to study complexity in high energy physics comes from holographic conjectures of complexity. In particular, Susskind et al [8, 9] conjectured that complexity can be used to probe the physics behind the black hole horizons via "complexity = volume" and "complexity = action" proposals. Following these works, circuit complexity has been computed using different techniques and even computed in the context of quantum field theory and cosmology [1, 20–22]. How-

ever, one issue with these calculations is that the notion of complexity is very ambiguous depending on the choices of gates, reference and target gates, and arbitrary tolerance. Fortunately, Krylov's complexity is free of such choices, therefore, making it an ideal candidate to study in holographic and QFT settings.

Our goal in this paper is to study Krylov complexity and chaos in de Sitter Cosmology with effective sound speed, and gain quantum information theoretic insights about cosmological evolution and structure formation. The reason to include the sound speed is to make our calculations as general as possible. While holography is mostly studied on the Anti de Sitter background, we seem to live in a de Sitter one. This also gives us a strong motivation to see if those holographic conjectures holds for de Sitter case too [2, 23, 24]. Particularly, we study the Krylov complexity and chaos on the scalar cosmological perturbations on an expanding Friedmann-Lemaitre-Robertson-Walker (FLRW) background. Scalar perturbations on an expanding background can naturally be described by the two mode squeezing operator. Modes inside the horizon are frozen while mode exiting the horizon are highly squeezed. After obtaining expressions for K-complexity and chaos for general perturbations, we apply to a simple model of de Sitter expansion where we obtain explicit expressions.

Squeezed states and squeezing operator is an extremely important subject in quantum optics with applications from quantum computing, quantum cryptography and even in gravitational physics [25–45]. For review on the fundamentals of squeezed states, we refer to vast literatures on [46–56]. From the cosmological point, the concept of squeezed states was introduced by Grishchuk and Sidorov [57, 58] on the inflationary cosmology where

they analysed the features of relic gravitons and phenomena such as particle creation and black-hole evaporation. Andreas Albrecht et al. [59] also used the two-mode squeezed state formalism to understand the inflationary cosmology and the amplification process of quantum fluctuations during the inflation period. For applications of squeezed state formalism in High energy physics and in cosmology see [21, 22, 60–80].

The structure of the paper is as follows:

- In Section II, we give a review of Krylov complexity. We review how Krylov complexity and Lancoz coefficients can be computed using Lancoz algorithms. Furthermore, we will see that Lancoz coefficients can characterize the chaotic properties of the system. For systems with symmetry, the computations of Lancoz coefficients and Krylov complexity can be simplified significantly.
- In Section III, we give a review of cosmological perturbations and obtain the quadratic Hamiltonian. In order to make the calculations as general as possible, we also include the effective sound speed  $c_s$ . With this quadratic hamiltonian, we obtain two-mode squeezed state formalism in section III A. We obtain a set of differential equations for squeezing parameters  $r_k$  and  $\phi_k$ . In Section III B, we obtain the expression for Krylov complexity and Chaos for two-mode squeezing operator. We obtain the Krylov complexity to be  $\sinh^2 r_k$  and lancoz coefficients  $\alpha n$ . Lancoz coefficients grow linearly with n showing that the system is chaotic in nature.
- In Section IV, we apply these calculations to the de Sitter background for different effective sound speed.
- In Section V, we give the conclusion of our work and give up future prospects.

### II. Review of Krylov complexity

In this section, we will give a brief overview of operator growth and Krylov complexity. There are different notions of complexity in literature. One approach getting popular in the high energy physics section is Nielsen's geometric approach of complexity [81–84]. The interest in Krylov complexity has its origins in the certain shortcomings of Nielsen's measure. Particularly, Nielsen's complexity measure is dependent on the choice of gates, choice of reference, and target states and tolerance. This makes it very difficult to define it properly in the context of QFT or holography. In contrast, Krylov complexity is well defined and is independent of these choices. These features make it well suited for application to QFTs and holography. Furthermore, Krylov complexity and Lancoz coefficients obtained from Krylov complexity can be

used to characterize the chaotic systems. For a detailed overview of Krylov complexity, we would like to refer to [11].

Consider a quantum Hamiltonian H and timedependent Heisenberg operator  $\mathcal{O}(t)$ . The time evolved operator is described by the Heisenberg equation:

$$\partial_t \mathcal{O}(t) = i[H, \mathcal{O}(t)] \tag{1}$$

where, [A, B] = AB - BA is the commutator. Denoting  $\mathcal{O}(0) = \mathcal{O}$ , the formal solution of Heisenberg equation is given by

$$\mathcal{O}(t) = e^{iHt} \mathcal{O}e^{-iHt} \tag{2}$$

Using the Baker-Campbell-Hausdorff (BCH) formula

$$e^{X}Ye^{-X} = \sum_{n=0}^{\infty} \frac{\mathcal{L}_{X}^{n}Y}{n!}$$
 (3)

where  $\mathcal{L}_X$  is the Liouvillian super-operator defined as  $\mathcal{L}_X Y = [X, Y]$ , we can obtain the time evolution series for  $\mathcal{O}(t)$  as:

$$\mathcal{O}(t) = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \mathcal{L}_H^n \mathcal{O}$$

$$= \mathcal{O} + it[H, \mathcal{O}] + \frac{(it)^2}{2!} [H, [H, \mathcal{O}]]$$

$$+ \frac{(it)^3}{3!} [H, [H, [H, \mathcal{O}]]] + \dots$$
(4)

With the time evolution the spreading of initial operators occurs and this means more complicated nested commutators need to be accounted. This gives a notion of complexity of the Heisenberg operator as a function of time. Krylov complexity quantifies this growth in a precise manner. In general, if the Hamiltonian is chaotic, the nested commutators for the operator  $\mathcal O$  will be given by increasingly complex operators.

In following, we will drop the subscript on Liouvillian super-operator  $\mathcal{L}_H$  as we will be only focusing on hamiltonian H and represent the repeated action of  $\mathcal{L}$  as  $\tilde{\mathcal{O}}_n = \mathcal{L}^n \mathcal{O}$ . Then, the time evolution series for  $\mathcal{O}(t)$  can be written as:

$$\mathcal{O}(t) = e^{i\mathcal{L}t}\mathcal{O} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \mathcal{L}^n \mathcal{O} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \tilde{\mathcal{O}}_n$$
 (5)

We can now interpret (5) as the Schrodinger's time evolution where  $\mathcal{O}(t)$  plays the role of "operator's wave functions", and the Liouvillian  $\mathcal{L}$  as the Hamiltonian. Then we associate  $|\mathcal{O}\rangle$  with the Hilbert space vectors corresponding to the operator  $\mathcal{O}$  as:

$$\mathcal{O} \equiv |\tilde{\mathcal{O}}\rangle, \mathcal{L}^1 \mathcal{O} \equiv |\tilde{\mathcal{O}}_1\rangle, \mathcal{L}^2 \mathcal{O} \equiv |\tilde{\mathcal{O}}_2\rangle, \mathcal{L}^3 \mathcal{O} \equiv |\tilde{\mathcal{O}}_3\rangle, \dots$$
 (6)

It is not necessary that these operators form an orthonormal basis a prior. However, starting from these basis

 $|\tilde{\mathcal{O}}_n\rangle$ , we can use a version of Gram–Schmidt orthogonalization procedure called Lanczos algorithm to construct orthonormal basis, known as Krylov basis  $|\mathcal{O}_n\rangle$ . For this, we need a choice of inner product of these operators. One natural choice is Wightman norm:

$$(A|B) = \langle e^{H\beta/2} A^{\dagger} e^{-H\beta/2} B \rangle_{\beta} \tag{7}$$

where  $\langle \dots \rangle_{\beta} = \text{Tr}\{e^{-\beta H} \dots\}/\text{Tr}\{e^{-\beta H}\}$  is the thermal expectation value at temperature  $1/\beta$ .

In order to obtain the Krylov basis using the Lancoz algorithm, we can use the fact that first two operators in  $|\tilde{\mathcal{O}}_n\rangle$  are orthogonal with respect to (7). So, we can include them in Krylov basis as:

$$|\mathcal{O}_0| := |\tilde{\mathcal{O}}_0| = |\mathcal{O}|, |\mathcal{O}_1| := b_1^{-1} \mathcal{L}|\tilde{\mathcal{O}}_0|$$
 (8)

where  $b_1 = \sqrt{(\tilde{\mathcal{O}}_0 \mathcal{L} | \mathcal{L} \tilde{\mathcal{O}}_0)}$  normalized the vector. Then, we can construct the next states iteratively as:

$$|A_n\rangle = \mathcal{L}|\mathcal{O}_{n-1}\rangle - b_{n-1}|\mathcal{O}_{n-2}\rangle \tag{9}$$

followed by normalization:

$$|\mathcal{O}_n| = b_n^{-1}|A_n|, b_n = \sqrt{(A_n|A_n)}$$
 (10)

We need to run this algorithm until  $b_n$  hits zero, then in addition to a full orthonomal basis called Krylov basis as well coefficients  $b_n$  which are called Lancoz coefficients. These lancoz coefficients are extremely useful and characterize the chaos of the system.

Once we obtain the Krylov basis, we can represent the time evolved operator  $\mathcal{O}(t)$  as:

$$|\mathcal{O}(t)\rangle = e^{i\mathcal{L}t}|\mathcal{O}\rangle = \sum_{n} i^n \phi_n(t)|\mathcal{O}_n\rangle$$
 (11)

where the amplitudes  $\phi_n(t)$  are real, and  $|\phi_n|^2$  can be thought up of probabilities which sums up to one.

$$\sum_{n} |\phi_n|^2 = \sum_{n} p_n = 1 \tag{12}$$

In order to obtain these amplitudes, we can think of (11) as a form of Schrodinger equation. Then, we take the partial derivative on (11) to obtain

$$\partial_t | \mathcal{O}(t)) = i \mathcal{L} | \mathcal{O}(t)) = \sum_n i^n \partial_t \phi_n(t) | \mathcal{O}_n)$$
 (13)

Applying the action of Liouvillian on Krylov basis

$$\mathcal{L}|\mathcal{O}_n\rangle = b_n|\mathcal{O}_{n-1}\rangle + b_{n+1}|\mathcal{O}_{n+1}\rangle \tag{14}$$

on (13), we obtain

$$\partial_t | \mathcal{O}(t)) = \sum_n i^n (b_n \phi_{n-1} - b_{n+1} \phi_{n+1}) | \mathcal{O}_n)$$
 (15)

Now, matching the coefficients appropriately on (13), we obtain the time evolution of amplitude as

$$\partial_t \phi_n(t) = b_n \phi_{n-1} - b_{n+1} \phi_{n+1} \tag{16}$$

This equation can be solved with the knowledge of Lanczos coefficients  $b_n$  and with initial condition  $\phi_n(0) = \delta_{n0}$ . Once we have obtained the expression for amplitudes  $\phi_n$ , we can then finally give the expression for Krylov complexity.

Krylov complexity/K-complexity is given by:

$$K = \sum_{n} n|\phi_n|^2 \tag{17}$$

One crucial advantage of the Lancoz algorithm we discussed above is that it also has a potential to capute the chaotic properties of the system. It was conjecture that Lancoz coefficients in a quantum system is bounded linearly as

$$b_n \le \alpha n + \gamma \tag{18}$$

where  $\alpha$  is the operator growth rate and  $\gamma$  is the constant depending on the operator. These two parameters are usally obtained from the Hamiltonian of the system. For extremely chaotic system, Krylov complexity grows exponentially fast with an exponent  $\gamma$  given by:

$$\gamma = 2\alpha \tag{19}$$

In literature, this exponent has also been associated with Lyapunov exponent. For example, at finite temperature  $T=1/\beta$ , one obtains  $\alpha=\pi/\beta$ . In [10], it was conjectured that this bound the lyapunov exponent, i.e. maximal chaotic system.

In a certain class of systems which enjoys symmetry, Krylov complexity can be computed analytically using the techniques developed in [11]. For these systems with symmetry group, action of Liouvillian  $\mathcal L$  on the Krylov basis can be seen as an action of raising and lowering operators

$$\mathcal{L} = \alpha (L_+ + L_-) \tag{20}$$

The parameter  $\alpha$  is dependent on the system we are considering, and influence the chaotic properties of the system. Furthermore, we can read off the Lancoz coefficients immediately with this approach from the action of ladder operators on Krylov basis.

$$\alpha L_+|\mathcal{O}_n\rangle = b_{n+1}|\mathcal{O}_{n+1}\rangle, \alpha L_-|\mathcal{O}_n\rangle = b_n|\mathcal{O}_{n-1}\rangle$$
 (21)

There are several examples of symmetries explored in [11] such as SL(2,R), SU(2). For us, SL(2,R) will be the most relevant.

#### ш. Krylov complexity of Cosmological perturbations

Now that we have explored the concept of Krylov complexity and lancoz coefficients, we will now apply it to the scalar cosmological perturbations. The operator of interest for our case is two mode squeezing operator. For a detailed review of cosmological quantum perturbations and quantum fields in curved space time, we refer to [85, 86].

We consider a spatially flat Friedmann-Lemaitre-Robertson-Walker (FLRW) metric:

$$ds^{2} = -dt^{2} + a(t)^{2}d\vec{x}^{2} = a(\tau)^{2}(-d\tau^{2} + d\vec{x}^{2})$$
 (22)

We will consider linear scalar field fluctuation  $\varphi(x) = \varphi_0(t) + \delta\varphi(x)$  on this background metric to obtain the metric

$$ds^{2} = a(\tau)^{2} \left( -(1+2\psi(x,\tau))d\tau^{2} + (1-2\psi(x,\tau))d\vec{x}^{2} \right)$$
 (23)

We will define a curvature perturbation term  $R = \psi + \frac{H}{\dot{\varphi}_0} \delta \varphi$ . Here, dot indicates derivative with respect to cosmic time t, and  $H = \frac{\dot{a}}{a}$ . If we insert these conditions into the total action and expand to second order, the action becomes [85]:

$$S = \frac{1}{2} \int dt d^3x a^3 \frac{\varphi_0^2}{c_s^2 H^2} \left[ \dot{R}^2 - c_s^2 \frac{1}{a^2} (\partial_i R)^2 \right]$$
 (24)

where  $c_s = \sqrt{\dot{p}/\dot{\rho}}$  the effective sound speed of the effective fluid. Here, p and  $\rho$  corresponds to the effective pressure and density of the effective fluid. For more details on effective sound speed, we refer to the literature on [59]. The effective sound speed is bounded by one to maintain the causality. Cosmological observations restrict the lower bound at  $c_s = 0.024$  [79]. So, we obtain the bound to be  $0.024 \le c_s \le 1$ . Physically,  $c_s = 1$  describes a single scalar field slow roll model while  $c_s < 1$  describes a wide class of non-canonical scalar field theories.

If we instead expand up to the third order, we will get non-gaussian terms [87] too, which are also very interesting to study. For our purpose, we will restrict to second order i.e. only up to gaussian states.

We will define Mukhanov variable  $\nu \equiv zR$ , where  $z \equiv \frac{a\sqrt{2}\epsilon}{c_s}$ , with  $\epsilon = -\frac{\dot{H}}{H^2} = 1 - \frac{H'}{H^2}$ . Here, the prime ' indicates derivative with respect to conformal time. With this, the action (24) becomes:

$$S = \frac{1}{2} \int d\tau d^3x \left[ \nu'^2 - \frac{c_s^2}{a^2} (\partial_i \nu)^2 + \frac{z''}{z} \nu^2 \right]$$
 (25)

Each mode will evolves independently. These modes satisfy the harmonic oscillator equation with time dependent effective mass from time dependence of the background. We can then quantize this harmonic oscillator according to the standard quantization technique of the harmonic oscillator. So, we will promote these perturbations to quantum fields and expand them to fourier series.

$$\hat{\nu}(\tau, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} \hat{\nu}_k(\tau) \exp\left(i\vec{k}.\vec{x}\right)$$
 (26)

We then define the usual creation and annihilation operators:

$$\hat{v}_k = \frac{1}{\sqrt{2k}} (\hat{a}_k + \hat{a}_{-k}) , \hat{v}'_k = -i\frac{k}{2} (\hat{a}_k + \hat{a}_{-k})$$
 (27)

With this, the quadratic hamiltonian becomes:

$$\hat{H} = \frac{1}{2} \int d^3k \hat{\mathcal{H}}_k$$

$$= \frac{1}{2} \int d^3k \left[ \Omega_k (\hat{a}_k \hat{a}_k^{\dagger} + \hat{a}_{-k}^{\dagger} \hat{a}_{-k}) - i\beta_k (\hat{a}_k \hat{a}_{-k} - \hat{a}_k^{\dagger} \hat{a}_{-k}^{\dagger}) \right]$$
(28)

where,

$$\Omega_k = \frac{k}{2} (1 + c_s^2)$$

$$\beta_k = \sqrt{\left(\frac{k}{2} (1 - c_s^2)\right)^2 + \left(\frac{z'}{z}\right)^2}$$
(29)

#### A. Squeezed states formalism

The first term in the hamiltonian (28) represents free particle hamiltonian. The second term shows the interaction between the quantum perturbation and the expanding background. Given this quadratic hamiltonian  $\hat{\mathcal{H}}_k$ , the unitary evolution  $\mathcal{U}_k$  can be factorized into product of two mode rotation operator  $\hat{R}_k(\beta_k)$  and two mode squeezing operator  $\hat{S}_k(r_k, \phi_k)$  [59]:

$$\mathcal{U}_k = \hat{S}_k(r_k, \phi_k) \hat{R}_k(\beta_k) \tag{30}$$

The two mode rotation operator  $\hat{R}_k(\beta_k)$  in terms of rotational parameter is given by:

$$\hat{R}_k(\beta_k) = \exp\left[-i\beta_k(\tau)(\hat{a}_k\hat{a}_k^{\dagger} + \hat{a}_{-k}^{\dagger}\hat{a}_{-k})\right]$$
(31)

while two-mode squeeze operator  $\hat{S}_k(r_k, \phi_k)$  in terms of squeezing parameter  $r_k(\tau)$  and squeezing angle  $\phi_k$  is given by:

$$\hat{S}_k(r_k, \phi_k) = \exp\left[\frac{r_k(\tau)}{2} \left(e^{-2i\phi_k(\tau)} \hat{a}_k \hat{a}_{-k} - e^{2i\phi_k(\tau)} \hat{a}_{-k}^{\dagger} \hat{a}_k \dagger\right)\right]$$
(32)

Since the rotation operator only changes the phase, we will ignore the rotation operator hereon as it doesn't have much consequences. When the two-mode squeezing operator acts on the vacuum, it gives squeezed vacuum states

$$|SQ(k,\tau)\rangle = \hat{S}_k(r_k,\phi_k) |0_k,0_{-k}\rangle$$

$$= \frac{1}{\cosh r_k} \sum_{n=0}^{\infty} e^{-2in\phi_k} \tanh^n r_k |n_k,n_{-k}\rangle$$
(33)

where,

$$|n_k, n_{-k}\rangle = \left[\frac{1}{n!} (a_k^{\dagger} a_{-k}^{\dagger})^n\right] |0_k, 0_{-k}\rangle$$
 (34)

The two-mode squeezed vacuum is normalized:

$$\langle SQ(k,\tau)|SQ(k,\tau)\rangle$$

$$= \frac{1}{\cosh^2 r_k} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{-2i(n-m)\phi_k} \tanh^{m+n} r_k \delta_{m,n}$$

$$= \frac{1}{\cosh^2 r_k} \sum_{n=0}^{\infty} \tanh^{2n} r_k = 1$$
(35)

The full wave function corresponding to all modes can be obtained straightforwardly as a tensor product of each k:

$$|SQ(\tau)\rangle = \bigotimes_k |SQ(k,\tau)\rangle$$
 (36)

One can obtain the time evolution of the squeezing parameters  $r_k(\tau)$ ,  $\phi_k(\tau)$  via Schrödinger equation:

$$i\frac{d}{d\tau}|SQ(k,\tau)\rangle = \vec{\mathcal{H}}|SQ(k,\tau)\rangle$$
 (37)

This gives us a set of differential equations:

$$\frac{dr_k}{d\tau} = -\beta_k \cos(2(\tilde{\phi_k} - \phi_k)) \tag{38}$$

$$\frac{d\phi_k}{d\tau} = \Omega_k + \beta_k \coth(2r_k) \sin(2(\tilde{\phi_k} - \phi_k))$$
 (39)

where,

$$\tilde{\phi_k} = -\frac{\pi}{2} + \frac{1}{2} \tan^{-1} \left[ \frac{\Omega_k}{2} \left( \frac{z'}{z} \right) \left( \frac{1 - c_s^2}{1 + c_s^2} \right) \right] \tag{40}$$

For a stationary background where z is constant, no squeezing occurs at r=0. These set of differential equations (38) can be solved for particular scale factor  $a(\tau)$  and obtain the solution for squeezing parameters  $r_k(\tau), \phi_k(\tau)$ . However, it is not always the case that nice analytical expression can be obtained, and one will have to rely on numerical methods. For an exponentially expanding de Sitter background with  $c_s=1$ , such analytical expression exists and this makes it easier to study concepts like Krylov complexity and chaos. We will look at it in detail on section IV. For other sound speed, we will rely on numerical tools.

### B. Complexity and chaos

The hamiltonian of interest from (28) is

$$\hat{H}_k = \frac{-\beta_k}{2} (\hat{a}_k \hat{a}_{-k} - \hat{a}_k^{\dagger} \hat{a}_{-k}^{\dagger}) \tag{41}$$

Comparing with the SL(2,R) Liouvillian operator (20),  $\mathcal{L} = \alpha(L_+ + L_-)$ , we can associate

$$\alpha = \frac{-\beta_k}{2}, L_+ = \hat{a}_k^{\dagger} \hat{a}_{-k}^{\dagger}, L_- = \hat{a}_k \hat{a}_{-k}$$
 (42)

For the two mode squeezing operator, the Krylov basis is the standard two-oscillator Fock space

$$|\mathcal{O}_n| = |n_k, n_{-k}\rangle = \left[\frac{1}{n!} (a_k^{\dagger} a_{-k}^{\dagger})^n\right] |0_k, 0_{-k}\rangle$$
 (43)

The Lancoz coefficients can be computed by the action of ladder operators on Krylov basis as  $\alpha L_{-}|\mathcal{O}_{n}) = b_{n}|\mathcal{O}_{n-1})$ . Since  $\hat{a}_{k}|n_{k}\rangle = \sqrt{n}\,|(n-1)_{k}\rangle$  we obtain lancoz coefficients as

$$b_n = \alpha n \tag{44}$$

These lancoz coefficients grow linearly with n showing that this system is chaotic in nature. Exact chaotic nature depends also on the coefficient  $\alpha$  coming from the Hamiltonian (28).

Then in the Krylov basis, we can write the Heisenberg's operator state as

$$|\mathcal{O}(t)| = \sum_{n} i^{n} \phi_{n}(t) |\mathcal{O}_{n}|$$

$$= \frac{1}{\cosh r_{k}} \sum_{n=0}^{\infty} e^{-2in\phi_{k}} \tanh^{n} r_{k} |n_{k}, n_{-k}\rangle \quad (45)$$

So the operators wave function are given by

$$\phi_n = \frac{e^{-2in\phi_k} \tanh^n r_k}{\cosh r_k} \tag{46}$$

and they sum to 1 as:

$$\sum_{n=0}^{\infty} |\phi_n|^2 = \frac{1}{\cosh^2 r_k} \sum_{n=0}^{\infty} \tanh^{2n} r_k = 1$$
 (47)

Using the operator wave function, we can now compute the Krylov complexity

$$K = \sum_{n} |\phi_n|^2$$

$$= \sum_{n=0}^{\infty} n \frac{\tanh^{2n} r_k}{\cosh^2 r_k} = \sinh^2 r_k$$
(48)

where, we used the identity

$$\sum_{m=0}^{\infty} mz^m = z/(1-z)^2 \tag{49}$$

for  $|z| \leq 0$ . Indeed, Krylov complexity for our model saturates the bound of maximum complexity growth proposed in [88]. The reason for this is two mode squeezed

states satisfies the SL(2,R) symmetry group structure, and it is argued in [88] that this group structure belonging to generalized coherent states has maximal complexity growth.

Finally for the two mode squeezed formalism of cosmological perturbations, we can give an explicit expression of Lancoz coefficients  $b_n$  and Krylov complexity K as:

$$b_n = \left| \frac{-\beta_k}{2} \right| n, K = \sinh^2 r_k \tag{50}$$

The Krylov complexity depending on the squeezing parameters  $r_k$  can now be obtained by solving the differential equations (38). For low amount of squeezing i.e.  $r \ll 1$ , we see that the Krylov complexity  $K \approx 0$ . This makes sense as for low amount of squeezing, the evolved operator would be similar to the initial operator thus demonstrating an operational reasoning to the concept of Krylov complexity.

For this type of Lanczos coefficients growing linearly with n in (50), the Krylov complexity grows exponentially fast in  $r_k$  with an exponent, which can also be interpreted as a Lyapunov exponent,

$$\lambda = 2\alpha = |\beta_k| = \sqrt{\left(\frac{k}{2}(1 - c_s^2)\right)^2 + \left(\frac{z'}{z}\right)^2},$$
 (51)

The lyapunov exponent has a interesting structure. For  $c_s = 1$ , the lyapunov exponent is just  $\lambda = \left| \frac{z'}{z} \right|$ , and is independent of mode vectors k. However for different effective sound speed  $c_s$  than 1, the lyapunov exponent is dependent on the mode vectors k too.

The linear growth of Lancoz coefficients in (50) indicates that this system is chaotic in nature. The origin of chaos comes from the fact that scalar cosmological perturbation behaves like an inverted harmonic oscillator at large scales. Since inverted harmonic oscillator are chaotic in nature, this feature is reflected on the expression for Lancoz coefficients and Krylov complexity in (50).

Interestingly, the expression for Krylov complexity obtained in (50) is equal to the average particle number in each mode:

$$\langle \hat{n}_k \rangle = \langle \hat{n}_{-k} \rangle = \sinh^2 r_k = K$$
 (52)

Since volume of a system, V, is proportional to number of particles n, we can see that the Krylov complexity is also proportional to volume. This matches to the complexity equals volume conjecture in the context of AdS/CFT.

We can also easily obtain the expression for K-entropy

 $S_K$  defined in [15] as

$$S_K = -\sum_{n=0}^{\infty} |\phi_n|^2 \ln|\phi_n|^2$$

$$= -\sum_{n=0}^{\infty} \frac{\tanh^{2n} r_k}{\cosh^2 r_k} \ln \frac{\tanh^{2n} r_k}{\cosh^2 r_k}$$

$$= -\sum_{n=0}^{\infty} \frac{\tanh^{2n} r_k}{\cosh^2 r_k} \left( \ln(\tanh^{2n} r_k) - \ln(\cosh^2 r_k) \right)$$

$$= \ln(\cosh^2 r_k) \cosh^2 r_k - \ln(\sinh^2 r_k) \sinh^2 r_k$$
(53)

# iv. Application to de Sitter Cosmology

Now that we have studied the Krylov complexity and chaos for the general cosmolgical perturbations, we will apply it in the context of exponentially expanding de Sitter background. There are several motivations in choosing the de Sitter background in particular. Usually obtaining analytical solutions for (38) is a difficult task and have to rely on numerical techniques, however for de Sitter background, there exists an exact expression for squeezing parameters  $r_k$  and  $\phi_k$  which makes it easier to study chaos and complexity. While this is one reason for this choice, the more important reason is that original motivation to study complexity and chaos comes from various conjectures in AdS/CFT. Since the universe we live in is de Sitter in naure rather than Anti- de Sitter, checking these conjectures for the de Sitter background is also an equally important task. This has motivated us to study complexity and chaos in this space.

For an exponentially expanding de Sitter background, the scale factor  $a(\tau)$  is given by

$$a(\tau) = \frac{-1}{H\tau} \tag{54}$$

where  $-\infty < \tau < 0$  so that  $z'/z = -1/\tau$ . For effective sound speed  $c_s = 1$ , we can obtain the exact solution to squeezing parameters to the differential equations (38). However, for other effective sound speeds, we will rely on the numerical methods.

## A. Effective sound speed: $c_s = 1$

For  $c_s = 1$ , the exact solutions to squeezing parameters to the differential equations (38) are

$$r_k(\tau) = -\sinh^{-1}\left(\frac{1}{2k\tau}\right)$$

$$\phi_k(\tau) = -\frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{1}{2k\tau}\right)$$
(55)

During early times  $k|\tau|\gg 1$ , so the modes are inside the horizon and squeezing parameters  $r_k$  is almost zero  $r_k\approx -1/(2k\tau)\ll 1$ . During this limit, squeezing angle is also constant at  $\phi_k\approx -\pi/4$ . At late times  $k|\tau|\ll 1$ , the modes are outside the horizon. During this limit, the system behaves like an inverted harmonic oscillator and the squeezing grows with time  $r_k\approx |ln(-k\tau)|\gg 1$ 

Given the expression for the squeezing parameters, we can now obtain an exact expressions of Krylov complexity K, Lancoz coefficients  $b_n$ , and Lyapunov exponents for de Sitter space with  $c_s = 1$  as

$$K = \sinh^2 r_k = \frac{1}{4k^2 \tau^2} \tag{56}$$

$$b_n = \frac{-z'}{2z}n = \frac{n}{2\tau} \tag{57}$$

$$\lambda = \frac{-z'}{z} = \frac{1}{\tau} \tag{58}$$

During early times  $k|\tau|\gg 1$ , Krylov complexity is almost zero,  $K\approx 0$ , as expected. Similarly, Lyapunov exponent also has a very low value  $\lambda\ll 1$  during this limit. For late times,  $k|\tau|\ll 1$  Krylov complexity grows exponentially  $K\gg 1$  and Lyapunov exponent is also much larger  $\lambda\gg 1$ . This is a strong feature of a chaotic system. It is interesting to see that Krylov complexity doesn't saturate in time, and rather keeps on increasing. This has to do with the fact that the expression for scale factor (54) is time dependent. So, the increase in chaotic nature of the de-Sitter background is primarily due to the time dependence of scale factor.

In figure 1, we have plotted the Krylov complexity for different wave numbers as a function of conformal time  $\tau$  for  $c_s = 1$ . Krylov complexity grows exponentially with  $\tau$  showing that the system is chaotic in nature. During early times, complexity is inversely proportional to the wave number while for late times the difference due to wave number is less. This exponential growth in complexity signifies chaos which is captured by lyapunov exponent in figure 2.

We can also make an comparision of the Krylov complexity with Nielsen's geometric complexity. In [21], the geometric complexity for cosmological perturbations was obtained to be

$$C = \left| \ln \left| \frac{1 + \exp(-2i\phi_k(\tau)) \tanh r_k(\tau)}{1 - \exp(-2i\phi_k(\tau)) \tanh r_k(\tau)} \right| + \left| \tanh^{-1}(\sin(2\phi_k(\tau)) \sinh(2r_k(\tau))) \right|$$
(59)

From hindsight, it looks like this measure of complexity captures the physics of the system in more detail than Krylov's complexity. In particular, geometric complexity is dependent on the squeezing angle too while Krylov's complexity is not. But one has to understand that Nielsen's geometric complexity has lots of ambiguities such as arbitrary choices of gates, reference and target states. This makes it very difficult to study in the

context of cosmological evolution. Nielsen's measure is a good approach while constructing optimal quantum circuit in lab, but a difficult choice for cosmology and holography.

In [10], quantum chaos was conjectured to be bounded from above by the temperature of the system. We can also relate the lyapunov exponent we have obtained with the temperature. In particular, temperature of the expanding universe T, is related to Hubble constant by

$$T \approx H/2\pi$$
. (60)

For de-Sitter background at time  $\tau_0$ 

$$|\tau_0| = 1/H_{dS} \tag{61}$$

Then, we obtain the Lyapunov exponent at  $\tau_0$  as

$$\lambda_{\tau_0} = 2\pi T \tag{62}$$

Therefore, chaos in de-Sitter space saturates the bound conjectured in [10]. Interestingly this bound on cosmological complexity also matches with previous results obtained via other complexity measures [21, 22].

Finally, we can also give an expression for K-entropy (53)

$$S_K = \ln\left(1 + \frac{1}{4k^2\tau^2}\right) \left(1 + \frac{1}{4k^2\tau^2}\right) - \ln\left(\frac{1}{4k^2\tau^2}\right) \frac{1}{4k^2\tau^2}$$
(63)

During early times, the difference between complexity and K- entropy is less but for late times, the difference is huge. This shows that, complexity can grow even after system has achieved saturation.

# B. Effective sound speed: $0.024 \le c_s \le 1$

For other effective sound speed, we can give an exact expression Lancoz coefficients  $b_n$ , and Lyapunov exponents:

$$b_n = \frac{n}{2} \left( \sqrt{\left(\frac{k}{2} (1 - c_s^2)\right)^2 + \left(\frac{1}{\tau}\right)^2} \right)$$
 (64)

$$\lambda = \sqrt{\left(\frac{k}{2}(1 - c_s^2)\right)^2 + \left(\frac{1}{\tau}\right)^2} \tag{65}$$

Interestingly unlike the case for  $c_s=1$ , the lyapunov exponent is also dependent on the mode vectors k. Since Lancoz coefficients grows linearly with n, the system is chaotic too. In 2, we have plotted Lyapunov exponent for values of k=1,0.1,0.01 and  $c_s=1.0,0.1,0.024$ . The lyapunov exponent is bounded from below by  $c_s=1$ . For other  $c_s$ , we can see that lyapunov exponent is strongly dependent on k. For example, lyapunov exponent for

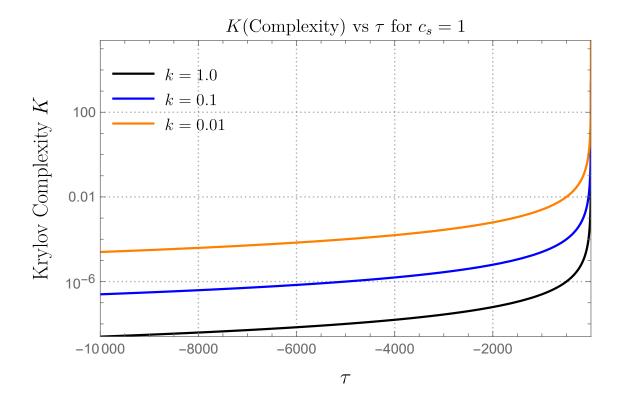


FIG. 1. Krylov complexity as a function of conformal time  $\tau$  for exponentially expanding de Sitter universe with different wave numbers k. Krylov complexity grows exponentially with  $\tau$  which is a sign of chaotic system

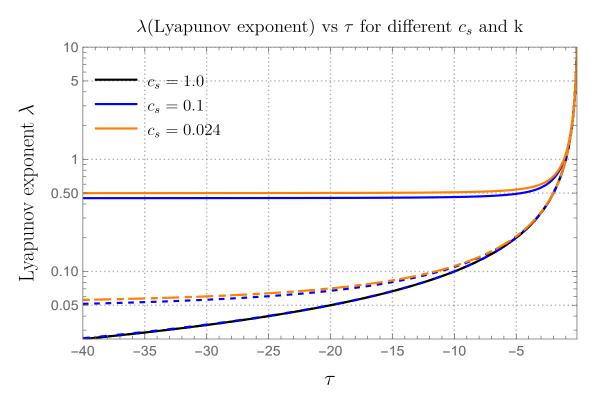


FIG. 2. Lyapunov exponent as function of conformal time  $\tau$  for different values of  $c_s$  and k. For each color, solid line belongs to k = 1, dashed line to k = 0.1 and dashed medium to k = 0.01.

 $c_s = 0.1$  with k = 1 is significantly higher than for  $c_s = 0.024$  with k = 0.01.

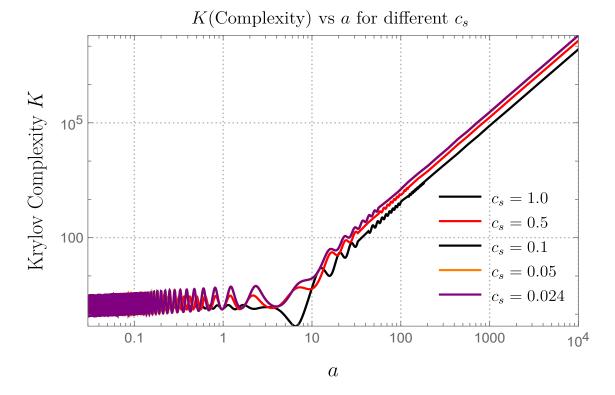


FIG. 3. Krylov complexity as function of scale factor a for different values of  $c_s$ 

For computing Krylov complexity, we will rely on the numerical tools to obtain the solutions to squeezing parameters for differential equations (38). In order to make the numerical solution easier, instead of using the conformal time  $\tau$  as the dynamical variable as in (38), we will perform the change in variable from  $\tau$  to  $a(\tau)$ 

$$\tau \longrightarrow a(\tau): \qquad \frac{d}{d\tau} = \frac{d}{da(\tau)} \frac{da(\tau)}{d\tau} = a'(\tau) \frac{d}{da(\tau)} \quad (66)$$

Consequently, differential equations (38) can be recast in terms of  $a(\tau)$  as:

$$\frac{dr_k(a)}{da} = -\frac{\beta_k(a)}{a'} \cos\left(2(\phi_k(a) - \phi_k(a))\right), \tag{67}$$

$$\frac{d\phi_k(a)}{da} = \frac{\Omega_k}{a'} - \frac{\beta_k(a)}{a'} \coth 2r_k(a) \sin\left(2(\phi_k(a) - \phi_k(a))\right)$$
(68)

For numerical solutions, we will also fix the boundary conditions at late time scale  $\tau = \tau_0$  where  $a(\tau_0) = 1$ , and the squeezing parameters are fixed to be  $r_k(a(\tau_0)) = \phi_k(a(\tau_0)) = 1$ . In Fig. 3, we have plotted Krylov complexity as a function of scale factor for different values of effective sound speed with the solutions of squeezing parameters obtained numerically. The mode vector k is fixed to be 1 for all sound speeds. Like Lyapunov exponent, we can see that complexity is bounded from below by  $c_s = 1.0$ .

#### v. Conclusion

In our work, we studied Krylov complexity and chaos for cosmological perturbations using squeezed states formalism and applied it to the de Sitter background. The main conclusions are as follows:

 We have obtained an explicit relation for Krylov complexity and chaos for cosmological perturbations

$$K = \sinh^2 r_k, \lambda = \sqrt{\left(\frac{k}{2}(1 - c_s^2)\right)^2 + \left(\frac{1}{\tau}\right)^2}$$

Interesting, Krylov complexity is equal to averal particle number in each mode. Since volume is proportional to number of particles, the Krylov complexity is also proportional to volume.

• For de Sitter background with  $c_s = 1$ , the expressions for complexity and chaos are

$$K = \sinh^2 r_k = \frac{1}{4k^2\tau^2}$$
$$b_n = \frac{-z'}{2z}n = \frac{n}{2\tau}$$
$$\lambda = \frac{-z'}{z} = \frac{1}{\tau}$$

This lyapunov exponent can also be written as  $\lambda = 2\pi T$  which saturates the bound conjectured

in [10]. For other sound speed, we rely on numerical tools and found that both Krylov complexity and lyapunov exponent are bounded from below by values for  $c_s = 1$ .

In this work, we mainly focused on de Sitter background because of it's simplicity as well as it's wide applications. There are several other interesting and realistic cosmological backgrounds such as inflation, radiation dominated models, cosmological islands where these concepts can be further explored. Our work can also be seen as studying complexity and chaos on quantum fields in curved background. These concepts can be further explored in quantum fields and in holography. We saw that for squeezed states formalism, Krylov complexity is proportional to volume of the system. It would be interesting to see if it has any relevance to "Complexity = Volume" conjectures in holography.

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- [1] S. Chapman and G. Policastro, "Quantum computational complexity from quantum information to black holes and back," Eur. Phys. J. C 82 no. 2, (2022) 128, arXiv:2110.14672 [hep-th].
- [2] S. Chapman, D. A. Galante, and E. D. Kramer, "Holographic complexity and de Sitter space," *JHEP* 02 (2022) 198, arXiv:2110.05522 [hep-th].
- [3] T. Faulkner, T. Hartman, M. Headrick, M. Rangamani, and B. Swingle, "Snowmass white paper: Quantum information in quantum field theory and quantum gravity," in 2022 Snowmass Summer Study. 3, 2022. arXiv:2203.07117 [hep-th].
- [4] E. Shaghoulian and L. Susskind, "Entanglement in De Sitter Space," arXiv:2201.03603 [hep-th].
- [5] A. Bhattacharyya, L. K. Joshi, and B. Sundar, "Quantum Information Scrambling: From Holography to Quantum Simulators," arXiv:2111.11945 [hep-th].
- [6] K. Adhikari, S. Choudhury, S. Kumar, S. Mandal, N. Pandey, A. Roy, S. Sarkar, P. Sarker, and S. S. Shariff, "Circuit Complexity in Z<sub>2</sub> EEFT," arXiv:2109.09759 [hep-th].

- [7] A. R. Brown and L. Susskind, "Second law of quantum complexity," *Phys. Rev. D* 97 no. 8, (2018) 086015, arXiv:1701.01107 [hep-th].
- [8] L. Susskind, "Entanglement is not enough," Fortsch. Phys. 64 (2016) 49-71, arXiv:1411.0690 [hep-th].
- [9] A. R. Brown, D. A. Roberts, L. Susskind, B. Swingle, and Y. Zhao, "Complexity, action, and black holes," *Phys. Rev. D* 93 no. 8, (2016) 086006, arXiv:1512.04993 [hep-th].
- [10] J. Maldacena, S. H. Shenker, and D. Stanford, "A bound on chaos," *JHEP* 08 (2016) 106, arXiv:1503.01409 [hep-th].
- [11] P. Caputa, J. M. Magan, and D. Patramanis, "Geometry of Krylov complexity," *Phys. Rev. Res.* 4 no. 1, (2022) 013041, arXiv:2109.03824 [hep-th].
- [12] D. E. Parker, X. Cao, A. Avdoshkin, T. Scaffidi, and E. Altman, "A Universal Operator Growth Hypothesis," *Phys. Rev. X* 9 no. 4, (2019) 041017, arXiv:1812.08657 [cond-mat.stat-mech].
- [13] D. A. Roberts, D. Stanford, and A. Streicher, "Operator growth in the SYK model," *JHEP* 06 (2018) 122, arXiv:1802.02633 [hep-th].
- [14] E. Rabinovici, A. Sánchez-Garrido, R. Shir, and J. Sonner, "Operator complexity: a journey to the edge of Krylov space," *JHEP* 06 (2021) 062, arXiv:2009.01862 [hep-th].
- [15] J. L. F. Barbón, E. Rabinovici, R. Shir, and R. Sinha, "On The Evolution Of Operator Complexity Beyond Scrambling," *JHEP* 10 (2019) 264, arXiv:1907.05393 [hep-th].
- [16] S.-K. Jian, B. Swingle, and Z.-Y. Xian, "Complexity growth of operators in the SYK model and in JT gravity," *JHEP* 03 (2021) 014, arXiv:2008.12274 [hep-th].
- [17] A. Dymarsky and A. Gorsky, "Quantum chaos as delocalization in Krylov space," *Phys. Rev. B* 102 no. 8, (2020) 085137, arXiv:1912.12227 [cond-mat.stat-mech].
- [18] A. Dymarsky and M. Smolkin, "Krylov complexity in conformal field theory," *Phys. Rev. D* 104 no. 8, (2021) L081702, arXiv:2104.09514 [hep-th].
- [19] K. Adhikari, S. Choudhury, and A. Roy, "Krylov Complexity in Quantum Field Theory," arXiv: 2204.02250 [hep-th].
- [20] R. Jefferson and R. C. Myers, "Circuit complexity in quantum field theory," JHEP 10 (2017) 107, arXiv:1707.08570 [hep-th].
- [21] A. Bhattacharyya, S. Das, S. Shajidul Haque, and B. Underwood, "Cosmological Complexity," *Phys. Rev.* D 101 no. 10, (2020) 106020, arXiv:2001.08664 [hep-th].
- [22] A. Bhattacharyya, S. Das, S. S. Haque, and B. Underwood, "Rise of cosmological complexity: Saturation of growth and chaos," *Phys. Rev. Res.* 2 no. 3, (2020) 033273, arXiv:2005.10854 [hep-th].
- [23] A. Reynolds and S. F. Ross, "Complexity in de Sitter Space," Class. Quant. Grav. 34 no. 17, (2017) 175013, arXiv:1706.03788 [hep-th].
- [24] Y.-S. An, R.-G. Cai, L. Li, and Y. Peng, "Holographic complexity growth in an FLRW universe," *Phys. Rev. D* 101 no. 4, (2020) 046006, arXiv:1909.12172 [hep-th].
- [25] C. M. Caves and B. L. Schumaker, "New formalism for two-photon quantum optics. i. quadrature phases and squeezed states," *Phys. Rev. A* 31 (May, 1985)

- 3068-3092. https: //link.aps.org/doi/10.1103/PhysRevA.31.3068.
- [26] V. Dodonov, "'nonclassical' states in quantum optics: A 'squeezed' review of the first 75 years," *Journal of Optics B: Quantum and Semiclassical Optics* 4 (01, 2002) R1.
- [27] K. Zelaya, S. Dey, and V. Hussin, "Generalized squeezed states," *Physics Letters A* 382 no. 47, (2018) 3369–3375.
- [28] N. C. Menicucci, P. van Loock, M. Gu, C. Weedbrook, T. C. Ralph, and M. A. Nielsen, "Universal quantum computation with continuous-variable cluster states," *Phys. Rev. Lett.* 97 (Sep, 2006) 110501.
- [29] F. R. Cardoso, D. Z. Rossatto, G. P. L. M. Fernandes, G. Higgins, and C. J. Villas-Boas, "Superposition of two-mode squeezed states for quantum information processing and quantum sensing," *Phys. Rev. A* 103 (Jun, 2021) 062405.
- [30] M. Hillery, "Quantum cryptography with squeezed states," Phys. Rev. A 61 (Jan, 2000) 022309.
- [31] C. M. Caves, "Quantum-mechanical noise in an interferometer," *Phys. Rev. D* 23 no. 8, (Apr., 1981) 1693–1708.
- [32] S. L. Braunstein and P. van Loock, "Quantum information with continuous variables," *Rev. Mod. Phys.* 77 (Jun, 2005) 513–577.
- [33] H. Yonezawa and A. Furusawa, "Continuous-variable quantum information processing with squeezed states of light," *Optics and Spectroscopy* 108 (02, 2010) 288–296.
- [34] S. L. Braunstein and P. Van Loock, "Quantum information with continuous variables," *Reviews of modern physics* 77 no. 2, (2005) 513.
- [35] H. Vahlbruch, M. Mehmet, K. Danzmann, and R. Schnabel, "Detection of 15 db squeezed states of light and their application for the absolute calibration of photoelectric quantum efficiency," *Phys. Rev. Lett.* 117 (Sep. 2016) 110801.
- [36] Y. Yamamoto and H. A. Haus, "Preparation, measurement and information capacity of optical quantum states," *Rev. Mod. Phys.* 58 (Oct, 1986) 1001–1020.
- [37] K. Adhikari, S. Choudhury, H. N. Pandya, and R. Srivastava, "PGW Circuit Complexity," arXiv:2108.10334 [gr-qc].
- [38] K. Ando and V. Vennin, "Bipartite temporal Bell inequalities for two-mode squeezed states," *Phys. Rev.* A 102 no. 5, (2020) 052213, arXiv:2007.00458 [quant-ph].
- [39] J. Martin, A. Micheli, and V. Vennin, "Discord and Decoherence," arXiv:2112.05037 [quant-ph].
- [40] H. Vahlbruch, S. Chelkowski, K. Danzmann, and R. Schnabel, "Quantum engineering of squeezed states for quantum communication and metrology," *New Journal of Physics* 9 no. 10, (Oct, 2007) 371–371.
- [41] P. M. Anisimov, G. M. Raterman, A. Chiruvelli, W. N. Plick, S. D. Huver, H. Lee, and J. P. Dowling, "Quantum metrology with two-mode squeezed vacuum: Parity detection beats the heisenberg limit," *Phys. Rev. Lett.* 104 (Mar, 2010) 103602.
- [42] V. Giovannetti, S. Lloyd, and L. Maccone, "Advances in quantum metrology," *Phys. Rev. Lett.* 96 (02, 2011).
- [43] J. P. Dowling, "Quantum optical metrology the lowdown on high-noon states," *Contemporary Physics* 49 no. 2, (2008) 125–143.

- [44] C. Xu, L. Zhang, S. Huang, T. Ma, F. Liu, H. Yonezawa, Y. Zhang, and M. Xiao, "Sensing and tracking enhanced by quantum squeezing," *Photon. Res.* 7 no. 6, (Jun, 2019) A14–A26.
- [45] M. Riedel, P. Böhi, Y. Li, T. Haensch, A. Sinatra, and P. Treutlein, "Atom-chip-based generation of entanglement for quantum metrology," *Nature* 464 (03, 2010) 1170–3.
- [46] B. L. Schumaker and C. M. Caves, "New formalism for two-photon quantum optics. ii. mathematical foundation and compact notation," *Phys. Rev. A* 31 (May, 1985) 3093–3111.
- [47] M. Teich, "Squeezed states of light (A)," Journal of the Optical Society of America A (01, 1987) 10.
- [48] C. Fabre, G. Grynberg, and A. Aspect, Introduction to Quantum Optics: From the Semi-classical Approach to Quantized Light. 09, 2010.
- [49] R. Loudon and P. Knight, "Squeezed light," Journal of Modern Optics 34 no. 6-7, (1987) 709-759.
- [50] U. Andersen, T. Gehring, C. Marquardt, and G. Leuchs, "30 years of squeezed light generation," *Physica Scripta* 91 (11, 2015).
- [51] O. Rosas-Ortiz, "Coherent and squeezed states: Introductory review of basic notions, properties, and generalizations," *Integrability, Supersymmetry and Coherent States* (2019) .
- [52] B. L. Schumaker and C. M. Caves, "New formalism for two-photon quantum optics. 2. Mathematical foundation and compact notation," *Phys. Rev. A* 31 (1985) 3093–3111.
- [53] A. Garcia-Chung, "Squeeze operator: a classical view," arXiv:2003.04257 [math-ph].
- [54] H. Vahlbruch, S. Chelkowski, B. Hage, A. Franzen, K. Danzmann, and R. Schnabel, "Coherent control of vacuum squeezing in the gravitational-wave detection band," *Phys. Rev. Lett.* **97** (Jul, 2006) 011101.
- [55] S. S. Y. Chua, B. J. J. Slagmolen, D. A. Shaddock, and D. E. McClelland, "Quantum squeezed light in gravitational-wave detectors," *Classical and Quantum Gravity* 31 no. 18, (Sep. 2014) 183001.
- [56] S. Choudhury, A. Mazumdar, and S. Pal, "Low & High scale MSSM inflation, gravitational waves and constraints from Planck," *JCAP* 07 (2013) 041, arXiv:1305.6398 [hep-ph].
- [57] L. P. Grishchuk and Y. V. Sidorov, "Squeezed quantum states in theory of cosmological perturbations," in 5th Seminar on Quantum Gravity. 1990.
- [58] L. P. Grishchuk, "Quantum Mechanics of the Primordial Cosmological Perturbations," in a talk given at the Sixth Marcel Grossmann Meeting. Kyoto,1991.
- [59] A. Albrecht, P. Ferreira, M. Joyce, and T. Prokopec, "Inflation and squeezed quantum states," *Phys. Rev. D* 50 (1994) 4807–4820, arXiv:astro-ph/9303001.
- [60] K. Hasebe, "Sp(4; R) Squeezing for Bloch Four-Hyperboloid via The Non-Compact Hopf Map," J. Phys. A 53 no. 5, (2020) 055303, arXiv:1904.12259 [quant-ph].
- [61] S. Choudhury and S. Pal, "Fourth level MSSM inflation from new flat directions," JCAP 04 (2012) 018, arXiv:1111.3441 [hep-ph].
- [62] S. Choudhury and S. Pal, "DBI Galileon inflation in background SUGRA," Nucl. Phys. B 874 (2013) 85-114, arXiv:1208.4433 [hep-th].
- [63] S. Choudhury and S. Pal, "Brane inflation in

- background supergravity," *Phys. Rev. D* **85** (2012) 043529, arXiv:1102.4206 [hep-th].
- [64] S. Choudhury, T. Chakraborty, and S. Pal, "Higgs inflation from new Kähler potential," Nucl. Phys. B 880 (2014) 155-174, arXiv:1305.0981 [hep-th].
- [65] S. Choudhury and S. Panda, "COSMOS-e'-GTachyon from string theory," Eur. Phys. J. C 76 no. 5, (2016) 278, arXiv:1511.05734 [hep-th].
- [66] S. Choudhury, "COSMOS-e<sup>7</sup>- soft Higgsotic attractors," Eur. Phys. J. C 77 no. 7, (2017) 469, arXiv:1703.01750 [hep-th].
- [67] S. Akhtar, S. Choudhury, S. Chowdhury, D. Goswami, S. Panda, and A. Swain, "Open Quantum Entanglement: A study of two atomic system in static patch of de Sitter space," *Eur. Phys. J. C* 80 no. 8, (2020) 748, arXiv:1908.09929 [hep-th].
- [68] S. Choudhury, S. Panda, and R. Singh, "Bell violation in primordial cosmology," *Universe* 3 no. 1, (2017) 13, arXiv:1612.09445 [hep-th].
- [69] S. Choudhury, S. Panda, and R. Singh, "Bell violation in the Sky," Eur. Phys. J. C 77 no. 2, (2017) 60, arXiv:1607.00237 [hep-th].
- [70] S. Choudhury and S. Panda, "Entangled de Sitter from stringy axionic Bell pair I: an analysis using Bunch-Davies vacuum," Eur. Phys. J. C 78 no. 1, (2018) 52, arXiv:1708.02265 [hep-th].
- [71] M. B. Einhorn and F. Larsen, "Squeezed states in the de Sitter vacuum," *Phys. Rev. D* 68 (2003) 064002, arXiv:hep-th/0305056.
- [72] S. Choudhury and S. Panda, "Quantum entanglement in de Sitter space from stringy axion: An analysis using  $\alpha$  vacua," *Nucl. Phys. B* **943** (2019) 114606, arXiv:1712.08299 [hep-th].
- [73] D. Baumann and L. McAllister, *Inflation and String Theory*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 5, 2015. arXiv:1404.2601 [hep-th].
- [74] J. Grain and V. Vennin, "Canonical transformations and squeezing formalism in cosmology," JCAP 02 (2020) 022, arXiv:1910.01916 [astro-ph.CO].
- [75] L. Grishchuk, H. A. Haus, and K. Bergman, "Generation of squeezed radiation from vacuum in the cosmos and the laboratory," *Phys. Rev. D* 46 (1992) 1440–1449.
- [76] P. Bhargava, S. Choudhury, S. Chowdhury, A. Mishara,

- S. P. Selvam, S. Panda, and G. D. Pasquino, "Quantum aspects of chaos and complexity from bouncing cosmology: A study with two-mode single field squeezed state formalism," *SciPost Phys. Core* 4 (2021) 026, arXiv:2009.03893 [hep-th].
- [77] S. Choudhury, S. Chowdhury, N. Gupta, A. Mishara, S. P. Selvam, S. Panda, G. D. Pasquino, C. Singha, and A. Swain, "Circuit Complexity From Cosmological Islands," *Symmetry* 13 (2021) 1301, arXiv:2012.10234 [hep-th].
- [78] K. Adhikari, S. Choudhury, S. Chowdhury, K. Shirish, and A. Swain, "Circuit complexity as a novel probe of quantum entanglement: A study with black hole gas in arbitrary dimensions," *Phys. Rev. D* 104 no. 6, (2021) 065002, arXiv:2104.13940 [hep-th].
- [79] S. Choudhury, A. Mukherjee, N. Pandey, and A. Roy, "Causality Constraint on Circuit Complexity from  $\mathcal{COSMOEFT}$ ," arXiv:2111.11468 [hep-th].
- [80] J. Martin and V. Vennin, "Real-space entanglement in the Cosmic Microwave Background," arXiv:2106.15100 [gr-qc].
- [81] M. A. Nielsen, "A geometric approach to quantum circuit lower bounds.".
- [82] M. A. Nielsen, "Quantum computation as geometry," Science 311 no. 5764, (Feb, 2006) 1133-1135. http://dx.doi.org/10.1126/science.1121541.
- [83] M. R. Dowling and M. A. Nielsen, "The geometry of quantum computation," *Quantum Info. Comput.* 8 no. 10, (Nov., 2008) 861–899.
- [84] M. A. Nielsen, M. R. Dowling, M. Gu, and A. C. Doherty, "Optimal control, geometry, and quantum computing," Phys. Rev. A 73 (Jun, 2006) 062323. https://link.aps.org/doi/10.1103/PhysRevA.73.062323.
- [85] V. Mukhanov, H. Feldman, and R. Brandenberger, "Theory of cosmological perturbations," *Physics Reports* 215 no. 5, (1992) 203–333.
- [86] V. Mukhanov and S. Winitzki, Introduction to quantum effects in gravity. Cambridge University Press, 6, 2007.
- [87] J. M. Maldacena, "Non-Gaussian features of primordial fluctuations in single field inflationary models," *JHEP* 05 (2003) 013, arXiv:astro-ph/0210603.
- [88] N. Hörnedal, N. Carabba, A. S. Matsoukas-Roubeas, and A. del Campo, "Ultimate Physical Limits to the Growth of Operator Complexity," arXiv:2202.05006 [quant-ph].