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Cosmological Models and Singularities in General Relativity

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Abstract

This is a thesis on general relativity. It analyzes dynamical properties of Einstein's field equations in cosmology and in the vicinity of spacetime singularities in a number of different situations. Different techniques are used depending on the particular problem under study; dynamical systems methods are applied to cosmological models with spatial homogeneity; Hamiltonian methods are used in connection with dynamical systems to find global monotone quantities determining the asymptotic states; Fuchsian methods are used to quantify the structure of singularities in spacetimes without symmetries. All these separate methods of analysis provide insights about different facets of the structure of the equations, while at the same time they show the relationships between those facets when the different methods are used to analyze overlapping areas.

The thesis consists of two parts. Part I reviews the areas of mathematics and cosmology necessary to understand the material in part II, which consists of five papers. The first two of those papers uses dynamical systems methods to analyze the simplest possible homogeneous model with two tilted perfect fluids with a linear equation of state. The third paper investigates the past asymptotic dynamics of barotropic multi-fluid models that approach a 'silent and local' space-like singularity to the past. The fourth paper uses Hamiltonian methods to derive new monotone functions for the tilted Bianchi type II model that can be used to completely characterize the future asymptotic states globally. The last paper proves that there exists a full set of solutions to Einstein's field equations coupled to an ultra-stiff perfect fluid that has an initial singularity that is very much like the singularity in Friedman models in a precisely defined way.

List of Accompanying Papers

I P. Sandin and C. Uggla, *Bianchi Type I Models with Two Tilted Fluids*, Class. Quantum Grav. **25** (2008), 225013.

II P. Sandin, *Tilted Two-Fluid Bianchi Type I Models*, Gen. Rel. Grav. **41** (2009), 2707.

III P. Sandin and C. Uggla, *Perfect Fluids and Generic Spacelike Singularities*, Class. Quantum Grav. **27** (2010), 025013.

IV S. Hervik, W. C. Lim, P. Sandin, and C. Uggla, *Future Asymptotics of Tilted Bianchi Type II Cosmologies*, Class. Quantum Grav. **27** (2010), 185006.

V M. Heinzle and P. Sandin, *The Initial Singularity of Ultrastiff Perfect Fluid Spacetimes Without Symmetries*, arXiv:1105.1643.

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Contents

Part I

A Companion to the Papers

Notation

Latin indices from the beginning of the alphabet, a, b, c, and so on generally run over four spacetime indices 0, 1, 2, 3, where v^0 denotes the time component of the vector v^a. An exception to this convention occurs in paper V and the related section in chapter 5 where they label spatial coordinate indices from 1 to 3.

Greek indices from the beginning of the alphabet, α, β, γ, and so on generally run over three spatial indices 1, 2, 3, and are used to label components relative an orthonormal frame, except in paper V where they label spacetime components from 1 to 4.

Greek indices from the middle of the alphabet, μ, ν, and so on are used to label spacetime coordinate components.

Latin indices from the middle of the alphabet, i, j, k, and so on are used to label spatial coordinate components, or as in paper IV, components relative a group invariant frame.

Repeated upper and lower indices are summed over, unless otherwise indicated.

The metric has signature $- + + +$.

Units for which $8\pi G = 1$ and $c = 1$, where G is the gravitational constant and c is the speed of light, are used throughout the thesis.

Vectors and tensors are represented by symbols in bold font, $\mathbf{x}$, $\mathbf{0}$, $\mathbf{g}$, $\mathbf{T}$ for example, where the dimension and rank should be discernable from the context.

Chapter 1

Introduction

A cosmological model is a mathematical representation of the universe at some averaging scale. It describes the geometry of space and time and the distribution and properties of matter in the universe within the framework of some physical theory (most commonly Newtonian or Einsteinian theories of gravity).

The most important guide for choosing a good model for spacetime and matter should be the observational data collected by land and space based observatories, which puts some constraints on permissable models. What evidence about the geometry of the universe can be made from observations? According to Malcolm MacCallum in 1973 [67] there were three main deductions about geometry observations could bring us at that time. The scientific community has since then acquired a wealth of new observational data but the three conclusions are still considered valid. These are listed below as points 1–3, together with the observational data found in support of them. At present time, two more major observations have been made which can be added to that list. These are given below as points 4 and 5.

1. *The universe is expanding.* This conclusion is supported by several different pieces of evidence. The first is the *velocity-distance relation* or *magnitude-redshift relation.* As first noted by Edwin Hubble in 1929 [49][1],

[1]Hubble drew his conclusions on the basis of a very limited set of data points with a considerable spread, but he considered his result "fairly definite [...] for such scanty material, so poorly distributed". He emphasized that the linear relation only should be viewed as "a first approximation representing a restricted range in distance." and interestingly enough speculated that the velocity-distance relation might indicate a de Sitter type cosmology with an accelerating universe.

measurements of the redshift of the light from galaxies show that there is a general relation between the distance to a galaxy and its velocity, where a galaxy is moving away from us with a radial velocity that is proportional to the distance from us. The velocity-distance relation is in itself not enough to rule out other scenarios, like the "steady state" theories proposed by Bondi, Gold [9] and Hoyle [48] for example, but other observations do give further support to the expanding picture. The perhaps most important of these comes from the observation of the *cosmic microwave background* (CMB) that indicates that the universe once was in a very hot and dense state, and has since expanded and cooled. The existence of a background of microwave radiation with a black-body spectrum in an expanding universe was first theoretically predicted by Gamow (1946) [30], and Alpher and Herman (1949) [1], but their predictions did not result in any attempt to try to observe it, and the subsequent discovery was made by chance by Penzias and Wilson (1965) [74] when they tried to understand the unexpected noise in their radio receiver. The details of the spectrum of the microwave background is now arguably the most important observational source of information about the early universe, and have resulted in the Nobel Prize in physics in 1978 and 2006.

2. *The universe is isotropic about us.* The most convincing support for this conclusion comes from the observation that the microwave background is isotropic about us to about one part in a hundred thousand (if one disregards the anisotropy-effect from the earth's relative motion with respect to the background) [92], [93]. It was however pointed out in [71] that an isotropic CMB does not necessarily imply a isotropic universe (although it does in most circumstances). Other independent, but weaker, constraints on the anisotropies can be obtained by observing that the distribution of extragalactic radio- [66] and x-ray sources is approximately isotropic about us.

3. *The universe is spatially homogeneous.* Assuming that we do not occupy any special position in the universe and thus that the universe looks isotropic about all points in space, one can draw the conclusion that the universe is spatially homogeneous. The homogeneity is a feature of the universe at large scales; the inhomogeneities are large on galactic scales, but on scales $\gg$100 megaparsec (Mpc), the inhomogeneities are evenly distributed, and result in a homogeneous universe. The currently largest inhomogeneity one has detected is a void, a fairly empty volume of space, of order of $\sim$140 Mpc [87], but on larger scales than that the universe appears increasingly homogeneous.

4. *Most of the matter in the universe is of an unknown form.* Observations

of the rotation velocities of galaxies and velocities of galaxies in galaxy clusters show that most of the matter in galaxies and clusters is not in the form of stars, gas, or dust but in some form that does not emit light. This conclusion is further supported by observations of acoustic oscillations in the CMB (and similar acoustic signatures in the large scale clustering of galaxies) [98], [25], observations of x-ray emissions from galaxy clusters [38], [10], and by measuring the amount of gravitational lensing by which foreground clusters affect the light from more distant galaxies [109]. It is currently believed that this matter is different from any hitherto discovered type of matter, and it is normally called *dark matter*.

5. *The expansion is accelerating.* Observations of supernova explosions of type Ia, from the spectra of which one believes one can determine the absolute brightness and thereby determine the distance, have led to the conclusion that the universe is not only expanding, but also accelerating [76], [89]. This is also supported by the detailed observations of the structure of the CMB [98]. The cause and mechanism of this accelerated expansion is unknown, but is a popular subject of theoretical speculations. The expansion can be modeled by adding a term proportional to the metric in the Einstein Field Equations for gravity, what is normally known as the cosmological constant[2]. In analogy with the dark matter cognomen, the accelerated expansion is commonly described as a *dark energy*[3].

The topic of this thesis will not be the causes of the accelerating expansion, or the nature of the dark matter, although those are questions that should be, and are, addressed elsewhere. It will in the following be assumed that the accelerated expansion can be modeled by a cosmological constant, and in terms of gravitational properties, dark matter is assumed to obey the same laws as ordinary matter. The models studied could therefore be considered as models of dark matter or ordinary matter, or both.

Cosmological models that obey the first three assumptions above were first described by Friedman[4] [29], and then later analyzed from a geometrical perspective by Robertson [81], [82] and Walker [105]. These models will here be referred

[2]The cosmological constant was first introduced by Einstein [23] as a way to obtain a static universe, but was abandoned by him when it was discovered that the universe is expanding, only later to be revived in light of the new observations of an accelerating universe.

[3]The names dark matter/energy are a bit misleading. It may sound like dark matter is completely black, absorbing all light, but in reality it is totally transparent, not interacting with light or normal matter at all, except through its gravitational pull (and possibly through weak interactions that are of no relevance on astronomical scales). More appropriate names would be invisible matter/energy.

[4]Александер Александерович Фриедман's last name is sometimes translated Friedmann and sometimes Friedman.

to as FRW-models[5]. The FRW-models may or may not include a cosmological constant (Friedman considered models with the value of the cosmological constant left undetermined) and they can thus be reconciled with all five points above. They are widely used by cosmologists and astronomers as the background by which they interpret their observations. All observations obtained so far are consistent with a large scale structure of the spacetime geometry consistent with the FRW-models with a cosmological constant since the *time of last scattering*[6]. We cannot probe further into the past than that time by any currently available observational means, although there exists proposals of future observatories, based on neutrinos [50] and gravitational waves [62], that can see through the opaque beginnings of time. The structure of the universe before the time when photons began to propagate freely can presently only be inferred from theoretical considerations. The most detailed predictions comes from calculations describing nucleosynthesis and comparisons with the relative abundances of the elements observed in the universe. The received view states that there was a period of nucleosynthesis, when the matter inhabiting the universe today was created, which was preceded by another period of accelerated expansion, what is called the era of *inflation*; see Weinberg (2008) [107] for an up to date account on the current state of physical cosmology.

This thesis is based on a series of papers that investigate the dynamics of cosmological models that do not – a priori – obey the constraints of homogeneity and isotropy above. For models that are homogeneous but not isotropic the dynamics can be formulated as a dynamical system, where the state of the system characterizes the properties of the universe. In this framework one can investigate questions like whether such models will evolve into isotropic states or not, and under what circumstances this will happen. Using bifurcation theory one can use parameter dependent matter models, and by varying the parameters continuously find exactly when the dynamics change qualitatively. By studying models that are close to FRW-universes at some stage in their evolution one can find what models that are compatible with the observational results, and by studying models under the early dense stages of the universe one can make statements about the evolution of the universe from a time where no observational data exists. The difference from the mainstream analysis of the early universe as described in [107] lies in the focus on the spacetime geometry rather than the matter model. Instead of studying what happens with the matter in an expanding FRW-universe one can find out whether an expanding universe that

[5]Sometimes called FLRW-models, where the L refers to Georges Lemaître who also studied them in the early 30's [59].

[6]The time of last scattering is the idealized moment of time when the matter permeating the universe became transparent and the photons emitted by the hot matter could traverse the space freely. It is these photons that now reach us in the form of the cosmic microwave background radiation.

is close but not exactly FRW will remain so, and whether the matter will affect the dynamics or not, and if so, what properties of matter will be important.

All of the models studied in this thesis are solutions of Einstein's field equations that are ill defined in a certain sense. The variables describing the properties of the spacetimes – the metric components, the curvature, and so on – become infinite at some time in the past; a solution found on a patch of spacetime cannot be extended indefinitely to the past without breaking down and forming a *singularity*. This is not only a pathology of the solutions under study here, but says something about the physics of gravitating systems.

Singularities signal the breakdown of known physical laws and point to situations where the nature of space, time and matter is not adequately understood. It has been believed that this can be resolved by finding a theory of quantum gravity that does for general relativity what quantum field theory has done for electrodynamics; but finding such a theory has been more difficult than first expected. At the present time there are several different proposals for theories of quantum gravity that differ quite a lot on both conceptual and mathematical levels. None have so far obtained any observational support in favor of any other so the proponents of the different theories are at present resorted to rest on aesthetic or ideological grounds.

In this thesis it is investigated how much one can say about singularities within the present theory of general relativity. Although the solutions become infinite at the singularity there may be some information still as to *how* the solutions become singular. For expanding spatially homogeneous models it is possible to introduce new variables that are scaled with the overall expansion in such a way that the new variables are finite even at the singularity. The singular behavior is characterized by a single function, and the remaining finite variables can be analyzed as a dynamical system. Inhomogeneous models are not directly susceptible to dynamical systems methods since the dynamics is governed by PDEs rather than ODEs, but in special situations, such as in the asymptotic limit in an approach to a space-like spacetime singularity, the equations can effectively be reduced to ODEs for a large class of spacetimes. Two different ways to study such situations are investigated in papers III and V.

The rest of part I of this thesis is written in a way as to introduce the necessary means to understand the papers in part II for someone with a background in physics, but outside this particular field. Chapter 2 introduces the necessary mathematics needed by introducing the basic concepts of dynamical systems theory. Chapter 3 formulates the equations governing the dynamics of the matter and geometry of the universe in a way as to be susceptible to analysis by the means of chapter 2. Chapter 4 introduces the concept of Fuchsian reduction and its application in relativistic cosmology. Chapter 5 discusses the methods used in the papers and presents some examples of the analysis to illustrate in a concrete way how the work was performed. It also makes comparisons between the papers where the analysis differed.

Part of chapter I, most of chapters II and III, except for the section on Bianchi cosmology, as well as the first two sections of chapter V, have previously been published together with the first two papers and an early version of paper III under the title *The Asymptotic States of Perfect Fluid Cosmological Models* [88], to obtain the *Degree of Licentiate of Philosophy* in accordance with the provisions of the Higher Education Ordinance, Swedish Code of Statutes (1993:100).

Chapter 2

Dynamical Systems

This chapter gives a brief introduction to the mathematical theory of continuous *dynamical systems*, or in other words: Systems of autonomous coupled ordinary differential equations. Discrete dynamical systems can also be defined in a similar way but such systems will not be covered here, see instead the books by Clark Robinson [83] or Robert Devaney [21]. This chapter is no substitute for an introductory course to dynamical systems; theorems will be stated without proofs, and beyond the most basic concepts and most important theorems, only areas which have been shown to be useful in research in cosmology will be covered. The purpose of this chapter is mainly to make the reader familiar with the concepts and terminology used in the following chapters and the accompanying papers; the interested reader is instead referred to Lawrence Perko's book [75] for a more thorough introduction to continuous dynamical systems, and then to Guckenheimer & Holmes [32] and Hirsch & Smale [47] for an in-depth treatment of the subject.

A rough description of a dynamical system is something that has a *state space* which parameterizes the *states* of a physical (or purely mathematical) system. Different states are described by different points in this state space. Usually the state space is defined as a subset of a finite dimensional Euclidian space. In addition there also exists a law that evolves the points in the state space in time. This law can either be a discrete time evolution law that successively maps points from one place to another in discrete jumps, or a continuous evolution that can be described as a *flow* on the state space. As mentioned above, only the latter kind will be considered here. For the system to be considered a Dynamical System it is also necessary that no other input than a point's position in the state space is required to determine its future evolution, the evolution law can not be explicitly dependent on time for example.

What was just described can formally be formulated as a system of first order *ordinary differential equations* (ODEs) where the time derivatives of points in the state space are equal to some function on the state space. The function cannot be explicitly dependent on time since the evolution of points can only be dependent on the position in the state space, i.e. the system of differential equations is *autonomous.*

An n-dimensional system of first order, autonomous, ODEs can be expressed in the form:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \tag{2.1}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$, $\dot{\mathbf{x}} \equiv \frac{dx}{dt}$, and where t usually denotes time. Solving the system is to find all *curves* on the state space who's time derivative obeys equation (2.1) on some time interval. Formally we define:

A *solution to the differential equation* (2.1) *on an interval* I is a differentiable function $\mathbf{x}(t)$ which for all $t \in I$ and $\mathbf{x}(t) \in \mathbb{R}^n$ satisfies

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t)).$$

Given an $\mathbf{x}_0 \in \mathbb{R}^n$, $\mathbf{x}(t)$ is *a solution of the initial value problem*

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}), \\ \mathbf{x}(t_0) &= \mathbf{x}_0, \end{aligned} \tag{2.2}$$

on an interval I if $t_0 \in I$, $\mathbf{x}(t_0) = \mathbf{x}_0$ and $\mathbf{x}(t)$ is a solution of the differential equation (2.1).

One more condition is usually required for a system of differential equations like (2.1) to be deemed a dynamical system, and it is that a solution exists for all $t \in \mathbb{R}$, i.e. $I = \mathbb{R}$. It can however be shown that for every system of equations like (2.2) there exists a *topological equivalent*[7] system of equations that has a solution for all $t \in \mathbb{R}$, this is what is called the *Global Existence Theorem* (a proof can be found in [75]).

Flows and Orbits

Consider a differential equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ that has solutions $\{\mathbf{x}(t)\}$, where $\mathbf{x}(0) = \mathbf{x}_0$, which are defined for all $t \in \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^n$, then the *flow* of

[7] See note on page 18.

the equation is defined as the one-parameter family of maps $\{\phi_t\}_{t\in\mathbb{R}}$ of $\mathbb{R}^n$ into itself such that

$$\phi_t(\mathbf{x}_0) = \mathbf{x}(t) \quad \forall\, \mathbf{x}_0 \in \mathbb{R}^n. \tag{2.3}$$

Properties of the flow:

- The flow of a differential equation forms a commutative group of maps of $\mathbb{R}^n$ into itself.
- If the function $\mathbf{f}$ in the system (2.1) is $C^1(\mathbb{R}^n)$ then the corresponding flow $\{\phi_t\}$ consists of C^1 maps.

Formally it is the flow (2.3) of a system of equations like (2.1) that is called a dynamical system, but in an informal sense also the system of equations go under that name, or in an even more general sense, any physical system that evolves in time. In this text a 'dynamical system' will refer to either the flow or the system of equations.

The image of a solution of the initial value problem (2.2) is called an *orbit*, or *trajectory*, through $\mathbf{x}_0$ and is here denoted by $\gamma(\mathbf{x}_0)$. Alternatively the orbit can be defined as the set of all points that can be mapped to (or from) $\mathbf{x}_0$ by an element of the flow $\{\phi_t\}$. Orbits can be categorized into several different types:

Equilibrium points are orbits that consist of a single point $\gamma(\mathbf{x}_0) = \mathbf{x}_0$. These points satisfy $\mathbf{f}(\mathbf{x}_0) = \mathbf{0}$ in eq. (2.1) and corresponds to an equilibrium state of the physical system described by the equation. Equilibrium points are alternatively called *fixed points*, *stationary points*, or *critical points*.

Periodic orbits are closed curves in $\mathbb{R}^n$.

Recurrent orbits describe physical systems that for all times will return arbitrarily close to to an earlier state. Formally an orbit $\gamma(\mathbf{x}_0)$ is recurrent if for all neighborhoods $N(\mathbf{x}_0)$ and all $T \in \mathbb{R}\ \exists\ t > T$ such that $\phi_t(\mathbf{x}_0) \in N(\mathbf{x}_0)$.

Homoclinic orbits connect equilibrium points to themselves.

Heteroclinic orbits connect equilibrium points to other equilibrium points.

Invariant Sets and Limit Sets

An *invariant set* is defined as a set $U \in \mathbb{R}^n$ that is mapped into itself by the flow $\{\phi_t\}$, i.e. $\phi_t(\mathbf{x}) \in U\ \forall\ \mathbf{x} \in U,\ t \in \mathbb{R}$. One can think of invariant sets as describing subsystems of the physical system. Finding invariant sets simplifies the description of the dynamical system since they can be studied separately. Often they have lower dimension than the original system and are therefore simpler to analyze. Information about the complete system can afterwards be obtained from the solutions on the lower dimensional invariant sets, especially if one can show that orbits approach an invariant subset for large positive or negative t. This behavior warrants the introduction of yet another concept, the *limit set*. The behavior of non-linear systems of equations are often too complicated for solutions to be found explicitly; the system may however stabilize after some time and it may then be possible to find solutions valid for large t. Solving Newton's equations for the solar system with the initial conditions relevant 4.5 billion years ago is a difficult task for example, but now the system has stabilized and the orbits of the planets are near Keplerian. The ω-*limit* set is a mathematical description of this behavior.

Definition. *A point* $\mathbf{p} \in \mathbb{R}^n$ *is an* ω*-limit point of the trajectory* $\gamma(\mathbf{x}_0)$ *if there exists a sequence* $t_n \to \infty$ *such that* $\lim_{n\to\infty} \phi_{t_n}(\mathbf{x}_0) = \mathbf{p}$*. The set of all* ω*-limit points of* $\gamma(\mathbf{x}_0)$ *is called the* ω*-limit set of* $\gamma(\mathbf{x}_0)$*.*

One can similarly define the α-limit using a sequence $t_n \to -\infty$. Even if one cannot solve the equations explicitly it may be possible to find the ω- and α-limits of some set of initial conditions, or at least constrain the limit sets to lie within some subset of the state space and thereby obtain information about the asymptotic states of the physical system. The limit sets can be equilibrium points or periodic orbits, or networks of equilibrium points connected by homoclinic or heteroclinic orbits, or even more complicated sets when the dimension of the system is higher than 2.

2.1 Linear Systems

The solutions of linear systems of equations is an important ingredient for understanding non-linear systems, and it will therefore be helpful to study these systems first.

If $\mathbf{f}$ is a linear function, i.e. $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$, where A is an $n \times n$ matrix of real numbers, then the ODE (2.1) is linear.[8] It can be shown that the initial value

[8]Some would call $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ a linear, homogeneous function to distinguish it from the function $\mathbf{f}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$, but here and elsewhere in this text such functions are called *affine functions*, and the term *linear* is reserved for the homogenous variety.

problem

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x}, \\ \mathbf{x}(0) &= \mathbf{x}_0, \end{aligned} \tag{2.4}$$

has the solution $\mathbf{x}(t) = \mathrm{e}^{At}\mathbf{x}_0$, where e^{At} is defined by the Taylor series

$$\mathrm{e}^{At} = I + At + \frac{1}{2!}(At)^2 + \frac{1}{3!}(At)^3 + \ldots \quad ,$$

where I is the unit matrix. Thus to solve the system of equations one only has to compute the matrix e^{At}. If the matrix A is diagonal, the system of equations is uncoupled and consists of n linear equations of the kind

$$\begin{aligned} \dot{x}_1 &= A_{11}x_1, \\ \vdots \;\; & \;\; \vdots \qquad \vdots \\ \dot{x}_n &= A_{nn}x_n, \end{aligned}$$

for which the general solutions can be written

$$\begin{aligned} x_1(t) &= c_1\mathrm{e}^{A_{11}t}, \\ \vdots \;\; & \;\; \vdots \qquad \vdots \\ x_n(t) &= c_n\mathrm{e}^{A_{nn}t}, \end{aligned}$$

or in matrix form

$$\mathbf{x}(t) = \begin{pmatrix} \mathrm{e}^{A_{11}t} & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \mathrm{e}^{A_{nn}t} \end{pmatrix} \mathbf{c}.$$

For coupled systems with real distinct eigenvalues one can diagonalize the matrix A and then find a solution of the form

$$\mathbf{x}(t) = \sum_{i=1}^{n} c_i \mathrm{e}^{\lambda_i t}\mathbf{v}_i,$$

where λ_i are the eigenvalues and $\mathbf{v}_i$ the eigenvectors of the matrix A. For coupled systems with complex or several identical real eigenvalues it is not possible to diagonalize the matrix completely, although one can always reduce it to its *Jordan canonical form*, which can be used to give the explicit time dependence of the solution (see Hirsch and Smale [47]). Often it is more useful to obtain a pictorial representation of the solution in terms of a *phase portrait* where the flow is visualized as directed curves on the state space.

Classification of Equilibrium Points

The origin, $\mathbf{0}$, is an equilibrium point for all linear systems. It is classified after the values of the eigenvalues of the defining matrix A. If the real parts of all eigenvalues are positive the equilibrium point is called *a source* or *an unstable fixed point*; if the real parts of all eigenvalues are negative it is called *a sink* or *a stable fixed point*; if some eigenvalues have negative real part and some positive real part the equilibrium point is called *a saddle*; if the real part is zero for any of the eigenvalues the point is called *a center*. These names refer to the behavior of the dynamical system in the vicinity of the fixed points. In the neighborhood of a source the flow of the system is directed away from the fixed point; if the fixed point is a sink it is directed towards the fixed point. A center has at least one direction where the flow is neither too or from the fixed point.

To each eigenvalue corresponds an eigenvector (if the eigenvalue is complex it corresponds to two real vectors). The eigenvectors span three invariant subspaces of the state space, *the stable subspace*, *the unstable subspace*, and *the center subspace*; E^u, E^s and E^c respectively.

Definition. *Let* $\lambda_j = a_j + ib_j$ *be a complex eigenvalue of the matrix* A *in* (2.4) *and* $\mathbf{w}_j = \mathbf{u}_j + i\mathbf{v}_j$ *the corresponding complex eigenvector. Then*

$$\begin{aligned} E^s &= span\{\mathbf{u}_j, \mathbf{v}_j \mid a_j < 0\}, \\ E^c &= span\{\mathbf{u}_j, \mathbf{v}_j \mid a_j = 0\}, \\ E^u &= span\{\mathbf{u}_j, \mathbf{v}_j \mid a_j > 0\}. \end{aligned}$$

Together the stable, center, and unstable subspaces span the full state space: $\mathbb{R}^n = E^u \oplus E^s \oplus E^c$ (see Perko [75] for a simple proof). All solutions that start in E^s will approach $\mathbf{0}$ as $t \to \infty$, and all solutions that start in E^u will approach $\mathbf{0}$ as $t \to -\infty$. In other words: $\mathbf{0}$ is the ω-limit of E^s and the α-limit of E^u. If $\mathbf{0}$ is a sink then $E^s = \mathbb{R}^n$ and $E^u = E^c = \{\mathbf{0}\}$; and likewise $E^u = \mathbb{R}^n$ and $E^s = E^c = \{\mathbf{0}\}$ if $\mathbf{0}$ is a source. If none of the eigenvalues have vanishing real part the equilibrium point is called *hyperbolic*; sinks, saddles, and sources are all hyperbolic fixed points.

2.2 The Hartman-Grobman Theorem

Leaving now the linear equations for the general case (2.1) once again, we shall see how the flow in the neighborhood of an equilibrium point can be approximated by the flow of a linear system of equations.

If $\mathbf{a}$ is an equilibrium point then the Taylor expansion to first order of the defining function $\mathbf{f}$ in (2.1) around $\mathbf{a}$ gives (since $\mathbf{f}(\mathbf{a}) = \mathbf{0}$)

$$f^i(\mathbf{x}) \approx \left(x^j - a^j\right) \left(\frac{\partial f^i}{\partial x^j}\right)_{\mathbf{x}=\mathbf{a}} .$$

The linearization of the dynamical system (2.1) is then obtained by replacing the defining function by its first order Taylor expansion

$$\dot{\mathbf{y}} = \mathrm{D}\mathbf{f}(\mathbf{a})\mathbf{y}, \tag{2.5}$$

where $\mathbf{y} = \mathbf{x} - \mathbf{a}$ and $\mathrm{D}\mathbf{f}(\mathbf{a}) = \left(\frac{\partial f^i}{\partial x^j}\right)_{\mathbf{x}=\mathbf{a}} = \left(\frac{\partial f^i}{\partial y^j}\right)_{\mathbf{y}=\mathbf{0}}$. Equilibrium points of non-linear systems can be classified in the same way as for linear systems by considering the eigenvalues of the matrix $\mathrm{D}\mathbf{f}(\mathbf{a})$. While hyperbolic linear systems only have a fixed point at the origin, non-linear systems may have many fixed points, say $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, ..., and they are then classified by the matrices $\mathrm{D}\mathbf{f}(\mathbf{a})$, $\mathrm{D}\mathbf{f}(\mathbf{b})$, $\mathrm{D}\mathbf{f}(\mathbf{c})$, and so on.

For non-linear dynamical systems there is an analogy to the stable, center and unstable subspaces in linear Dynamical Systems. This is proved by the *Stable Manifold Theorem* for non-linear systems.

Theorem 2.2.1 (The Stable Manifold Theorem)**.** *Let N be a neighborhood of a fixed point* $\mathbf{a}$ *of the equation* (2.1) *where* $\mathbf{f}$ *is at least $C^1(N)$, and let ϕ_t be the flow of the system. Suppose* $\mathbf{a}$ *is hyperbolic and that* $\mathrm{D}\mathbf{f}(\mathbf{a})$ *has k eigenvalues with negative real part and n-k eigenvalues with positive real part. Then there exists a k-dimensional differential manifold S tangent to the stable subspace E^s of the linear system* (2.5) *at* $\mathbf{a}$*, and a (n-k)-dimensional differential manifold U tangent to the unstable subspace E^u of* (2.5) *such that*

$$\begin{array}{ll} \phi_t(S) \in S \;\; \forall t \geq 0\,; & \lim_{t\to\infty} \phi_t(S) = \mathbf{a}, \\ \phi_t(U) \in U \;\; \forall t \leq 0\,; & \lim_{t\to-\infty} \phi_t(U) = \mathbf{a}. \end{array}$$

Proof See Perko [75].

S and U are called the local stable and unstable manifolds of $\mathbf{a}$; they can be extended outside the neighborhood N in Theorem (2.2.1) by application of the flow on S and U.

Definition. *The global stable and unstable manifolds of an equilibrium point* $\mathbf{a}$ *of* (2.1) *are defined by*

$$W^s(\mathbf{a}) = \bigcup_{t\leq 0} \phi_t(S)\,; \quad W^u(\mathbf{a}) = \bigcup_{t\geq 0} \phi_t(U)\,.$$

Thus the neighborhoods of hyperbolic fixed points of a non-linear dynamical system (2.1) are very similar to the neighborhood of the origin of the corresponding linearized system (2.5). For a linear system like (2.4) one can always find the explicit solution $\mathbf{x}(t) = \mathrm{e}^{At}\mathbf{x}_0$, but in the non-linear case it is not as easy. But the theorem after which this subsection is named states that in a neighborhood of a critical point there exists a solution that is similar, in a precisely defined way, to the solution of the corresponding linearized system, namely a solution that is *topologically conjugate*:

Theorem 2.2.2 (Hartman-Grobman Theorem). *If* $\mathbf{a}$ *is a hyperbolic fixed point of eq.* (2.1)*, then there exists a homeomorphism*[9] *H from some neighborhood U of* $\mathbf{a}$ *to some neighborhood V of the origin such that*

$$H \circ \phi_t(\mathbf{x}) = \mathrm{e}^{At} H(\mathbf{x}),$$

where ϕ_t is the flow of (2.1) *and* $A = \mathrm{D}\mathbf{f}(\mathbf{a})$.

Proof. See the book by Philip Hartman himself [33].

The solution curves of the non-linear system and the ones of its linearization are thus in one-to-one correspondence near a hyperbolic fixed point, and the time direction and parametrization is preserved by the map as well.[10]

2.3 Center Manifolds

The previous sections have been focused on the behavior of dynamical systems in the neighborhood of hyperbolic fixed points, but many systems have fixed points which are not hyperbolic. Like in the linear case there exists an invariant subset associated with the eigenvectors with eigenvalues located on the imaginary axis, called the *center manifold*, denoted by W^c. The behavior on the center manifold is truly non-linear and can be complicated to solve, and in this thesis no center manifold analysis have been made in the problems considered, with one trivial exception – the transversally hyperbolic sets described below; the discussion of center manifolds here will therefore be short.

There exists a theorem analogous to the Hartman-Grobman theorem that is relevant for non-hyperbolic equilibrium points. It is called *the reduction principle*

[9] Two sets U and V are said to be homeomorphic if there exists a continuous, one-to-one map from U onto V with a continuous inverse. Such a map is called a homeomorphism.

[10] A homeomorphism that only preserves the time direction and not the parametrization, $H \circ \phi_{\tau(t,\mathbf{x})}(\mathbf{x}) = \mathrm{e}^{At} H(\mathbf{x})$, is called a *topological equivalency*, a slightly weaker relation than topological conjugacy.

[68], or sometimes *the Shoshitaishvili theorem*[11] [18]:

Theorem 2.3.1. *Suppose that* $\mathbf{f}$ *in* (2.1) *is at least* $C^1(N)$ *in a neighborhood* N *of the origin,* $\mathbf{0}$*, where* $\mathbf{f}(\mathbf{0}) = \mathbf{0}$ *(if the fixed point is located elsewhere it can be moved to the origin by a constant translation). The* $n \times n$ *matrix* $\mathrm{D}\mathbf{f}(\mathbf{0})$ *can then be written as* $\mathrm{diag}[A, B]$ *where* A *is a square matrix with* c *eigenvalues on the imaginary axis and* B *is a square matrix with* $(n-c)$ *eigenvalues with non-zero real part. The system is topologically conjugate to the system*

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + \mathbf{F}(\mathbf{x}), \\ \dot{\mathbf{y}} &= B\mathbf{y}, \end{aligned}$$

for $\mathbf{x} \in \mathbb{R}^c$*,* $\mathbf{y} \in \mathbb{R}^{(n-c)}$ *in a neighborhood of* $\mathbf{0}$*, where* $\mathbf{F}(\mathbf{0}) = \mathbf{0}, \mathrm{D}\mathbf{F}(\mathbf{0}) = \mathbf{0}$*.*

The theorem thus tells us that the system decouples into one part that contains all the non-linear behavior and one that behaves similar to a linear system in the neighborhood of the equilibrium point. One must still solve a non-linear problem, but the dimension of the problem has been reduced to the dimension of the center manifold.

A special case of center manifolds that occurs sometimes is when one has one- two- or multi-dimensional sets of fixed points. The eigenvalues corresponding to eigendirections within these sets are then naturally identical to zero and all of the fixed points are classified as non-hyperbolic; but the non-linear part of the equation related to the directions within the 'center-manifold' is trivial, the matrix A and vector field $\mathbf{F}$ of Theorem 2.3.1 are both identically zero. The behavior in the neighborhood of these fixed point manifolds are then determined solely by the behavior in directions transversal to the manifold, i.e. by the matrix B of Theorem 2.3.1. The fixed point are then called *transversally hyperbolic* and the problem can be considered as a linear problem in the neighborhood of the fixed point set, but one where the matrix B normally is dependent on the position of the fixed point within the set. The stability properties can thus be different on different parts of the fixed point set.

2.4 Monotonicity Principle

The Hartman-Grobman theorem and the reduction principle only tells us that there exists a neighborhood of an equilibrium point where the dynamics can

[11] Proved first by Šošitaĭšvili in the more general, parameter dependent, form of the theorem that that normally bears his name, see V. I. Arnold [4] p. 269, also [94] for an English translation of the theorem and [95] for a translation of the proof.

be faithfully mimicked by a linear system (at least off the center manifold in the case of non-hyperbolic fixed points), but give no information about what happens far away from equilibrium. A useful help for obtaining results about the global behavior of the Dynamical System is if one can find functions on some invariant subset of the state space that are always decreasing or increasing in value. Then the following proposition places strong restrictions on the behavior on this subset.

Proposition 2.4.1 (Monotonicity principle). *Let $\{\phi_t\}$ be a flow on $\mathbb{R}^n$, and S an invariant set of $\{\phi_t\}$. Let $Z : S \to \mathbb{R}$ be a C^1-function whose range is the interval (a, b), where $a \in \mathbb{R} \cup \{-\infty\}$, $b \in \mathbb{R} \cup \{+\infty\}$ and $a < b$. If Z is monotone decreasing on orbits in S, then for all $x \in S$,*

$$\begin{aligned}\omega(x) &\subseteq \{s \in \bar{S}\backslash S | \lim_{y\to s} Z(y) \neq b\},\\ \alpha(x) &\subseteq \{s \in \bar{S}\backslash S | \lim_{y\to s} Z(y) \neq a\}.\end{aligned}$$

Proof See appendix A in the paper by LeBlanc et al. [58].

If one wants to find the ω- and α-limits of a set of initial data of a dynamical system, then a monotone function on an invariant subset tells us that the limit sets of the subset can be found on the boundary of this subset.

2.5 Systems with Constraints

If the variables of the dynamical system are not independent, but some fixed relation exists between them, the system of differential equations (2.1) must be supplemented by a set of constraints

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}),\\ \Phi_i(\mathbf{x}) &= 0, \qquad i = 1 \ldots k.\end{aligned} \tag{2.6}$$

For the constrained system to be consistent, the constraints must be propagated by the equation

$$\dot{\Phi}_i = C^j{}_i \Phi_j, \tag{2.7}$$

where the $C^j{}_i$ are functions of the state space, otherwise one has to add additional constraints until relation (2.7) is satisfied. Assuming then that (2.7) is satisfied and also that the constraints are linearly independent (which one can assume without loss of generality) the set of constraints define an $(n-k)$-dimensional submanifold of $\mathbb{R}^n$, on which the orbits of the system lie. If a constraint can be solved analytically everywhere for one of the variables, it can be used to eliminate this variable (and constraint) from the state space. Eliminating all of the constraints in this way amounts to finding a global coordinate system for the reduced submanifold. This can in general not be done, one normally obtains only a coordinate system defined in some coordinate patch by elimination of the constraints, which is not suitable for describing the global behavior of the dynamical system. Instead one keeps the constraints and solve them only locally in the neighborhood of fixed points.

Assume $\mathbf{x} = \mathbf{a}$ is a fixed point of the system (2.6), then one can linearize the constraints in the neighborhood of this fixed point by making a first order Taylor expansion

$$\Phi_i(\mathbf{x}) \approx \Phi_i(\mathbf{a}) + \left(x^j - a^j\right) \left(\frac{\partial \Phi_i}{\partial x^j}\right)_{\mathbf{x}=\mathbf{a}},$$

which gives the linear equation (since $\Phi_i(\mathbf{a}) = 0$):

$$\mathrm{D}\Phi_i(\mathbf{a})\mathbf{y} = 0, \tag{2.8}$$

where $\mathbf{y} = \mathbf{x} - \mathbf{a}$ and $\mathrm{D}\Phi_i(\mathbf{a}) = \left(\frac{\partial \Phi_i}{\partial x^j}\right)_{\mathbf{x}=\mathbf{a}} = \left(\frac{\partial \Phi_i}{\partial y^j}\right)_{\mathbf{y}=\mathbf{0}}$. Eq. (2.8) is a linear equation and can be used to eliminate one of the variables at the fixed point. Solving all of the constraints in this way reduces the system to the tangent space of the physical manifold, where one can use the linearization of the ODE to obtain the stability properties of the fixed point. A problem that may occur is that the gradient of the constraint function may be identically zero at the fixed point, $\mathrm{D}\Phi_i(\mathbf{a}) = 0$; one can then not use the linear approximation to solve for one of the variables. This occurs in papers I and II at the fixed point set named *the Kasner circle.*

Chapter 3

Cosmological Models

This chapter is meant as an introduction to the field of relativistic cosmology, introducing certain methods and areas of general relativity that are not part of a first course on the subject, in particular orthonormal frames, 1+3 splitting of spacetime, expansion normalized variables, and anisotropic Bianchi cosmologies.

In a relativistic cosmological model space and time is described by a four-dimensional differential manifold $\mathcal{M}$ with a Lorentzian metric $\mathbf{g}$, and matter by a symmetric tensor $\mathbf{T}$ – the stress-energy tensor (or energy-momentum tensor). The dynamical laws are the *Einstein Field Equations* (EFE)[12] that relate the curvature of spacetime, described by the Einstein tensor $\mathbf{G}$, to the energy-momentum tensor:

$$G_{ab} = T_{ab}. \tag{3.1}$$

In addition one must further specify the properties of matter by making some assumption of the type of matter model one wishes to use, possibly adding equations governing the interaction of the matter with itself or with other forms of matter, as for example Maxwell's equations for electromagnetic fields, or the equation of state for a perfect fluid, or the Vlasov equation for a kinetic gas; one then ends up with a system of coupled partial differential equations relating the different matter types with each other and the geometry of spacetime.

[12] For an introductory course in Einstein's general theory of relativity one could start with Bernard Schutz's book [91]; the mathematically inclined could thereafter continue with the book by Robert Wald [104], and for a "rigorous, full-year course at the graduate level" is the extensive but very readable classic *Gravitation* [70] by Misner, Thorne and Wheeler a good choice.

3.1 Choosing a Frame

The fundamental objects in general relativity are all represented as tensors (or spinors) on the tangent space of the manifold, and to extract information such as the energy density of matter at some point, or "the gravitational potential"[13], one must specify a basis (also called a frame).

Let $\{\mathbf{e}_a\}$ be a basis of vector fields and $\{\boldsymbol{\omega}^a\}$ the dual[14] basis of one-forms, where $a = 0, 1, 2, 3$. A vector field, $\mathbf{v}$, can then be expressed in this frame as

$$\mathbf{v} = v^a \mathbf{e}_a,$$

where v^a are the components of the vector field relative to the basis. The components of the metric $\mathbf{g}$ relative to this basis can be obtained by acting with the metric on the basis vectors

$$g_{ab} = \mathbf{g}(\mathbf{e}_a, \mathbf{e}_b),$$

and the metric can conversely be expanded by summing the metric components over the tensor product of the dual basis

$$ds^2 \equiv \mathbf{g} = g_{ab}\, \boldsymbol{\omega}^a \otimes \boldsymbol{\omega}^b,$$

where the common notation ds^2 for the metric is introduced. The notation ds^2 emphasizes the role of the metric to represent "infinitesimal squared distance" and with this notation the metric is normally called *the line element*.

Any set of linearly independent vectors that span the tangent space is permissable as a frame, but two choices stand out as especially convenient:

1. The coordinate frame. In a given coordinate chart $\{x^\mu\}$, $\mu = 0, 1, 2, 3$, one can choose the partial derivatives with respect to the coordinate functions $\{\partial/\partial x^\mu\}$ as a basis, with the dual basis being the coordinate one-forms $\{\mathrm{d}x^\mu\}$.

2. The orthonormal frame (also *tetrad*, or *vierbein*). On a metric space one can choose a basis where the four vector fields are mutually orthogonal and of unit length (with the time-like basis vector of negative unit length). In this frame the metric components are given by

$$\mathbf{g}(\mathbf{e}_a, \mathbf{e}_b) = \eta_{ab},$$

[13] The Newtonian gravitational potential can be defined for a metric "close to flatness" as the time-time component of the object $\frac{1}{4}(\frac{1}{2}\gamma\, \eta_{\mu\nu} - \gamma_{\mu\nu})$, where $\gamma = \gamma^\mu{}_\mu$, and $\gamma_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ is the deviation from the Minkowski metric, $\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. See for example Wald [104] p. 74–75.

[14] $\boldsymbol{\omega}^a$ and $\mathbf{e}_b$ satisfy the duality relation $\boldsymbol{\omega}^a(\mathbf{e}_b) = \delta^a{}_b$, where $\delta^a{}_b$ is the Kronecker delta.

where $\eta_{ab} = \mathrm{diag}(\text{-}1, 1, 1, 1)$, and the line element is given by

$$ds^2 = \eta_{ab}\, \boldsymbol{\omega}^a \otimes \boldsymbol{\omega}^b.$$

The components $\Gamma^a{}_{bc}$ of the Levi-Civita connection[15] (or *covariant derivative*), ∇, relative the frame $\{\mathbf{e}_a\}$ are defined through the relation

$$\nabla_{\mathbf{e}_b} \mathbf{e}_a = \Gamma^c{}_{ab} \mathbf{e}_c. \tag{3.2}$$

The $\Gamma^a{}_{bc}$ are called *the connection coefficients*[16]. They can be expressed in terms of derivatives of the metric components and commutators of the frame fields:

$$\Gamma_{abc} = \tfrac{1}{2} \left[\mathbf{e}_b(g_{ac}) + \mathbf{e}_c(g_{ba}) - \mathbf{e}_a(g_{bc}) + c^d{}_{cb} g_{ad} + c^d{}_{ac} g_{bd} - c^d{}_{ba} g_{cd} \right], \tag{3.3}$$

where $\Gamma_{abc} = g_{ad}\Gamma^d{}_{bc}$, and $c^a{}_{bc}(x^i) = \boldsymbol{\omega}^a([\mathbf{e}_b, \mathbf{e}_c])$ are called the *commutator functions* (also *commutator coefficients* [70]).

In the coordinate frame the commutator functions are identically zero (since partial derivatives commute) and the connection coefficients take the form

1. $$\Gamma_{\mu\nu\rho} = \tfrac{1}{2} \left[\partial_\nu(g_{\mu\rho}) + \partial_\rho(g_{\nu\mu}) - \partial_\mu(g_{\nu\rho}) \right], \tag{3.4}$$

where $\partial_\mu \equiv \partial/\partial x^\mu$. The connection coefficients are then called *Christoffel symbols*[16], which are symmetric in the last two indices, $\Gamma_{\mu\nu\rho} = \Gamma_{\mu(\nu\rho)}$.

In the orthonormal frame on the other hand, the metric components are constant and the connection coefficients take the form

2. $$\Gamma_{abc} = \tfrac{1}{2} \left[c^d{}_{cb} \eta_{ad} + c^d{}_{ac} \eta_{bd} - c^d{}_{ba} \eta_{cd} \right]. \tag{3.5}$$

The connection coefficients are then called *Ricci rotation coefficients*[16], which are anti-symmetric in the first two indices, $\Gamma_{abc} = \Gamma_{[ab]c}$. In this case there is a one-to-one correspondence between the connection coefficients and the commutator functions, where the inverse relation is given by

$$c^a{}_{bc} = -\left[\Gamma^a{}_{bc} - \Gamma^a{}_{cb} \right]. \tag{3.6}$$

[15] A Levi-Civita connection is the unique affine connection that is: 1. Compatible with the metric, 2. Torsion free [86].

[16] The terminology in this area is a bit confusing. Misner et al. [70] uses the name *connection coefficients*, but Schutz [90] calls them *Christoffel symbols*. Another name is *Christoffel symbols of the second kind* (see Mathworld.com [108] for references, there Christoffel symbols of the second kind are mapped to Christoffel symbols of the *first kind* by the metric, $\Gamma_{abc} = g_{ad}\Gamma^d{}_{bc}$, but here no distinction is made between the two). Here the name Christoffel symbol is reserved for the components relative a coordinate basis, as in [70] and Wald [104]. The components relative an orthonormal frame are called *Ricci rotation coefficients*.

For a completely general frame there are n^3 independent commutator coefficients (or in four dimensions 64). In a coordinate basis the symmetry in the two last indices reduce the number to $n^2(n+1)/2$ (or 40 in 4D) independent Christoffel symbols, and in the orthonormal frame there are as many Ricci rotation coefficients as there are independent commutator functions, i.e. $n^2(n-1)/2$ (or 24 in 4D).

From relation (3.2) one can show that the covariant derivative acting on a vector $\mathbf{v}$ produces a (1,1) tensor $\nabla\mathbf{v}$ with components $v^a{}_{;b} = \mathbf{e}_b(v^a) + \Gamma^a{}_{cb}v^c$, where sometimes the notation $\nabla_b v^a$ for $v^a{}_{;b}$ is used.[17]

The Riemann curvature tensor is defined from the connection through the curvature operator, $\mathcal{R}(\mathbf{u},\mathbf{v}) = [\nabla_\mathbf{u}, \nabla_\mathbf{v}] - \nabla_{[\mathbf{u},\mathbf{v}]}$, as $R^a{}_{bcd} = \boldsymbol{\omega}^a(\mathcal{R}(\mathbf{e}_c, \mathbf{e}_d)\mathbf{e}_b)$.[18] Its components of can be written in terms of the connection coefficients and their derivatives

$$R^a{}_{bcd} = \mathbf{e}_c(\Gamma^a{}_{bd}) - \mathbf{e}_d(\Gamma^a{}_{bc}) + \Gamma^a{}_{fc}\Gamma^f{}_{bd} - \Gamma^a{}_{fd}\Gamma^f{}_{bc} - \Gamma^a{}_{bf}c^f{}_{cd}.$$

Through contractions of the Riemann curvature tensor one can define the Ricci tensor and the Ricci scalar,

$$R_{ab} = R^c{}_{acb}, \qquad R = R^a{}_a,$$

and in terms of these objects the Einstein tensor is defined as

$$G_{ab} := R_{ab} - \tfrac{1}{2}Rg_{ab}.$$

In a coordinate basis the fundamental variables are the 10 metric components, the EFE are then second order partial differential in these variables and the gauge group is the group of diffeomorphisms on the spacetime manifold.

In an orthonormal basis the metric components are trivial, the fundamental variables are instead the 16 tetrad components (also called *frame functions*), $e_a{}^\mu$, relating the tetrad basis fields to a coordinate basis:

$$\mathbf{e}_a = e_a{}^\mu \partial/\partial x^\mu = e_a{}^\mu\partial_\mu, \qquad \boldsymbol{\omega}^a = e^a{}_\mu \,\mathrm{d}x^\mu, \qquad (\mu = 0,1,2,3)$$

where the tetrad components $e_a{}^\mu(x^\nu)$ and their inverse components $e^a{}_\mu(x^\nu)$ satisfy the following duality relations and orthogonality conditions

$$\begin{array}{rclrcl} e_a{}^\mu\, e^a{}_\nu &=& \delta^\mu{}_\nu, & \eta_{ab}\, e^a{}_\mu\, e^b{}_\nu &=& g_{\mu\nu}, \\ e_a{}^\mu\, e^b{}_\mu &=& \delta^b{}_a, & g_{\mu\nu}\, e_a{}^\mu\, e_b{}^\nu &=& \eta_{ab}. \end{array}$$

[17] The index on ∇_b now is a representation of its status as a (0,1)-differential operator, something different from when a vector is used as an index as in $\nabla_\mathbf{u}$ which is a scalar differential operator related to ∇_b as $\nabla_\mathbf{u} = u^b\nabla_b$.

[18] The definition of $R^a{}_{bcd}$ is the same as in Hawking & Ellis [35], Misner et al. [70] and Schutz [91], but different in sign from other textbooks, such as Weinberg's [106].

In addition an extra $SO(3,1)$ gauge freedom is introduced through the choice of orthonormal frame.

One precondition to the use of dynamical systems methods on a system of differential equations is that the equations are of first order. To this end the 24 commutator functions $c^a{}_{bc}$ are elevated to the status of independent variables, which make the EFE first order partial differential equations in these variables. The defining relation of the commutator functions become 24 differential equations:

$$[\mathbf{e}_a, \mathbf{e}_b] = c^c{}_{ab}\,\mathbf{e}_c \;\Rightarrow\; e_a{}^\nu \partial_\nu\, e_b{}^\mu - e_b{}^\nu \partial_\nu\, e_a{}^\mu = c^c{}_{ab}\, e_c{}^\mu. \tag{3.7}$$

The Jacobi identities become integrability conditions for the equations above,

$$\mathbf{e}_{[c} c^d{}_{ab]} - c^d{}_{e[c} c^e{}_{ab]} = 0, \tag{3.8}$$

and give first order equations for some of the commutator functions. This first order approach is called *the orthonormal frame formalism* [102].[19]

3.2 Spacetime to Space and Time

For computational reasons it may be advantageous to introduce a preferred time-like direction, even though the kinematical structure does not provide one to start with. The direction of time might be physically motivated, e.g. by a time-like vector field $\mathbf{u}$ describing the average motion of matter, or defined as being the direction orthogonal to a preferred space-like hypersurface foliation Σ for example; or it may be a purely mathematical construct.

Assume $\mathbf{u}$ is a unit vector field determining some time-like fibration of the spacetime manifold, physical or otherwise. The orthonormal frame can be aligned with the fibers such that $\mathbf{e}_0 = \mathbf{u}$. All tensors can then be split into components along $\mathbf{u}$ and components orthogonal to $\mathbf{u}$ (just by taking their components relative the orthonormal basis), i.e. $\mathbf{v} = v^a \mathbf{e}_a = v^0 \mathbf{e}_0 + v^\alpha \mathbf{e}_\alpha$ for a vector $\mathbf{v}$, where $\{\mathbf{e}_\alpha\}$ $(\alpha = 1,2,3)$ is the spatial triad orthogonal to $\mathbf{e}_0$. The introduction of the vector field $\mathbf{u}$ can be viewed as a partial frame gauge fixation, since the

[19] This is of course not the only way to obtain first order equations; one could for example use a coordinate frame and use the metric components together with the Christoffel symbols as a kind of generalized momenta as variables (this is normally known as the "Palatini method" but was actually first investigated by Einstein [24]. See Ferraris *et al.* [28] for the historical developments of the method.), or as in the ADM formalism [5] use the 3-metric on a spatial foliation together with the extrinsic curvature as variables, which yields a formulation that is first order only in the time derivative, not in the spatial derivatives. This is the formulation used in paper V.

existence of a preferred direction of time reduces the gauge group in the equations involving only tetrad components from the Lorentz group $SO(1,3)$ to the subgroup $SO(3)$ of spatial rotations preserving the form of the spatial metric components.

Splitting the Commutator Functions

A 1+3 split of the commutator functions is commonly written in terms of the following variables (see Ellis & MacCallum [26], Wainwright & Ellis [102], or Ellis & van Elst [27])[20]:

$$(\theta,\, \sigma_{\alpha\beta},\, \dot{u}_\alpha,\, \omega_\alpha,\, \Omega_\alpha,\, n_{\alpha\beta},\, a_\alpha), \tag{3.9}$$

where

$\theta = c^\alpha{}_{0\alpha} = u^a{}_{;a}$ is the *expansion* of the congruence $\mathbf{u}$.

$\sigma_{\alpha\beta} = c_{\langle\alpha\beta\rangle 0}$ (where $c_{abc} = g_{ad}c^d{}_{bc}$, and $\langle ... \rangle$ is used to denote trace-free symmetrization[21] of indices) is the *shear tensor*. The shear tensor can be defined covariantly as the trace-free symmetric part of the spatial projection of the covariant derivative of $\mathbf{u}$; $\sigma_{ab} = h^c{}_{\langle a}h^d{}_{b\rangle}u_{c;d}$, where $h_{ab} = g_{ab}+u_a u_b$. The $\sigma_{\alpha\beta}$ are then the spatial components of this tensor, all the others are zero since $\sigma_{ab}u^a = \sigma_{ab}u^b = \sigma_{a\mu}\mathbf{e}_0{}^\mu = 0$.

$\dot{u}_\alpha = c^0{}_{0\alpha} = u_{\alpha;b}u^b$ is the *acceleration vector*. Note that $\mathbf{u}$ is a geodesic congruence when $\dot{u}_\alpha = 0$.

$\omega^\alpha = \frac{1}{2}\epsilon^{\alpha\beta\gamma}\omega_{\beta\gamma} = \frac{1}{4}\epsilon^{\alpha\beta\gamma}c^0{}_{\beta\gamma}$ is the *vorticity vector*, and $\omega_{\beta\gamma}$ is the equivalent *vorticity tensor*. The vorticity vector (or tensor) describes the rotation of the congruence $\mathbf{u}$. The congruence is hyper-surface normal if the vorticity is zero. The vorticity tensor can be defined covariantly as the antisymmetric part of the spatial projection of the covariant derivative of $\mathbf{u}$; $\omega_{ab} = h^c{}_{[a}h^d{}_{b]}u_{c;d}$.

$\Omega^\alpha = -(\frac{1}{2}\epsilon^\alpha{}_\beta{}^\gamma c^\beta{}_{0\gamma} + \omega^\alpha) = \frac{1}{2}\epsilon^{\alpha bcd}u_b e_c{}^\mu e_{d\mu;f}u^f$ is the local angular velocity of the spatial frame with respect to a Fermi-propagated frame.

$n^{\alpha\beta}$ and a^α are derived from the spatial components of the commutator functions by first mapping them to a 3×3 matrix with the totally antisymmetric tensor, $c^{\alpha\beta} = c^\alpha{}_{\gamma\delta}\epsilon^{\beta\gamma\delta}$, and then separating it into a symmetric

[20] There are some differences in the literature as of how to define the variables Ω^α and ω^α. Here the conventions of [27] are followed, but in [102] both Ω^α and ω^α are defined with the opposite sign. In van Elst & Uggla [101] only ω^α is defined with the opposite sign.

[21] $A_{\langle\alpha\beta\rangle} = A_{(\alpha\beta)} - \frac{1}{3}\delta_{\alpha\beta}A^\gamma{}_\gamma$

and a antisymmetric part, and finally mapping the antisymmetric part to a three-vector, $c^{\alpha\beta} = c^{(\alpha\beta)} + c^{[\alpha\beta]} = n^{\alpha\beta} + \epsilon^{\alpha\delta\gamma} a_\gamma$. The decomposition is due to Engelbert Shücking, and was introduced as a way to classify the three-dimensional structure constants of real Lie algebras (as described in Krasiński et al. [57]).

The scalar θ, the five components of the trace-free symmetric matrix $\sigma_{\alpha\beta}$, the three components of each of $\dot{u}_\alpha$, ω_α, Ω_α, a_α, and the six components of $n_{\alpha\beta}$ completely characterize the 24 commutator functions $c^a{}_{bc}$. Expanding the commutator equations in terms of the decomposed commutator functions one obtains:

$$[\,\mathbf{e}_0, \mathbf{e}_\alpha\,] = \dot{u}_\alpha\,\mathbf{e}_0 - [\,\tfrac{1}{3}\theta\,\delta_\alpha{}^\beta + \sigma_\alpha{}^\beta + \epsilon_\alpha{}^\beta{}_\gamma\,(\omega^\gamma + \Omega^\gamma)\,]\,\mathbf{e}_\beta, \tag{3.10a}$$

$$[\,\mathbf{e}_\alpha, \mathbf{e}_\beta\,] = 2\epsilon_{\alpha\beta\gamma}\,\omega^\gamma\,\mathbf{e}_0 + (2a_{[\alpha}\,\delta_{\beta]}{}^\gamma + \epsilon_{\alpha\beta\delta}\,n^{\delta\gamma})\,\mathbf{e}_\gamma. \tag{3.10b}$$

From the commutator equations (3.10b) one can immediately see that a vanishing ω^α implies that the algebra $[\,\mathbf{e}_\alpha, \mathbf{e}_\beta\,]$ closes and that the spatial frame becomes hyper-surface forming by Frobenius' theorem (see Schutz [90] p. 81).

Splitting the Energy-Momentum Tensor

To relate the variables (3.9), describing the properties of the geometry and the frame, to the energy-momentum tensor through the EFE one must make a similar split of T^{ab} w.r.t. the time-like congruence $\mathbf{u}$:

$$T_{ab} = \rho\,u_a\,u_b + 2q_{(a}\,u_{b)} + p\,h_{ab} + \pi_{ab}, \tag{3.11}$$

where $\rho = T_{ab}u^a u^b$ is the *energy density* relative to $\mathbf{u}$, $q^a = -T_{bc}u^b h^{ca}$ is interpreted as the *relativistic momentum density* or *energy flux* relative to $\mathbf{u}$, $p = \frac{1}{3}T_{ab}h^{ab}$ is the *isotropic pressure*, and $\pi_{ab} = T_{cd}h^c{}_{\langle a}h^d{}_{b\rangle}$ is the *anisotropic pressure*, or *stress-tensor*. From the definitions of the variables (ρ, q_a, p, π_{ab}) it follows that they satisfy the properties

$$q_a u^a = 0, \quad \pi_{ab}u^a = \pi_{ab}u^b = 0, \quad \pi^a{}_a = 0, \quad \pi_{ab} = \pi_{(ab)}.$$

The tensors q_a and π_{ab} can be completely characterized by their spatial components, q_α and $\pi_{\alpha\beta}$, since neither has any components along $\mathbf{u} = \mathbf{e}_0$. The ten components of the tensor T_{ab} are thus expressed in the two scalars ρ and p, the three components of the spatial vector q_α, and the five components of the trace-free, symmetric spatial stress-tensor $\pi_{\alpha\beta}$.

The Einstein Field Equations

Writing the EFE in the form

$$R_{ab} = T_{ab} - \tfrac{1}{2}T^c{}_c g_{ab},$$

expressing the Ricci tensor in the variables (3.9), the energy-momentum tensor in the variables (3.11), and then projecting them onto a 00-part, a 0α-part, a $(00) + (11) + (22) + (33)$-part, and an $\langle\alpha\beta\rangle$-part one obtains the equations

$$\mathbf{e}_0(\theta) = -\tfrac{1}{3}\theta^2 - \sigma^2 + 2\omega^2 + (\mathbf{e}_\alpha + \dot{u}_\alpha - 2a_\alpha)\dot{u}^\alpha - \tfrac{1}{2}(\rho + 3p), \tag{3.12a}$$

$$\begin{aligned} q_\alpha =& \tfrac{2}{3}\mathbf{e}_\alpha(\theta) - (\mathbf{e}_\beta - 3a_\beta)\sigma_\alpha{}^\beta - \epsilon_\alpha{}^{\gamma\delta}\sigma_\gamma{}^\beta n_{\beta\delta} \\ & - \epsilon_\alpha{}^{\beta\gamma}(\mathbf{e}_\beta + 2\dot{u}_\beta - a_\beta)\omega_\gamma - n_\alpha{}^\beta\omega_\beta, \end{aligned} \tag{3.12b}$$

$$\rho = \tfrac{1}{3}\theta^2 - \sigma^2 + \omega^2 - 2\omega_\alpha\Omega^\alpha + \tfrac{1}{2}{}^3R, \tag{3.12c}$$

$$\begin{aligned} \mathbf{e}_0(\sigma_{\alpha\beta}) =& -\theta\sigma_{\alpha\beta} - 2\epsilon^{\gamma\delta}{}_{\langle\alpha}\sigma_{\beta\rangle\gamma}\Omega_\delta - 2\omega_{\langle\alpha}\Omega_{\beta\rangle} - {}^3S_{\alpha\beta} + \pi_{\alpha\beta} \\ & + (\mathbf{e}_{\langle\alpha} + \dot{u}_{\langle\alpha} + a_{\langle\alpha})(\dot{u}_{\beta\rangle}) - \epsilon^{\gamma\delta}{}_{\langle\alpha}n_{\beta\rangle\gamma}\dot{u}_\delta, \end{aligned} \tag{3.12d}$$

where

$${}^3S_{\alpha\beta} = b_{\langle\alpha\beta\rangle} + 2\epsilon^{\gamma\delta}{}_{\langle\alpha}\, n_{\beta\rangle\delta}\, a_\gamma + \mathbf{e}_\gamma(\delta^\gamma{}_{\langle\alpha}\, a_{\beta\rangle} + \epsilon^\gamma{}_{\langle\alpha}{}^\delta\, n_{\beta\rangle\delta}), \tag{3.13a}$$

$${}^3R = -\tfrac{1}{2}b^\alpha{}_\alpha - 6a^2 + 4\mathbf{e}_\alpha\, a^\alpha, \tag{3.13b}$$

$$b_{\alpha\beta} = 2n_{\alpha\gamma}\, n^\gamma{}_\beta - n^\gamma{}_\gamma\, n_{\alpha\beta}. \tag{3.13c}$$

The first equation is the *Raychaudhuri equation* [77], from which one immediately can see that all non-spinning, geodesic, time-like congruences in a universe where the matter obeys the strong energy condition[22] have a forever decreasing expansion. The objects ${}^3R$ and ${}^3S_{\alpha\beta}$ have no physical meaning in general but when the congruence is irrotational they represent the scalar and trace-free part of the 3-curvature of the hyper-surfaces spanned by the spatial frame.

The Jacobi Identities

A 1+3-split of the Jacobi identities can be obtained by mapping them onto a 4×4-matrix-system with the 4-dimensional totally antisymmetric tensor and then projecting it onto directions parallel or orthogonal to $\mathbf{u}$. Let

$$(\mathbf{e} \times c)^{ab} := \epsilon^{acde}\mathbf{e}_c c^b{}_{de}, \qquad (c \times c)^{ab} := \epsilon^{acde}c^b{}_{fc}c^f{}_{de};$$

the Jacobi identities are then expressed through the equation

$$(\mathbf{e} \times c)^{ab} - (c \times c)^{ab} = 0.$$

[22] The strong energy condition: $\rho + 3p \geq 0$.

Separating it into a 00-component, 0α- and $\alpha 0$-components, symmetric $\alpha\beta$-components, and anti-symmetric $\alpha\beta$-components results in the following equations:

$$\mathbf{e}_\alpha \omega^\alpha - (\dot{u}_\alpha + 2a_\alpha)\omega^\alpha = 0, \tag{3.14a}$$

$$\mathbf{e}_\beta n^{\beta\alpha} + \epsilon^{\alpha\beta\gamma}\mathbf{e}_\beta a_\gamma + 2\theta\omega^\alpha + 2\sigma^\alpha{}_\beta\omega^\beta - 2n^\alpha{}_\beta a^\beta - 2\epsilon^{\alpha\beta\gamma}\omega_\beta\Omega_\gamma = 0, \tag{3.14b}$$

$$2\mathbf{e}_0\omega^\alpha - \epsilon^{\alpha\beta\gamma}\mathbf{e}_\beta\dot{u}_\gamma + n^{\alpha\beta}\dot{u}_\gamma + \epsilon^{\alpha\beta\gamma}a_\beta\dot{u}_\gamma + \tfrac{4}{3}\theta\omega^\alpha - \sigma^\alpha{}_\beta\omega^\beta + 2\epsilon^{\alpha\beta\gamma}\omega_\beta\Omega_\gamma = 0, \tag{3.14c}$$

$$\begin{aligned}&\mathbf{e}_0 n^{\alpha\beta} + \tfrac{1}{3}\theta n^{\alpha\beta} - 2\sigma^{(\alpha}{}_\gamma n^{\beta)\gamma} + 2\epsilon^{\gamma\delta(\alpha}n^{\beta)}{}_\gamma(\omega_\delta + \Omega_\delta) + (\mathbf{e}_\gamma + \dot{u}_\gamma)\sigma^{(\alpha}{}_\delta\epsilon^{\beta)\gamma\delta}\\ &- [\mathbf{e}^{(\alpha} + \dot{u}^{(\alpha}][\omega^{\beta)} + \Omega^{\beta)}] + [\mathbf{e}_\gamma + \dot{u}_\gamma][\omega^\gamma + \Omega^\gamma]\delta^{\alpha\beta} = 0,\end{aligned} \tag{3.14d}$$

$$2\mathbf{e}_0 a_\alpha - [\mathbf{e}_\beta - 2a_\beta + \dot{u}_\beta][\tfrac{1}{3}\theta\delta^\beta{}_\alpha + \sigma^\beta{}_\alpha + \epsilon_\alpha{}^{\beta\gamma}(\omega_\gamma + \Omega_\gamma)] = 0, \tag{3.14e}$$

where the three components of the anti-symmetric matrix equation have been mapped to a three-component vector equation with $\epsilon_{\alpha\beta\gamma}$ in (3.14e).

Energy-Momentum Conservation

From the definition of the Riemann tensor follows the Bianchi identities[23]:

$$R^a{}_{b[cd;e]} = 0, \tag{3.15}$$

which contracted once and twice gives, respectively

$$R^a{}_{bcd;a} = 2R_{b[c;d]} \qquad \text{and} \qquad 2R^a{}_{d;a} = R_{;d}.$$

Applying the twice contracted identity to the EFE results in local energy momentum conservation[24]

$$T^{ab}{}_{;b} = 0. \tag{3.16}$$

One can use equation (3.16) to obtain evolution equations for ρ and q_α along the congruence $\mathbf{u}$. These equations do not provide any further dynamical input, since they can in principle be derived from equations (3.12). Splitting (3.16) into its time- and spatial components yields:

$$\mathbf{e}_0\rho = -\theta(\rho + p) - \pi_{\alpha\beta}\sigma^{\alpha\beta} - (\mathbf{e}_\alpha + 2\dot{u}_\alpha - 2a_\alpha)q^\alpha, \tag{3.17a}$$

$$\begin{aligned}\mathbf{e}_0 q_\alpha = &- \tfrac{4}{3}\theta q_\alpha - \sigma_\alpha{}^\beta q_\beta - \epsilon_\alpha{}^{\beta\gamma}(\omega_\gamma - \Omega_\gamma)q_\beta + \epsilon_\alpha{}^{\beta\gamma}n_\beta{}^\delta\pi_{\delta\gamma}\\ &- \mathbf{e}_\alpha p - (\mathbf{e}_\beta + \dot{u}_\beta - (\rho + p)\dot{u}_\beta - 3a_\beta)\pi_\alpha{}^\beta.\end{aligned} \tag{3.17b}$$

[23] A proof can be found in most textbooks on general relativity, for example Wald [104] p. 39–40.

[24] The reasoning should actually be reversed. The need for local energy conservation was the original motivation for the form (3.1) of the EFE (see Einstein [22]).

The Bianchi identities do not provide evolution equations for the rest of the components of the energy-momentum tensor, $(p, \pi_{\alpha\beta})$; to determine their evolution it is necessary to further specify the properties of matter.

Frame Equations

To complete the picture one also want to be able to relate the frame variables to a coordinate system. Coordinates naturally adapted to a given $1+3$ decomposition of a spacetime is given by the threading approach (Boersma & Dray [7]) where the metric takes the form

$$ds^2 = -M^2 dt^2 + M^2 M_i dt dx^i + (h_{ij} - M_i M_j) dx^i dx^j,$$

where now t is a coordinate along our reference congruence $\mathbf{u}$, M is the *threading lapse function*, M_i are the components of the *threading lapse one form*, and h_{ij} are the components of the *threading metric*. In these coordinates the orthonormal frame vectors take the form

$$\mathbf{u} = \mathbf{e}_0 = M^{-1}\partial_t, \qquad \mathbf{e}_\alpha = e_\alpha{}^i (M_i \partial_t + \partial_i) \equiv M_\alpha \partial_t + e_\alpha{}^i \partial_i. \tag{3.18}$$

This amounts to a partial fixing of the tetrad component functions, $e_a{}^\mu$, where $e_0{}^t = M^{-1}$, $e_\alpha{}^t = M_\alpha$, and where

$$e_0{}^i = 0.$$

From (3.7) one obtains equations for the non-vanishing tetrad component functions:

$$\mathbf{e}_0\, e_\alpha{}^i = -[\,\tfrac{1}{3}\theta\, \delta_\alpha{}^\beta + \sigma_\alpha{}^\beta + \epsilon_\alpha{}^\beta{}_\gamma\, (\omega^\gamma + \Omega^\gamma)]\, e_\beta{}^i, \tag{3.19a}$$

$$\mathbf{e}_0\, M_\alpha = -[\,\tfrac{1}{3}\theta\, \delta_\alpha{}^\beta + \sigma_\alpha{}^\beta + \epsilon_\alpha{}^\beta{}_\gamma\, (\omega^\gamma + \Omega^\gamma)\,]\, M_\beta + (\mathbf{e}_\alpha + \dot{u}_\alpha) M^{-1}, \tag{3.19b}$$

$$n_\alpha{}^\beta\, e_\beta{}^i = \epsilon_\alpha{}^{\gamma\beta}\, (\mathbf{e}_\gamma - a_\gamma) e_\beta{}^i, \tag{3.19c}$$

$$2M^{-1}\omega_\alpha = [\epsilon_\alpha{}^{\gamma\beta}\, (\mathbf{e}_\gamma - a_\gamma) - n_\alpha{}^\beta]\, M_\beta. \tag{3.19d}$$

3.3 Modeling Matter as a Perfect Fluid

In choosing a matter model one must weigh the advantages of a detailed and highly accurate description of the matter in the universe against the disadvantages of computational complexity such a model is equipped with. A simpler and more coarse model may be enough to model the most important features, but it may be difficult on the other hand to see when the simple model breaks down as a useful description of reality, especially when modeling phenomena far from experimental realization.

A reasonably simple model of the matter content of the universe on large scales is that of a *ideal* or *perfect* fluid, where the viscosity and heat flow can be neglected. The energy-momentum tensor is described by the fluid's four velocity $\hat{\mathbf{u}}$, its energy density $\hat{\rho}$, and pressure $\hat{p}$ (in the rest frame of $\hat{\mathbf{u}}$):

$$T_{ab} = (\hat{\rho} + \hat{p})\hat{u}_a\hat{u}_b + \hat{p}\, g_{ab}. \tag{3.20}$$

Hatted objects are used to distinguish the fluid's velocity, energy density and pressure from the energy density ρ, and pressure p, measured with respect to the congruence $\mathbf{u}$. So far $\mathbf{u}$ has only been a purely mathematical construct to be used as a reference along which one can measure time. If no physical structure other than the perfect fluid exists on the spacetime the most natural choice would be to pick the fluid velocity $\hat{\mathbf{u}}$ as the reference congruence, in which one could dispense with the hats and equate (3.20) with (3.11) where the shear and momentum density terms are dropped.

If one wants to keep the option open to measure time along another congruence than the fluid's velocity congruence one keep the hats and make a 1+3 split of the energy-momentum tensor (3.20) along $\mathbf{u}$, which will result in shear and momentum density terms as measured by an observer moving with $\mathbf{u}$. In such a reference frame the fluid is said to be *tilted*, while when the reference and fluid congruences are aligned the fluid is called *non-tilted* or *comoving*[25].

For a tilted fluid it is convenient to express the fluid's velocity with a space-like vector orthogonal to $\mathbf{u}$ representing the three-velocity of the fluid in the rest spaces of $\mathbf{u}$ (a three-vector is sufficient to determine the four-velocity since the constraint $\mathbf{u}\cdot\mathbf{u} = -1$ reduces the number of independent components to three). Let

$$\hat{u}^a = \gamma(u^a + v^a), \qquad u_a v^a = 0, \qquad \gamma \equiv 1/\sqrt{1 - v^2}, \tag{3.21}$$

[25] When the reference congruence is normal to a space-like hypersurface the non-tilted models are also called *orthogonal* models (see e.g. Wainwright & Ellis [103]).

where γ is the usual Lorentz factor[26,27]. From (3.11) one can compute

$$\begin{array}{rclrcl} \rho & = & \gamma^2(1+wv^2)\hat{\rho}, & q^a & = & (1+w)(1+wv^2)^{-1}\rho v^a, \\ p & = & w\rho + \frac{1}{3}(1-3w)q_a v^a, & \pi_{ab} & = & q_{\langle a}v_{b\rangle}, \end{array} \tag{3.22}$$

where $w \equiv \hat{p}/\hat{\rho}$. The perfect fluid have five degrees of freedom $(\hat{\rho}, \hat{p}, v^\alpha)$ while the conservation equations (3.16) determines evolution equations of only four of them; to completely specify the system one must specify an equation of state for the fluid. In the following it shall be assumed that the fluid obeys a *barotropic* state equation where the pressure is dependent on the energy density only,

$$\hat{p} = \hat{p}(\hat{\rho}). \tag{3.23}$$

Through the relations (3.22) and knowledge of the function (3.23) it is possible to completely determine the properties of the fluid with the variables (ρ, v^α), and from (3.17) together with (3.22) one can obtain an evolution equation for v^α:

$$\mathbf{e}_0 v^\alpha = \frac{3\theta(1+wv^2)}{\rho(1+w)}\left[\left(\delta^\alpha{}_\beta + \left(\frac{2c_s^2}{1-c_s^2v^2}\right)v^\alpha v_\beta\right)\mathbf{e}_0 q^\beta - v^\alpha\left(\frac{1+c_s^2}{1-c_s^2v^2}\right)\mathbf{e}_0\rho\right],$$

where $c_s^2 \equiv d\hat{p}/d\hat{\rho}$ is the square of the speed of sound in the fluid when $d\hat{p}/d\hat{\rho}$ is non-negative and less than, or equal to, one.

3.4 Scaling the Variables with the Expansion

The expansion of a family of observers can be interpreted as a change in a relative length scale determined by the observers, as the fluid congruence determine the length scale in FRW-models,

$$\theta = 3\dot{l}/l,$$

[26] In flat spacetime the orthonormal basis can also be a coordinate basis and the relation (3.21) can be expressed as a relation between partial derivatives

$$\partial_{\hat{t}} = \gamma(\partial_t + \mathbf{v}\cdot\boldsymbol{\partial}) \qquad \Rightarrow \qquad d\hat{t} = \gamma(dt - \mathbf{v}\cdot\mathbf{dx}),$$

which, for constant $\mathbf{v}$, can be integrated to yield the standard Lorentz transformation relation

$$\hat{t} = \gamma(t - \mathbf{v}\cdot\mathbf{x}).$$

[27] Sometimes in the literature on perfect fluids in curved spacetimes, Γ is used instead to denote the Lorentz factor while γ is reserved for the adiabatic index C_p/C_V (in for example Uggla et al. [100] or the papers in Part II of this thesis).

where the $\dot{}$ refers to the derivative along the congruence. A positive expansion at the present implies that the length scale was smaller in the past (i.e. the spatial distance between any two fibres was smaller). If $\dot{l}$ was constant one could conclude that l was zero a time $T = l/\dot{l}$ ago. The Raychaudhuri equation shows that for any non-rotating, geodesic, time-like congruence, $\dot{l}$ was actually larger before if the strong energy condition holds (which is not necessarily the case in the presence of a cosmological constant). This conclusion is relevant both for the fluid congruence in FRW as well as in anisotropic generalizations, which means that the length scale for such models was zero at a time closer to the present. The time where l was zero is referred to as the *Big Bang* and indicates that the universe started as a spacetime singularity. Originally it was assumed that the singularity in the past of FRW-models was an artifact of the exact symmetry of those models and that a singularity would not occur in more realistic models, but in 1965 Penrose [73] proved the existence of singularities in models of collapsing matter without any assumptions of symmetry, and then Hawking [34] and Geroch [31] produced similar proofs of singularities in cosmological models.[28] These proofs gave very little information of the nature of the singularities though. Lifshitz and Khalatnikov [61] provided heuristical arguments for that the approach of a cosmological singularity would in general be characterized by a metric where "The evolution of the metric proceeds through successive periods (call them eras) which condense towards $t = 0$. During every era the spatial distances in two directions oscillate, and in the third direction decrease monotonically".

The study of the structure of singularities and the evolution of cosmological models have been studied more recently by the methods of orthonormal frames and dynamical systems, as described above. Wainwright & Hsu [103] introduced variables in Bianchi cosmology that were scaled with the expansion of the reference congruence to produce a system which remained finite even as one approaches a initial singularity; Uggla et al. [100] then used these variables to study inhomogeneous models. For spatially homogeneous models it was possible to formulate the equations determining the evolution of the variables describing the matter and the geometry of spacetime as a constrained system of ODEs on a state space, i.e. a *dynamical system.* The initial singularity could then be described by the properties of the past asymptotic states of the dynamical system.

Variables very similar to those in [100] can be obtained through a slightly different viewpoint proposed by Röhr and Uggla [84] where the singularity is mapped to the boundary of a spacetime that is conformally related to the original one in a manner somewhat analogous to Penrose's "conformal infinity" [72]. This

[28] See Hawking & Ellis [35] ch. 8 for a discussion of the singularity theorems.

approach was used in paper III.

3.4.1 Normalized Variables Through a Conformal Rescaling of the Metric

Let $\tilde{\mathbf{g}}$ be an unphysical metric related to the physical metric $\mathbf{g}$ through a conformal transformation,

$$\mathbf{g} = \Psi^2 \tilde{\mathbf{g}},$$

and let $\{\tilde{\mathbf{e}}\}$ be an orthonormal basis and $\tilde{\nabla}$ a connection compatible with the unphysical metric. Using the commutator functions of the frame as the basic variables,

$$[\tilde{\mathbf{e}}_a, \tilde{\mathbf{e}}_b] = \tilde{c}^c{}_{ab}\, \tilde{\mathbf{e}}_c,$$

and making a $1+3$ split with respect to some time-like reference congruence one obtains

$$[\,\tilde{\mathbf{e}}_0, \tilde{\mathbf{e}}_\alpha\,] = \dot{\tilde{u}}_\alpha\, \tilde{\mathbf{e}}_0 - [\,\tfrac{1}{3}\tilde{\theta}\,\delta_\alpha{}^\beta + \tilde{\sigma}_\alpha{}^\beta + \epsilon_\alpha{}^\beta{}_\gamma\,(\tilde{\omega}^\gamma + \tilde{\Omega}^\gamma)\,]\,\tilde{\mathbf{e}}_\beta, \tag{3.24a}$$

$$[\,\tilde{\mathbf{e}}_\alpha, \tilde{\mathbf{e}}_\beta\,] = 2\epsilon_{\alpha\beta\gamma}\,\tilde{\omega}^\gamma\,\tilde{\mathbf{e}}_0 + (2\tilde{a}_{[\alpha}\,\delta_{\beta]}{}^\gamma + \epsilon_{\alpha\beta\delta}\,\tilde{n}^{\delta\gamma})\,\tilde{\mathbf{e}}_\gamma. \tag{3.24b}$$

The relation between the variables obtained from $c^a{}_{bc}$ and $\tilde{c}^a{}_{bc}$ are

$$\begin{array}{llll} \tilde{\theta} + 3r_0 = \Psi\theta, & \tilde{\sigma}_{\alpha\beta} = \Psi\sigma_{\alpha\beta}, & \dot{\tilde{u}}_\alpha + r_\alpha = \Psi\dot{u}_\alpha, & \tilde{\omega}_\alpha = \Psi\omega_\alpha, \\ \tilde{n}^{\alpha\beta} = \Psi n^{\alpha\beta}, & \tilde{a}_\alpha - r_\alpha = \Psi a_\alpha, & \tilde{\Omega}^\alpha = \Psi\Omega^\alpha, & \end{array} \tag{3.25}$$

where $r_a = \tilde{\mathbf{e}}_a\Psi/\Psi$, $a = 0,1,2,3$, and where the objects associated with $\mathbf{g}$ ($\tilde{\mathbf{g}}$) are calculated in an ON-frame of $\mathbf{g}$ ($\tilde{\mathbf{g}}$).[29]

In the above Ψ is a completely arbitrary function on the spacetime manifold. In cosmological models the Raychaudhuri equation shows that θ, which carries the physical dimension of $[\text{length}]^{-1}$, diverges to the past under quite general circumstances. To study the vicinity of a singularity characterized by such a divergence it is advantageous to find variables that stay finite even as θ diverges. Any conformal factor Ψ that asymptotically scales as the inverse of θ should be a natural choice for an anzats to obtain finite variables. In the literature on physical cosmology the inverse of the Hubble scalar, $H = \theta/3 = \dot{l}/l$, is often used to scale the variables, and to comply with the standard variables we choose it as the appropriate conformal factor; thus we set $\Psi = H^{-1}$ to define

[29] The $_\alpha$ on the left hand side of equations (3.25) refers to components relative a frame $\{\tilde{\mathbf{e}}_\alpha\}$ whereas the $_\alpha$ on the right hand side refers to components relative a frame $\{\mathbf{e}_\alpha\}$.

the conformally Hubble normalized variables:

$$\mathcal{M} = HM, \qquad \mathcal{M}_\alpha = H^{-1}M_\alpha, \qquad E_\alpha{}^i = H^{-1}e_\alpha{}^i, \tag{3.26a}$$
$$\Sigma_{\alpha\beta} = H^{-1}\sigma_{\alpha\beta}, \qquad W_\alpha = H^{-1}\omega_\alpha, \qquad \dot{U}_\alpha = H^{-1}\dot{u}_\alpha - r_\alpha, \tag{3.26b}$$
$$R_\alpha = H^{-1}\Omega_\alpha, \qquad A_\alpha = H^{-1}a_\alpha + r_\alpha, \qquad N^{\alpha\beta} = H^{-1}n^{\alpha\beta}, \tag{3.26c}$$

where we also have included the frame components of the frame $\{\boldsymbol{\partial}_a\} = \{H^{-1}\mathbf{e}_a\}$. The conformally Hubble normalized variables in (3.26) are all written in capital letters (or calligraphic script when the un-normalized variable is denoted by a capital letter) instead of with a ˜ to emphasize the choice of a particular conformal factor.

In addition to the Hubble scalar, which measures the expansion, another parameter, which measures whether the expansion is speeding up or slowing down is often used in the cosmological literature, this is the deceleration parameter,

$$q = -\ddot{l}l/(\dot{l})^2.$$

Rewriting this in terms of the Hubble scalar and its time derivative,

$$1 + q = -\boldsymbol{\partial}_0 H/H = r_0,$$

one can express the normalized Hubble scalar through (3.25) as

$$\tilde{H} = H^{-1}H - r_0 = -q.$$

The full set of variables describing the commutator functions $\tilde{c}^a{}_{bc}$ are then

$$(q, \Sigma_{\alpha\beta}, \dot{U}_\alpha, W_\alpha, N^{\alpha\beta}, A_\alpha, R^\alpha).$$

In the cosmological literature the energy density in the universe is often described by the *density parameter* Ω, defined by

$$\Omega = \frac{\rho}{3H^2}.$$

To obtain dimensionless variables for the full energy-momentum tensor one therefore uses the same normalization also for the pressure, energy flux, and anisotropic pressure:

$$(\Omega, P, Q_\alpha, \Pi_{\alpha\beta}) = (\rho, p, q_\alpha, \pi_{\alpha\beta})/3H^2. \tag{3.27}$$

The Dimensionless Equations

Expressing the equations (3.12), (3.14), and (3.17) in the variables (3.26) and (3.27) one obtains a system of equations where the physical length scale is factored out. The equations describe the evolution of the commutator functions and matter *relative* the overall expansion of the time lines. The equations are of two types – those that only contain spatial frame derivatives, and those that also contain time frame derivatives of the variables. The latter are called *evolution equations* and the first *constraint equations*. Note that the differential operators $\boldsymbol{\partial}_0$ and $\boldsymbol{\partial}_\alpha$ are not simply partial derivatives but differential operators that themselves evolve with time, and may include combinations of all partial derivatives. The separation into evolution and constraints may at this stage therefore might seem a bit arbitrary, but we will see below a situation where the notions really becomes well and uniquely defined.

Einstein field equations:

$$\begin{aligned} 3Q_\alpha =&(3\delta_\alpha{}^\gamma\, A_\beta + \epsilon_\alpha{}^{\delta\gamma}\, N_{\delta\beta})\,\Sigma^\beta{}_\gamma - (\boldsymbol{\partial}_\beta + 2r_\beta)\,\Sigma_\alpha{}^\beta \\ &- [\mathbf{C}_\alpha{}^\beta + 2\epsilon_\alpha{}^{\gamma\beta}(\dot U_\gamma + r_\gamma)]\,W_\beta - 2r_\alpha, \qquad (3.28a)\\ \Omega =&1 - \Sigma^2 + \tfrac{1}{6}\,{}^3\mathcal{R} + \tfrac{1}{3}W^2 - \tfrac{2}{3}W_\alpha R^\alpha - \tfrac{1}{3}(2\boldsymbol{\partial}_\alpha - 4A_\alpha + r_\alpha)\,r^\alpha, \qquad (3.28b)\\ \boldsymbol{\partial}_0\,\Sigma_{\alpha\beta} =&-(2-q)\Sigma_{\alpha\beta} + 2\epsilon^{\gamma\delta}{}_{\langle\alpha}\,\Sigma_{\beta\rangle\delta}\,R_\gamma - {}^3\mathcal{S}_{\alpha\beta} + 3\Pi_{\alpha\beta} - 2W_{\langle\alpha}\,R_{\beta\rangle} \\ &+ (\boldsymbol{\partial}_{\langle\alpha} + \dot U_{\langle\alpha} + A_{\langle\alpha})\,\dot U_{\beta\rangle} + 2(\boldsymbol{\partial}_{\langle\alpha} - r_{\langle\alpha} + A_{\langle\alpha})\,r_{\beta\rangle} \\ &- \epsilon^{\gamma\delta}{}_{\langle\alpha}\,N_{\beta\rangle\gamma}\,(\dot U_\delta + 2r_\delta), \qquad (3.28c) \end{aligned}$$

Jacobi identities:

$$\begin{aligned} &(\boldsymbol{\partial}_\alpha - \dot U_\alpha - 2A_\alpha)\,W^\alpha = 0, \qquad (3.29a)\\ &A_\beta\,N^\beta{}_\alpha - \tfrac{1}{2}\boldsymbol{\partial}_\beta\,(\epsilon_\alpha{}^{\beta\gamma}A_\gamma + N_\alpha{}^\beta) - (F_\alpha{}^\beta - 2q\delta_\alpha{}^\beta + 2\Sigma_\alpha{}^\beta)\,W_\beta = 0, \qquad (3.29b)\\ &\boldsymbol{\partial}_0\,W_\alpha = (F_\alpha{}^\beta + q\delta_\alpha{}^\beta + 2\Sigma_\alpha{}^\beta)\,W_\beta + \tfrac{1}{2}\mathbf{C}_\alpha{}^\beta\,\dot U_\beta, \qquad (3.29c)\\ &\boldsymbol{\partial}_0\,N^{\alpha\beta} = (3q\delta_\gamma{}^{(\alpha} - 2F_\gamma{}^{(\alpha})N^{\beta)\gamma} + \epsilon^{\gamma\delta(\alpha}(\boldsymbol{\partial}_\gamma + \dot U_\gamma)F_\delta{}^{\beta)}, \qquad (3.29d)\\ &\boldsymbol{\partial}_0\,A_\alpha = F_\alpha{}^\beta\,A_\beta + \tfrac{1}{2}(\boldsymbol{\partial}_\beta + \dot U_\beta)(3q\delta_\alpha{}^\beta - F_\alpha{}^\beta). \qquad (3.29e) \end{aligned}$$

The symbols $\mathbf{C}_\alpha{}^\beta$ and $F_\alpha{}^\beta$ denote collections of frequently occurring combinations of terms, and are defined by

$$\begin{aligned} F_\alpha{}^\beta &= q\,\delta_\alpha{}^\beta - \Sigma_\alpha{}^\beta - \epsilon_\alpha{}^\beta{}_\gamma\,(W^\gamma + R^\gamma),\\ \mathbf{C}_\alpha{}^\beta &= \epsilon_\alpha{}^{\gamma\beta}\,(\boldsymbol{\partial}_\gamma - A_\gamma) - N_\alpha{}^\beta, \end{aligned}$$

and the terms ${}^3\mathcal{S}_{\alpha\beta}$, ${}^3\mathcal{R}$, and $B_{\alpha\beta}$ are the dimensionless versions of the variables (3.13):

$$\begin{aligned} {}^3\mathcal{S}_{\alpha\beta} &= B_{\langle\alpha\beta\rangle} + 2\epsilon^{\gamma\delta}{}_{\langle\alpha}\, N_{\beta\rangle\delta}\, A_\gamma + \boldsymbol{\partial}_\gamma(\delta^\gamma{}_{\langle\alpha}\, A_{\beta\rangle} + \epsilon^\gamma{}_{\langle\alpha}{}^\delta\, N_{\beta\rangle\delta}), \\ {}^3\mathcal{R} &= -\tfrac{1}{2}B^\alpha{}_\alpha - 6A^2 + 4\boldsymbol{\partial}_\alpha\, A^\alpha, \\ B_{\alpha\beta} &= 2N_{\alpha\gamma}\, N^\gamma{}_\beta - N^\gamma{}_\gamma\, N_{\alpha\beta}. \end{aligned}$$

The deceleration parameter q is given in terms of the other variables through the Raychaudhuri equation

$$q = 2\Sigma^2 + \tfrac{1}{2}(\Omega + 3P) - \tfrac{2}{3}\, W^2 - \tfrac{1}{3}[\boldsymbol{\partial}_\alpha + \dot{U}_\alpha - 2(A_\alpha - r_\alpha)]\, (\dot{U}^\alpha + r^\alpha), \quad (3.32)$$

and W^2 and Σ^2 are contractions of the shear and rotation, and are defined by $\Sigma^2 = \frac{1}{6}\Sigma_{\alpha\beta}\Sigma^{\alpha\beta}$, and $W^2 = W_\alpha W^\alpha$.

The twice contracted Bianchi identities give evolution equations for some of the normalized matter variables

Matter equations:

$$\boldsymbol{\partial}_0\, \Omega = (2q-1)\,\Omega - 3P + 2A_\alpha\, Q^\alpha - \Sigma_{\alpha\beta}\Pi^{\alpha\beta} - [\boldsymbol{\partial}_\alpha + 2(\dot{U}_\alpha + r_\alpha)]\, Q^\alpha, \quad (3.33a)$$

$$\begin{aligned} \boldsymbol{\partial}_0\, Q_\alpha = {} & (F_\alpha{}^\beta - (2-q)\,\delta_\alpha{}^\beta)\, Q_\beta + (3\delta_\alpha{}^\gamma\, A_\beta + \epsilon_\alpha{}^{\delta\gamma}\, N_{\delta\beta})\,\Pi^\beta{}_\gamma \\ & + 2\epsilon_\alpha{}^{\beta\gamma}\, W_\gamma\, Q_\beta - (\boldsymbol{\partial}_\beta + \dot{U}_\beta + 2r_\beta)\,(P\delta_\alpha{}^\beta + \Pi_\alpha{}^\beta) - \dot{U}_\alpha\,\Omega - r_\alpha(\Omega - 3P). \end{aligned} \quad (3.33b)$$

Specializing to a perfect fluid one obtains from (3.33) equations for the energy density and three-velocity of the fluid.

Perfect fluid equations:

$$\boldsymbol{\partial}_0\, \Omega = (2q - 1 - 3w)\,\Omega + [(3w-1)\, v_\alpha - \Sigma_{\alpha\beta}\, v^\beta + 2(A_\alpha - \dot{U}_\alpha - r_\alpha) - \boldsymbol{\partial}_\alpha]\, Q^\alpha, \quad (3.34a)$$

$$\begin{aligned} \boldsymbol{\partial}_0 v_\alpha = {} & \bar{G}_-^{-1}\, \left[(1-v^2)(3c_s^2 - 1 - c_s^2\, A^\beta\, v_\beta) + (1-c_s^2)(A^\beta + \Sigma_\gamma{}^\beta\, v^\gamma)\, v_\beta\right] v_\alpha \\ & - [\Sigma_\alpha{}^\beta + \epsilon_\alpha{}^{\beta\gamma}\,(R_\gamma + N_\gamma{}^\delta\, v_\delta)]\, v_\beta - A_\alpha\, v^2 + \epsilon_\alpha{}^{\beta\gamma} W_\gamma\, v_\beta \\ & - (\delta_\alpha{}^\beta - v_\alpha\, v^\beta)\dot{U}_\beta - (1+w)^{-1}(1-v^2)[(1-w)\delta_\alpha{}^\beta - 4w\, c_s^2\, \bar{G}_-^{-1}\, v_\alpha v^\beta]r_\beta \\ & - \left(\frac{v}{Q}\right) \left[(\delta_\alpha{}^\beta + 2c_s^2\, \bar{G}_-^{-1}\, v_\alpha v^\beta)\boldsymbol{\partial}_\gamma(P\delta_\beta{}^\gamma + \Pi_\beta{}^\gamma) - (1+c_s^2)\bar{G}_-^{-1}\, v_\alpha\, \boldsymbol{\partial}_\beta Q^\beta\right], \end{aligned} \quad (3.34b)$$

where

$$Q^\alpha = (1+w)(G_+)^{-1}\,\Omega\, v^\alpha, \quad G_\pm = 1 \pm w\, v^2, \quad \bar{G}_\pm = 1 \pm c_s^2\, v^2.$$

From the frame equations (3.19) finally, evolution equations and constraints are obtained for the Hubble normalized shift- and lapse functions, and spatial frame components,

Frame equations:

$$\boldsymbol{\partial}_0\,\mathcal{M}_\alpha = F_\alpha{}^\beta\,\mathcal{M}_\beta + (\boldsymbol{\partial}_\alpha + \dot{U}_\alpha)\mathcal{M}^{-1}, \tag{3.35a}$$

$$\boldsymbol{\partial}_0\,E_\alpha{}^i = F_\alpha{}^\beta\,E_\beta{}^i, \tag{3.35b}$$

$$\mathbf{C}_\alpha{}^\beta\mathcal{M}_\beta = 2\mathcal{M}^{-1}W_\alpha, \tag{3.35c}$$

$$\mathbf{C}_\alpha{}^\beta\,E_\beta{}^i = 0. \tag{3.35d}$$

The system of equations (3.28) - (3.35) involves the space of functions

$$\mathbf{X} = (E_\alpha{}^i, \mathcal{M}, \mathcal{M}_\alpha, r_\alpha, W_\alpha, R_\alpha, \dot{U}_\alpha, \Sigma_{\alpha\beta}, A_\alpha, N_{\alpha\beta}, \Omega, v_\alpha). \tag{3.36}$$

Note that there are no evolution equations for the variables $(R_\alpha, \dot{U}_\alpha, r_\alpha, \mathcal{M})$. The quantities $\dot{U}_\alpha$ and $\mathcal{M}$ can be related to the other variables and their derivatives algebraically through the constraints (3.28a), (3.29a), and (3.35c) when W_α is non-vanishing:

$$\begin{aligned}\dot{U}_\alpha\ &= W^{-1}\big\{W_\alpha(\boldsymbol{\partial}_\beta - 2A_\beta)W^\beta - W^\beta(\boldsymbol{\partial}_{[\alpha} - A_{[\alpha} + 2r_{[\alpha})W_{\beta]} + W^\beta N_{\sigma[\alpha}\Sigma^\sigma{}_{\beta]} \\ &\quad - \tfrac{1}{2}W^\beta\epsilon_{\beta\alpha\gamma}\left[\Sigma^{\gamma\delta}A_\delta - 3Q^\gamma - (\boldsymbol{\partial}_\sigma + 2r_\sigma)\Sigma^{\gamma\sigma} + N^{\gamma\sigma}W_\sigma - 2r^\gamma\right]\big\}, \\ \mathcal{M}^{-1}\ &= \tfrac{1}{2}W^{-2}W^\alpha\mathbf{C}_\alpha{}^\beta\mathcal{M}_\beta.\end{aligned}$$

whereas only the part of R_α that is parallel to W_α is related to the other variables in a similar fashion through (3.28b). Equations for r_α can be obtained by writing the commutator equations in operator form

$$(\boldsymbol{\partial}_\alpha + \dot{U}_\alpha)\boldsymbol{\partial}_0 - (\delta_\alpha{}^\beta\,\boldsymbol{\partial}_0 - F_\alpha{}^\beta)\,\boldsymbol{\partial}_\beta = 0, \tag{3.38a}$$

$$2W_\alpha\,\boldsymbol{\partial}_0 - \mathbf{C}_\alpha{}^\beta\,\boldsymbol{\partial}_\beta = 0, \tag{3.38b}$$

and applying them to $\ln(H)$:

$$\boldsymbol{\partial}_0 r_\alpha \quad = \quad F_\alpha{}^\beta\,r_\beta + (\boldsymbol{\partial}_\alpha + \dot{U}_\alpha)(q+1), \tag{3.39a}$$

$$0 \quad = \quad \mathbf{C}_\alpha{}^\beta\,r_\beta - 2(q+1)\,W_\alpha. \tag{3.39b}$$

If the reference congruence $\mathbf{u}$ is chosen to be orthogonal to a spatial foliation of the spacetime, then W_α is identical to zero, and for consistency with (3.29c) also $\mathbf{C}_\alpha{}^\beta\dot{U}_\beta$ must vanish, but this is consistent with the other equations since they no longer constrain $\dot{U}_\alpha$ when W_α vanishes. In this sense both W_α and $\dot{U}_\alpha$

are gauge variables, dependent on how one chooses to foliate the spacetime with time-like curves. If instead $\mathbf{u}$ represents the 4-velocity of a physical object in the spacetime, like a fluid for example, then both quantities represent physical properties of this object, the rotation and acceleration of the fluid elements in this case, and setting them to zero is then a physical statement about the dynamics of the fluid.

3.4.2 Asymptotic Silence and Locality

Asymptotic silence towards a singularity is heuristically defined in Lim et al. [65] as the shrinking of particle horizons to zero size towards the singularity along any timeline (that does not become null) that approaches it; a singularity with this property is called *asymptotically silent*. The notion was introduced by Uggla et al. [100] with the belief that this would imply that spatial inhomogeneities would asymptotically have superhorizon scale and would not be relevant to the asymptotical dynamics. Uggla et al. [100] links the formation of shrinking particle horizons to the condition ${E_\alpha}^i \to 0$, which by Andersson *et al.* [3] is used to define asymptotic silence instead. There the notion of asymptotic silence was separated from the condition that inhomogeneities would have superhorizon scale, something that requires that spatial frame derivatives of all the variables in $\mathbf{X}$ vanishes, ${E_\alpha}^i \partial_i \mathbf{X} \to 0$. This condition is called *asymptotic locality*. It may appear that the first condition would imply the second but it is possible that spatial coordinate derivatives of $\mathbf{X}$ may blow up and counteract the vanishing of ${E_\alpha}^i$, something that have been shown to occur in numerical computations ([3], Lim [63]), so called "recurrent spike formation". This occurs along some timelines where the dynamics is not "local" in the sense above even though the singularity is asymptotically silent. Still we conjecture in paper III that the conditions of asymptotic silence and locality hold for most timelines of a generic class of models and hence that the study of these conditions are relevant for the understanding of singularities under quite general circumstances.

From equations (3.28) - (3.39) one finds that there exists an invariant subset of the total space $\mathbf{X}$ where ${E_\alpha}^i = \dot{U}_\alpha = r_\alpha = W_\alpha = \mathcal{M}_\alpha = 0$, i.e. systems with this property retains it for all times. This invariant set is called the *silent boundary*, or *local boundary* [100], and the conjecture that asymptotic locality will hold in a limit is in the dynamical systems formulation a conjecture that the asymptotic state of the system will be contained in this subset.

Equations on the Silent Boundary

On the silent boundary, the spatial frame derivatives of the functions in $\mathbf{X}$ vanishes, and the dynamics of the system is the same as if the state space functions were independent of the spatial coordinates. They are not, but the spatial coordinates have ceased to play a dynamical rôle, and are now reduced to passive indices labeling a separate system of equations at each spatial point. Of the remaining variables, $\mathcal{M}$ and R_α can now be specified completely arbitrarily and are considered as gauge variables. The remaining space of functions, which now can be regarded as a state space, consists of

$$\mathbf{S} = (\Sigma_{\alpha\beta}(x^i), N_{\alpha\beta}(x^i), A^\alpha(x^i), \Omega(x^i), v^\alpha(x^i)), \tag{3.40}$$

and the reduced system of equations on the silent boundary is a system of ordinary differential equations on a constrained state space of functions, where $\boldsymbol{\partial}_0 = \mathcal{M}^{-1}\partial_t \equiv \partial_\tau$ is a partial derivative with respect to the dimensionless time variable τ. Now the separation of equations into evolution and constraints really become just that and the system can be analyzed by the methods described in chapter 2. The equations are those of the spatially homogeneous *Bianchi models*:

Evolution equations:

$$\boldsymbol{\partial}_0\,\Sigma_{\alpha\beta} = -(2-q)\Sigma_{\alpha\beta} + 2\epsilon^{\gamma\delta}{}_{\langle\alpha}\,\Sigma_{\beta\rangle\delta}\,R_\gamma - {}^3\mathcal{S}_{\alpha\beta} + 3\Pi_{\alpha\beta}, \tag{3.41a}$$

$$\boldsymbol{\partial}_0\,A_\alpha = F_\alpha{}^\beta\,A_\beta, \tag{3.41b}$$

$$\boldsymbol{\partial}_0\,N^{\alpha\beta} = (3q\delta_\gamma{}^{(\alpha} - 2F_\gamma{}^{(\alpha})N^{\beta)\gamma}. \tag{3.41c}$$

Constraint equations:

$$0 = 1 - \Sigma^2 - \Omega_k - \Omega, \tag{3.41d}$$

$$0 = (3\delta_\alpha{}^\gamma\,A_\beta + \epsilon_\alpha{}^{\delta\gamma}\,N_{\delta\beta})\,\Sigma^\beta{}_\gamma - 3Q_\alpha, \tag{3.41e}$$

$$0 = A_\beta\,N^\beta{}_\alpha, \tag{3.41f}$$

where

$$q = 2\Sigma^2 + \tfrac{1}{2}(\Omega + 3P), \qquad P = w\Omega + \tfrac{1}{3}(1-3w)Q_\alpha v^\alpha,$$

$$Q_\alpha = (1+w)G_+^{-1}\Omega v_\alpha, \qquad \Pi_{\alpha\beta} = Q_{\langle\alpha}v_{\beta\rangle},$$

$${}^3\mathcal{S}_{\alpha\beta} = B_{\langle\alpha\beta\rangle} + 2\epsilon^{\gamma\delta}{}_{\langle\alpha}\,N_{\beta\rangle\delta}\,A_\gamma, \qquad B_{\alpha\beta} = 2N_{\alpha\gamma}\,N^\gamma{}_\beta - N^\gamma{}_\gamma\,N_{\alpha\beta},$$

$${}^3\mathcal{R} = -\tfrac{1}{2}B^\alpha{}_\alpha - 6A^2, \qquad \Omega_k = -\tfrac{1}{6}\,{}^3\mathcal{R}.$$

Perfect fluid equations:

$$\boldsymbol{\partial}_0\,\Omega = (2q-1-3w)\,\Omega + [(3w-1)\,v_\alpha - \Sigma_{\alpha\beta}\,v^\beta + 2A_\alpha]\,Q^\alpha, \tag{3.41g}$$

$$\begin{aligned}\boldsymbol{\partial}_0 v_\alpha = {}& \bar{G}_-^{-1}\,\left[(1-v^2)(3c_s^2 - 1 - c_s^2\,A^\beta\,v_\beta) + (1-c_s^2)(A^\beta + \Sigma_\gamma{}^\beta\,v^\gamma)\,v_\beta\right]v_\alpha \\ & - [\Sigma_\alpha{}^\beta + \epsilon_\alpha{}^{\beta\gamma}\,(R_\gamma + N_\gamma{}^\delta\,v_\delta)]\,v_\beta - A_\alpha\,v^2.\end{aligned} \tag{3.41h}$$

3.5 Bianchi Cosmologies

The Bianchi cosmologies are spatially homogeneous spacetimes with a three-parameter isometry group acting on spatial slices. They can be described dynamically by the dynamical system above, with the only difference that the state space variables **S** are now actually independent of the spatial coordinates. The variables $(N_{\alpha\beta}, A^\alpha)$ are are directly related to the structure constants of the Lie algebra corresponding to the group of motions on the surfaces of homogeneity. Following Schücking and Behr [57] and Ellis & MacCallum [26] the Bianchi cosmologies are usually first classified into class A ($A_\alpha = 0$) and class B ($A_\alpha \neq 0$), and then further into subgroups using the eigenvalues of the matrix $N_{\alpha\beta}$. In class B there exists a constant of motion, h, defined through the relation $A_\alpha A_\beta = \frac{1}{2} h \epsilon_{\alpha\rho\sigma}\epsilon_{\beta\gamma\delta} N^{\rho\gamma} N^{\sigma\delta}$. If $N_{\alpha\beta}$ and A_α are given in an eigenbasis of $N_{\alpha\beta}$,

$$(N_{\alpha\beta}) = \mathrm{diag}(N_1, N_2, N_3), \qquad (A_\alpha) = (A, 0, 0),$$

then the classification is given by the table 3.1,[30] where $A^2 = hN_2N_3$. The

Table 3.1: Classification of Bianchi cosmologies.

	Class A ($A=0$)						Class B ($A \neq 0$)			
Group type	I	II	VI_0	VII_0	VIII	IX	V	IV	VI_h	VII_h
N_1	0	+	0	0	−	+	0	0	0	0
N_2	0	0	+	+	+	+	0	0	+	+
N_3	0	0	−	+	+	+	0	+	−	+
h	-	-	0	0	0	0	-	-	−	+

Bianchi cosmologies are anisotropic generalizations of the homogeneous FRW-cosmologies, and orthonormal frame methods have been very useful in the study of them because of the close connection between the orthonormal frame variables

[30] The table is adapted from Table 1.1 in [102].

and the structure constants of the Lie algebra of the Killing vector fields of the isometry group. The different group types correspond to different invariant subsets of the state space **S**, defined above, and arranges the different Bianchi types into a hierarchy of increasing complexity.

The Bianchi models have been studied extensively since the late 60's as examples of exact solutions (see Stephani *et al.* [97]), and from a dynamical systems perspective at least since 1971 by Collins [17], and developed further by Bogoyavlensky [8], Rosquist/Jantzen [85], and others. The book by Coley [12], and the collaborative work edited by Wainwright and Ellis [102] give detailed accounts of the uses of dynamical systems in Bianchi cosmologies. Worth mentioning is the proof by Ringström [80] that the past asymptotic states of Bianchi type IX models with an orthogonal perfect fluid, obeying the strong and weak energy condition, generically are contained on the closure of the union of the vacuum Bianchi type I and II subsets. The important role of these two Bianchi models in connection with the early state of Bianchi type IX models and possibly also with the physical universe itself merits them with a closer description here.

Vacuum Models of Bianchi Type I (The Kasner Solutions)

If the energy-momentum tensor vanishes and the isometry group on the spatial slices is the trivial one, then one obtains the Kasner [53] solutions. The metric is usually written as

$$ds^2_{Kasner} = -dt^2 + t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2, \tag{3.42}$$

where $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^2 = 1$. In terms of the dynamical system (3.41) they correspond to fixed points where $N_{\alpha\beta} = A_\alpha = \Omega = 0$, and $\Sigma_{\alpha\beta} = \text{const}$ (the v_α's are ignored here since they are irrelevant in a vacuum model, and the R_α's can be put to zero). The only dynamics occurs in the overall expansion, given by the equation

$$\partial_\tau H = -3H,$$

where τ is the dimensionless time parameter along the congruence generated by $\boldsymbol{\partial}_0 = \mathcal{M}^{-1}\partial_t = H^{-1}\partial_t = \partial_\tau$. This equation have the solution

$$H = H_0 e^{-3\tau} = \frac{1}{3t}.$$

The conditions on the Kasner exponents p_α defines a circle in the space spanned by $(p_1,\ p_2,\ p_3)$, or equivalently the space spanned by the Hubble normalized shear eigenvalues $\Sigma_\alpha = 3p_\alpha - 1$ (fig. 3.1). The Kasner exponents can be represented in parametric form:

$$p_1 = \frac{-u}{1+u+u^2}, \quad p_2 = \frac{1+u}{1+u+u^2}, \quad p_3 = \frac{u(1+u)}{1+u+u^2}, \tag{3.43}$$

where the Kasner parameter u usually is taken to vary in the interval $[1, \infty]$, for which the Kasner exponents assume all their values and are ordered according to $p_1 \leq p_2 \leq p_3$, corresponding to sector (123) in fig. 3.1. If one extends the domain to the entire real line one can parameterize the entire circle of fixed points.[31] This is usually not done however (except in a recent paper by Damour and Lecian [20]), instead one keeps the interval of u bounded and switch the ordering of the definition of the Kasner exponents in terms of the Kasner parameter on each new sector. Each value of $u \in [1, \infty]$ then corresponds to six points on the circle (except $u = 1$ that correspond to the three points Q_α, and $u = \infty$ that correspond to the three points T_α), but each multiplet of points correspond to a unique physical state, the only difference being the labeling of the different axes. The circle therefore gives a redundant representation of the Kasner solutions. The extended definition of the Kasner parameter is nevertheless a convenient representation when the Kasner solutions are viewed as a subclass of a larger set of solutions, where the permutation symmetry of the axes is not necessarily present.

The importance of the Kasner circle lies in its connection with the dynamics at early times close to the initial singularity in a Big Bang cosmology. If one linearizes the dynamical system (3.41) with $w < 1$ at the Kasner fixed points, one finds that at each point (with the exception of the Taub points T_α) there is a single negative eigenvalue of the linearized system. The stable manifold corresponding to the negative eigenvalue describes a vacuum model where a single eigenvalue of the matrix $(N_{\alpha\beta})$ is non-zero, i.e. a vacuum model of Bianchi type II.

Vacuum Models of Bianchi Type II

In terms of the dynamical system (3.41) the vacuum Bianchi type II models define invariant subsets where $N_2 = N_3 = A_\alpha = \Omega = 0$, and N_1 is either positive or negative (or similarly with N_2 and N_3). Denote these sets $\mathcal{B}^{\text{vac.}}_{N_\alpha^\pm}$. In

[31] The transformations $u \mapsto u,\ 1/u,\ -(1+u),\ -u/(1+u),\ -1/(1+u),\ -(1+u)/u$ form a representation of the symmetric group S_3 as permutations of the Kasner exponents, and each of the corresponding intervals $[1, \infty], [0, 1], [-\infty, -2], [-1, -1/2], [-1/2, 0], [-2, -1]$ is a suitable domain for the parametrization of the Kasner exponents.

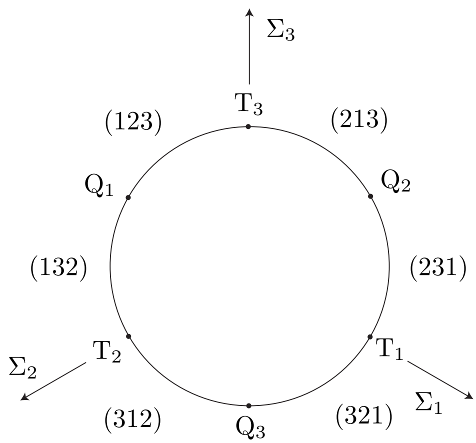

Figure 3.1: The Kasner circle, defined in terms of the shear eigenvalues. The circle is divided into six sectors ($\alpha\beta\gamma$) where the numbers denote order of the shear eigenvalues (sector (123) has $\Sigma_1 < \Sigma_2 < \Sigma_3$, and so on). Separating the sections are points where the two of the shear eigenvalues coincide. These points correspond to special solutions, that are isotropic in the plane of these eigendirections. The points Q_α correspond to the LRS Kasner solutions, where two of the Kasner exponents are equal but non-zero. The points T_α correspond to the Taub form of flat spacetime, where two of the Kasner exponents are zero.

a Fermi transported ($R_\alpha = 0$) Σ and N-eigenframe, the equations reduces to:

$$\partial_\tau \Sigma_1 = -\tfrac{1}{6} N_1^2 (\Sigma_1 + 4), \tag{3.44a}$$

$$\partial_\tau \Sigma_2 = -\tfrac{1}{6} N_1^2 (\Sigma_2 - 2), \tag{3.44b}$$

$$\partial_\tau \Sigma_3 = -\tfrac{1}{6} N_1^2 (\Sigma_3 - 2), \tag{3.44c}$$

$$\partial_\tau N_1 = 2(\Sigma^2 + \Sigma_1) N_1, \tag{3.44d}$$

with the constraints

$$\Sigma_1 + \Sigma_2 + \Sigma_3 = 0, \qquad 2\Sigma_1^2 + 2\Sigma_2^2 + 2\Sigma_3^2 + N_1^2 = 12. \tag{3.44e}$$

The constraints (3.44e) bound the absolute values of the shear variables strictly away from 2 on the proper Bianchi type II subsets. This results in a monotone decreasing Σ_1 and monotone increasing Σ_2 and Σ_3. A monotone function on a bounded codomain must have a limit value on the closure of the codomain, in particular one has $\lim_{\tau\pm\infty} \Sigma_1 \in [-2,\ 2]$. Eq. (3.44a) shows that this implies that $\lim_{\tau\pm\infty} N_1 = 0$ – the vacuum Bianchi type II orbits both start and end at the Kasner circle. The figure 3.2 shows the projection of the orbits on the plane in shear-space defined by the first constraint in (3.44e).

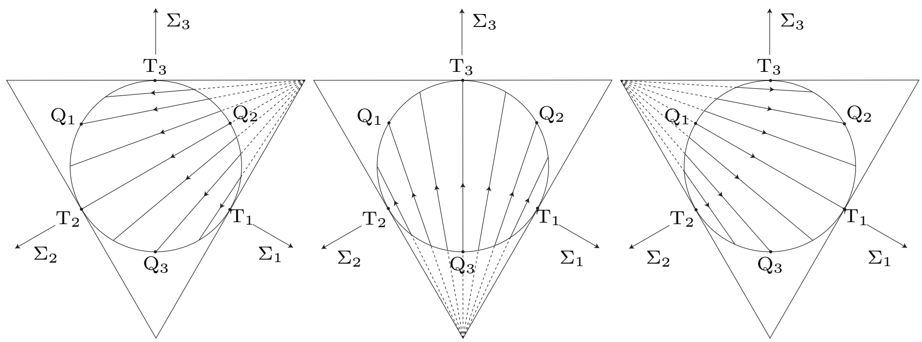

Figure 3.2: Projection of the vacuum type II orbits onto shear eigenvalue space. Arrows point in the reverse time direction.

The Mixmaster Attractor

One can view the set of orbits as a map from the Kasner circle to itself. Usually one considers this map in the reverse time direction since its importance lies in the $\tau \to -\infty$ limit. The one-dimensional stable manifold of each Kasner point connects it to another Kasner point and so on, in a never ending sequence (unless it finally connects to a Taub point), see fig. 3.3. In terms of the Kasner parameter u, the map defines a sequence $\{u_i\}$ that obeys the simple rule

$$\begin{aligned} u_{i+1} &= u_i - 1, && \text{if } u_i \geq 2, \\ u_{i+1} &= \frac{1}{u_i - 1}, && \text{if } u_i \leq 2. \end{aligned}$$

If the sequence is infinite and non-periodic it corresponds to an infinite heteroclinic network on the Kasner circle. Other possible types of sequences are finite (if the u_i's are rational numbers) or eventually periodic (see Heinzle and Uggla [36] for a detailed account on the different possible types of sequences and their statistical properties).

The importance of this heteroclinic network and corresponding Kasner map lies in its conjectured property as a template for the past dynamics for most orbits in the space of Bianchi type VIII and type IX-models, and in extension as an asymptotic limit of "almost all" cosmological models close to the singularity, including inhomogeneous models. For orthogonal models of Bianchi type IX

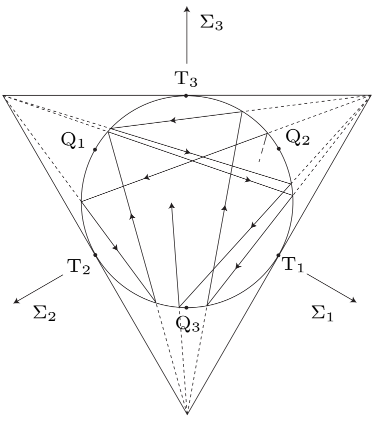

Figure 3.3: A few orbits of one of the possible heteroclinc networks on the Kasner circle. The arrows are directed towards the past.

it has been proved that the α-limit of almost all orbits is contained in the closure of the set defined by the union of all the different vacuum Bianchi type II subsets of the same sign and the Kasner circle (Ringström [80]), although it was not established definitely whether the α-limit sets coincide with this set or with subsets of it. The set is called the "Mixmaster attractor" since it gives rise to a so called Mixmaster oscillatory singularity (Wainwright & Ellis [102], Misner [69]). Recent work by Liebscher et al. [60], Béguin [6], and Reiterer & Trubowitz [78] have provided more detailed results on Mixmaster dynamics and its relevance for Bianchi models of type IX.

The Kasner solutions and the Mixmaster attractor are believed to be important models close to the initial singularity not only for homogeneous Bianchi cosmologies but for also for more general, inhomogeneous cosmologies, assuming that the matter obeys certain restrictions. Other matter sources will have different asymptotic states, see e.g. paper V in this thesis. But even defining in what way an inhomogeneous cosmology is close to a given solution requires some sort of measure on the space of all solutions, something that is much more complicated when dealing with spatially dependent objects rather than with spatial constants. The dynamical systems approach have as of yet not provided any general proofs about the asymptotic structure of inhomogeneous spacetimes without a priori imposing simplifying assumptions. One way to define "closeness" of different inhomogeneous cosmologies is provided by the Fuchsian method, which is used in paper V to prove statements about the initial singularity of inhomogeneous ultra-stiff perfect fluid cosmologies. The Fuchsian method is described in the next chapter.

3.5.1 Tilted Bianchi Cosmologies

Tilted Bianchi cosmologies, where the fluid velocity is not normal to surfaces of homogeneity, had not been studied as extensively as the non-tilted models until the beginning of the new millennium, although they are now quite well described in terms of dynamical systems essentially like (3.41), at least in terms of the existence of fixed points and their stability properties.

First of the tilted models to be fully described by means of dynamical systems were the non-rotational type V models with a linear equation of state (by Hewitt and Wainwright [46] in 1992), then followed studies of: the tilted Bianchi type II models in 2001 (Hewitt et al. [45]); the type VI_0 models in 2004 (Hervik [39]); the late time behavior of type IV, and VII_h models in 2005 (Coley and Hervik [14], Hervik et al. [41]); the late time behavior of type VII_0 in 2006 (Hervik et al. [42], Lim et al. [64]); the late time behavior of type VIII (Hervik and Lim 2006 [40]); the late time behavior of type VI_h models (Hervik et al. 2007 [43], Coley and Hervik 2008 [15]); and the the late time behavior of the exceptional type $\mathrm{VI}_{-1/9}$ model (Hervik et al. [44] 2008). The main focus of these investigations was the late time behavior of the models, and in particular whether future stable fixed points exist for them. Fixed points that are future stable were found for all the models except for a small range in parameter space (h, w) of some of the class B models where the future attractor instead had the structure of a periodic orbit (type IV, IV_h) or a torus (type VII_h). The fixed point analysis was supported by numerical results, and in some cases monotone functions that provided global results for subranges of parameter space and on invariant subspaces, but global analytical results are lacking in most cases.

In paper IV in this thesis we use Hamiltonian methods to provide new monotone functions for the tilted type II model and thereby prove that the future local attractors found in [45] also are global attractors.

The tilted Bianchi models can be generalized to multi-fluid matter sources in a straight forward manner by assuming that the fluids are non-interacting, i.e. that each fluid is separately conserved. The state space is extended in three dimensions for each extra fluid, and even finding all possible fixed points quickly become a laborious task. The tilted two-fluid type VI_0 model was partially analyzed by dynamical systems methods by Coley and Hervik [13], where they found all future stable fixed points of the system. Bianchi type I models have not been studied in connection with tilted fluids for the simple reason that the constraint equation (3.41e) forces the momentum density to vanish in the Bianchi type I case, thereby making the Bianchi type I model incompatible with tilted fluids. When considering multi-fluid models however, this constraint only states that the total momentum density of all the fluids together must vanish.

The tilted two-fluid Bianchi type I model can be considered as a subspace of the two-fluid type VI_0 model, and this subset is studied in detail in paper I and II in this thesis.

Chapter 4

Fuchsian Reduction

Another mathematical concept that is used in this thesis is that of the Fuchsian reduction, or of a Fuchsian system of equations. The concept of Fuchsian reduction has successfully been applied to the study of singularities in inhomogeneous cosmology in the past decade, where the aim has been to make rigorous mathematical statements about the structure of the singularity and the dynamics in its vicinity. This chapter reviews the basic idea of Fuchsian reduction and states a useful theorem and then describes the uses in General Relativity, with emphasis on spacetime singularities.

4.1 Fuchsian Reduction in an Analytic Setting

The denomination "Fuchsian" in Fuchsian reduction is derived from Fuchs[32] theorem and Fuchsian differential equations, where it essentially has come to denote differential equations for which all formal power series are convergent. Here, as in the cosmological literature, the term Fuchsian equation will have a more specific meaning, which will be described below. The "reduction" in the same term describes the representation of a singular solution $\mathbf{u}$ of a nonlinear differential equation in the form

$$\mathbf{u} = \mathbf{s} + t^m \mathbf{v}, \qquad (m > 0)$$

where $\mathbf{s}$ is a known function, singular at $t = 0$, and where $\mathbf{v}$ represents the regular part of $\mathbf{u}$. The reduction is useful when the solution $\mathbf{u}$ is unknown but

[32] Lazarus Immanuel Fuchs 1833-1902

expected from various considerations to have the same asymptotic behavior at $t = 0$ as $\mathbf{s}$. If it can be shown that the remaining part really is regular, then this expectation is confirmed. The trick to show this is to obtain an equation for $\mathbf{v}$ that has the particular form

$$t\partial_t \mathbf{v} + A(x)\mathbf{v} = f(t, x, \mathbf{v}, \mathbf{v}_x). \qquad (4.1)$$

Here $\mathbf{v}(t, x)$ takes values in $\mathbb{R}^k$, and $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ (or $\mathbb{C}^n$), but t is real and positive; A is a matrix-valued function of x, independent of t, and f is a function of $t, x, \mathbf{v}$ and the spatial derivatives of $\mathbf{v}$, $\mathbf{v}_x$, that takes values in $\mathbb{R}^k$. If the functions A and f satisfies certain regularity conditions one calls this a Fuchsian system of equations. The regularity conditions can actually be of different types, e.g. one can consider Fuchsian systems in analytic or smooth settings, and corresponding to the different conditions are theorems about the existence of solutions to the system. Only the most regular type of Fuchsian system will be considered here. For this system there also exists the strongest theorem, which will now be stated.

Theorem 4.1.1 (Kichenassamy & Rendall 1998 [55]). *If f is continuous in t, analytic in the other variables for $t > 0$, and vanishes like some power of t as $t \to 0$; A is analytic in x and satisfies the condition that $t^{A(x)}$ is bounded on compact subsets of x-space near $t = 0$, then the equation* (4.1) *has a unique solution $\mathbf{v}$ defined near $t = 0$ which is continuous in t and analytic in x and tends to zero as $t \to 0$.*

The positivity condition on $A(x)$ is satisfied if $A(x)$ has no eigenvalues with negative real part or purely imaginary eigenvalues, and that the dimension of the kernel of A equals the number of zero eigenvalues, i.e. no zero eigenvalues give rise to non-diagonal Jordan blocks.

The solution of the Fuchsian system that the theorem guarantees gives through the reduction a unique one-to-one correspondence between solutions $\mathbf{u}$ of the original system and the functions $\mathbf{s}$. In the limit $t \to 0$, this one-to-one map becomes the identity map. The generality of an obtained solution $\mathbf{u}$, counted as the number of free functions that can be given as initial data, depends on the generality of the leading order functions $\mathbf{s}$.

Generalizations of the above theorem exists where the differential operator $D = t\partial_t$ is replaced with an operator depending on several "time variables", $N = \sum_{k=0}^{l}(t_k + kt_{k-1})\partial/\partial t_k$ (see Kichenassamy [54, thm. 4.5]), or when the right hand side of eq. (4.1) also contains a term $g(t, x, \mathbf{v})t\partial_t\mathbf{v}$ where g fulfills the same regularity conditions as f (see Andersson & Rendall [2]).

The equation (4.1) is of first order and partial derivatives with respect to t only appears through the scale invariant operator $D = t\partial_t$. To reduce an equation of higher order to this form it is necessary to introduce auxiliary variables representing the derivatives of the variables occurring in the original equation, and often to make a space-dependent shift of the time coordinate such that the singularity under study appears at $t = 0$. It is also necessary to "extract" the divergent behavior by splitting the solution into a sum of terms, i.e. by making a formal series solution ansatz where the first terms should contain the divergent parts. The Kichenassamy/Rendall theorem is then used to show that this formal solution is an actual solution. What the theorem does not do is to provide any information on how to find the leading order terms. In physically motivated problems one can sometimes guess the leading order behavior from physical arguments or intuition, and then just use the theorem as confirmation of that guess. An indirect way of guessing the leading order terms is to suppose they are exact solutions to a different set of equations whose solutions have the same asymptotics close to $t = 0$. This has been the route taken in problems relating to general relativity, although it is not at all obvious how to find these equations. In most cases the equations are found by removing dependence of such variables that are expected to be irrelevant close to the singularity. When the system includes constraint equations (equations that do not include any derivatives with respect to the time variable t) one must also check that the new set of constraints obtained are preserved under time evolution.

4.2 Fuchsian Reduction and Spacetime Singularities

Fuchsian methods were first applied to general relativity in the context of constructing large classes of spacetimes with Cauchy horizons (see Rendall [79] for a review of the uses of Fuchsian methods in general relativity up to 2004). Their use in the study of spacetime singularities began with Gowdy models with spatial topology T^3 (Kischenassamy & Rendall [55]).

Gowdy models are vacuum spacetimes with compact spatial topology and two commuting spatial Killing vectors, thus having essentially only one degree of spatial inhomogeneity. In addition they have a discrete symmetry, characterized by the vanishing of the so called "twist constants"[33]. The spatial topology can be that of T^3, $S^2 \times S^1$, S^3 or a lens space $L(p, q)$ (Ståhl [96]), and the Einstein equations reduce to two equations of two unknown functions of two variables, plus two decoupled equations arising from the Gauss and Codazzi constraints.

[33] The twist constants are defined from the two Killing vector fields ξ, χ through the relations $\kappa_\xi = \epsilon^{abcd}\xi_a\chi_b\nabla_c\xi_d$ and $\kappa_\chi = \epsilon^{abcd}\xi_a\chi_b\nabla_c\chi_d$.

In the T^3-case one has:

The Gowdy metric:

$$ds^2 = e^{\lambda(t,\theta)/2}t^{-1/2}(-dt^2 + d\theta^2) + t[e^{P(t,\theta)}(dy + Q(t,\theta)dz)^2 + e^{-P(t,\theta)}dz^2].$$

The field equations:

$$D^2Q - t^2Q_{\theta\theta} = -2(DQDP - t^2Q_\theta P_\theta), \tag{4.2a}$$

$$D^2P - t^2P_{\theta\theta} = e^{2P}[(DQ)^2 - t^2Q_\theta^2], \tag{4.2b}$$

$$\lambda_\theta = 2(P_\theta DP + e^{2P}Q_\theta DQ), \tag{4.2c}$$

$$D\lambda = (DP)^2 + t^2P_\theta^2 + e^{2P}[(DQ)^2 - t^2Q_\theta^2], \tag{4.2d}$$

where $D = t\partial_t$. In [55] it was proven through Fuchsian methods that a solution of the Gowdy system (4.2), analytic in θ, could be expanded like

$$P(t,\theta) = k(\theta)\ln t + \phi(\theta) + t^\epsilon u(t,\theta), \tag{4.3}$$

$$Q(t,\theta) = Q_0(\theta) + t^{2k(\theta)}(\psi(\theta) + v(t,\theta)), \tag{4.4}$$

for some constant $\epsilon > 0$ and analytic functions $0 < k(\theta) < 1$, $\phi(\theta)$, $Q_0(\theta)$, and $\psi(\theta)$, where the functions u and v vanish with t. The condition on k is called "the low velocity condition"[34], and such solutions are called "asymptotically velocity term dominated" (AVTD) (or just "velocity term dominated", VTD). The function $k(\theta)$ can be related to the generalized Kasner exponents of the Kasner solution. If one chooses $Q = 0$, $P = k\ln t$ for any real constant value of k, the metric reduces as $ds^2_{Gowdy} \to t^{(k^2+3)/2}ds^2_{Kasner}$ in terms of the form (3.42) of the Kasner line element given in the previous section. This is of course a line element equivalent to (3.42), the difference only being a simple rescaling of the time coordinate. An AVTD solution can therefore be viewed as asymptotic to a different Kasner solution at each spatial point. The corresponding Kasner exponents are $(k^2-1)/(k^2+3)$, $2(1-k)/(k^2+3)$, and $2(1+k)/(k^2+3)$. The asymptotic velocity k is related to the Kasner parameter u through $k = (u-1)/(u+1)$ (see eq. (3.43)). Negative k could be treated analogously in [55], which gives an asymptotic value of k between -1 and 1. This corresponds to two sectors on the Kasner circle, namely the ones where the direction of inhomogeneity has the smallest Kasner parameter.

Ståhl [96] used Fuchsian methods to study the Gowdy models with $S^2 \times S^1$ and S^3 topologies, but the existence of symmetry axes where the defining Killing vectors vanish due to the imposed isotropy complicated the analysis. Any solution

[34] The equations (4.2) can be interpreted as a harmonic-like map from 1 + 1 Mikowski space to 1 + 1 hyperbolic space with metric $dP^2 + e^{2P}dQ^2$. The "velocity", defined as $\nu(t,\theta) = ((DP)^2 + \exp(2P)(DQ)^2)^{1/2}$, approaches $|k(\theta)|$ in the limit of small t.

must respect the isotropy about the symmetry axis, and if the past asymptotic state of a point on the axis is to be represented by a Kasner solution, it must be a locally rotationally symmetric (LRS) Kasner solution, i.e. either a Taub point or the opposite LRS point corresponding to the symmetry axis. In [96] this symmetry axis corresponded to the asymptotic velocities of -1 or 3 while all points off the symmetry axis could be treated like the T^3-case and had asymptotic velocities between 0 and 1 as above. This seems to contradict the existence of any continuous solution, $k(\theta)$, asymptotic to the Kasner solution for the $S^2 \times S^1$ and S^3 models, although it is unclear whether this is an artefact of the parametrization of the metric in [96] or an actual property of the models.

Isenberg & Kichenassamy [51] applied the Fuchsian method to so called polarized T^2-symmetric models. They are a subclass of the full class of T^2-symmetric models, which are like the Gowdy models but where the twist constants do not vanish. Like the Gowdy models the polarized T^2-symmetric models where shown to have solutions depending on the maximal set of free functions asymptotic to the Kasner solutions.

If the isometry group of the T^2 models is reduced to a single $U(1)$-group acting on a compact spatial dimension one obtain the $U(1)$ symmetric vacuum spacetimes. These models can be divided into classes of decreasing generality: generic, half-polarized, and polarized, and like the Gowdy models come with several different compact spatial topologies. Isenberg & Moncrief [52] used Fuchsian methods to show that the polarized and half-polarized $U(1)$-models with spatial topology T^3 have AVTD behavior. This result was later extended by Choquet-Bruhat & Isenberg [11] to all half polarized[35] $U(1)$-bundles over $\Sigma \times R$, where Σ is any compact surface.

In all cases described here, the Fuchsian method have been able to prove that analytic solutions of the equations under consideration have VTD (Kasner-like) behavior in the limit when $t \to 0$. Such dynamics is not, however, expected to happen in generic models. The generic behavior is supposed instead to be captured by the Mixmaster model, and so far such complicated oscillating dynamics have been beyond what the Fuchsian method can handle. Therefore the Fuchsian method have not been applied to generic T^2 symmetric or $U(1)$ symmetric models, since numerical studies and heuristic analytical results show that these models do exhibit oscillatory dynamics close to the singularity as the Mixmaster model predicts. Only special cases have singularities where the oscillatory behavior is suppressed, such as:

[35] The condition of half polarization can be interpreted geometrically. The asymptotic solutions can be interpreted as geodesics in the hyperbolic plane described in note 34 and the condition of half polarization as picking only geodesics that at $t = 0$ emanate from a particular point in the plane. (see [11, sec. 5.1.2]).

- Models with some of the gravitational degrees of freedom turned off, e.g. by reasons of symmetry. All of the models above are of this type.
- Models with more than four spacetime dimensions.
- Models with special matter sources.

Damour et al. [19] used Fuchsian methods to show that Einstein-vacuum models of dimension larger than 10 have Kasner-like singularities. The same was proved for spacetimes of lower dimension if the matter source consists of a scalar field, or a scalar field and several p-form fields with certain restrictions on the couplings between the scalar field and the p-form fields. This paper generalized, in terms of the number of dimensions studied and the inclusion of p-forms in the matter content, the analysis of Andersson & Rendall [2] where the 4-dimensional Einstein-scalar field system without symmetries was first studied with Fuchsian methods. This paper also included an analogous analysis of the Einstein-Euler system with a stiff fluid, which also could be proved to exhibit Kasner-like behavior close to the singularity.

The analysis of Andersson and Rendall can also be applied to the Einstein-Euler system with an ultra-stiff fluid, a system that have been predicted through heuristical arguments to have a singularity structure similar to the initial singularity in the Friedman universe. That this is the case is proved in the last paper included in this thesis. For further explanatory comments, see the next chapter.

Chapter 5

Comments on Accompanying Papers

Three of the five papers (I, II, and IV) in this thesis investigate homogeneous Bianchi models with tilted perfect fluids. The first two are studies of Bianchi type I models with two perfect fluids with linear equations of state where the focus has been on finding the past and future asymptotic states of the system. Paper IV combines Hamiltonian methods with the dynamical systems formulation to find monotone functions that can be used to determine the global dynamics fully. Paper III investigates the dynamics of more general cosmological models with multiple perfect fluids, with a general barotropic equation of state, where the past asymptotic state of the system is described by a local and silent singularity as defined above. The last paper uses Fuchsian methods to prove the existence of a large class of solutions to the Einstein equations coupled to an ultra-stiff fluid with a initial singularity of VTD-type.

5.1 Papers I and II

In these papers we investigate Bianchi type I models that contain two perfect fluids that are tilted with respect to the hypersurfaces of homogeneity, and in addition, a cosmological constant in paper I. To be compatible with type I the total energy flux of the two fluids must vanish, but this turns out to be a constraint that is preserved under time evolution and hence the model is self-consistent. The tilted two-fluid type I model is contained as a subspace of the tilted two-fluid type VI_0 model studied by Coley and Hervik [13], but they did not analyze it in their paper. We restrict ourselves to type I, but perform a rather thorough analysis that illustrates how the complexity of the dynamical system increases with the complexity of the source.

The kinematical structure of the problem closely resembles that of a Bianchi type I model with a magnetic field, studied in [58], and we therefore choose our variables in a way analogous to that paper. The resulting variables are not identical to those in [58], due mainly to the necessity to fix the frame rotation variables R_α with a different sign. The resulting dynamical system take the form:

Evolution equations:

$$\Sigma'_+ = -(2-q)\Sigma_+ + 3\Sigma_A^2 - Q_{(1)}v_{(1)} - Q_{(2)}v_{(2)}, \tag{5.1a}$$

$$\Sigma'_A = -(2-q+3\Sigma_+ + \sqrt{3}\Sigma_B)\Sigma_A, \tag{5.1b}$$

$$\Sigma'_B = -(2-q)\Sigma_B + \sqrt{3}\Sigma_A^2 - 2\sqrt{3}\Sigma_C^2, \tag{5.1c}$$

$$\Sigma'_C = -(2-q-2\sqrt{3}\Sigma_B)\Sigma_C, \tag{5.1d}$$

$$v'_{(i)} = (G_-^{(i)})^{-1}(1-v_{(i)}^2)(3w_{(i)} - 1 + 2\Sigma_+)v_{(i)}, \tag{5.1e}$$

$$\Omega'_{(i)} = (2q - 1 - 3w_{(i)})\Omega_{(i)} + (3w_{(i)} - 1 + 2\Sigma_+)Q_{(i)}v_{(i)}, \tag{5.1f}$$

$$\Omega'_\Lambda = 2(1+q)\Omega_\Lambda. \tag{5.1g}$$

where $i = 1, 2$ denote the two fluids.

Constraint equations:

$$0 = 1 - \Sigma^2 - \Omega_{(1)} - \Omega_{(2)} - \Omega_\Lambda, \tag{5.2a}$$

$$0 = Q_{(1)} + Q_{(2)}, \tag{5.2b}$$

where

$$q = 2\Sigma^2 + \tfrac{1}{2}(\Omega_\mathrm{m} + 3P_\mathrm{m}) - \Omega_\Lambda\,, \qquad G_\pm^{(i)} = 1 \pm wv_{(i)}^2,$$

$$Q_{(i)} = (1+w)(G_+^{(i)})^{-1}\Omega_{(i)}v_{(i)}, \qquad \Omega_\mathrm{m} = \Omega_{(1)} + \Omega_{(2)},$$

$$\Sigma^2 = \Sigma_+^2 + \Sigma_A^2 + \Sigma_B^2 + \Sigma_C^2, \qquad P_\mathrm{m} = P_{(1)} + P_{(2)},$$

and $'$ denotes differentiation w.r.t. the time parameter τ related to the vector field $\boldsymbol{\partial}_0 = d/d\tau$.

One of the five variables determining the Hubble normalized shear (in the paper denoted by ϕ) have decoupled from the other variables and is left out of the dynamical system. The remaining four are denoted $(\Sigma_+, \Sigma_A, \Sigma_B, \Sigma_C)$; the variables $(\Omega_{(i)}, v_{(i)}, \Omega_\Lambda)$ are the Hubble normalized energy densities of the two fluids, their three-velocities, and the Hubble normalized cosmological constant. The equation of state parameters $w_{(i)}$ take values from (and including) 0 to (and not including) 1. In paper I the inequality $w_{(2)} < w_{(1)}$ is imposed whereas in paper II the equality $w_{(2)} = w_{(1)}$ is studied instead. The separation into two papers is made because of a qualitative difference in the dynamics between the two cases, and to keep the papers reasonably short.

The analysis of the system consists of mainly three parts:

- Try to find constants of motion or monotone functions to solve, or asymptotically solve, part of the system.
- Finding all the fixed points of the system and linearizing the equations in their neighborhood to obtain the qualitative dynamics on an open set around the fixed points.
- Solve the equations numerically for randomly chosen initial data to study if the conclusions from the local stability analysis holds globally.

5.1.1 The Linear Analysis

The linear analysis was performed on the subset where $\Omega_\Lambda = 0$ since the asymptotic states can easily be found globally to the future when Ω_Λ is non-zero, and resides on the subset $\Omega_\Lambda = 0$ towards the past. The Gauss constraint (5.2a) was used to solve for $\Omega_{(2)}$ while the Codazzi constraint (5.2b) was left unsolved since it cannot be solved globally for any of the variables, instead it was solved locally at each fixed point for one of the variables.

Example 1. In paper I where $w_{(2)} < w_{(1)}$, there exists a FRW fixed point with the softer fluid acting as an extremely tilted test fluid:

$$\mathrm{F}^{10}_{01}: \ \Omega_{(1)} = 1, \ v_{(2)} = -1, \ v_{(1)} = \Omega_{(2)} = \Sigma_+ = \Sigma_A = \Sigma_B = \Sigma_C = 0.$$

At F^{10}_{01}, the linearization of the constraint (5.2b) is

$$(1 + w_{(1)})v_{(1)} + (\Omega_{(1)} - 1) = 0,$$

and one can use it to solve for $\Omega_{(1)}$. Reinserting this value for $\Omega_{(1)}$ into the remaining equations of (5.1) and linearizing at F^{10}_{01} one obtains the matrix equation

$$\begin{bmatrix} \Sigma_+ \\ \Sigma_A \\ \Sigma_B \\ \Sigma_C \\ v_{(1)} \\ v_{(2)} \end{bmatrix}' = \frac{1}{2} \begin{bmatrix} -3(1-w_{(1)}) & 0 & 0 & 0 & -2(1-w_{(1)}) & 0 \\ 0 & -3(1-w_{(1)}) & 0 & 0 & 0 & 0 \\ 0 & 0 & -3(1-w_{(1)}) & 0 & 0 & 0 \\ 0 & 0 & 0 & -3(1-w_{(1)}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(3w_{(1)}-1) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{2(1-3w_{(2)})}{1-w_{(2)}} \end{bmatrix} \begin{bmatrix} \Sigma_+ \\ \Sigma_A \\ \Sigma_B \\ \Sigma_C \\ v_{(1)} \\ v_{(2)} \end{bmatrix},$$

which immediately gives the eigenvalues

$$\lambda_{1,2,3,4} = -\tfrac{3}{2}(1 - w_{(1)}), \quad \lambda_5 = 3w_{(1)} - 1, \quad \lambda_6 = \frac{2(1 - 3w_{(2)})}{1 - w_{(2)}}.$$

The fixed point F^{10}_{01} is non-hyperbolic when either of the state parameters $w_{(i)}$ take the radiation value 1/3, and a saddle-point otherwise.

Not all of the fixed points are as easily diagonalized as the FRW-points, particularly those where the fluid velocities take a non-zero, non-extreme, tilt value have linearizations that are dauntingly complex, but with the help of computer algebra software (or a great deal of patience) one can extract the eigenvalues. The eigenvalues of these points are long and are not written out explicitly in the papers but as an example of how they look we can take a look at the fixed point $\mathrm{TW}_{1v^*_{(2)}}$ in paper I:

Example 2. For values of $w_{(2)}$ between 1/2 and 3/5 there exists a fixed point $\mathrm{TW}_{1v^*_{(2)}}$, which is defined by:

$$\begin{aligned} \mathrm{TW}_{1v^*_{(2)}}: \quad & \Sigma_+ = -\tfrac{1}{2}(3w_{(2)} - 1), \qquad \Sigma_A = \sqrt{\tfrac{3}{2}(1 - w_{(2)})(2w_{(2)} - 1)}, \\ & \Sigma_B = \sqrt{3}(2w_{(2)} - 1), \qquad \Sigma_C = 0, \\ & v_{(1)} = 1, \qquad v_{(2)} = v^*_{(2)} = -\frac{(1 - w_{(2)})(15w_{(2)} - 7)}{-25w^2_{(2)} + 18w_{(2)} - 1}, \\ & \Omega_{(1)} = B(w_{(2)}), \qquad \Omega_{(2)} = 1 - \tfrac{1}{4}(3w_{(2)} - 1)(15w_{(2)} - 7) - B(w_{(2)}), \end{aligned}$$

where

$$B(w_{(2)}) = -\frac{3(1 - w_{(2)})(7 - 15w_{(2)})(25w^2_{(2)} - 18w_{(2)} + 1)}{32(5w^2_{(2)} - 5w_{(2)} + 1)}.$$

Linearizing the constraint (5.2b) at any point P one obtains:

$$\begin{bmatrix} -2\Sigma_+ v_{(2)} \\ -2\Sigma_A v_{(2)} \\ -2\Sigma_B v_{(2)} \\ -2\Sigma_C v_{(2)} \\ \left(\frac{(1+w_{(1)})G_-^{(1)}G_+^{(2)}}{(1+w_{(2)})(G_+^{(1)})^2}\right)\Omega_{(1)} \\ \left(\frac{G_-^{(2)}}{G_+^{(2)}}\right)(1-\Sigma^2-\Omega_{(1)}) \\ \frac{(1+w_{(1)})G_+^{(2)}}{(1+w_{(2)})G_+^{(1)}}v_{(1)} \end{bmatrix}^T_{|P} (\mathbf{S}-\mathbf{S}^*) = 0,$$

where $\mathbf{S} = \left[\Sigma_+ \ \Sigma_A \ \Sigma_B \ \Sigma_C \ v_{(1)} \ v_{(2)} \ \Omega_{(1)}\right]^T$ and $\mathbf{S}^*$ is $\mathbf{S}$ calculated at P. Inserting the values of $\mathrm{TW}_{1v^*_{(2)}}$ one obtains:

$$\begin{bmatrix} \frac{\left(25\,w_{(2)}^2+1-18\,w_{(2)}\right)\left(1-3\,w_{(2)}\right)\left(1-w_{(2)}\right)\left(7-15\,w_{(2)}\right)}{54\,w_{(2)}^2+1+12\,w_{(2)}-260\,w_{(2)}^3+225\,w_{(2)}^4} \\ \frac{\left(25\,w_{(2)}^2+1-18\,w_{(2)}\right)\left(1-w_{(2)}\right)\left(7-15\,w_{(2)}\right)\sqrt{18\,w_{(2)}-12\,w_{(2)}^2-6}}{54\,w_{(2)}^2+1+12\,w_{(2)}-260\,w_{(2)}^3+225\,w_{(2)}^4} \\ 2\sqrt{3}\,\frac{\left(25\,w_{(2)}^2+1-18\,w_{(2)}\right)\left(1-w_{(2)}\right)\left(7-15\,w_{(2)}\right)\left(2\,w_{(2)}-1\right)}{54\,w_{(2)}^2+1+12\,w_{(2)}-260\,w_{(2)}^3+225\,w_{(2)}^4} \\ 0 \\ \frac{3}{32}\,\frac{\left(25\,w_{(2)}^2+1-18\,w_{(2)}\right)\left(1-w_{(2)}\right)\left(7-15\,w_{(2)}\right)\left(-1+w_{(1)}\right)}{\left(1+w_{(1)}\right)\left(5\,w_{(2)}^2+1-5\,w_{(2)}\right)} \\ \frac{3}{32}\,\frac{\left(25\,w_{(2)}^2+1-18\,w_{(2)}\right)^2\left(-1285\,w_{(2)}^4-682\,w_{(2)}^2+1594\,w_{(2)}^3-1+85\,w_{(2)}+225\,w_{(2)}^5\right)}{\left(-35\,w_{(2)}^4+66\,w_{(2)}^2-206\,w_{(2)}^3+1+13\,w_{(2)}+225\,w_{(2)}^5\right)\left(5\,w_{(2)}^2+1-5\,w_{(2)}\right)} \\ 8\,\frac{75\,w_{(2)}^4-135\,w_{(2)}^3+80\,w_{(2)}^2-17\,w_{(2)}+1}{54\,w_{(2)}^2+1+12\,w_{(2)}-260\,w_{(2)}^3+225\,w_{(2)}^4} \end{bmatrix}^T (\mathbf{S}-\mathbf{S}^*) = 0.$$

The constraint can be used to solve for any variable but Σ_C. Using Maple™ to solve for $\Omega_{(1)}$, linearize the reduced system, and find the eigenvalues we obtain:

$$\lambda_{\Sigma_1} = -\tfrac{3}{2}(5-9w_{(2)}),\ \lambda_2 = -6\frac{w_{(1)}-w_{(2)}}{1-w_{(1)}},\ \lambda_{3,4,5,6} = -\frac{3}{4}(1-w_{(2)})\left(1 \pm \sqrt{\frac{B \pm A}{C}}\right),$$

where

$$
\begin{aligned}
A &= \big(793 + 41254\, w_{(2)} - 767027\, w_{(2)}^2 + 5061056\, w_{(2)}^3 - 15973294\, w_{(2)}^4 \\
&\quad +21782564\, w_{(2)}^5 + 6015506\, w_{(2)}^6 - 55878280\, w_{(2)}^7 + 54125125\, w_{(2)}^8 \\
&\quad +8587750\, w_{(2)}^9 - 38089375\, w_{(2)}^{10} + 12975000\, w_{(2)}^{11} + 2250000\, w_{(2)}^{12}\big)^{1/2}, \\
B &= 48225\, w_{(2)}^6 - 195310\, w_{(2)}^5 + 268519\, w_{(2)}^4 - 162868\, w_{(2)}^3 + 42575\, w_{(2)}^2 \\
&\quad -2942\, w_{(2)} - 247, \\
C &= \Big(1 - 85\, w_{(2)} + 682\, w_{(2)}^2 - 1594\, w_{(2)}^3 + 1285\, w_{(2)}^4 - 225\, w_{(2)}^5\Big)\left(1 - w_{(2)}\right).
\end{aligned}
$$

The eigenvalues $\lambda_{3,4,5,6}$ are plotted in figure 5.1. One can see that the fixed point is a sink for any value of $w_{(2)}$ between 1/2 and 5/9.

Example 3. In both paper I and II there exist closed one-parameter families of fixed points corresponding to the vacuum Kasner solutions [53]. Several different representations of these solutions exist, with different values of the three-velocities of the fluids, which act as test fields that can either be zero or one. For the representation where both fluids are aligned with the normal to the hypersurfaces of homogeneity, $\mathrm{K}_{00}^{\bigcirc}$, the linearization of the constraint (5.2b) yields:

$$\frac{\partial}{\partial \mathbf{S}} \left(Q_{(1)} + Q_{(2)}\right)_{|\mathrm{K}_{00}^{\bigcirc}} = (0,0,0,0,0,0,0).$$

Thus we have that the constraint is completely degenerated at the fixed points and cannot be used to eliminate any of the variables. Linearizing the system without solving the constraint means that we have one unphysical degree of freedom left. Normally then the linear analysis is not enough to determine the dynamics in a neighborhood of the fixed points, but in this case a part of $\mathrm{K}_{00}^{\bigcirc}$ has a complete set of positive eigenvalues, even without imposing the constraint, which means that this part must be a source also in the constrained subspace.

5.1.2 Numerical Computations

The linear analysis can give a qualitative understanding of the dynamics in a neighborhood of the hyperbolic fixed points through the Hartman-Grobman theorem, and the neighborhoods of fixed point sets like the Kasner circles through the reduction principle, but the theorems do not determine a precise qualitative asymptotic behavior, like an analytical asymptotic solution, or what will happen with any given initial set of data. Theorems 5.1 – 5.4 in paper I show that systems where one of the fluids are softer than radiation isotropizes to the

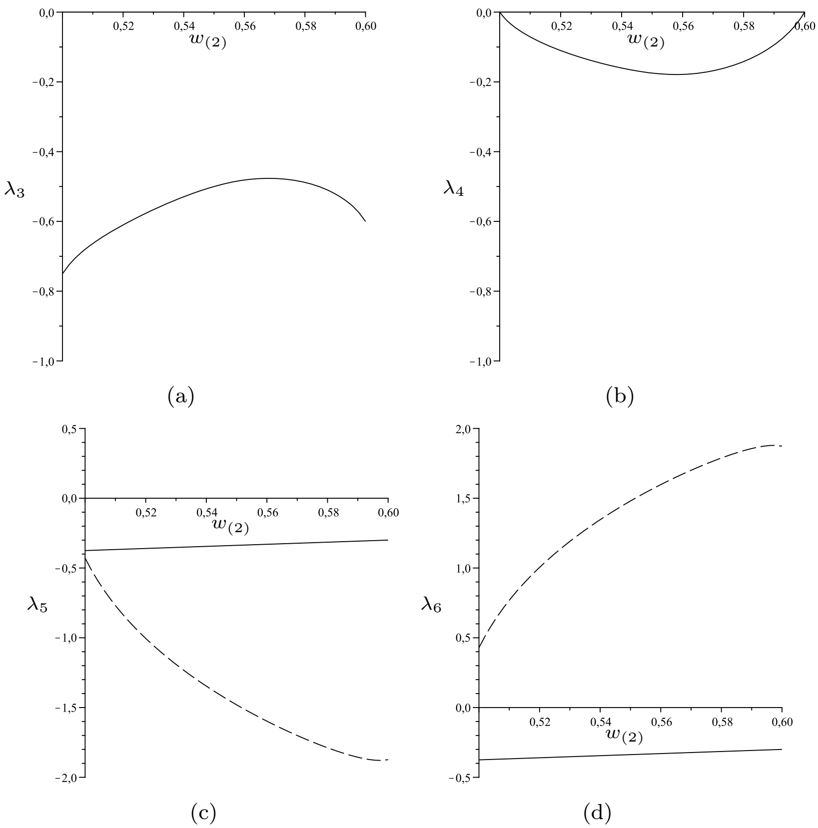

Figure 5.1: The real parts of the eigenvalues $\lambda_{3,4,5,6}$ are plotted with a solid line; the imaginary parts are shown with a dashed line.

future, and systems where both are stiffer than radiation do not. Given that there only exists one sink and one source for a given value of the parameters $w_{(i)}$ one would like to think that all points (except for a set of measure zero) would emanate from a neighborhood of the source and end up at the sink eventually, but to confirm this we solve the system numerically for a number of different data points to see if they all follow this pattern.

Example 1. In paper I, the linear analysis show that $\text{TW}_{1v^*_{(2)}}$ is a future attractor of a neighborhood around that point for values of $w_{(2)}$ between 1/2 and 5/9. Using MATLAB® to solve the equations (5.1), (5.2) numerically for many different, randomly chosen, points with the internal MATLAB ODE solvers, one finds that the local attractor also seems to be a global attractor. In solving the equations, $\Omega_{(2)}$ is given through the Gauss constraint (5.2a), and the Codazzi constraint (5.2b) is only imposed on the initial data. To confirm that the constraint (5.2b) is preserved by the evolution, $Q_{(1)} + Q_{(2)}$ is also computed and plotted. Figure 5.2 shows the evolution of the 8 state space variables $\mathbf{S} = \big(\Sigma_+, \Sigma_A, \Sigma_B, \Sigma_C, v_{(1)}, v_{(2)}, \Omega_{(1)}, \Omega_{(2)}\big)$, and the constraint function $Q_{(1)} + Q_{(2)}$ for 4 randomly chosen points.

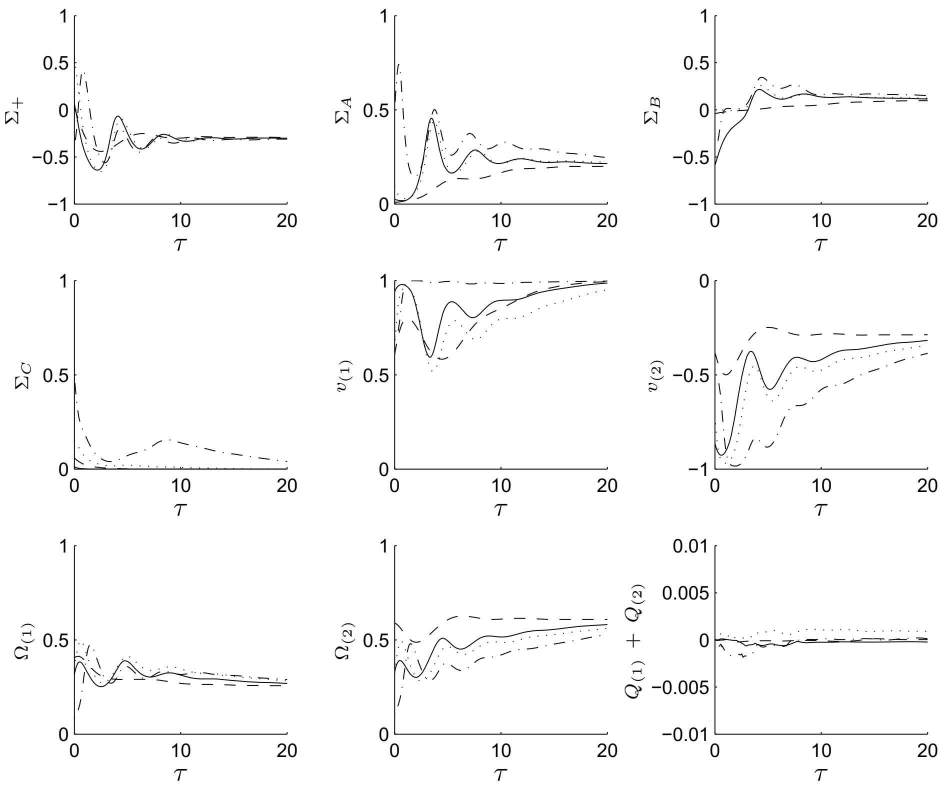

Figure 5.2: The graphs show the future evolution of the state space variables of 4 randomly chosen points that are evolved in time for equations of state parameters $w_{(1)} = 20/36$ and $w_{(2)} = 19/36$. All of them approach the sink $\mathrm{TW}_{1v^*_{(2)}}$. The time parameter τ is related to the vector field $\boldsymbol{\partial}_0$ according to $\boldsymbol{\partial}_0 = d/d\tau$.

The numerical computations show that the constraint (5.2b) is marginally stable; initially the points pick up computational errors and are thrown off the constraint surface by a few parts in a thousand, but then the constraint function stabilizes and stays close to zero in the future evolution, in comparison to the other the other variables.

Example 2. The linear analysis of the 4 Kasner circles in paper I suggests that when $w_{(1)} > w_{(2)} > 2/3$, then the past attractor is made up of sections of the Kasner circles $\mathrm{K}^{\bigcirc}_{11}$, $\mathrm{K}^{\bigcirc}_{01}$, and $\mathrm{K}^{\bigcirc}_{00}$ (defined by $\Omega_{(1)} = \Omega_{(2)} = \Sigma_A = \Sigma_C = 0,\ \Sigma_+^2 + \Sigma_B^2 = 1$, and $(v_{(1)} = -v_{(2)} = 1)$, $(v_{(1)} = 0,\ v_{(2)} = -1)$, and $(v_{(1)} = v_{(2)} = 0))$ respectively; connected by the lines of fixed points $\mathrm{KL}^{+}_{v_{(1)}1}$ and $\mathrm{KL}^{+}_{0v_{(2)}}$, (defined by $\Omega_{(1)} = \Omega_{(2)} = \Sigma_A = \Sigma_C = 0$, and

$$\Sigma_+ = \tfrac{1}{2}(1 - 3w_{(1)}),\ \Sigma_B = \sqrt{1 - \Sigma_+^2},\ 0 \leq v_{(1)} \leq 1,\ v_{(2)} = -1$$

and

$$\Sigma_+ = \tfrac{1}{2}(1 - 3w_{(2)}),\ \Sigma_B = \sqrt{1 - \Sigma_+^2},\ v_{(1)} = 0,\ -1 \leq v_{(2)} \leq 0$$

respectively, see appendix A and B in paper I).

Computing the evolution of the system numerically for 4 randomly chosen points for equations of state parameters $w_{(1)} = 0.9$ and $w_{(2)} = 0.8$ one finds that the shear and density parameters stabilize after about 15 units of time, see figure 5.3.

The variables $v_{(1)}$ and $v_{(2)}$ also seem to have stabilized at the extreme values $0, \pm 1$, but comparing with the linear analysis one finds that one points has approached a saddle fixed point (the solid line). Evolving the system further makes the state point leave the saddle point and approach the local source (since the system is evolved backwards in time), see figure 5.4.

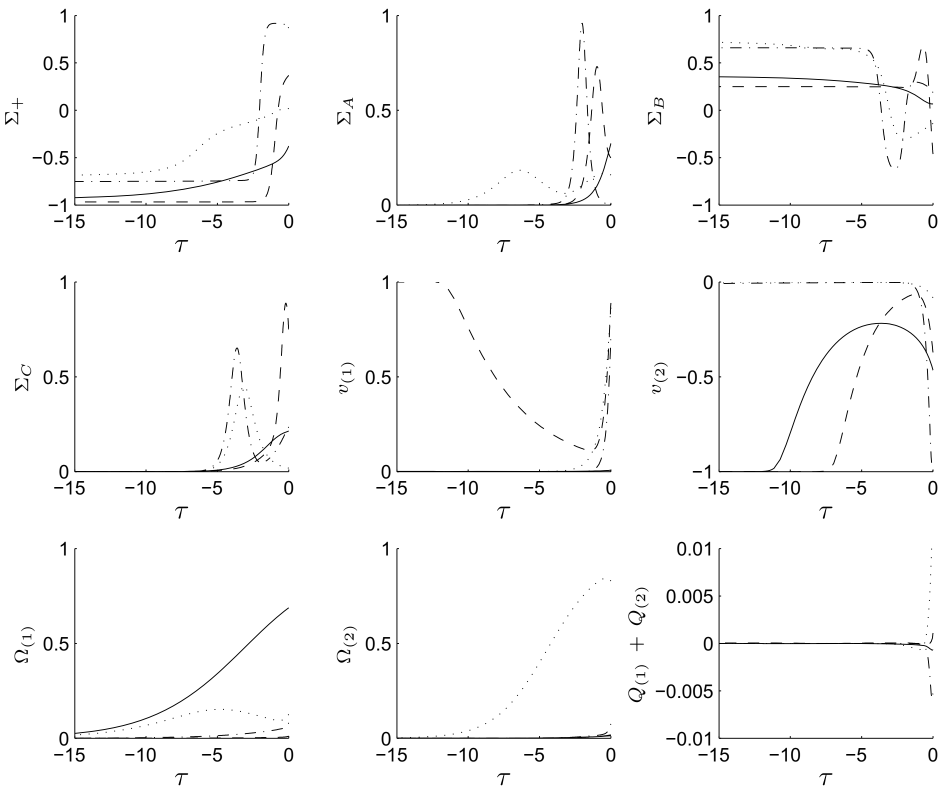

Figure 5.3: The numerically computed evolution of four randomly chosen points backwards in time for 15 units of time. The Codazzi constraint (5.2b) is only imposed on the initial data and the constraint function is plotted in the last frame to check that the condition $Q_{(1)} + Q_{(2)} = 0$ is preserved.

5.2 Paper III

In this paper we study the past asymptotic dynamics of cosmologies with an arbitrary number of tilted perfect fluids, with arbitrary barotropic equations of state. Under the assumption of asymptotic silence and locality as described in section 3.4.2 we derive results about the relationship between properties of matter, such as the stiffness of the fluids, and the asymptotic geometry.

Unexpectedly, the evolution equations for general barotropic fluids are structurally identical, on the silent boundary, to the evolution equations of perfect

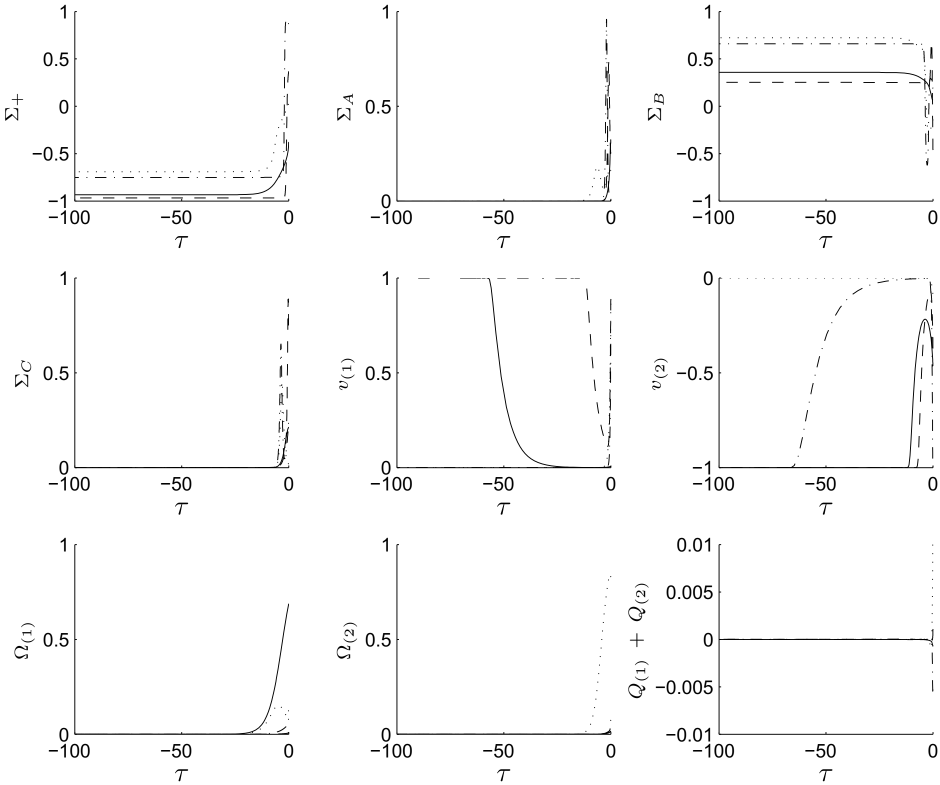

Figure 5.4: Numerical evolution of the same initial data as in figure 5.3 backwards in time for 100 units of time. All points approach the local sources, which also are global sources.

fluids with linear equations of state with the equation of state parameter w replaced by the square of the speed of sound c_s^2 in the medium defined by the fluids. This result also holds for Bianchi cosmologies.

A fruitful way to analyze the motion of the fluids is to split the fluid velocities v_α into their speed, $v = (v_\alpha v^\alpha)^{1/2}$, and direction $c_\alpha = v_\alpha / v$ (relative the local rest spaces defined by the orthonormal frame). On the type I subset one finds that the equations governing the evolution of the direction decouples from the speed, and under the assumption that the asymptotic state either is a 'vacuum solution' ($\Omega = 0$) or described by the Jacobs or Friedman solutions ($\Omega = 1$), depending on the stiffness of the stiffest equation of state of the fluids, one can find the asymptotic state of the fluid. In the asymptotic vacuum case one finds that the fluids will align (or anti-align) with the eigendirection of the shear with the largest eigenvalue (see corollary 3.3 in paper III). One can compare this analysis with the one done in paper I, which treats a similar although simpler problem, but in a different frame, and other variables associated to the different choice of frame.

5.2.1 Spatial Frame Choices

Since only two fluids are present in paper I, they have to be anti-aligned with respect to each other to obey the Codazzi constraint (5.2b), and thus they define a preferred direction in space. The spatial frame is chosen such that the time derivative of the fluid velocities always stay proportional to their velocity; the frame is 'corotating' with the fluids in the sense that the frame is rotating relative a gyroscopically fixed frame in such a way that it is always aligned with the fluids. The 'rotation' should not be confused with vorticity; the vorticity of the fluids is always zero on the Bianchi type I subset (see [56]). This frame choice removes two of the degrees of freedom from the velocities but instead the shear becomes non-diagonal.

One can compare the two resulting systems of equations on the vacuum boundary (one can of course compare the two systems off the vacuum boundary but no analysis of the system was performed off the vacuum boundary in paper III because it was there conjectured that the past asymptotic state would be contained on the vacuum boundary):

Corotating frame (paper I and II):

$$v^{\alpha}_{(i)} = (0, 0, v_{(i)}), \qquad R_1 = -\Sigma_{23}, \qquad R_2 = \Sigma_{31}, \qquad R_3 = 0.$$

Make the variable substitutions

$$\Sigma_+ = \tfrac{1}{2}(\Sigma_{11}+\Sigma_{22}), \quad \Sigma_{31}+\mathrm{i}\,\Sigma_{23} = \sqrt{3}\Sigma_A\, e^{\mathrm{i}\phi}, \quad \Sigma_- + \frac{\mathrm{i}}{\sqrt{3}}\Sigma_{12} = (\Sigma_B+\mathrm{i}\,\Sigma_C)e^{2\mathrm{i}\phi},$$

where $\Sigma_- = (\Sigma_{11} - \Sigma_{22})/(2\sqrt{3})$, to obtain the system of equations:

$$\boldsymbol{\partial}_0\Sigma_+ = 3\Sigma_A^2, \tag{5.5a}$$
$$\boldsymbol{\partial}_0\Sigma_A = -(3\Sigma_+ + \sqrt{3}\Sigma_B)\Sigma_A, \tag{5.5b}$$
$$\boldsymbol{\partial}_0\Sigma_B = \sqrt{3}\Sigma_A^2 - 2\sqrt{3}\Sigma_C^2, \tag{5.5c}$$
$$\boldsymbol{\partial}_0\Sigma_C = 2\sqrt{3}\Sigma_B\Sigma_C, \tag{5.5d}$$
$$\boldsymbol{\partial}_0\phi = -\Sigma_C, \tag{5.5e}$$
$$\boldsymbol{\partial}_0 v = G_-^{-1}(1-v^2)(3w-1+2\Sigma_+)v, \tag{5.5f}$$

where

$$\Sigma_+^2 + \Sigma_A^2 + \Sigma_B^2 + \Sigma_C^2 = 1.$$

In paper III, where a shear eigenframe was used, the same physical system is expressed through the system of equations (where we can assume there are only two fluids for comparison with the system above):

Shear eigenframe (paper III):

$$\Sigma_{\alpha\beta} = \mathrm{diag}(\Sigma_1, \Sigma_2, \Sigma_3), \qquad R_\alpha = 0.$$

Make the variable substitutions

$$(\Sigma_1, \Sigma_2, \Sigma_3) = (3p_1 - 1, 3p_2 - 1, 3p_3 - 1),$$

to obtain the equations

$$\boldsymbol{\partial}_0\, p_\alpha = 0, \tag{5.6a}$$
$$\boldsymbol{\partial}_0 v = 3\bar{G}_-^{-1}\,(1-v^2)\,\left[c_s^2 - p_1\, c_1^2 - p_2\, c_2^2 - p_3\, c_3^2\right] v, \tag{5.6b}$$
$$\boldsymbol{\partial}_0\, c_1 = 3[(p_2 - p_1)c_2^2 + (p_3 - p_1)c_3^2]\, c_1, \tag{5.6c}$$
$$\boldsymbol{\partial}_0\, c_2 = 3[(p_3 - p_2)c_3^2 + (p_1 - p_2)c_1^2]\, c_2, \tag{5.6d}$$
$$\boldsymbol{\partial}_0\, c_3 = 3[(p_1 - p_3)c_1^2 + (p_2 - p_3)c_2^2]\, c_3, \tag{5.6e}$$

where

$$p_1^2 + p_2^2 + p_3^2 = 1, \qquad p_1 + p_2 + p_3 = 1, \qquad c_1^2 + c_2^2 + c_3^2 = 1.$$

In the corotating frame, the dynamics is given in terms of the shear variables and the fluid speed, while in the shear eigenframe, the shear variables are constant and the dynamics is given in terms of the fluid speed and direction. The result of the analysis is the same: In the corotating frame one can show that Σ_A

and Σ_C must vanish asymptotically to the past. This in turn implies that $\boldsymbol{\partial}_0\phi \to 0$, which means that the frame can be transformed to an asymptotic shear eigenframe by choosing $\phi \to 0$. One concludes that the asymptotic state of the shear variables is confined to the closure of the part of the Kasner circle that is a source, which is the sector (213), and the two points T_3 and Q_2, given in figure 3, appendix B, in paper II. This is the part of the Kasner circle where $\Sigma_{22} \leq \Sigma_{11} \leq \Sigma_{33}$. One thus finds that the fluid, which defines the $\boldsymbol{\partial}_3$-direction, will asymptotically align or anti-align with the eigendirection of the shear with the largest eigenvalue, which is exactly the same conclusion as in paper III.

The corotating frame choice only works for the particular case where there is only one special direction along which to attach the frame, as is the case with two anti-aligned fluids or with the similar problem with a magnetic field as described in [58]; it is useful since the fluid analysis is simpler. On the vacuum subset though, the motion of the fluid is easy to solve in a shear eigenframe, and one can find the past asymptotic states for an arbitrary number of fluids, and find that they will all be aligned or anti-aligned.

5.3 Paper IV

The main result of this paper is that the future asymptotic global dynamics of tilted Bianchi type II models is completely described. This was possible due to the construction of several hitherto unknown monotone functions that were sufficiently restrictive to pinpoint the future attractors for all investigated values of the equation of state parameter w.

The first of the monotone functions was found by using Hamiltonian methods, a mode of procedure described in a paper by Heinzle and Uggla [37]. Without going into detail one can say that they show (for non-tilted perfect fluid Bianchi models of type A) that the existence of certain symmetries (isometries or homothetic symmetries w.r.t. the minisuperspace metric) of the Hamiltonian lead to conserved quantities or monotone functions in the dynamical systems formulation of the same system. The tilted type II models share certain features with the class of spacetimes they studied and it is reasonable to believe that their method also extends to more complicated systems such as these.

In our paper we did not perform a similarly thorough analysis to be able to state under exactly what circumstances a monotone function can be constructed for the model under investigation. Instead we first analyzed an invariant subspace of the model (the orthogonally transitive case) where the Hamiltonian already was known (Uggla et al. [99]). Then we used this Hamiltonian to identify a function that is monotone decreasing to the future. Showing that it was possible to express this function in Hubble normalized variables confirmed the existence of a monotone function in the dynamical system. Based on this monotone function we could construct a set of monotone quantities valid in the full state space and covering all values of the equation of state parameter w. In this section I will present some of these calculations in detail, as well as adding clarifying comments that could not fit into the original paper due to restrictions of space.

New Hubble Normalized Variables in Two Steps

The basic Hubble normalized equations for the tilted Bianchi models of class A are given by

Evolution equations

$$\Sigma'_{\alpha\beta} = -(2-q)\Sigma_{\alpha\beta} - 2\epsilon^{\gamma\delta}{}_{\langle\alpha}\,\Sigma_{\beta\rangle\gamma}\,R_\delta - {}^3\mathcal{R}_{\langle\alpha\beta\rangle} + 3\Pi_{\alpha\beta}, \tag{5.7a}$$

$$(N_{\alpha\beta})' = qN_{\alpha\beta} + 2\Sigma^\gamma{}_{(\alpha}N_{\beta)\gamma} - 2\epsilon^{\gamma\delta}{}_{(\alpha}N_{\beta)\gamma}R_\delta, \tag{5.7b}$$

$$\Omega' = (2q-1)\,\Omega - 3P - \Sigma_{\alpha\beta}\Pi^{\alpha\beta}, \tag{5.7c}$$

$$v' = -G_-^{-1}\,(1-v^2)\,\left[1 - 3w + \Sigma_{\alpha\beta}\,c^\alpha c^\beta\right] v, \tag{5.7d}$$

$$c'_\alpha = [\delta_\alpha{}^\beta - c_\alpha c^\beta][q\,c_\beta - \Sigma_\beta{}^\gamma\,c_\gamma - \epsilon_{\beta\gamma\delta}R^\delta c^\gamma - v\epsilon_\beta{}^\gamma{}_\delta\,N^{\delta\zeta}\,c_\zeta\,c_\gamma]. \tag{5.7e}$$

Constraint equations

$$0 = 1 - \Sigma^2 - \Omega_\mathrm{k} - \Omega, \tag{5.8a}$$

$$0 = \epsilon_\alpha{}^{\delta\gamma}\,N_{\delta\beta}\,\Sigma^\beta{}_\gamma - 3Q_\alpha, \tag{5.8b}$$

where

$$\Omega_\mathrm{k} = \tfrac{1}{12}{}^3\mathcal{R}^\alpha{}_\alpha, \quad {}^3\mathcal{R}_{\alpha\beta} = 2N_{\alpha\gamma}\,N^\gamma{}_\beta - N^\gamma{}_\gamma\,N_{\alpha\beta} - N^{\gamma\delta}N_{\gamma\delta}\,\delta_{\alpha\beta} + \tfrac{1}{2}\,(N^\gamma{}_\gamma)^2\,\delta_{\alpha\beta}.$$

The variables are the standard Wainwright-Hsu-variables except that the three-velocity v_α is split into its norm and a directional unit vector c_α: $v_\alpha = vc_\alpha$.

For the type II models the matrix $(N_{\alpha\beta})$ only has one independent component. It is possible do diagonalize it, usually to the form $(N_{\alpha\beta}) = \mathrm{diag}(\mathrm{N}, 0, 0)$, by setting the frame rotation coefficients to $R_2 = \Sigma_{13}$, and $R_3 = -\Sigma_{12}$ and then making a time-independent frame rotation. From equation (5.8b) one then has $c_1 = 0$. It is also possible to align the spatial frame with the fluid velocity by making a suitable choice of the remaining frame rotation coefficient R_1. This is what is done in Hewitt et al. where it is chosen such that the fluid is aligned with the frame vector $\mathbf{e}_3$. Instead of making this frame choice we introduce variables that are invariant under rotations in the 2-3-plane in two steps. If we use capital Latin indices $A, B, ... = 2, 3$, define the rotation invariant shear variable $\check{\Sigma}^2 = \frac{1}{3}(\Sigma_{12}^2 + \Sigma_{13}^2) = \frac{1}{3} R_A R^A$, and solve the constraint equation (5.8a) for Ω, the evolution equations reduce to

Evolution equations

$$\Sigma'_{AB} = -(2-q)\Sigma_{AB} - 2R_1\,(\Sigma_{AC} - \tfrac{1}{2}\Sigma^B{}_B\delta_{AC})\epsilon_B{}^C + \tfrac{1}{3}N^2\,\delta_{AB} - 2R^2\,(\epsilon_{AC}\,c^C)(\epsilon_{BD}\,c^D) - N\,R\,v\,(c_A\,c_B - \tfrac{1}{3}\delta_{AB}), \tag{5.9a}$$

$$(\check{\Sigma}^2)' = -2[2 - q + \Sigma_{AB}\,c^A\,c^B - 2\Sigma_A{}^A)\,\check{\Sigma}^2, \tag{5.9b}$$

$$N' = (q - 2\Sigma_A{}^A)\,N, \tag{5.9c}$$

$$v' = -G_-^{-1}\,(1 - v^2)\,\left[1 - 3w + \Sigma_{AB}\,c^A c^B\right]v, \tag{5.9d}$$

$$c'_A = -(\delta_A{}^B - c_A\,c^B)[\Sigma_B{}^C\,c_C + R_1\epsilon_{BC}\,c^C]. \tag{5.9e}$$

The symmetric matrix (Σ_{AB}) have three independent components that we want to express in terms of variables that are invariant under rotations in the 2-3-plane. One obvious such variable is the trace, so we split Σ_{AB} into its trace and trace-free parts: $\Sigma_{AB} = \Sigma_{\langle AB\rangle} + \Sigma_+\delta_{AB}$, where $\Sigma_+ = \frac{1}{2}\Sigma^A{}_A$. The two independent components of $\Sigma_{\langle AB\rangle}$ can be extracted by contracting with the fluid direction vector c^A and its orthogonal complement $\epsilon^A{}_C c^C$, and we therefore define the variables $\bar{\Sigma} = \frac{1}{\sqrt{3}}\Sigma_{\langle AB\rangle}c^A c^B$ and $\tilde{\Sigma} = \frac{1}{\sqrt{3}}\Sigma_{\langle AB\rangle}\epsilon^B{}_C c^A c^C$.[36] When the evolution equations are expressed in these variables the equation (5.9e), expressing the fluid motion w.r.t. the frame, becomes redundant, and we are left with a system of scalar equations under rotations in the 2-3-plane.

Evolution equations:

$$\Sigma'_+ = -(2-q)\Sigma_+ - 3\check{\Sigma}^2 + 4\Omega_k + \tfrac{1}{2}(1+w)G_+^{-1}v^2\Omega, \tag{5.10a}$$

$$\bar{\Sigma}' = -(2-q)\bar{\Sigma} - 2\sqrt{3}\tilde{\Sigma}^2 + \sqrt{3}\check{\Sigma}^2 + \tfrac{\sqrt{3}}{2}(1+w)G_+^{-1}v^2\Omega, \tag{5.10b}$$

$$(\tilde{\Sigma}^2)' = -2(2 - q - 2\sqrt{3}\bar{\Sigma})\tilde{\Sigma}^2, \tag{5.10c}$$

$$(\check{\Sigma}^2)' = -2[2 - q - 3\Sigma_+ + \sqrt{3}\bar{\Sigma}]\check{\Sigma}^2, \tag{5.10d}$$

$$\Omega'_k = 2(q - 4\Sigma_+)\Omega_k, \tag{5.10e}$$

$$(v^2)' = 2G_-^{-1}\,(1 - v^2)[3w - 1 - \Sigma_+ - \sqrt{3}\bar{\Sigma}]v^2. \tag{5.10f}$$

[36] In paper IV we contracted with the projection operator $(c^A c^B - \frac{1}{2}\delta^{AB})$ instead of $c^A c^B$ but the result is the same, the contraction with δ^{AB} does not contribute to any of the variables since $\Sigma_{\langle AB\rangle}$ is both symmetric and trace-free.

Constraint equation:

$$4\check{\Sigma}^2\Omega_k - (1+w)^2 G_+^{-2} v^2 \Omega^2 = 0. \tag{5.10g}$$

The Hamiltonian for the Orthogonally Transitive Tilted Bianchi Type II Models

The orthogonally transitive tilted Bianchi type II models are subcases of the fully tilted Bianchi type II models where the fluid velocity only has one independent component u_3, and where the metric can be given in a form where it has only one non-zero off-diagonal component $g_{12} = g_{21}$,

$$ds^2 = -N^2 dt^2 + g_{11}(t)(\hat{\omega}^1)^2 + 2g_{12}(t)\hat{\omega}^1\hat{\omega}^2 + g_{22}(t)(\hat{\omega}^2)^2 + g_{33}(t)(\hat{\omega}^3)^2. \tag{5.11}$$

The $\hat{\omega}^i$'s are the one-forms generating the Lie algebra of the type II models, usually chosen such that

$$d\hat{\omega}^1 = -\hat{n}_1\hat{\omega}^2 \wedge \hat{\omega}^3, \quad d\hat{\omega}^2 = 0, \quad d\hat{\omega}^3 = 0.$$

It is convenient to introduce the metric variables $(\beta^1, \beta^2, \beta^3, \theta^3)$ through the relations

$$g_{11} = e^{2\beta^1}, \quad g_{12} = -\sqrt{2}\theta^3 e^{2\beta^1}, \quad g_{22} = e^{2\beta^2} + 2(\theta^3)^2 e^{2\beta^1}, \quad g_{33} = e^{2\beta^3}, \tag{5.12}$$

with the inverse

$$g^{11} = e^{-2\beta^1} + 2(\hat{n}_1\theta^3)^2 e^{-2\beta^2}, \quad g^{12} = \sqrt{2}\theta^3 e^{-2\beta^2}, \quad g^{22} = e^{-2\beta^2}, \quad g_{33} = e^{-2\beta^3}. \tag{5.13}$$

The usefulness of these variables is seen when the Hamiltonian function is expressed in terms of them. The Hamiltonian is defined as the time-time component of the difference of the Einstein tensor and the energy-momentum tensor, and can naturally be split into a gravitational and a fluid part. The gravitational part can be further split into a kinetic part and a potential part while the fluid part only contains potential terms.

$$\tilde{\mathcal{H}} := 2N\, g^{\frac{1}{2}} n^a\, n^b (G_{ab} - T_{ab}) = 2\tilde{N} g \big(k^i{}_j k^j{}_i - (\operatorname{tr} k)^2 - {}^3R + G_+\Gamma^2\tilde{\rho}\big),$$

where $\tilde{N} = N g^{-1/2}$.

The kinetic (gravitational) part consists of the first two terms quadratic in the extrinsic curvature. Expressed in the variables $(\beta^1, \beta^2, \beta^3, \theta^3)$ it can be written as

$$\tilde{N}T := \tilde{N}g\big(-(\operatorname{tr} k)^2 + k^i{}_j k^j{}_i \big) = \tilde{N}^{-1}\big(\mathcal{G}_{ij}\dot{\beta}^i\dot{\beta}^j + e^{2(\beta^1-\beta^2)}(\dot{\theta}^3)^2 \big),$$

where

$$\mathcal{G}_{ij} = \begin{pmatrix} 0 & -1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}, \qquad \mathcal{G}^{ij} = \tfrac{1}{2}\begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \tag{5.14}$$

are the so called minisuperspace metric and its inverse.

The gravitational potential is identified with the term proportional to the three-curvature

$$U_g := -2g^3R = (\hat{n}_1)^2 e^{4\beta^1}, \tag{5.15}$$

and the fluid potential is

$$U_f := gG_+\Gamma^2\tilde{\rho}.$$

One can use the constant of motion $l = \tilde{n}\sqrt{g}\Gamma$ (see eq. (A.14) in paper IV) to solve for $\tilde{\rho}$ in the fluid potential

$$U_f = gG_+\Gamma^2\tilde{\rho} = l^{1+w}G_+\Gamma^{1-w}e^{(1-w)(\beta^1+\beta^2+\beta^3)}. \tag{5.16}$$

Likewise one can implicitly express the fluid velocity v^2 in terms of metric variables and Taub's circulation one-form ((A.15) in paper IV), $t_a = \tilde{\mu}\tilde{u}_a = (1+w)\tilde{n}^w\tilde{u}_a = (1+w)(lg^{-1/2}\Gamma^{-1})^w\tilde{u}_a$. The orthogonally transitive models have only one non-zero fluid component, v_3 and therefore only one non-zero component t_3. It is for this reason that the component $\hat{t}_3 = e^{\alpha}{}_3 t_\alpha = e^{2\beta^3}t_3$ in eq. (A.15) is conserved ($\hat{t}_{i=1,2} = 0 \Rightarrow \hat{C}^k{}_{3j}\,\hat{t}_k\,\hat{t}^j = 0$). It then follows exactly as through (B.11-14) that [37]

$$F := v^2\Gamma^{2(1-w)} = (1+w)^{-2}l^{-2w}\hat{t}_3^2\exp[2w(\beta^1+\beta^2) - 2(1-w)\beta^3], \tag{5.17}$$

and therefore that v^2 is a function of the particular combination of metric variables $(w(\beta^1+\beta^2) - (1-w)\beta^3)$.

The Hamiltonian is independent of the variable θ^3, and its corresponding momenta, $\pi_\theta := \partial\tilde{\mathcal{H}}/\partial\dot{\theta}^3 = 2\tilde{N}^{-1}e^{2(\beta^1-\beta^2)}\dot{\theta}^3$,[38] must therefore be conserved.

[37] There is a missprint in paper IV, eq. (B.13). Instead of $\exp[w(\beta^1+\beta^2) - (1-w)\beta^3]$ it should say $\exp[2w(\beta^1+\beta^2) - 2(1-w)\beta^3]$.

[38] The defining relation $\pi_i := \partial\mathcal{L}/\partial\dot{\beta}^i$ for some Lagrangian on the form $\mathcal{L}(\beta^i, \dot{\beta}^i) = \mathcal{G}_{kl}\dot{\beta}^k\dot{\beta}^l - U(\beta^i)$ have the relations $\pi_i = 2\mathcal{G}_{il}\dot{\beta}^l$ and $\mathcal{H} := \dot{\beta}^i\pi_i - \mathcal{L} = \mathcal{G}_{kl}\dot{\beta}^k\dot{\beta}^l + U(\beta^i)$ and thereby also $\pi_i = \partial\mathcal{H}/\partial\dot{\beta}^i$ when $\mathcal{H}$ is expressed as a function of the configuration space variables $(\beta^i, \dot{\beta}^i)$.

Hence one can solve $\dot{\theta}^3$ in terms of the metric variables and the constant momentum and consider the corresponding term in the Hamiltonian as a potential term instead, $T = T_d + U_c$, where

$$U_c = \frac{1}{4} e^{-2(\beta^1 - \beta^2)} (\pi_\theta)^2 \tag{5.18}$$

is the 'centrifugal potential' and T_d is the remaining kinetic part, thus reducing the independent variables to just the diagonal degrees of freedom, $(\beta^1, \beta^2, \beta^3)$. It can be useful to introduce the variable $\beta^0 = \frac{1}{3}(\beta^1 + \beta^2 + \beta^3)$ and its conjugate momentum $\pi_0 = \pi_1 + \pi_2 + \pi_3$. The Hamiltonian then, finally, is written as:

$$\tilde{\mathcal{H}} = \frac{\tilde{N}}{4}[\mathcal{G}^{ij}\pi_i\pi_j + (\pi_\theta)^2 e^{-2(\beta^1-\beta^2)} + 4(\hat{n}_1)^2 e^{4\beta^1} + 4l^{1+w}G_+\Gamma^{1-w}e^{3(1-w)\beta^0}].$$

At this point it is also useful to specify the lapse such that the time coordinate becomes dimensionless. This is done by setting $NH = 1$, or equivalently $\tilde{N} = -12\pi_0^{-1}$. With this special choice of lapse we denote the time coordinate as τ instead of t.

The Monotone Function

It is then the main point that if there exists a scale symmetry transformation of the potential $U := U_c + U_g + U_f$: $c^i \frac{\partial U}{\partial \beta^i} = rU$, where the c^is are constants, then the function $(c^i \pi_i)$ has the time derivative

$$(c^i \dot{\pi_i}) = (c^i \dot{\pi}_i) = -(c^i \frac{\partial \tilde{\mathcal{H}}}{\partial \beta^i}) = 12\pi_0^{-1}(c^i \frac{\partial (T+U)}{\partial \beta^i}) = 12r\pi_0^{-1}U,$$

thus *if* U/π_0 has a definite sign then this function is monotone. The potential U as it is defined above is a strictly positive function, thus the function is monotone if π_0 has a definite sign.

Another, more complicated function is

$$M := (c^i \pi_i) \exp\left[-\frac{r}{2}\frac{\mathcal{G}_{ij}c^i \beta^j}{\mathcal{G}_{kl}c^k c^l}\right].$$

Its time derivative is

$$\dot{M} = \left[\frac{(c^j\,\dot{\pi}_j)}{(c^i\,\pi_i)} - \frac{r}{2}\frac{\mathcal{G}_{ij}c^i\,\dot{\beta}^j}{\mathcal{G}_{kl}c^k\,c^l}\right] M = \left[-\frac{c^i}{(c^j\,\pi_j)}\frac{\partial\tilde{H}}{\partial\beta^i} - \frac{r}{2}\frac{\mathcal{G}_{ij}c^i}{\mathcal{G}_{kl}c^k\,c^l}\frac{\partial\tilde{H}}{\partial\pi_j}\right]M$$
$$= 12r\pi_0^{-1}\left[\frac{U}{(c^j\,\pi_j)} + \frac{\mathcal{G}_{ij}c^i}{2\mathcal{G}_{kl}c^k\,c^l}\frac{\partial T}{\partial\pi_j}\right] M = 12r\pi_0^{-1}\left[-\frac{T_d}{(c^j\,\pi_j)} + \frac{c^i\pi_i}{4\mathcal{G}_{kl}c^k\,c^l}\right]M$$
$$= 3r\pi_0^{-1}\left[-\mathcal{G}^{ij} + \frac{c^ic^j}{\mathcal{G}_{kl}c^k\,c^l}\right]\pi_i\pi_j\exp\left[-\frac{r}{2}\frac{\mathcal{G}_{ij}c^i\,\beta^j}{\mathcal{G}_{kl}c^k\,c^l}\right]. \tag{5.19a}$$

If c^i is a timelike vector w.r.t the minisuperspace metric then $\left[-\mathcal{G}^{ij} + \frac{c^ic^j}{\mathcal{G}_{kl}c^k\,c^l}\right]$ is positive semidefinite. The function is then monotone if π_0 has a definite sign. In paper IV we use the function M^2 instead. It has the derivative $\dot{M^2} = 2M\dot{M}$ and is thus monotone if M is strictly positive or negative.

The potentials (5.15), (5.16), (5.18) above have a homothetic symmetry for $\mathbf{c} = (1-w, 3(1-w), 4w)$ and $r = 4(1-w)$. The (minisuperspace-) norm of this vector is $|c|^2_{\mathcal{G}} = -2(1-w)(3+13w)$, and a monotone function can therefore be constructed for the orthogonally transitive subcase in the range $-3/13 < w < 1$ where $\mathbf{c}$ is timelike.

The Monotone Function in Hubble Normalized Variables

To be useful in the dynamical systems approach the monotone function must be translated into the Hubble normalized variables (5.10) derived from the orthonormal frame. The group invariant frame can be related to the orthonormal frame through a purely time-dependent linear transformation, $\omega^\alpha = e^\alpha{}_i\hat{\omega}^i := D^\alpha{}_j S^j{}_i\hat{\omega}^i$:[39]

$$\begin{bmatrix}\omega^1\\ \omega^2\\ \omega^3\end{bmatrix} = \begin{bmatrix}e^{\beta^1} & 0 & 0\\ 0 & e^{\beta^2} & 0\\ 0 & 0 & e^{\beta^3}\end{bmatrix}\cdot\begin{bmatrix}1 & -\sqrt{2}\theta^3 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}\hat{\omega}^1\\ \hat{\omega}^2\\ \hat{\omega}^3\end{bmatrix}, \tag{5.20}$$

which put the line element in the form

$$ds^2 = -N^2dt^2 + e^{2\beta^1}(\hat{\omega}^1 - \sqrt{2}\theta^3\hat{\omega}^2)^2 + e^{2\beta^2}(\hat{\omega}^2)^2 + e^{2\beta^3}(\hat{\omega}^3)^2.$$

[39] The transformation between the group invariant frame $\hat{\omega}^i$ and the on-frame $\hat{\omega}^\alpha$ is split into a diagonal scaling D and a special unit determinant transformation S. There is a missprint in eq. (B.4) in paper IV, where the matrix component $-\sqrt{2}\hat{n}_1\theta^3(x^0)$ should be just $-\sqrt{2}\theta^3(x^0)$.

The orthonormal frame obeys the algebra (on a given slice $dt = 0$)

$$d\omega^1 = -e^{\beta^1-\beta^2-\beta^3}\hat{n}_1\omega^2\wedge\omega^3, \quad d\omega^2 = 0, \quad d\omega^3 = 0,$$

and we have that

$$H = -\tfrac{1}{3}(\operatorname{tr} k) = -\tfrac{1}{12}e^{-3\beta^0}\pi_0, \tag{5.21a}$$

$$\begin{aligned}(\Sigma^\alpha{}_\beta) &= (\sigma^\alpha{}_\beta)/H = (e^\alpha{}_i(\tfrac{1}{3}\operatorname{tr} k\delta^i{}_j - k^i{}_j)e^j{}_\beta)/H = \\ &= -\frac{1}{3}\begin{pmatrix} -2\dot\beta^1+\dot\beta^2+\dot\beta^3 & (3/\sqrt{2})\dot\theta^3 & 0 \\ (3/\sqrt{2})\dot\theta^3 & \dot\beta^1-2\dot\beta^2+\dot\beta^3 & 0 \\ 0 & 0 & \dot\beta^1+\dot\beta^2-2\dot\beta^3\end{pmatrix} \\ &= 2\pi_0^{-1}\begin{pmatrix} -2\pi_1+\pi_2+\pi_3 & (3/\sqrt{2})e^{-(\beta^1-\beta^2)}\pi_\theta & 0 \\ (3/\sqrt{2})e^{-(\beta^1-\beta^2)}\pi_\theta & \pi_1-2\pi_2+\pi_3 & 0 \\ 0 & 0 & \pi_1+\pi_2-2\pi_3\end{pmatrix},\end{aligned} \tag{5.21b}$$

$$N_1 = e^{\beta^1-\beta^2-\beta^3}\hat{n}_1/H = -12\hat{n}_1\pi_0^{-1}e^{2\beta^1}. \tag{5.21c}$$

The Hamiltonian is proportional to eq. (3.41d) :

$$\tilde{\mathcal{H}} = 6g^{1/2}NH^2(\Sigma^2+\Omega_k+\Omega-1) = 6g^{1/2}H(\Sigma^2+\Omega_k+\Omega-1) =$$

$$-\frac{1}{2}\pi_0(\Sigma^2-1+\Omega_k+\Omega) = \tilde{N}(T+U_g+U_f) = -12\pi_0^{-1}(T+U_g+U_f),$$

and we have the relations

$$T = \frac{\pi_0^2}{24}(\Sigma^2-1), \quad U_g = \frac{\pi_0^2}{24}\Omega_k, \quad U_f = \frac{\pi_0^2}{24}\Omega.$$

The homothetic property of the potential, $c^i\partial_{\beta^i}U = 4(1-w)U$, allows one write it on the form

$$U = \exp\left[4(1-w)\frac{\mathcal{G}_{ij}c^i\beta^j}{|c|^2_{\mathcal{G}}}\right]V, \tag{5.22}$$

where $c^i\partial_{\beta^i}V = 0$. The monotone function can in turn be expressed in terms of V through the Hamiltonian constraint, and V can be expressed in terms of the scale invariant variables (5.21). We have

$$T+U_g+U_f = 0 \iff \exp\left[4(1-w)\frac{\mathcal{G}_{ij}c^i\beta^j}{|c|^2_{\mathcal{G}}}\right] = -\frac{T}{V_g+V_f} = \frac{\pi_0^2}{24}\frac{1-\Sigma^2}{V_g+V_f},$$

which gives

$$M = (c^i\pi_i)^2\exp\left[-4(1-w)\frac{\mathcal{G}_{ij}c^i\beta^j}{|c|^2_{\mathcal{G}}}\right] = 24\left(\frac{c^i\pi_i}{\pi_0}\right)^2\frac{V_g+V_f}{1-\Sigma^2}.$$

It can be read of directly from eq. (5.21b) and the definition of π_0 that $\pi_1 = \frac{\pi_0}{6}(2 - \Sigma_{11})$ and so on, and we have

$$\left(\frac{c^i\,\pi_i}{\pi_0}\right) = \tfrac{4}{3} - \tfrac{1}{6}(1-w)\Sigma_{11} - \tfrac{1}{2}(1-w)\Sigma_{22} - \tfrac{2}{3}w\Sigma_{33}.$$

To be able to express it in terms of the variables (5.10) we note that the variables (5.21) correspond to a frame choice where the fluid is aligned with the 3-direction, so we have the identification $\Sigma_+ = \frac{1}{2}(\Sigma_{33}+\Sigma_{22})$, $\bar{\Sigma} = \frac{1}{2\sqrt{3}}(\Sigma_{33}-\Sigma_{22})$, which gives

$$\left(\frac{c^i\,\pi_i}{\pi_0}\right) = \tfrac{4}{3} - \tfrac{1}{6}(1+3w)\Sigma_+ + \tfrac{\sqrt{3}}{6}(3-7w)\bar{\Sigma}.$$

The second factor can be reset in scale invariant variables by considering the quotients

$$\frac{V_g}{V_f} = \frac{\Omega_k}{\Omega} = \frac{U_g}{U_f} = \frac{(\hat{n}_1)^2}{l^{1+w}}\exp\left[(3+w)\beta^1 - (1-w)(\beta^2+\beta^3)\right], \qquad (5.23)$$

and by using the Hamiltonian constraint once again

$$\frac{V_g+V_f}{1-\Sigma^2} = \frac{V_g/V_f+1}{1-\Sigma^2}V_f = \frac{\Omega_k/\Omega+1}{\Omega+\Omega_k}V_f = \frac{V_f}{\Omega}. \qquad (5.24)$$

From (5.22) and (5.16) we have

$$V_f = l^{1+w}G_+\Gamma^{1-w}e^{-\left[(3-8w+13w^2)\beta^1-(1+4w-13w^2)\beta^2+(5-13w)(1-w)\beta^3\right]/(3+13w)},$$

which can be split into factors proportional to the scale invariant combinations (5.17) and (5.23):

$$\begin{aligned} V_f &= l^{1+w}G_+\Gamma^{1-w}\left[e^{-(1-w)\left[(3+w)\beta^1-(1-w)(\beta^2+\beta^3)\right]}e^{2(3-7w)\left[w(\beta^1+\beta^2)-(1-w)\beta^3\right]}\right]^{\frac{1}{(3+13w)}} \\ &= l^{1+w}G_+\Gamma^{1-w}\left[\left(\frac{l^{1+w}\Omega_k}{(\hat{n}_1)^2\Omega}\right)^{-(1-w)}\left(\frac{(1+w)l^w v\Gamma^{(1-w)}}{\hat{t}_3}\right)^{2(3-7w)}\right]^{\frac{1}{(3+13w)}}, \end{aligned}$$

which gives the monotone function (B.25) in paper IV.

5.4 Paper V

Finding the Reduced System

An important step in the Fuchsian reduction is to find the correct ansatz for the asymptotic solution. In this paper we do this by assuming that the asymptotic solution is a general solution to a different set of equations. The new set of equations are found by removing those variables that are assumed to be negligible close to the singularity from the original equations.

What variables can we assume are negligible then? Studies of ultra-stiff spatially inhomogeneous models by Coley & Lim [16] have shown that, under the assumption that the past asymptotic state is contained on the silent boundary, one has $\Sigma_{\alpha\beta}, N_{\alpha\beta}, A_\alpha, v_\alpha \to 0$ as $t \to 0^+$, in terms of the variables (3.40); which is to say that the only dominating variables are the expansion θ (or Hubble scalar $H = \frac{1}{3}\theta$) and the energy density, all other variables are negligible close to the singularity. We want to translate this into the variables used in paper V. The variables $N_{\alpha\beta}, A_\alpha$ determine the Hubble normalized spatial curvature, and to say that they vanish is to say that the spatial curvature variables ${}^3R^a{}_b$ are small compared to $(\mathrm{tr}\, k)^2$.[40] That $\Sigma_{\alpha\beta}$ also vanishes is to say that $\sigma^a{}_b$ is small compared to $(\mathrm{tr}\, k)$ asymptotically, and finally that v^α vanishes is the same as $e^\alpha{}_i v^i \to 0$, that is to say that the v^i are small compared to the $e^\alpha{}_i$. It follows that $v^2 = v_i v^i = v_\alpha v^\alpha \to 0$. The corresponding variables in paper V are the three spatial components of the four velocity, denoted u^a in paper V. That the three-velocity vanishes implies that $u^2 = {}^3g_{ab}u^a u^b \to 0$ as $t \to 0^+$.

The Einstein field equations (the Euler equations are left out here), given in a initial value formulation, are stated in terms of the variables $(g_{ab}, (\mathrm{tr}\, k), \sigma^a{}_b, \mu, u_a)$ as a system of evolution equations

$$\partial_t g_{ab} = 2g_{ac}\big(\sigma^c{}_b - \tfrac{1}{3}(\mathrm{tr}\, k)\, \delta^c{}_b\big), \tag{5.25a}$$

$$\partial_t \sigma^a{}_b = (\mathrm{tr}\, k)\sigma^a{}_b - \big(R^a{}_b - \tfrac{1}{3}R\delta^a{}_b\big) + (1+w)\mu(u^a u_b - \tfrac{1}{3}u^2\delta^a{}_b), \tag{5.25b}$$

$$\partial_t(\mathrm{tr}\, k) = R + (\mathrm{tr}\, k)^2 - \mu[(1+w)u^2 + \tfrac{3}{2}(1-w)], \tag{5.25c}$$

and a system of constraints

$$R - \sigma^a{}_b\sigma^b{}_a + \tfrac{2}{3}(\mathrm{tr}\, k)^2 = 2\mu\big(1 + (1+w)u^a u_a\big), \tag{5.26a}$$

$$-\nabla_a\sigma^a{}_b - \tfrac{2}{3}\nabla_b(\mathrm{tr}\, k) = \mu(1+w)(1+u_a u^a)^{1/2}u_b. \tag{5.26b}$$

From the above considerations, one ansatz for the new set of equations would be to discard the terms containing $\sigma^a{}_b, u_a, R^a{}_b$ from the constraint equations and

[40] The positions of the indices are important here since each raising or lowering of an index requires a contraction with the metric or its inverse, which themselves carry dimensional weight and thus affects the behavior at the approach to the singularity. It is the mixed form of the three-curvature that corresponds to the quantities (3.13) constructed from the orthonormal frames. When the slicing defining k_{ab} is identified with the planes spanned by the spatial orthonormal frame fields one also has the relation $\theta = -(\mathrm{tr}\, k)$ which gives the conclusion above.

right hand sides of the evolution equations. This does not turn out to be a good ansatz however. By dropping the shear terms one loses too much structure in the equations to obtain an asymptotic solution. The resulting set of equations is no longer self-consistent even, since the constraint equations are not preserved under time evolution. One way to remedy this is to keep the shear terms in the equations. This ansatz is the one used in [2] to study the stiff case, although the equations are here split into shear and $(\mathrm{tr}\, k)$. The resulting set of equations is then

$$\partial_t\, {}^0g_{ab} = 2\, {}^0g_{ac}\big(\, {}^0\sigma^c{}_b - \tfrac{1}{3}(\mathrm{tr}\ {}^0k)\delta^c{}_b\big), \tag{5.27a}$$

$$\partial_t\, {}^0\sigma^a{}_b = (\mathrm{tr}\ {}^0k)\, {}^0\sigma^a{}_b, \tag{5.27b}$$

$$\partial_t(\mathrm{tr}\ {}^0k) = (\mathrm{tr}\ {}^0k)^2 + \tfrac{3}{2}(w-1)\, {}^0\mu, \tag{5.27c}$$

and

$$-\, {}^0\sigma^a{}_b\, {}^0\sigma^b{}_a + \tfrac{2}{3}(\mathrm{tr}\ {}^0k)^2 = 2\, {}^0\mu, \tag{5.28a}$$

$$-\, {}^0\nabla_a\, {}^0\sigma^a{}_b - \tfrac{2}{3}\, {}^0\nabla_b(\mathrm{tr}\ {}^0k) = {}^0\mu(1+w)\, {}^0u_b, \tag{5.28b}$$

where the variables are appended with a ^{0} to emphasize that they are not to be interpreted as solutions of the Einstein field equations but as an ansatz for the leading order terms of a solution close to $t = 0$. The constraint equations are now preserved under time evolution. To obtain the ansatz for the leading order terms one now just need to solve this system.

A Comparison Between the Stiff and the Ultra-Stiff Models

The inhomogeneous ultra-stiff cosmological model studied in paper V exhibits many similarities, but naturally also some differences, with the stiff models, described by Andersson and Rendall [2]. In this section I take the opportunity to comment on some of them.

One very important similarity of both systems (and actually all systems that have been solved by Fuchsian methods) is that the spatial derivatives of the variables are dynamically irrelevant close to the singularity. The variables are in this case the three-metric and the extrinsic curvature, and that the spatial derivatives of these variables are irrelevant is to say that the spatial connection and spatial curvature are dynamically irrelevant close to the singularity. Only models that have this property can be hoped to be successfully analyzed by Fuchsian methods with these variables. Spatial gradients become important when a singularity is of Mixmaster type during the transitions that generate the Kasner map, which is why only models that have so called AVTD behavior have been analyzed so far.

Perhaps the most important difference between the two models is the structure of the asymptotic solution. The stiff model has an asymptotic structure resembling the anisotropic Jacob's solution while the ultra-stiff model have an asymptotic structure resembling the Friedman model. Technically this difference can be seen in the components of the extrinsic curvature, $k^a{}_b$, of a "simultaneous bang slicing". For the stiff models, all components of the extrinsic curvature have the same scaling with time to first order, namely as t^{-1}, while for the ultra-stiff models, the trace and the trace-free parts of the extrinsic curvature scale differently to leading order, where the trace-part is dominating over the trace-free part close to $t = 0$. The reason for this can be seen directly from the asymptotic system, eqs. (8b) and (8c), in paper V:

$$\partial_t \, {}^0\sigma^a{}_b = (\mathrm{tr}\ {}^0\!k)\, {}^0\sigma^a{}_b,$$
$$\partial_t(\mathrm{tr}\ {}^0\!k) = (\mathrm{tr}\ {}^0\!k)^2 + \tfrac{3}{2}(w-1)\, {}^0\mu.$$

When $w = 1$, the energy decouples from the evolution of $(\mathrm{tr}\ {}^0\!k)$ and both the trace and the trace-free parts are governed by structurally identical equations. For $w > 1$ on the other hand, the energy density makes a positive contribution to the time derivative of $(\mathrm{tr}\ {}^0\!k)$. In an expanding universe ($(\mathrm{tr}\ {}^0\!k) < 0$) this implies that the trace will have a steeper slope as a function of time in an ultra-stiff cosmology than in a stiff cosmology, and hence diverge faster as $t \to 0^+$. In particular it will diverge faster than the shear in the equation above.

In one sense the ultra-stiff model is much simpler than the stiff model, since the asymptotic solution is dominated by the scalar function $(\mathrm{tr}\ {}^0\!k)$, which is a spatial constant to leading order, instead of the full set ${}^0\!k^a{}_b$, which not all can be chosen to be spatial constants by a choice of foliation. In the stiff case, the spatial dependence of the leading order requires spatially dependent exponents in the Fuchsian ansatz, which complicates the analysis considerably.

In another sense the ultra-stiff cosmology is more complicated since the asymptotic system is much harder to solve when the arbitrary parameter w enters the equations. In paper V we only managed to find a closed-form solution for the asymptotic system when $w = 3$. This does not turn out to be a serious problem though since series solutions could be found for general $w > 1$ and all estimates works the same anyway.

5.5 My Contribution to the Papers

Since the papers I, III, IV, and V are written in collaboration with co-authors it is of particular importance to specify my contributions to the different papers.

Paper I This paper was an offshoot of paper III which was initialized when I discovered that it was possible to have two (or more) tilted fluids in Bianchi type I when their energy flux cancel each other. I performed the linear analysis, and did all numerical computations that was used to support the linear analysis. The choice of variables and frame was done in collaboration and the structure of the paper and text was developed through continuous discussions.

Paper III I formulated the conjectures and proved most of the theorems. I also derived the equation for the velocity of the perfect fluid and the equations for the electric and magnetic parts of the Weyl tensor, and the equations for the particle number densities. As in paper I the text and the structure of the paper was developed through continuous discussions.

Paper IV I constructed the new partially rotation invariant dynamical system, expressed the function derived from the Hamiltonian in terms of them and showed that it was monotone on the entire state space for a limited range of the equation of state parameter.

Paper V I proved that the reduced system was of Fuchsian type for the special case when $w = 3$, and showed symmetry for the solution. The final text and structure of the paper was produced in collaboration with the co-author.

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Cosmological Models and Singularities in General Relativity

This is a thesis on general relativity. It analyzes dynamical properties of Einstein's field equations in cosmology and in the vicinity of spacetime singularities in a number of different situations. Different techniques are used depending on the particular problem under study; dynamical systems methods are applied to cosmological models with spatial homogeneity; Hamiltonian methods are used in connection with dynamical systems to find global monotone quantities determining the asymptotic states; Fuchsian methods are used to quantify the structure of singularities in spacetimes without symmetries. All these separate methods of analysis provide insights about different facets of the structure of the equations, while at the same time they show the relationships between those facets when the different methods are used to analyze overlapping areas.

The thesis consists of two parts. Part I reviews the areas of mathematics and cosmology necessary to understand the material in part II, which consists of five papers. The first two of those papers uses dynamical systems methods to analyze the simplest possible homogeneous model with two tilted perfect fluids with a linear equation of state. The third paper investigates the past asymptotic dynamics of barotropic multi-fluid models that approach a 'silent and local' space-like singularity to the past. The fourth paper uses Hamiltonian methods to derive new monotone functions for the tilted Bianchi type II model that can be used to completely characterize the future asymptotic states globally. The last paper proves that there exists a full set of solutions to Einstein's field equations coupled to an ultra-stiff perfect fluid that has an initial singularity that is very much like the singularity in Friedman models in a precisely defined way.

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