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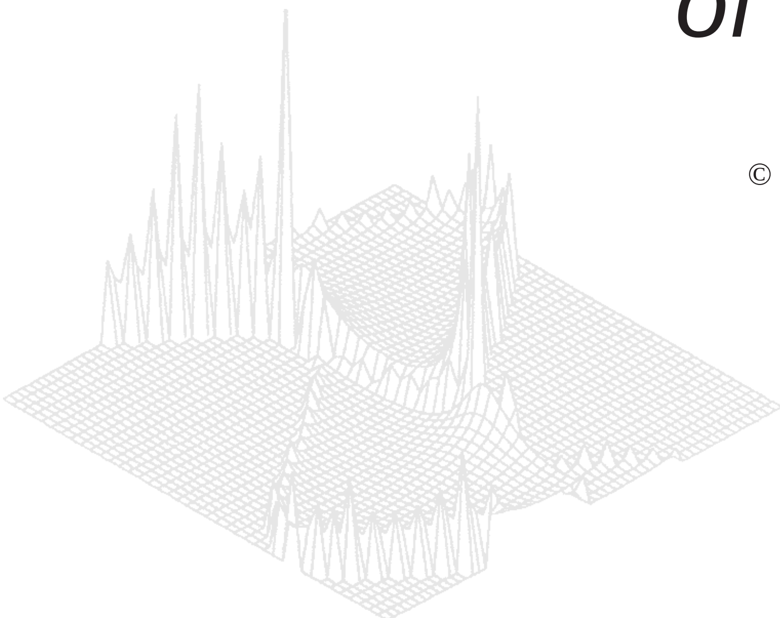

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Cosmological Particle Production and Causal Thermodynamics

M. K. Mak and T. Harko

Department of Physics, University of Hong Kong,
Pokfulam, Hong Kong, China.
email: mkmak@vtc.edu.hk

Abstract

The full linear causal Israel–Stewart–Hiscock theory of bulk viscous processes in relativistic cosmological fluids is reformulated as an effective phenomenological theory for describing particle production processes in the early universe. Explicit expressions for the particle balance law and particle production rates are obtained that relate the particle creation rate to the bulk viscous (creation) pressure. The general formalism is applied to the case of a full causal cosmological fluid with bulk viscosity coefficient proportional to the Hubble function. In this case the general solution of the gravitational field equations can be expressed in an exact parametric form. For an appropriate choice of the physical parameters, the dynamics of the universe can be modelled as starting from a vacuum quasi-Minkowskian geometry, followed by an inflationary period but ending in a non-inflationary phase. The influence of the matter creation processes on the evolution of the universe and the behaviour of the energy density, temperature and entropy are investigated.

1. Introduction

Dissipative processes are supposed to play a fundamental role in the evolution and dynamics of the early universe. Over thirty years ago Misner (1966) suggested that the observed large-scale isotropy of the universe is due to the action of the neutrino viscosity, which was effective when the universe was about one second old. Bulk viscosity may arise in many physical phenomena during the evolution of the early universe. Some of the physical processes that generate bulk viscosity could be the evolution of cosmic strings due to their interaction with each other and with the surrounding matter, the classical description of the (quantum) particle production phase, interaction between matter and radiation, quark and gluon plasma viscosity, different components of dark matter, etc. see Chimento and Jakubi (1996). The first attempts at creating a theory of relativistic dissipative fluids were those of Eckart (1940) and Landau and Lifshitz (1987). These theories are now known to be pathological in several respects. Regardless of the choice of equation of state, all equilibrium states in these theories are unstable, and signals may be propagated through the fluid at velocities exceeding the speed of light (Israel 1976). The problems arise due to the first-order nature of the theory, i.e. it considers only first-order deviations from equilibrium, leading to parabolic differential equations and hence to infinite speeds of propagation for heat flow and viscosity, in contradiction with the principle of causality. While such

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paradoxes appear particularly glaring in relativistic theory, infinite propagation speeds already constitute a difficulty at the classical level, since one does not expect thermal disturbances to be carried faster than some (suitably defined) mean molecular speed. Conventional theory is thus applicable only to phenomena that are 'quasi-stationary', i.e. slowly varying on space and time scales characterised by mean free path and mean collision time (Israel 1976). This is inadequate for many phenomena in high-energy astrophysics and relativistic cosmology that involve steep gradients or rapid variations. The deficiencies can be traced to the fact that the conventional theories (both classical and relativistic) make overly restrictive hypotheses concerning the relation between the fluxes and densities of entropy, energy and particle number.

A relativistic second-order theory was found by Israel (1976) and developed by Israel and Stewart (1976) into what is called 'transient' or 'extended' irreversible thermodynamics. In this theory the deviations from equilibrium (bulk stress, heat flow and shear stress) are treated as independent dynamical variables, leading to a total of 14 dynamical fluid variables to be determined. However, Hiscock and Lindblom (1989) and Hiscock and Salmonson (1991) have shown that most versions of the causal second-order theories omit certain divergence terms. The truncated causal thermodynamics of bulk viscosity leads to pathological behaviour in the late universe, while the solutions of the full causal theory are well behaved for all times. Therefore, the best currently available theory for analysing dissipative processes in the universe is the full Israel–Stewart–Hiscock causal thermodynamics.

On the other hand, it has been suggested by Zeldovich (1970) and later by Murphy (1973) and Hu (1982) that the introduction of viscosity in the cosmological fluid is nothing but a phenomenological description of the effect of creation of particles by the non-stationary gravitational field of the expanding cosmos. A non-vanishing particle production rate is equivalent to a bulk viscous pressure in the cosmological fluid, or, from a quantum point of view, with a viscosity of the vacuum. This is due to the simple circumstance that any source term in the energy balance of a relativistic fluid may be formally rewritten in terms of an effective bulk viscosity. Quantum corrections of the macroscopic stress–energy tensor can be described by a viscous-type pressure that is a polynomial function of the expansion factor (Vereshkov *et al*. 1977). Barrow (1988) has considered creation processes in string-driven inflation within a fluid model, and concluded that this process may be described phenomenologically in terms of an effective bulk viscosity. From the point of view of the kinetic theory based on a Boltzmann-type equation, it follows that this simple phenomenological approach is compatible with the kinetic theory in homogeneous spacetimes but not in inhomogeneous ones (Triginer *et al*. 1996). A simple kinetic model describing particle production processes in the expanding universe and the equilibrium conditions for Maxwell–Boltzmann gas with variable particle number has been investigated by Zimdahl *et al*. (1996). The creation of particles is dynamically equivalent in this context to a non-vanishing bulk pressure, and exponential inflation is shown to become inconsistent with the second law of thermodynamics after a time interval of the order of the Hubble time.

Consequently, if it is possible to describe particle production processes consistently by means of an effective viscous pressure, one is able to study, at least on a simple classical phenomenological level, the impact of these processes on the early dynamics of the universe.

Zimdahl (1996) has considered in detail the possibility that the bulk viscous pressure of the full Israel–Stewart–Hiscock theory may also be interpreted as an effective description for particle production processes. The creation process leads to considerable changes in the thermodynamical behaviour of the universe. If the chemical potential of the newly created particles is zero, $\mu = 0$, then the non-vanishing bulk pressure Π associated with an increase in the number of fluid particles satisfies formally the same equation as in the case of the presence of a real dissipative bulk viscosity. The reheating process in inflationary universe models, considered as an out-of-equilibrium mixture of two interacting and reacting fluids, has been studied within the framework of causal irreversible thermodynamics by Zimdahl *et al*. (1997). The particle decay and creation rates are determined by the causal thermodynamics and are estimated by the authors at different stages of the reheating process. Zimdahl (1998) has defined generalised equilibrium states for cosmological fluids with particle production. The equivalence between the creation rate for particles with non-zero mass and an effective viscous fluid pressure follows as a consequence of the generalised equilibrium properties, and leads to the possibility of a power-law inflationary behaviour. The two distinct irreversible phenomena of matter creation and bulk viscosity, considered independently but giving rise to cross effects, have been applied to Friedmann–Robertson–Walker (FRW) cosmologies by Gariel and Le Denmat (1995). It was shown that the cross effects could play a non-negligible role in the early universe evolution. Gariel *et al*. (1997) accounted for bulk viscosity and matter creation in a simple cosmological fluid and studied the wavefront speed associated with the characteristics of the fluid. They have shown that power-law inflation can be a causal solution for FRW cosmologies and is thermodynamically admissible, whereas exponential inflation is not. Along the same lines, Gariel and Cissoko (1996) calculated, using the relativistic extended irreversible thermodynamics, wavefront velocities for a simple dissipative fluid with scalar effects including bulk viscous stresses and matter creation. Applying this method to a spatially flat FRW cosmological model, one obtains infinite wave speeds, so exponential inflation cannot be driven by bulk viscosity. Maartens and Mendez (1997) have proposed a nonlinear generalisation of the causal linear thermodynamics of bulk viscosity, incorporating the positivity of the entropy production rate and of the effective specific entropy. As applied to viscous fluid inflation (which is necessarily a far-from-equilibrium process), the nonlinear theory leads to thermodynamically consistent inflationary solutions for both exponential and power-law cases. Exact solutions for the nonlinear theory, with bulk viscosity coefficient proportional to the Hubble factor for models with barotropic temperature and ideal gas temperature, have been obtained by Chimento *et al*. (1997) and the asymptotic stability of the de Sitter or Friedmann solutions has been investigated.

Recently, Chimento and Jakubi (1997*a*) have found the exact general solution to the Einstein gravitational field equations in a homogeneous universe filled with a full causal viscous fluid source obeying the relation $\xi \sim \rho^{\frac{1}{2}}$. The solutions correspond to two different choices of the state equations for pressure, bulk viscosity coefficient, temperature and bulk relaxation time. The equation describing the dynamics of the universe in this case takes the form of a particular Painlevé–Ince equation that can be linearised by means of an appropriate transformation. The exact solution of the gravitational field equations is expressed in an exact parametric form as a two-parameter family. Depending on the values of the parameters, Chimento and Jakubi (1997*a*) classified their two-parameter families of solutions according to their number of singularities and obtained several scenarios for the dynamics of the universe. Thus in the case $p = (\gamma - 1)\rho$, $\xi \sim \rho^{\frac{1}{2}}$, $T \sim \rho^{r}$ and $\tau = \xi/\rho \sim \rho^{-\frac{1}{2}}$, the evolution of the universe could begin at a singularity and asymptotically approaches a Minkowskian spacetime. Alternative possibilities are asymptotic Friedmann or de Sitter behaviour, or an asymptotic Minkowski phase in the far past and asymptotically Friedmann, de Sitter or divergent behaviour at finite time in the future. All these two-parameter solutions violate the dominant energy condition or the strong energy condition for some time intervals. Chimento (1997) considered in detail the properties of the corresponding Painlevé–Ince equation, whose invariant form was reduced to a linear inhomogeneous ordinary second-order differential equation with constant coefficients by means of a non-local transformation.

The evolution equation of the bulk viscous universe can be also transformed into an Abel-type equation. Its general solution is represented in an exact closed parametric form and corresponds to a transition between two Minkowskian spacetimes connected by an inflationary period (Mak and Harko 1998*a*). New classes of exact solutions of the field equations can also be generated from some particular solutions of the Abel equation, leading to two classes of general solutions of the Einstein field equations corresponding to particular values of the parameters $b = \frac{1}{4}$ and $b = \frac{2}{9}$ entering the physical model. The solutions obtained are also represented mathematically in an exact parametric form and are interpreted physically as describing cosmological particle production (Mak and Harko 1998*b*). Recently we have presented new classes of exact causal viscous cosmologies in the case of $\xi \sim \rho^{s}$, where $s = \frac{1}{2} \pm 2[\frac{1}{2}(1-r)^{3/2}]$ (see Harko and Mak 1999).

Our intention in the present paper is to investigate in a systematic way the possibility that causal thermodynamics can be used as a phenomenological description of particle production in the cosmological fluid filling the very early universe. The causal bulk viscous inflationary solutions generally lead to a rapid increase of the energy density, temperature and entropy of the cosmological fluid due to the presence of intense dissipative processes, and this behaviour is in fact exactly what is expected from a theory describing particle production. The equivalence between the causal thermodynamics and particle production theories will be considered, leading to general expressions for the particle creation rates. As a toy-model in which the particle production rates can be explicitly obtained and their time behaviour explicitly analysed, we present the model of a cosmological fluid with bulk viscosity coefficient proportional to the square root of the density, $\xi \sim \rho^{\frac{1}{2}}$, or, equivalently, proportional to the Hubble factor H, $\xi \sim H$. In this case, with the use of a mathematical formalism different from that of Chimento and Jakubi (1997*a*) and Chimento (1997), and by means of an appropriate transformation, we reduce the equation describing the evolution of the dissipative-processes-dominated universe to an exactly integrable differential equation. The general solution (mathematically equivalent to that of Chimento) is expressed in an exact and easy-to-handle parametric form, and is physically interpreted as describing particle production in the early universe. The behaviours of the energy density, temperature, bulk viscosity coefficient, deceleration parameter, particle creation rate and the entropy are analysed.

The present paper is organised as follows. In Section 2 we formulate the full Israel–Stewart–Hiscock causal thermodynamics as a phenomenological theory describing particle production. The equations describing the dynamics of a causal bulk viscous fluid-filled FRW universe are obtained in Section 3. The general solution of the field equations, in the case of a bulk viscosity coefficient proportional to the Hubble function, is obtained in Section 4. Finally, in Section 5 we summarise and conclude our results.

2. Causal Thermodynamics as a Particle Production Theory

In this section we show that causal thermodynamics can be reformulated as a particle creation theory without introducing the supplementary hypothesis of an exterior source term in the particle balance equation. The particle creation rate naturally follows from the choice of a state equation of the newly produced matter, and is determined by the internal mathematical and physical structure of the theory.

The energy–momentum tensor of a relativistic fluid with bulk viscosity as the only dissipative phenomena is $(c = 8\pi G = 1)$

$$T_i^k = (\rho + p + \Pi)u_i u^k - (p + \Pi)\delta_i^k \,. \tag{1}$$

Here ρ is the energy density, u^i, $i = 0, 1, 2, 3$, the four-velocity $(u_i u^i = 1)$, p the equilibrium pressure and Π the bulk viscous pressure. The particle flow vector N^i is given by

$$N^i = nu^i \,, \tag{2}$$

where $n \geq 0$ is the particle number density.

Limiting ourselves to second-order deviations from equilibrium, the entropy flow vector S^i takes, in the framework of causal thermodynamics, the form (Israel 1976; Israel and Stewart 1976)

$$S^i = sN^i + \frac{R^i(N^i, T_i^k)}{T} = sN^i - \frac{\tau\Pi^2}{2\xi T}u^i \,, \tag{3}$$

where s is the entropy per particle, τ the relaxation time, T the temperature and ξ the coefficient of bulk viscosity.

In the case of a flat FRW spacetime with the line element

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \,, \tag{4}$$

the gravitational field equations take the form

$$3H^2 = \rho \,, \tag{5}$$

$$2\dot{H} + 3H^2 = -p - \Pi \,, \tag{6}$$

where $H = \frac{1}{3}u^i_{;i} = \dot{a}/a$ is the Hubble factor. We also introduce the deceleration parameter

$$q = \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{H}\right) - 1 = \frac{\rho + 3p + 3\Pi}{2\rho}\,. \tag{7}$$

From equations (5) and (6) or the conservation law $T^k_{i;k} = 0$ (where a semicolon represents the covariant with respect to the line element), we obtain

$$\dot{\rho} + 3H(\rho + p) = -3H\Pi\,. \tag{8}$$

We shall consider in the following that there is a change in the number of particles, due to matter creation processes with bulk viscous pressure playing the role of a 'creation pressure'. We shall first suppose as a toy model that the newly created particles obey equations of state of the form

$$\rho = \rho_0\left(\frac{n}{n_0}\right)^{\gamma} = kn^{\gamma}, \qquad p = (\gamma - 1)\rho\,, \tag{9}$$

where we have denoted $k = \rho_0/n_0^{\gamma}$ and $1 \leq \gamma \leq 2$. Using (9), equation (8) takes the form of a particle balance equation,

$$\dot{n} + 3Hn = \Gamma n\,, \tag{10}$$

where

$$\Gamma = -\frac{\Pi}{\gamma H} = \frac{H}{\gamma}[3\gamma - 2(q + 1)] \tag{11}$$

is the particle production rate proportional to the bulk viscous pressure. The condition $\Gamma \geq 0$ leads to the following restriction imposed on the deceleration parameter q:

$$q \leq \frac{3\gamma}{2} - 1\,. \tag{12}$$

This condition is always satisfied by inflationary cosmological models for which $q < 0$ and consequently $\rho + 3(p + \Pi) < 0$.

Combining the equation of state (9) with the Gibbs relation

$$T\mathrm{d}s = \mathrm{d}\left(\frac{\rho}{n}\right) + p\mathrm{d}\left(\frac{1}{n}\right), \tag{13}$$

we obtain

$$s = s_0 = \text{constant}\,. \tag{14}$$

Physically, equation (14) means that particles are created with constant entropy. By using the expressions given above, we obtain for the entropy production density the expression

$$S^i_{;i} = -\frac{\Pi}{T}\left[\frac{s_0 nT}{\gamma H} + \frac{\tau\dot{\Pi}}{\xi} + \frac{\tau\Pi}{2\xi}\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right)\right]. \tag{15}$$

(There is entropy production because of the enlargement of the phase space.)

The simplest way to guarantee $S^i_{;i} \geq 0$ and to maintain the form (3) of the entropy, thus ensuring causal and stable behaviour of the matter-creation-dominated thermodynamical system, implies for the particle creation rate Γ the evolution equation

$$\tau\dot{\Gamma} + \left[1 + \left(\frac{\dot{H}}{H} + \tfrac{1}{2}\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right)\right)\tau\right]\Gamma = \frac{s_0}{\gamma^2}\left(\frac{3}{k}\right)^{\frac{1}{\gamma}}\xi T H^{2\left(\frac{1}{\gamma}-1\right)} \tag{16}$$

leading to

$$S^i_{;i} = \frac{\gamma^2 H^2}{\xi T}\Gamma^2 = \frac{H^4[3\gamma - 2(q+1)]^2}{\xi T} \geq 0\,. \tag{17}$$

If the condition

$$3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T} = 0 \tag{18}$$

is satisfied, the equations governing the evolution of the particle creation rate and of the entropy production density become

$$\tau\dot{\Gamma} + \left[1 + \tau\frac{\dot{H}}{H}\right]\Gamma = \frac{s_0}{\gamma^2}\left(\frac{3}{k}\right)^{\frac{1}{\gamma}}\tau a^3 H^{2\left(\frac{1}{\gamma}-1\right)} \tag{19}$$

$$S^i_{;i} = \frac{\gamma^2 H^2}{\tau a^3}\Gamma^2 \geq 0\,. \tag{20}$$

Let us assume now that the newly created particles obey equations of state in the general form

$$\rho = \rho(n,T), \qquad p = p(n,T)\,, \tag{21}$$

according to which the particle number density n and the temperature T are the basic thermodynamic variables. By using the general relation

$$\frac{\partial\rho}{\partial n} = \frac{\rho + p}{n} - \frac{T}{n}\frac{\partial p}{QpT}\,, \tag{22}$$

equation (8) can again be rewritten in the form of a particle balance equation,

$$\dot{n} + 3Hn = \Gamma n\,, \tag{23}$$

where the particle creation rate Γ is given by

$$\Gamma = \frac{1}{\rho + p}\left[T\left(\frac{\dot{n}}{n}\frac{\partial\rho}{\partial T} - \frac{\dot{T}}{T}\frac{\partial\rho}{\partial T}\right) - 3H\Pi\right] = -\frac{(nT\dot{s} + 3H\Pi)}{\rho + p}\,. \tag{24}$$

The condition of non-negativity $\Gamma \geq 0$, imposes the following constraint on the deceleration parameter q:

$$q \leq \frac{T}{6H^3}\left(\frac{\dot{n}}{n}\frac{\partial p}{\partial T} - \frac{\dot{T}}{T}\frac{\partial\rho}{\partial T}\right) + \frac{p}{2H^2} + \tfrac{1}{2} = \frac{1}{2H^2}\left(p - \frac{nT\dot{s}}{3H}\right) + \tfrac{1}{2}\,. \tag{25}$$

The Gibbs equation (13) leads to

$$n\dot{s} = \frac{\dot{T}}{T}\frac{\partial\rho}{\partial T} - \frac{\dot{n}}{n}\frac{\partial p}{\partial T} = \frac{Q}{T}\,. \tag{26}$$

The entropy production density becomes

$$S^i_{;i} = -\frac{\Pi}{T}\left[\frac{\mu n\Gamma}{\Pi} + 3H + \frac{\tau\dot{\Pi}}{\xi} + \frac{\Pi T}{2}\left(\frac{\tau}{\xi T}u^i\right)_{;i}\right], \tag{27}$$

where we have introduced the chemical potential $\mu = Ts - (\rho + p)/n$.

The simplest way to guarantee $S^i_{;i} \geq 0$ and causal and stable behaviour implies for the particle creation rate Γ the following strongly nonlinear evolution equation:

$$\tau\dot{\Gamma} + \left[1 + \frac{\tau}{2}\left(\frac{2\dot{\sigma}}{\sigma} - \frac{2\dot{H}}{H} + F\right)\right]\Gamma =$$

$$\frac{\tau Q}{\sigma}\left(\frac{\dot{H}}{H} - \frac{\dot{Q}}{Q} - \frac{F}{2} - \frac{1}{\tau}\right) + \frac{9H^2\xi}{\sigma} - \frac{9H^2\xi\mu n\Gamma}{\sigma(Q + \sigma\Gamma)}. \tag{28}$$

Equation (28) also gives

$$S^i_{;i} = \frac{[nT\dot{s} + (\rho + p)\Gamma]^2}{9H^2\xi T} \geq 0\,. \tag{29}$$

In equation (28) we have denoted

$$\sigma = \rho + p \quad \text{and} \quad F = 3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\,.$$

Thus we have arrived at the result that the universe with particle production can be described by using the basic equations of the causal thermodynamics. These lead to explicit expressions for the particle creation rate and to the possibility of a phenomenological description of the matter creation processes (which are essentially quantum processes) at the classical level in the framework of this theory. The entire dynamics and evolution of the universe can be expressed, and are determined by the particle creation rate that in the present theory simply follows from the state equation of the newly created matter.

Finally, we shall consider the relationship between the present theory describing matter and entropy creation in the early universe and the theory proposed by Prigogine *et al.* (1988). This particle creation theory is based on the reinterpretation of the matter energy–stress tensor in the framework of the thermodynamics of open systems, leading to the modification of the adiabatic energy conservation law and thereby including irreversible matter creation. Calvao *et al.* (1992) have re-examined this phenomenological approach within a manifestly covariant formulation. The matter creation corresponds to an irreversible energy flow from the gravitational field to the created particle constituents, and involves the inclusion in the matter energy–momentum tensor of a supplementary creation pressure p_c given by

$$p_{\mathrm{c}} = -\frac{\rho + p}{3nH}(\dot{n} + 3Hn)\,. \tag{30}$$

Particle production is associated with entropy production, which is given by Calvao *et al.* (1992) in the form

$$S^i_{;i} = -\frac{3Hp_{\mathrm{c}}}{T}\left(1 + \frac{\mu\Gamma n}{3Hp_{\mathrm{c}}}\right) \geq 0\,. \tag{31}$$

By assuming a particle creation rate of the form

$$\Gamma = \frac{\alpha H^2}{n}, \qquad \alpha = \mathrm{constant}\,, \tag{32}$$

a three-stage cosmology has been obtained by Prigogine *et al.* (1988). The universe starts from an initial fluctuation (instability) of the vacuum and a creation period drives the cosmological system to a de Sitter space. The de Sitter space exists during the decay time of its constituents (second stage) and a phase transition turns the de Sitter space into the usual Robertson–Walker universe (third stage).

By using equations (10) and (11) in equation (30), it immediately follows that

$$p_c = \Pi\,. \tag{33}$$

The causal bulk viscous pressure Π from the present formalism acts as a creation pressure. In the formulation of Calvao *et al.* (1992) the particle production rate is related to the creation pressure, in the case of a relativistic fluid obeying an equation of state given by equation (9), by means of a phenomenological ansatz of the form $\Gamma \sim -\Pi H^{1-\frac{2}{\gamma}}$. In the particle creation formalism described in this paper the particle production rate is given by equation (11), $\Gamma \sim -\Pi H^{-1}$, and is independent of the equation of state of the newly produced matter. In the case of dust ($\gamma = 1$) the two particle creation rates are similar. The major difference between the theory presented in this paper and that of Prigogine *et al.* (1988) and Calvao *et al.* (1992) is the expression for entropy production. In our theory this takes a more general form, being quadratic in the creation pressure, $S^i_{;i} \sim p_c^2/\xi T$, and involves a new dynamical variable, the bulk viscosity coefficient.

3. Dynamics of Causal Bulk Viscous Universes

In the framework of the full causal thermodynamics, the causal evolution equation for the bulk viscous pressure Π is given by (Maartens 1995)

$$\tau\dot{\Pi} + \Pi = -3\xi H - \frac{\tau\Pi}{2}\left(3H + \frac{\dot{\tau}}{\tau} - \frac{\dot{\xi}}{\xi} - \frac{\dot{T}}{T}\right). \tag{34}$$

In order to close the system of equations (5), (6), (8) and (34), we have to give the equation of state for p and specify T, τ and ξ. Some authors have discussed the equations of state for a homogeneous isotropic viscous fluid. Lake (1982) considered a rather simplified equation of state given by the condition of the trace of the energy–momentum tensor being null. The analysis of the relativistic kinetic equation for some simple cases (Murphy 1973; Belinskii and Khalatnikov 1975; Belinskii *et al.* 1979) shows that in the asymptotic regions of small and large values of energy density, the viscosity coefficients can be approximated by power functions of the energy density. Definite requirements on the exponents of these functions are imposed. For small values of the energy density it is reasonable to consider large exponents, equal in the extreme case to unity. For large ρ, the power of the bulk viscosity coefficient should be considered as less than or equal to $\frac{1}{2}$. Thus we shall assume the following simple phenomenological laws (Belinskii and Khalatnikov 1975; Belinskii *et al.* 1979; Maartens 1995):

$$p = (\gamma - 1)\rho\,, \tag{35}$$

$$\xi = \alpha\rho^s\,, \tag{36}$$

$$T = \beta\rho^r\,, \tag{37}$$

$$\tau = \frac{\xi}{\rho} = \alpha\rho^{s-1}\,, \tag{38}$$

where $1 \le \gamma \le 2$, $\alpha \ge 0$, $\beta > 0$, $r \ge 0$ and $0 \le s \le \frac{1}{2}$ are constants. Equations (35)–(37) are standard in cosmological models, whereas (38) is a simple procedure to ensure that the speed of viscous pulses does not exceed the speed of light (Belinskii *et al*. 1979).

In the present model, equation (22) imposes the constraint

$$r = \frac{\gamma - 1}{\gamma}\,, \tag{39}$$

so that $0 \le r \le \frac{1}{2}$ for $1 \le \gamma \le 2$, a range of values that is usually considered in the physical literature (Chimento and Jakubi 1997*a*).

The growth of the total comoving entropy over a proper time interval $(t_0,\, t_1)$ is given by (Maartens 1995)

$$\Sigma_1 - \Sigma_0 = -\frac{3}{k_{\rm B}} \int_{t_0}^{t_1} \frac{\Pi a^3 H}{T} {\rm d}t\,, \tag{40}$$

where $k_{\rm B}$ is Boltzmann's constant.

The Israel–Stewart theory is derived under the assumption that the thermodynamical state of the fluid is close to equilibrium, which means that the non-equilibrium bulk viscous pressure should be small when compared to the local equilibrium pressure, that is (Maartens 1995)

$$|\Pi| \ll p = (\gamma - 1)\rho\,. \tag{41}$$

If this condition is violated then one is effectively assuming that the linear theory holds also in the nonlinear regime far from equilibrium. For a fluid description of the matter and in the presence of real viscosity, equation (41) must be satisfied (Maartens 1995).

With these assumptions the field equations and the causal evolution equation for the bulk viscosity lead to the following evolution equation for the Hubble function (Maartens 1995):

$$\ddot{H} + 3H\dot{H} + 3^{1-s}\alpha^{-1}H^{2-2s}\dot{H} - (1+r)H^{-1}\dot{H}^2 + \tfrac{9}{4}(\gamma - 2)H^3 + \tfrac{1}{2}3^{2-s}\alpha^{-1}\gamma H^{4-2s} = 0. \tag{42}$$

In the following we shall consider equation (42) in the particular case of a bulk viscous fluid with bulk viscosity coefficient proportional to the square root of the density. That is, we choose $s = \frac{1}{2}$, which corresponds to the extreme limit of high matter densities (Belinskii *et al*. 1979); we shall also use equation (39). With these assumptions equation (42) takes the form

$$\ddot{H} + 3\left(1 + \frac{1}{\sqrt{3}\alpha}\right)H\dot{H} - (1+r)H^{-1}\dot{H}^2 + \frac{9(2/\sqrt{3}\alpha - 1 + 2r)}{4(1-r)}H^3 = 0\,. \tag{43}$$

4. General Solution to the Field Equations for $\xi \sim H$

By means of the transformations (Chimento and Jakubi 1993)

$$H^2 = y, \qquad \eta = 3\left(1+\frac{1}{\sqrt{3}\alpha}\right)\int H\mathrm{d}t\,, \tag{44}$$

equation (43), describing the evolution of a causal bulk viscous fluid-filled universe with bulk viscosity coefficient proportional to the Hubble function, takes the form

$$\frac{\mathrm{d}^2 y}{\mathrm{d}\eta^2} + \frac{\mathrm{d}y}{\mathrm{d}\eta} - \frac{1+r}{2y}\left(\frac{\mathrm{d}y}{\mathrm{d}\eta}\right)^2 + \frac{2by}{1-r} = 0\,, \tag{45}$$

where we have denoted

$$b = \left(2r - 1 + \frac{2}{\sqrt{3}\alpha}\right)\bigg/ 4\left(1+\frac{1}{\sqrt{3}\alpha}\right)^2.$$

We shall consider in the following that $r \in [0, \frac{1}{2}]$ and α are independent (positive) parameters.

With the use of the mathematical substitution $\mathrm{d}y/\mathrm{d}\eta = yw(y)$, equation (45) is transformed into the following first-order differential equation for the unknown function $w(y)$:

$$y\frac{\mathrm{d}w}{\mathrm{d}y} = \left(\frac{r-1}{2}\right)w - 1 + \left(\frac{2b}{r-1}\right)\frac{1}{w}\,. \tag{46}$$

Equation (46) has the general solution

$$y = y_0[|(1-r)w + 1 + \Delta|]^{-\frac{1+\Delta^{-1}}{1-r}}[|(1-r)w + 1 - \Delta|]^{-\frac{1-\Delta^{-1}}{1-r}}\,, \tag{47}$$

where

$$\Delta = |\sqrt{1-4b}| \text{ and } 1 - 4b > 0, \text{ for all } r,\ \gamma \text{ and } \alpha\,, \tag{48}$$

and y_0 is a constant of integration. The condition $b < \frac{1}{4}$ is identically satisfied for the considered range of the physical parameters r, γ and α.

By introducing a parameter $\theta > 0$ by means of the transformation

$$\theta = \left[\frac{|(1-r)w + 1 - \Delta|}{2\Delta}\right]^{\frac{1+\Delta^{-1}}{2(1-r)}}\,, \tag{49}$$

where

$$\beta_0 = \frac{4(1-r)(2\Delta)^{\frac{1}{1-r}}}{3(1-\Delta)\beta_1\sqrt{y_0}}, \qquad n = \frac{2(1-r)}{1-\Delta^{-1}},$$

$$m = \frac{1+\Delta^{-1}}{2(1-r)} = \frac{1}{1-r} - \frac{1}{n}, \qquad l = \frac{2}{3\Delta\beta_1},$$

$$H_0 = \sqrt{y_0}(2\Delta)^{-\frac{1}{1-r}}, \qquad \rho_0 = 3H_0^2, \qquad \xi_0 = \sqrt{3}\alpha H_0,$$

$$T_0 = 3^r\beta H_0^{2r}, \qquad \tau_0 = \frac{\alpha}{H_0\sqrt{3}}, \qquad q_0 = \frac{1}{\beta_0 H_0},$$

$$S_0 = \frac{3^{1-r}}{\beta k_{\mathrm{B}}} a_0^3 H_0^{2(1-r)}, \qquad \beta_1 = 2\left(1 + \frac{1}{\sqrt{3}\alpha}\right),$$

and with y_0, t_0, Σ_0 and a_0 constants of integration, we can express the general solution of the gravitational field equation for a FRW universe filled with a causal bulk viscous cosmological fluid, whose bulk viscosity coefficient is proportional to the Hubble function, in the following exact parametric form:

$$t - t_0 = \beta_0 \int_{\theta_0}^{\theta} (\chi^n + 1)^{m-1} \mathrm{d}\chi, \tag{50}$$

$$H = \frac{H_0}{\theta(\theta^n + 1)^m}, \tag{51}$$

$$a = a_0(\theta^{-n} + 1)^l, \tag{52}$$

$$\rho = \frac{\rho_0}{\theta^2(\theta^n + 1)^{2m}}, \tag{53}$$

$$p = \frac{(\gamma - 1)\rho_0}{\theta^2(\theta^n + 1)^{2m}}, \tag{54}$$

$$\xi = \frac{\xi_0}{\theta(\theta^n + 1)^m}, \tag{55}$$

$$T = \frac{T_0}{\theta^{2r}(\theta^n + 1)^{2rm}}, \tag{56}$$

$$\tau = \tau_0\theta(\theta^n + 1)^m, \tag{57}$$

$$q = q_0[(1 + mn)\theta^n + 1] - 1, \tag{58}$$

$$\Sigma(\theta) - \Sigma_0(\theta_0) = S_0 \int_{\theta_0}^{\theta} \chi^{2r-3-3nl}(\chi^n + 1)^{2m(r-1)-1+3l}$$
$$\times [3\gamma H_0 \beta_0 - 2 - 2(1+nm)\chi^n] d\chi \,. \tag{59}$$

For $|\chi^n| < 1$, a condition that holds in the case of a negative n, for large values of χ, the integral (46) can be obtained in an exact form given by

$$t - t_0^{'} = \beta_0 \sum_{k=0}^{\infty} \frac{C_k^{m-1}}{1+nk} \theta^{1+nk} \,, \tag{60}$$

where

$$C_k^{m-1} = \frac{(m-1)(m-2)...(m-k)}{k!}, \qquad t_0^{'} = t_0 - \beta_0 \sum_{k=0}^{\infty} \frac{C_k^{m-1}}{1+nk} \theta_0^{1+nk} \,.$$

Hence for large values of the parameter θ, the general solution of the field equations can be expressed in an exact analytical form.

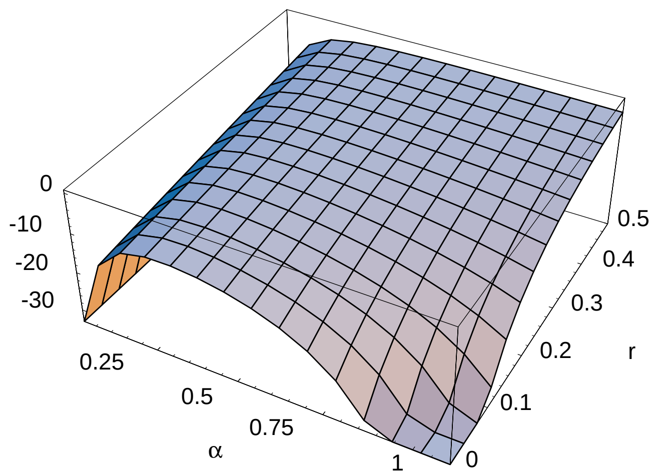

Fig. 1. Variation of the parameter n as a function of r and α.

5. Discussion and Final Remarks

The general solution of the gravitational field equations describing a universe filled with a bulk viscous cosmological fluid, given by equations (50)–(60), is mathematically equivalent to the solution previously obtained by Chimento and Jakubi (1997*a*). For a detailed analysis of its behaviour see Chimento and Jakubi (1997*a*). Our main purpose is to show that for some values of the physical parameters, the solution of the field equations obtained above can be interpreted as a physical model of matter creation in the very early universe. Consequently we shall restrict our analysis only to the case $n < 0$. The variation of the parameter n as a function of r and α is represented in Fig. 1. In Fig. 2 we have represented the variation of n as a function of α for different values of r. Generally $n < 0$ for $r \in [0, \frac{1}{2}]$ and $\alpha \in [0, 1\cdot 5]$.

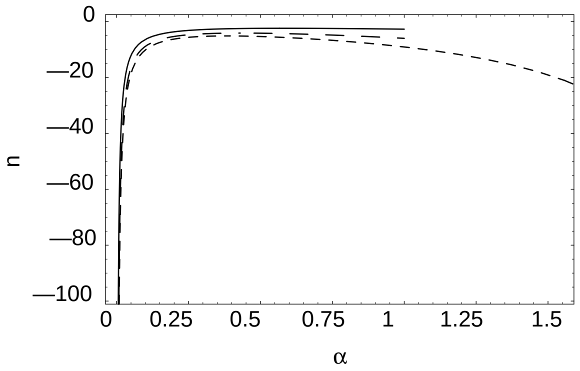

Fig. 2. Variation of the parameter n as a function of α for different values of r. Solid curve: $r = \frac{1}{2}$, long-dashed curve: $r = \frac{1}{3}$, short-dashed curve: $r = \frac{1}{4}$.

For these values of the physical parameters, the universe described by equations (50)–(60) starts its evolution at the moment $t = t_0$ from a non-singular state characterised by a finite non-zero value of the scale factor, $a(t_0) = a_0 \neq 0$. This initial Minkowskian geometry corresponds null values of the Hubble factor and of the energy density, $H(t_0) = 0$ and $\rho(t_0)$. For small times (and values of θ for which $\theta^{-n} \ll 1$) the geometry of the universe is approximately Minkowskian but during this era there is a slow increase in the energy density, the temperature and the bulk viscosity coefficient, due to a small increase in the Hubble function. The intensity of the bulk-viscous-type dissipative processes is increasing, and after a finite time spent in the quasi-Minkowskian era, accelerated expansion occurs, leading to a rapid increase of the energy density, temperature and bulk viscosity coefficient. The maximum value of the energy density is obtained for values of the parameter θ so that $d\rho/d\theta = 0$. From equation (53) we easily obtain

$$\theta_{\max} = \left(-\frac{1}{1+mn}\right)^{\frac{1}{n}}. \tag{61}$$

In order that both θ and the energy density ρ be real and positive-valued, it is necessary that $1 + mn < 0$ and $mn/(1+mn) > 0$. These conditions are both satisfied if and only if $b > 0$. In this case the maximum value of the energy density is given by

$$\rho_{\max} = \rho_0(|mn|)^{-2m}(|1+mn|)^{\frac{2}{1-r}}. \tag{62}$$

The condition $b > 0$ is fulfilled for all $r \in [0, \frac{1}{2}]$ for $\alpha \in [0, 1\cdot 3]$. For $r = \frac{1}{4}$, $\alpha \in [0, 2\cdot 4]$. For this range of values of r and α, and with $n < 0$, the maximum in the energy density, temperature and bulk viscosity coefficient is obtained at a time $t_{\max}$ given by

$$t_{\max} = t_0 + \beta_0 \int_{\theta_0}^{\theta_{\max}} (\chi^n + 1)^{m-1} \mathrm{d}\chi \,. \qquad (63)$$

After this time the energy density, temperature and bulk viscosity coefficient are monotonically decreasing functions. During the evolution of the bulk-viscous-fluid-filled universe, a large amount of comoving entropy is produced. The initial evolution of the universe is inflationary, with $q < 0$ for $t \geq t_0$. In the limit of large values of the parameter θ and for $n < 0$, we obtain from equation (58)

$$q_\infty = q_0 - 1 \,. \qquad (64)$$

If $q_0 > 1$ the causal bulk viscous cosmological fluid-filled universe ends in a non-inflationary era with the deceleration parameter $q > 0$. The non-inflationary long-time behaviour is obtained for a large range of values r and α. For example, for $r = \frac{1}{4}$, values of $\alpha < 0{\cdot}5$ lead for large times to $q = \text{const.} > 0$. Thus in the present model, and with a particular choice of the parameters, there is a natural solution of the 'graceful exit' problem of inflationary cosmologies. For other values of the parameters r and α, the universe experiences an inflationary behaviour for all $t \geq t_0$. The behaviour of the scale factor, energy density, deceleration parameter and entropy is represented for different values of r in Figs 3–6 respectively.

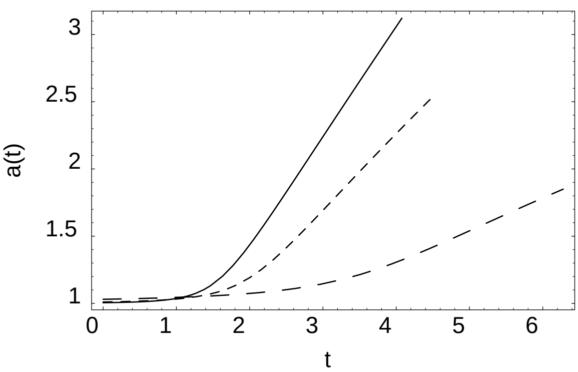

Fig. 3. Variation of the scale factor a as a function of the cosmological time t for $\alpha = 0{\cdot}4$ and for different values of r. Solid curve: $r = \frac{1}{4}$; long-dashed curve: $r = \frac{1}{2}$; short-dashed curve: $r = \frac{1}{3}$.

The dynamics of the high-density causal bulk viscous universe with bulk viscosity coefficient proportional to the Hubble function are inflationary for small times, leading at least for finite intervals of time to the violation of the strong energy condition

$$\rho + 3p_{\text{eff}} = \rho + 3p + 3\Pi < 0 \,, \qquad (65)$$

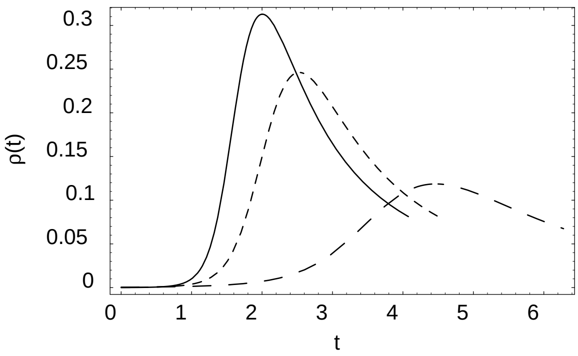

Fig. 4. Variation of the energy density ρ as a function of the cosmological time t for $\alpha = 0 \cdot 4$ and for different values of r. Solid curve: $r = \frac{1}{4}$; long-dashed curve: $r = \frac{1}{2}$; short-dashed curve: $r = \frac{1}{3}$.

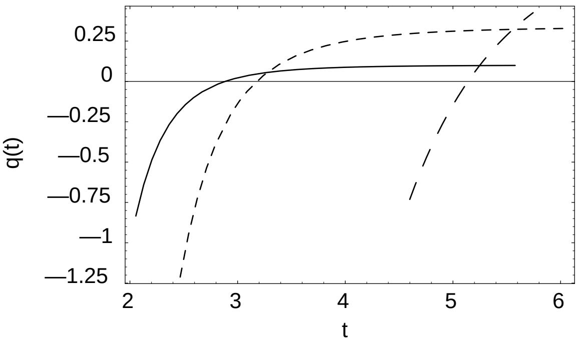

Fig. 5. Variation of the deceleration parameter q as a function of the cosmological time t for $\alpha = 0 \cdot 4$ and for different values of r. Solid curve: $r = \frac{1}{4}$; long-dashed curve: $r = \frac{1}{2}$; short-dashed curve: $r = \frac{1}{3}$.

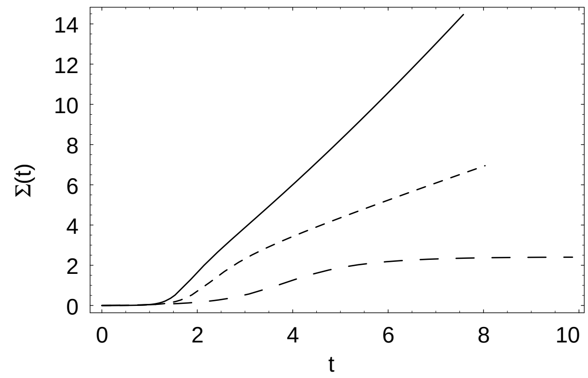

Fig. 6. Variation of the comoving entropy Σ as a function of the cosmological time t for $\alpha = 0 \cdot 4$ and for different values of r. Solid curve: $r = \frac{1}{4}$; long-dashed curve: $r = \frac{1}{2}$; short-dashed curve: $r = \frac{1}{3}$.

and of the near-to-equilibrium condition for the bulk viscous cosmological pressure $|\Pi| \ll p$. Thus this solution of the gravitational field equations is meaningless in the framework of the linear causal thermodynamic of dissipative processes (Maartens 1995; Maartens and Mendez 1997). But, according to the formalism developed in Section 2, we interpret the solution as describing, on a phenomenological classical level, particle production in a cosmological framework.

The evolution of the universe described by equations (50)–(60) starts, for $n < 0$, from a vacuum state with zero energy. Quite a few descriptions of the origin of the universe lead to the conjecture that the total energy of the universe should be zero. Tyron (1973) and Fomin (1975) assumed that if the net value of all conserved quantities of the universe, and in particular the total energy (gravitational plus material), is zero, then the universe might have arisen as a quantum fluctuation of the vacuum. Since the success of the inflationary cosmology, the idea of such fluctuations has been developed further and there have been attempts to show that the total energy of the universe is zero even from a purely classical point of view (Cooperstock 1994; Banerjee and Sen 1997). Quantum cosmological considerations favour a zero value of the quantum vacuum, and it has been suggested that the vacuum energy is indeed zero but we are currently in the midst of a phase transition where the universe is hung up in the false vacuum (Kraus and Turner 1995). Brout *et al.* (1978), Prigogine and Geheniau (1986) and Gunzig *et al.* (1987) have proposed cosmological models in which an inflationary de Sitter spacetime appears as a result of the quantum fluctuation in the conformal degree of freedom of an initial Minkowski spacetime vacuum. These authors have shown that dissipative processes lead to the possibility of cosmological models that start from empty conditions and gradually build up matter and entropy. In these models also the gravitational entropy takes a simple meaning, being associated with the entropy that is necessary to produce matter.

Due to the matter creation process being modelled classically and on a phenomenological level by means of bulk-viscous-type processes, particles are continuously added to spacetime. Assuming that the newly created particles obey an equation of state of the form $\rho \sim n^{\gamma}$, the explicit form and time evolution of the particle creation rate follows from equations (11), (51) and (58), and is given by

$$\Gamma = \frac{H_0}{\gamma} \frac{3\gamma - 2q_0[(1+mn)\theta^n + 1]}{\theta(\theta^n + 1)^m} \,. \tag{66}$$

The temporal behaviour of the particle creation rate is very similar to that of the energy density. For small times it is a monotonically increasing function of time, and for values of parameters leading to a real-valued, positive θ satisfying the equation $d\Gamma/d\theta = 0$, it has a maximum at $\theta = \theta_{\max}$. From equation (66) we obtain

$$\theta_{\max} = \left(\frac{\left(n - 2 + \frac{3\gamma}{2q_0}\right) \pm \sqrt{\left(n - 2 + \frac{3\gamma}{2q_0}\right)^2 - \frac{2(3\gamma - 2q_0)[n(1-m)-1]}{q_0(1+mn)}}}{2[1 - n(1-m)]} \right)^{\frac{1}{n}} , \tag{67}$$

and the maximum value of the particle creation rate can be obtained from equation (66). For $\theta > \theta_{\max}$ the particle creation rate decreases monotonically and tends to zero at large times. Particle creation is associated with a heating period in the early history of the universe, leading to a large increase in the temperature. The variation of the particle production rate for different values of r is represented in Fig. 7.

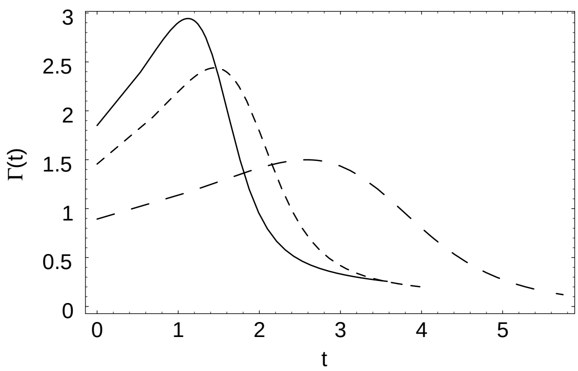

Fig. 7. Variation of the particle creation rate Γ as a function of the cosmological time t for $\alpha = 0{\cdot}4$ and for different values of r. Solid curve: $r = \frac{1}{4}$; long-dashed curve: $r = \frac{1}{2}$; short-dashed curve: $r = \frac{1}{3}$.

In the present paper we have analysed the process of creation of matter particles in the expanding universe, using an imperfect cosmological fluid model as well. We have proposed explicit expressions for the particle balance law and particle creation rate. The dynamics and evolution of the early universe are entirely determined by particle production processes, and the influence of particle production on geometry is essential. In the universe with particle production, the evolution of the universe starts from a quasi-Minkowskian era, in contrast to the familiar FRW description without particle production (Weinberg 1972). As a consequence of a large particle creation rate, inflationary behaviour is obtained. The evolution of the universe is determined for a given equation of state by the numerical values of a single parameter α, the proportionality coefficient relating the energy density to the bulk viscosity coefficient. Matter creation processes are naturally stopped after a finite interval of time and the universe ends in a non-inflationary era. A large amount of comoving entropy is produced during the evolution of the universe. The thermodynamic arrow of time, the direction in which the entropy increases, coincides with the cosmological arrow of time, the direction in which the universe is expanding (Goldwirth and Piran 1991). Thus matter-creation-driven inflation provides a way to align the cosmological arrow of time with the thermodynamic one.

Despite the fact that the present universe is matter-dominated, there are physical reasons to believe that the early universe was radiation-filled. If the quantum vacuum had decayed into massive particles via a particle creation mechanism, then in principle too many nucleon–anti-nucleon pairs would have been created continuously. Unless these pairs were separated from each other through an unknown mechanism, they would annihilate one other and produce observable gamma ray and neutrino fluxes (Mubarak and Ozer 1998), but these fluxes have not been experimentally detected. Therefore we can consider that the matter component of the present model consists of radiation, and in the above equations the most physically plausible value of γ is $\gamma = \frac{4}{3}$, leading via equation (39) to $r = \frac{1}{4}$.

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