Cosmological Perturbation Theory

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The linear perturbation theory of spatially homogeneous and isotropic universes is reviewed and reformulated extensively. In the first half of the article, a gauge-invariant formulation of the theory is carried out with special attention paid to the geometrical meaning of the perturbation. In the second half of the article, the application of the theory to some important cosmological models is discussed.

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Chapter I

Introduction

§ I-1. Brief historical survey

It is commonly accepted that large-scale inhomogeneous structures of the universe observed today, such as galaxies and clusters of galaxies, were formed as a result of growth of density fluctuations whose amplitudes had been very small in the early universe [see, e.g., Peebles (1980)]. The main purpose of developing cosmological perturbation theories is to examine the properties of primordial density fluctuations necessary to explain these observed structures of the universe and to clarify the origin and the evolutionary behavior of such density fluctuations. Roughly speaking, the necessary density fluctuations should have been either produced dynamically in the course of the evolution of the universe or present from the beginning determined simply by the initial condition of the universe. Since the former idea is much more attractive that the latter from the physical point of view, one generally looks for a mechanism of generating the necessary density fluctuations in an early stage of the universe.

It would be desirable if the necessary density fluctuations could have arisen after the recombination of hydrogens, $T_{\rm rec} \sim 4000$K, when their characteristic scale was well within the Hubble horizon and relativistic effects were no longer important so that we could analyze the generation and evolution processes by Newtonian theory. Unfortunately, however, no reasonable mechanism exists which could allow the localization of energy density on scales comparable to or greater than clusters of galaxies after the recombination time. This is primarily due to the smallness of the matter sound velocity which limits the scale over which the matter energy can be transported. Together with observational evidences, this implies that the necessary density fluctuations were already there at the recombination time with amplitude of order 10^{-3} at least.

According to the standard model of the universe, the cosmic expansion rate is always smaller than the rate of increase in the Hubble horizon size. Thus, for example, the size of a comoving region corresponding to a supercluster at present (~ 30Mpc) was comparable to the horizon at epoch shortly before the recombination time and was much greater than the horizon, say, at $T > 1$eV.

All these considerations imply that there must have been some kind of perturbations of appropriate amplitude on super-horizon scales, over which no causal contact was possible, in the early universe. Thus the investigation in possible sources of perturbations and the evolutionary behavior of them on super-horizon scales is one of the principal themes in cosmological perturbation theory.

The problem associated with perturbations on super-horizon scales is that the notion of density perturbations, for example, loses its direct physical significance due to the presence of coordinate gauge freedom inherent in general relativistic perturbation theories. On scales greater than the Hubble horizon size, the amplitude of perturbation in geometrical quantities is generally comparable to or even greater than that of a density perturbation and one can assign practically any value to the latter by a suitable gauge

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transformation. Thus when one discusses the generation and growth of perturbations on super-horizon scales, one must be very careful about the prescription of initial conditions or the analysis on the evolutionary behavior of perturbations. Otherwise, unphysical gauge modes would dominate over physical modes and lead to an incorrect conclusion.

The linear perturbation analysis of spatially homogeneous, isotropic cosmological models was pioneered by Lifshitz (1946). This work was then extended by Lifshitz and Khalatnikov (1963) with corrections of a couple of errors in the original paper. Although their analysis was entirely correct, their results were often misinterpreted and misused by a number of authors who subsequently considered the generation and growth of cosmological density perturbations on super-horizon scales. This unfortunate situation arose because too much attention was paid to the growth rate of the density perturbation without realizing that it essentially depends on the choice of coordinates. In addition the fact that the equations for density perturbations in the synchronous gauge, which was used in the analysis by Lifshitz and Khalatnikov, are too complicated to allow the elimination of unphysical gauge modes in general gave rise to a number of incorrect conclusions in the literature.

In order to eliminate unphysical gauge modes, Hawking (1966) developed a formulation which deals with the perturbation in the curvature tensor directly, avoiding explicit appearance of the perturbation in the metric tensor. Olson (1976) then extended this formalism and gave the perturbation equation in a simple closed form which is free from gauge modes. However, since the density perturbation is not directly dealt with in Olson's method, one gauge mode comes into the expression for the density perturbation amplitude which happened to cause a certain degree of ambiguity in the case of a cosmic fluid with zero pressure. Meanwhile, Harrison (1967) derived the equations for density perturbations in a longitudinal gauge which was found to be free from gauge modes. Also, Nariai (1969) succeeded in deriving the perturbation equation in the second order form, hence free from gauge modes, in a comoving gauge. Then Sakai (1969), extending the analysis of Nariai, investigated the evolutionary behavior of density perturbations extensively under various gauge conditions and clarified the gauge-dependence of the growth rate of the density perturbation.

In these earlier papers, however, only the equation for adiabatic perturbations without sources had been considered. Thus there had remained yet one problem; that is, what could play a role of sources on scales greater than the horizon and how large the amplitude of the generated density parturbations would be. In this connection, it had been neither clarified yet in which gauge, if ever, the amplitude of the density perturbation could be regarded as representing the true amplitude of the linear perturbation. That is, what is the criterion for the validity of linear perturbation analysis?

Concerning the source of density perturbations and the magnitude of resulting density perturbations, it was Press and Vishniac (1980) who made a thorough analysis first. They worked in the synchronous gauge but carefully eliminated two unphysical gauge modes associated with this gauge. This enabled them to reveal the source of some erroneous ideas about density perturbations on super-horizon scales explicitly. They also showed that an entropy perturbation can give rise to a density perturbation but the resulting amplitude of it when it comes within the horizon is of the same order of the initial amplitude of the entropy perturbation. Then Bardeen (1980) formulated the perturbation equations in a completely gauge-invariant way and gave a more general

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analysis on the inhomogeneous version of the equations governing the density perturbation as well as the other two types (vector and tensor) of perturbations. Further Bardeen discussed the criterion for the validity of linear perturbation analysis to some extent by comparing the perturbation amplitudes of several physical quantities under typical gauge conditions with each other.

Thus now we are in a good position to discuss cosmological perturbations without worrying about gauge ambiguity. In view of such a good position at present, we shall first extend Bardeen's gauge-invariant formalism and develop a rather complete theory of gauge-invariant cosmological perturbations. The theory is formulated for general $(n+1)$-dimensional spatially homogeneous and isotropic universes. This generalization has been motivated by the fact that there are growing interests now among particle physicists and cosmologists in Kaluza-Klein theories of unified gauge interactions, according to which the spacetime dimension of the universe might have been higher than four in the very early stage of the universe [Chodos and Detweiler(1980); Freund (1982)]. Then we shall apply our theory to several important cosmological situations and evaluate the behavior of perturbations. The plan of the paper, including a guide to readers who are interested in some specific problems, will be presented in detail in the next section. Basic geometrical notation used in this paper will be listed in §I-3.

§ I-2. Plan of the paper

The present article is intended to be an extensive review on the theory of cosmological perturbations. We take full advantage of Bardeen's gauge-invariant formalism, since it seems the most natural and conceptually clearest formalism for dealing with cosmological perturbations. However, we have attempted to be more general and complete both in respect of formulation and application of the theory. Consequently several new results have been obtained, which are also presented in this paper.

The paper is divided into two main parts: Part One concerns the gauge-invariant formulation of cosmological perturbations, while Part Two deals with applications of the theory to some specific cosmological situations. Part One is divided into Chapters II and III and Part Two is divided into Chapters IV, V and VI. Each chapter, in which a specific topic is dealt with, is further divided into several smaller sections.

In Chapter II, an extended version of Bardeen's gauge-invariant formalism is presented, which includes higher dimensional generalization of the theory (§§II-1~4) and derivation of gauge-invariant equations for a multi-component system (§II-5). In formulating the theory, special emphasis is laid on the geometrical meaning of both gauge-dependent and gauge-invariant perturbation variables. This helps us a great deal to interpret the physical meaning of a given perturbation and to discuss the validity of the linear perturbation theory, which will be the topic of Chapter III.

In Chapter III, comparison of the gauge-invariant formalism with those depending on particular choices of gauge is made (§III-1) in order to show the advantage of the former and to clarify the source of confusion existed previously in the physical interpretation of density perturbations on super-horizon scales (§III-2). In particular, we carefully investigate the validity of the linear perturbation theory in §III-2 based on the geometrical meaning of perturbation variables. Then the gauge-invariant measure of the linear perturbation amplitude is presented. However, the arguments in this chapter are restrict-

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ed to the case of a universe with the equation of state p/ρ=const.

In Chapter IV, the generation and evolutionary behavior of density perturbations on super-horizon scales in universes associated with transient phenomena are discussed extensively. Basic equations and the method of analysis are given in §IV-1. Then effects of transient phenomena on the generation and behavior of density perturbations are estimated (§§IV-2~5). This chapter also includes the presentation of the exact analytic solution for a density perturbation on super-horizon scales in a universe filled with radiation and a pressureless matter (§IV-4). The analyses given in this chapter are far more complete than those by Press and Vishniac and by Bardeen and confirm their results under very general situations.

In Chapter V, the behavior of perturbations in the baryon-photon system is investigated, taking into account the interaction of electrons with photons through Thomson scattering explicitly. Basic equations for this purpose are given in §V-1, using which perturbations on super-horizon scales (§V-2) and sub-horizon scales (§V-3) are analyzed. Summarizing the results, some implications of them are discussed in §V-4. Since this topic has been analyzed by many authors in various contexts, most of the results given in this chapter are not new. However our analysis differs from the conventional ones [see, e.g., Silk (1974)] in some respects. In particular, it is completely free from gauge ambiguity and it takes into accout dynamical degrees of the so-called isothermal perturbations explicitly, which were made possible by using the gauge-invariant equations for a multi-component system developed in §I-5. As a consequence, the new possibility of generating isothermal perturbations from adiabatic perturbations is raised and discussed (§V-2) in addition to the opposite process which has already been known.

In Chapter VI, perturbations in a universe dominated by a classical scalar field are discussed. This is a recent topic which is closely connected with grand unified theories of elementary particles, which predict the existence of cosmological phase transitions associated with spontaneous breakdown of symmetries. The perturbation equations for a multi-component system (§II-5) are first put into the form appropriate for the present purpose in §VI-1. Then the behavior of density perturbations in the so-called inflationary universe [Guth (1981); Sato (1981)], which has a stage dominated by potential energy of a classical scalar field, is discussed in detail (§§VI-2 and 3). In particular, the amplitude of density perturbations associated with the scalar field which are induced by those in radiation energy density existed initially is estimated carefully (§VI-3). Incidentally, the measure of the linear perturbation amplitude for an inflationary universe is found to be the same as the one obtained in § III-2 for a universe with a simple equation of state p/ρ =const. Some peculiar properties of perturbations associated with an oscillating scalar field are discussed in §VI-4, though no extensive analysis is made because of complexity of the perturbation equations for such a system.

Finally, Chapter VII is devoted to conclusions and comments on problems to be attacked in the future.

There are seven appendices in the end (Appendices A~G) where some basic ideas and formulas used in the text are derived or explained in detail.

We have made efforts to write each chapter as self-containedly as possible. However, since almost all of the dicussions given in Chapters III~VI are heavily based on the formalism developed in Chapter II, a good understanding of the topic of each chapter is possible only after Chapter II is learned. Nevertheless, readers who are interested only

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in specific applications of the gauge-invariant formalism (Chapters IV~VI) should not find it too difficult to read any one of them alone except for difficulty in identifying the symbols used to denote gauge-invariant variables. For convenience of such readers, these symbols are listed in Appendix G with their definitions and/or indication of the places they are defined. On the other hand, for readers who are mainly interested in the formulation and general consequences of the theory, Chapters II~IV suffice to be learned.

§ I-3. Basic notation

Basic geometrical notation used in this paper is as follows:

Spacetime dimension $n+1$ ($n \geq 3$ is assumed)

Tensor indices

Greek indices $(\alpha, \beta, \cdots, \mu, \nu, \cdots)$ run from 0 to n

Lattin indices $(a, b, \cdots, i, j, \cdots)$ run from 1 to n

Metric (see also Appendix A)

Total spacetime $g_{\mu\nu}$ with signature $(-, +, \cdots, +)$

Constant curvature n-space γ_{ij} with signature $(+, \cdots, +)$

Christoffel symbols

$$\Gamma^{\mu}{}_{\nu\lambda}=\frac{1}{2}g^{\mu\alpha}(g_{\alpha\nu,\lambda}+g_{\alpha\lambda,\nu}-g_{\nu\lambda,\alpha})$$

Riemann tensor

$$R^{\alpha}{}_{\beta\mu\nu}=\Gamma^{\alpha}{}_{\beta\nu,\mu}-\Gamma^{\alpha}{}_{\beta\mu,\nu}+\Gamma^{\alpha}{}_{\lambda\mu}\Gamma^{\lambda}{}_{\beta\nu}-\Gamma^{\alpha}{}_{\lambda\nu}\Gamma^{\lambda}{}_{\beta\mu}$$

$$R_{\mu\nu}=R^{\alpha}{}_{\mu\alpha\nu}$$

$$R=R^{\mu}{}_{\mu}$$

Einstein tensor

$$G_{\mu\nu}=R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$$

Derivatives

Covariant differentiation with respect to $g_{\mu\nu}$ $\quad {}_{;\mu}=\nabla_{\mu}$

Covariant differentiation with respect to γ_{ij} $\quad {}_{|i}={}^{s}\nabla_{i}$

Proper-time derivative $\quad \dot{}=d/dt$

Conformal time derivative $\quad {}'=d/d\eta$

Throughout the paper we adopt the units $c=\hbar=1$. The gravitational coupling constant is denoted by $\varkappa$ such that it appears in the Einstein equations as $G_{\mu\nu}=\varkappa^2 T_{\mu\nu}$. For spacetime dimension of four ($n=3$), $\varkappa^2=8\pi G$ where G is the usual Newtonian gravitational constant.

Some theorems in general relativity will be used in the text without mentioning particular references. However, if necessary, readers are referred to Misner, Thorne and Wheeler (1973) for basic knowledge of general relativity.

Within each chapter, equations are numbered with the section number. However, in the other chapters, they are referred to with the chapter number also. For example, the third equation of §2, Chapter II will be denoted as Eq. (2·3) in Chapter II, while as Eq. (II-2·3) in the other chapters.

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Chapter II

Gauge-Invariant Formalism

In order to develop a general and sufficiently useful formalism of linear perturbation theories in curved spacetime, we must restrict the background spacetime to those which belong to a certain special class. The class considered in this paper is a spatially homogeneous and isotropic spacetime (i.e., the so-called Robertson-Walker spacetime). There are two reasons for this restriction. One reason is technical; on such spacetime we can expand perturbations by harmonic functions on a constant curvature space and reduce the evolution equations for perturbations to a set of mutually decoupled ordinary differential equations. The other reason is that our real universe is, at least on the large scale, well described by a Robertson-Walker spacetime. Though the spacetime we live in is four dimensional, and applications of the perturbation theory we are going to develop are mainly to problems of the generation and evolution of perturbations in our universe, recent growing interest in higher dimensional cosmological models [Chodos and Detweiler (1980); Freund (1982)] has made us to expect that it sometimes becomes necessary to analyse the behavior of perturbations in higher dimensional spacetime. Hence we do not specify the spatial dimension but treat a general n-dimensional constant curvature space.

A Robertson-Walker spacetime is described by the metric

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a^2 d\sigma^2 , \qquad (0\cdot1)$$

where $a = a(t)$ is the cosmic scale factor depending only on time t and $d\sigma^2$ is the time-independent metric of an n-dimensional space Σ with a constant curvature K, which we call the invariant n-space, given by

$$d\sigma^2 = \gamma_{ij} dx^i dx^j = \frac{dr^2}{1-Kr^2} + r^2 d\Omega_{n-1}^2 , \qquad (0\cdot2)$$

in which $d\Omega_{n-1}^2$ is the metric of the $(n-1)$-dimensional Euclidean sphere whose explicit form is not needed in the following. The curvature tensor of this invariant n-space is given by

$${}^sR_{ijkl} = K(\gamma_{ik}\gamma_{jl} - \gamma_{il}\gamma_{jk}), \qquad (0\cdot3)$$

$${}^sR_{ij} = (n-1)K\gamma_{ij}, \qquad (0\cdot3\text{a})$$

$${}^sR = n(n-1)K . \qquad (0\cdot3\text{b})$$

Various geometrical quantities of the Robertson-Walker spacetime are recapitulated in Appendix A. Geometrical notation and sign conventions used in this paper are given in §I-3. It is assumed that a dot denotes differentiation with respect to the proper-time t defined by Eq. $(0\cdot1)$ and a prime denotes differentiation with respect to the conformal time η defined by

$$d\eta = dt/a . \qquad (0\cdot4)$$

In this chapter the components of vectors and tensors are those in the coordinates (η, x^i)

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unless otherwise stated.

The restriction of the background spacetime to the Robertson-Walker class obliges the energy-momentum tensor of the background matter to take a perfect fluid form

$$T_{\mu\nu}=(\rho+p)u_\mu u_\nu+pg_{\mu\nu}\,, \tag{0·5}$$

where ρ and p are functions depending only on time and $(u^\mu)=(a^{-1}, 0, \cdots, 0)$. The Einstein equations are reduced to the two equations

$$\left(\frac{\dot{a}}{a}\right)^2+\frac{K}{a^2}=\frac{2\chi^2}{n(n-1)}\rho\,, \tag{0·6}$$

$$\dot{\rho}=-n\frac{\dot{a}}{a}h\,, \tag{0·7}$$

where

$$h\equiv\rho+p\,. \tag{0·8}$$

It is assumed that a cosmological constant, if it exists, is contained in ρ and p.

§ II-1. Decomposition of perturbations and harmonic expansion

In order to expand perturbations by harmonic functions on the invariant n-space Σ, we first classify perturbations into three groups on the basis of their behavior under the transformation of space-coordinates x^i; the scalar type, vector type and tensor type.

A vector quantity v^i on Σ can be decomposed as

$$v^i=v_*{}^i+v^{|i}\,; \tag{1·1}$$

$$\varDelta v=v^i{}_{|i}\,, \tag{1·1a}$$

$$v_*{}^i{}_{|i}=0\,, \tag{1·1b}$$

where $\varDelta$ is defined by

$$\varDelta\equiv\gamma^{ij}\,{}^s\nabla_i{}^s\nabla_j\,. \tag{1·2}$$

Note that the elliptic equation for v, (1·1a), has a unique solution hence $\varDelta^{-1}$ exists always when Σ is compact and if v vanishes sufficiently rapidly at infinity when Σ is open, which is assumed throughout this article implicitly. From their transformation properties under spatial coordinate transformations, we call v and $v_*{}^i$ the scalar type and the vector type components of v^i, respectively. Similarly a symmetric second-rank tensor t_{ij} on Σ can be decomposed as[†)]

$$t^{ij}=t_*{}^{ij}+(t_*{}^{i|j}+t_*{}^{j|i})+(s^{|ij}-n^{-1}\gamma^{ij}\varDelta s)+n^{-1}t\gamma_{ij}; \tag{1·3}$$

$$t=t^j{}_j\,, \tag{1·3a}$$

$$s=\frac{n}{n-1}\varDelta^{-1}(\varDelta+nK)^{-1}\left(t^{ij}{}_{|ij}-\frac{1}{n}\varDelta t\right), \tag{1·3b}$$

†) Due to the symmetry of the tensor indices appearing in the Einstein equations we only need to consider symmetric second-rank tensors.

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$$t_*{}^i=[\varDelta+(n-1)K]^{-1}(\delta^i{}_j-{}^s\nabla^i\varDelta^{-1}{}^s\nabla_j)(t^{jm}{}_{|m}-n^{-1}t^{|j}), \tag{1·3c}$$

$$t_*{}^i{}_i=0\,, \tag{1·3d}$$

$$t_*{}^{ij}{}_{|j}=0\,. \tag{1·3e}$$

Raising or lowering indices of quantities defined on the invariant n-space Σ is done by γ^{ij} or γ_{ij}. Similar to the vector case, we call $t_*{}^{ij}$, $t_*{}^i$ and (s, t) the tensor type, vector type and scalar type components of t^{ij}, respectively. In this way we can decompose perturbations of various quantities into three types of components. As proved in Appendix B, in a Robertson-Walker spacetime, scalar, vector and symmetric second-rank tensor equations, if they are covariant with respect to the coordinate transformation in Σ, linear in unknown geometrical quantities and second order at most in the case of differential equations, are decomposed into groups of equations each of which contains only components of one type. Therefore we can study three types of components of perturbations in various quantities separately. We call them scalar, vector and tensor perturbations, respectively.[†]

Scalar quantities can be expanded by a complete set of scalar harmonic functions $\mathrm{Y}(\boldsymbol{x})$ satisfying the equation

$$(\varDelta+k^2)\mathrm{Y}=0\,, \tag{1·4}$$

where $-k^2$ represents an eigenvalue of the Laplace-Beltrami operator $\varDelta$ on Σ. The k^2 takes continuous values larger than or equal to $(n-2)^2|K|$ for $K\leqq 0$ and $k^2=l(l+n+1)K(l=0, 1, 2,\ldots)$ for $K>0$ [Vilenkin and Smorodinskii (1964)]. We omit the indices to distinguish different eigenfunctions since the explicit form of these functions are not used in this article, and since there exists no mode-mode coupling. From Eqs. (1·1) and (1·3) scalar type components of vectors on Σ are expanded by

$$\mathrm{Y}_i\equiv -k^{-1}\mathrm{Y}_{|i}\,, \tag{1·5}$$

and those of tensors are expanded by

$$\mathrm{Y}_{ij}\equiv k^{-2}(\mathrm{Y}_{|ij}-n^{-1}\gamma_{ij}\varDelta\mathrm{Y})=(k^{-2}\mathrm{Y}_{|ij}+n^{-1}\gamma_{ij}\mathrm{Y}) \tag{1·6}$$

and $\gamma_{ij}\mathrm{Y}$.

Similarly divergenceless vectors on Σ are expanded by a complete set of vector harmonic functions $\mathrm{Y}^{(1)}_i$ specified by

$$(\varDelta+k^2)\mathrm{Y}^{(1)}_i=0\,; \tag{1·7}$$

$$\mathrm{Y}^{(1)i}{}_{|i}=0\,. \tag{1·7a}$$

From Eq. (1·3) vector type components of tensors are expanded by $\mathrm{Y}^{(1)}_{ij}$ defined by

$$\mathrm{Y}^{(1)}_{ij}\equiv -(2k)^{-1}(\mathrm{Y}^{(1)}_{i|j}+\mathrm{Y}^{(1)}_{j|i}). \tag{1·8}$$

Finally tensor type components, namely divergenceless and traceless second-rank symmetric tensors on Σ, are expanded by a complete set of tensor harmonic functions $\mathrm{Y}^{(2)}_{ij}$

[†] Alternatively, the former two are frequently called irrotational and rotational perturbations, respectively, by referring to the behavior of the matter and the last one is called gravitational wave perturbations.

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specified by

$$(\Delta+k^2)Y^{(2)}_{ij}=0\,; \tag{1·9}$$

$$Y^{(2)i}{}_{i}=0\,, \tag{1·9a}$$

$$Y^{(2)|j}_{ij}=0\,. \tag{1·9b}$$

Some useful formulas for these harmonic functions are summarized in Appendix C.

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§ II-2. Perturbations of metric and energy-momentum tensor

As has been shown in §1, three types of perturbations, the scalar type, vector type and tensor type, completely decouple from each other dynamically. Hence we can treat each type of perturbations independently. Furthermore owing to the homogeneity and isotropy of the invariant n-space, there is no coupling among the expansion coefficients of harmonic functions with different eigenvalues in perturbation equations. Hence we omit the summation symbol as well as the eigenvalue indices of harmonic functions. We put a tilde on a perturbed quantity to distinguish it from the corresponding unperturbed background quantity.

First we consider scalar perturbations. By a spatial coordinate transformation, the components of the metric tensor $\tilde{g}_{00}$, $\tilde{g}_{0j}$ and $\tilde{g}_{ij}$ transform as a scalar, a vector and a tensor, respectively. Hence for a scalar perturbation the perturbed metric tensor $\tilde{g}_{\mu\nu}$ is generally expressed in terms of four independent functions of time A, B, H_L and H_T as

$$\tilde{g}_{00}=-a^2[1+2AY], \tag{2·1a}$$

$$\tilde{g}_{0j}=-a^2BY_j\,, \tag{2·1b}$$

$$\tilde{g}_{ij}=a^2[\gamma_{ij}+2H_LY\gamma_{ij}+2H_TY_{ij}]. \tag{2·1c}$$

Correspondingly, up to first order in A, B, H_L and H_T, $\tilde{g}^{\mu\nu}$ is written as

$$\tilde{g}^{00}=-a^{-2}[1-2AY], \tag{2·2a}$$

$$\tilde{g}^{0j}=-a^{-2}BY^j\,, \tag{2·2b}$$

$$\tilde{g}^{ij}=a^{-2}[\gamma^{ij}-2H_LY\gamma^{ij}-2H_TY^{ij}]. \tag{2·2c}$$

In the language of the $(n+1)$-formalism, A is interpreted as the amplitude of perturbation in the lapse function which represents the ratio of the proper-time distance to the coordinate-time distance between two neighboring constant time hypersurfaces. While, B is interpreted as the amplitude of a perturbation in the shift vector which represents the rate of deviation of a constant space-coordinate line from a line normal to a constant time hypersurface. Further from the equation

$$\det(\tilde{g}_{ij})=-a^{-2}(1-2nH_LY)\det(\gamma_{ij}), \tag{2·3}$$

H_L is interpreted as the amplitude of perturbation of a unit spatial volume. Finally H_T represents the amplitude of anisotropic distortion of each constant time hypersurface. The formulas for perturbations of various geometrical quantities are given in Appendix D.

In order to write down expressions for matter variables in terms of the harmonics, we must first choose appropriate variables which represent the matter at a perturbed state. For writing down the perturbed Einstein equations, it is most convenient to take the algebraically independent components of the energy-momentum tensor as such variables. Specifically we define the perturbed $(n+1)$-velocity of matter $\tilde{u}^\mu$ as the time-like eigenvector with unit norm of the perturbed energy-momentum tensor $\tilde{T}^{\mu\nu}$, and the perturbed proper density $\tilde{\rho}$ by the corresponding eigenvalue:

$$\tilde{T}^\mu{}_\nu \tilde{u}^\nu = -\tilde{\rho}\tilde{u}^\mu , \tag{2·4}$$

$$\tilde{u}_\mu \tilde{u}^\mu = -1 . \tag{2·5}$$

The remaining freedom is given by the spatial stress tensor

$$\tilde{\tau}_{\mu\nu} = \tilde{P}_\mu{}^\alpha \tilde{P}_\nu{}^\beta \tilde{T}_{\alpha\beta} , \tag{2·6}$$

where

$$\tilde{P}^\mu{}_\nu \equiv \delta^\mu{}_\nu + \tilde{u}^\mu \tilde{u}_\nu . \tag{2·7}$$

Note that $\tilde{\tau}_{\mu\nu}$ is orthogonal to $\tilde{u}^\mu$ from Eqs. (2·5) and (2·7):

$$\tilde{u}^\mu \tilde{\tau}_{\mu\nu} = 0 . \tag{2·8}$$

Since $\tilde{\rho}$ is a scalar, it is expressed as

$$\tilde{\rho} = \rho[1+\delta \mathrm{Y}]. \tag{2·9}$$

δ is the amplitude of a density perturbation. The independent degree of freedom of $\tilde{u}^\mu$ is three and represented by the spatial velocity $\tilde{v}^i \equiv \tilde{u}^i/\tilde{u}^0$. Since $\tilde{v}^i$ transforms as an n-space vector which vanishes in the unperturbed state, it is expressed as

$$\tilde{v}^i \equiv \tilde{u}^i/\tilde{u}^0 = v\mathrm{Y}^i , \tag{2·10}$$

where v is a function of time. From the normalization condition (2·5) and Eqs. (2·1a) ~(2·1c), $\tilde{u}^0$ is expressed to first order as

$$\tilde{u}^0 = a^{-1}(1-A\mathrm{Y}). \tag{2·10·a}$$

Correspondingly $\tilde{u}_\mu$ is expressed to the same order as

$$\tilde{u}_i = a(v-B)\mathrm{Y}_i , \tag{2·11}$$

$$\tilde{u}_0 = -a(1+A\mathrm{Y}). \tag{2·11a}$$

From Eqs. (2·10) and (2·11) the components of $\tilde{P}^\mu{}_\nu$ are expressed to first order as

$$\tilde{P}^0{}_0 = 0 , \tag{2·12a}$$

$$\tilde{P}^0{}_j = (v-B)\mathrm{Y}_j , \tag{2·12b}$$

$$\tilde{P}^j{}_0 = -v\mathrm{Y}^j , \tag{2·12c}$$

$$\tilde{P}^i{}_j = \delta^i{}_j . \tag{2·12d}$$

Hence to the same order $\tilde{\tau}_{\mu\nu}$ is expressed as

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$$\tilde{\tau}^0{}_0 = 0\,, \tag{2·13a}$$

$$\tilde{\tau}^0{}_j = p(v-B)Y_j\,, \tag{2·13b}$$

$$\tilde{\tau}^j{}_0 = -pvY_j\,, \tag{2·13c}$$

$$\tilde{\tau}^i{}_j = T^i{}_j\,. \tag{2·13d}$$

Since $\tilde{T}^i{}_j$ is a second-rank symmetric tensor with respect to spatial coordinate transformations, it is expressed as

$$\tilde{T}^i{}_j = p[\delta^i{}_j + \pi_L \delta^i{}_j + \pi_T Y^i{}_j]. \tag{2·14}$$

Here, π_L is interpreted as the amplitude of an isotropic pressure perturbation since

$$\tilde{p} \equiv \frac{1}{n}\tilde{\tau}^\mu{}_\mu = p(1+\pi_L Y). \tag{2·15}$$

Correspondingly π_T is interpreted as the amplitude of an anisotropic stress perturbation. The four functions of time v, δ, π_L and π_T completely describe the perturbed energy-momentum tensor. In fact Eq. (2·6) is rewritten as

$$\tilde{T}^\mu{}_\nu = \tilde{\rho}\tilde{u}^\mu \tilde{u}_\nu + \tilde{\tau}^\mu{}_\nu\,. \tag{2·16}$$

Substituting Eqs. (2·9)~(2·11) and (2·13)~(2·15) to this equation one finds the expressions for $\tilde{T}^0{}_0$, $\tilde{T}^0{}_j$ and $\tilde{T}^j{}_0$ in terms of these four quantities:

$$\tilde{T}^0{}_0 = -\rho[1+\delta Y], \tag{2·17}$$

$$\tilde{T}^0{}_j = (\rho+p)(v-B)Y_j\,, \tag{2·18}$$

$$\tilde{T}^j{}_0 = -(\rho+p)vY^j\,. \tag{2·19}$$

Note that v, δ, π_L and π_T are not necessarily the most fundamental variables representing the state of matter. For example, when the matter is composed of more than two components, these four quantities are expressed in terms of the corresponding quantities of individual components. The relation among such quantities are determined by the equations of motion for individual components. Such a case will be the topic discussed in §5.

It is quite easy to write down expressions for the corresponding variables for vector and tensor perturbations. The definitions of the matter variables are exactly the same. For a vector perturbation the perturbed quantities are written as

$$\tilde{g}_{00} = -a^2\,, \tag{2·20a}$$

$$\tilde{g}_{0j} = -a^2 B^{(1)} Y_j^{(1)}, \tag{2·20b}$$

$$\tilde{g}_{ij} = a^2[\gamma_{ij} + 2H_T{}^{(1)} Y_{ij}{}^{(1)}], \tag{2·20c}$$

$$\tilde{u}^j/\tilde{u}^0 = v^{(1)} Y^{(1)j}\,, \tag{2·21a}$$

$$\tilde{u}^0 = a^{-1}\,, \tag{2·21b}$$

$$\tilde{u}_j = a(v^{(1)} - B^{(1)}) Y_j^{(1)}, \tag{2·22a}$$

$$\tilde{u}_0 = -a\,, \tag{2·22b}$$

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$$\tilde{T}^0{}_0 = -\rho\,, \tag{2·23a}$$

$$\tilde{T}^j{}_0 = -(\rho+p)v^{(1)}Y^{(1)j}\,, \tag{2·23b}$$

$$\tilde{T}^0{}_j = (\rho+p)(v^{(1)}-B^{(1)})Y_i^{(1)}\,, \tag{2·23c}$$

$$\tilde{T}^i{}_j = p[\delta^i{}_j + \pi_T{}^{(1)}Y^{(1)i}{}_j]. \tag{2·23d}$$

Hence a vector perturbation is described by the two functions of time, $B^{(1)}$ and $H_T{}^{(1)}$ for the metric, and the two functions of time, $v^{(1)}$ and $\pi_T{}^{(1)}$ for the matter.

Similarly for a tensor perturbation, we find

$$\tilde{g}_{00} = -a^2\,, \tag{2·24a}$$

$$\tilde{g}_{0j} = 0\,, \tag{2·24b}$$

$$\tilde{g}_{ij} = a^2[\gamma_{ij} + 2H_T{}^{(2)}Y_{ij}{}^{(2)}], \tag{2·24c}$$

$$\tilde{u}^0 = a^{-1}\,, \qquad \tilde{u}^j = 0\,, \tag{2·25}$$

$$\tilde{u}_0 = -a, \qquad \tilde{u}_j = 0\,, \tag{2·26}$$

$$\tilde{T}^0{}_0 = -\rho\,, \tag{2·27a}$$

$$\tilde{T}^j{}_0 = \tilde{T}^0{}_j = 0\,, \tag{2·27b}$$

$$\tilde{T}^i{}_j = p[\delta^i{}_j + \pi_T{}^{(2)}Y^{(2)i}{}_j]. \tag{2·27c}$$

Thus a tensor perturbation is described by one function of time $H_T{}^{(2)}$ for the metric and one function of time $\pi_T{}^{(2)}$ for the matter. Note that no density or isotropic pressure perturbation is associated with vector or tensor perturbations. The formulas for the perturbed geometrical quantities for vector and tensor perturbations are also given in Appendix D.

§ II-3. Gauge-invariant variables

As stated in the Introduction, the variables representing perturbations introduced in §2 change their values under the change of correspondence between the perturbed world and the unperturbed background. The change of correspondence is formally expressed in terms of a coordinate transformation in the perturbed world, which is called a gauge transformation in order to distinguish it from a genuine coordinate transformation. In the linear perturbation theory it is necessary only to consider infinitesimal gauge transformations, which are classified into the scalar type and the vector type. There exists no tensor type gauge transformation. Infinitesimal transformations of each type are further expanded by the corresponding harmonic functions and different modes are decoupled from each other. Hence we can discuss the gauge transformation properties of perturbation variables for each mode independently. In the following we study the gauge transformation properties of perturbation variables and construct gauge-invariant variables for a scalar, vector and tensor perturbation in this order.

(1) Scalar perturbations

A scalar type infinitesimal gauge transformation $(\eta, \boldsymbol{x}) \to (\bar{\eta}, \bar{\boldsymbol{x}})$ is expressed as

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$$\bar{\eta}=\eta+T\mathrm{Y}\,, \tag{3·1a}$$

$$\bar{x}^j=x^j+L\mathrm{Y}^j\,, \tag{3·1b}$$

where T and L are arbitrary functions of time, being regarded as quantities of the same order as the perturbation variables. Since the comparison of a quantity on the perturbed world and that on the unperturbed background should be made at points with the same coordinate values, the change of the perturbed metric tensor under the transformation (3·1) is given by

$$\begin{aligned}\bar{\tilde{g}}_{\mu\nu}(\eta,\, x^j)&=\frac{\partial x^\alpha}{\partial\bar{x}^\mu}\frac{\partial x^\beta}{\partial\bar{x}^\nu}\tilde{g}_{\alpha\beta}(\eta-T\mathrm{Y},\, x^j-L\mathrm{Y}^j)\\ &\simeq\tilde{g}_{\mu\nu}(\eta,\, x^j)+g_{\alpha\nu}\delta x^\alpha{}_{,\mu}+g_{\alpha\mu}\delta x^\alpha{}_{,\nu}-g_{\mu\nu,\lambda}\delta x^\lambda\,.\end{aligned} \tag{3·2}$$

Thus to first order we obtain

$$\bar{A}=A-T'-(a'/a)T\,, \tag{3·3a}$$

$$\bar{B}=B+L'+kT\,, \tag{3·3b}$$

$$\bar{H}_L=H_L-(k/n)L-(a'/a)T\,, \tag{3·3c}$$

$$\bar{H}_T=H_T+kL\,. \tag{3·3d}$$

Since the gauge transformation (3·1) contains two arbitrary functions of time, two independent gauge-invariants can be constructed from A, B, H_L and H_T. One possible choice of such two invariants are

$$\mathcal{A}\equiv A-a^{-1}[(a^2/a')(H_L+(1/n)H_T)]'\,, \tag{3·4}$$

$$\mathcal{B}\equiv B+(a'/a)^{-1}k(H_L+(1/n)H_T)-k^{-1}H_T'\,. \tag{3·5}$$

As easily seen from the structure of Eqs. (3·3), any gauge-invariant which is constructed from A, B, H_L and H_T and their time-derivatives can be written as a linear combination of $\mathcal{A}$ and $\mathcal{B}$ and their time-derivatives with coefficients of arbitrary functions of time. Examples of such combinations are Bardeen's invariants Φ and Ψ defined by [Bardeen (1980)]

$$\Phi\equiv k^{-1}(a'/a)\mathcal{B}=H_L+n^{-1}H_T+k^{-1}(a'/a)(B-k^{-1}H_T'), \tag{3·6}$$

$$\begin{aligned}\Psi&\equiv\mathcal{A}+(ka)^{-1}(a\mathcal{B})'\\ &=A+k^{-1}(a'/a)(B-k^{-1}H_T')+k^{-1}(B'-k^{-1}H_T'').\end{aligned} \tag{3·7}$$

Another interesting combination is given by

$$\begin{aligned}C&\equiv-\mathcal{A}+k^{-1}(a'/a)^{-1}a(a'/a^2)'\mathcal{B}\\ &=-A+k^{-1}(a'/a)^{-1}a(a'/a^2)'B+(a'/a)^{-1}H_L'\\ &\quad+(a'/a)^{-1}\{n^{-1}-k^{-2}a(a'/a^2)'\}H_T'\,.\end{aligned} \tag{3·8}$$

Let us expose the geometrical meaning of these gauge-invariants. A choice of a time-coordinate determines a family of constant time hypersurfaces in the perturbed spacetime which we refer to as time slicing. With each time slicing there are three

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important geometrical quantities; namely, the intrinsic scalar curvature ${}^sR$ of each constant time hypersurface, the expansion rate θ_g and the shear σ_g of the unit vector field normal to this hypersurface. The suffix g is introduced to distinguish these quantities from the corresponding ones defined later for the matter $(n+1)$-velocity field. In order to relate these geometrical quantities to the gauge-invariant variables, it is necessary to find expressions for the former in terms of A, B, H_L and H_T.

First we consider the spatial scalar curvature. Since the perturbed intrinsic metric of a constant time hypersurface $\tilde{\Sigma}$ is given by $\tilde{g}_{ij}$, the perturbation of the spatial Christoffel symbol $\delta{}^s\Gamma^i{}_{jm}$ is given by

$$\begin{aligned}\delta{}^s\Gamma^i{}_{jm}&=(1/2)\gamma^{in}(\delta\gamma_{nj|m}+\delta\gamma_{nm|j}+\delta\gamma_{jm|n})\\&=-kH_L(\delta^i{}_j\mathrm{Y}_m+\delta^i{}_m\mathrm{Y}_j+\gamma_{jm}\mathrm{Y}^i)+H_T(\mathrm{Y}^i{}_{j|m}+\mathrm{Y}^i{}_{m|j}-\mathrm{Y}_{jm}{}^{|i}),\end{aligned}\tag{3·9}$$

where

$$\delta\gamma_{ij}\equiv 2H_L\gamma_{ij}\mathrm{Y}+2H_T\mathrm{Y}_{ij}\,.\tag{3·10}$$

Hence the perturbation of the spatial Riemannian curvature tensor ${}^sR^i{}_{jmn}$ is given by

$$\begin{aligned}\delta{}^sR^i{}_{jmn}&=\delta{}^s\Gamma^i{}_{jn|m}-\delta{}^s\Gamma^i{}_{jm|n}\\&=H_L(\delta^i{}_n\mathrm{Y}_{|jm}-\delta^i{}_m\mathrm{Y}_{|jn}+\mathrm{Y}^{|i}{}_{|n}\gamma_{jm}-\mathrm{Y}^{|i}{}_{|m}\gamma_{jn})\\&\quad+H_T(\mathrm{Y}^i{}_{n|jm}-\mathrm{Y}^i{}_{m|jn}+Y_{jm}{}^{|i}{}_{|n}-\mathrm{Y}_{jn}{}^{|i}{}_{|m}-{}^sR_{nm}{}^i{}_p\mathrm{Y}^p{}_j-{}^sR_{nmj}{}^p\mathrm{Y}^i{}_p).\end{aligned}\tag{3·11}$$

Contracting the second and third indices of $\delta{}^sR^i{}_{jmn}$ and using formulas for Y listed in Appendix C, one obtains

$$\begin{aligned}\delta{}^sR_{ij}&=\left[(2-n)k^2H_L+\left\{2(2n-1)K-\frac{n-2}{n}k^2\right\}H_T\right]Y_{ij}\\&\quad+\frac{2(n-1)}{n}\left\{k^2H_L+\frac{1}{n}(k^2-nK)H_T\right\}\gamma_{ij}Y\,,\end{aligned}\tag{3·12}$$

$$\delta{}^sR=2(n-1)a^{-2}(k^2-nK)\mathfrak{R}\mathrm{Y}\,,\tag{3·13}$$

where

$$\mathfrak{R}=H_L+\frac{1}{n}H_T\,,\tag{3·14}$$

which represents the amplitude of perturbation in the intrinsic curvature of a constant time hypersurface $\tilde{\Sigma}$.

The remaining two quantities θ_g and σ_g are both connected with the behavior of the vector normal to $\tilde{\Sigma}$. In general the covariant derivative of any time-like unit vector field V^μ can be decomposed uniquely as follows:

$$V_{\mu;\nu}=\omega_{\mu\nu}+\sigma_{\mu\nu}+n^{-1}\theta P_{\mu\nu}-a_\mu V_\nu\,,\tag{3·15}$$

where

$$P_{\mu\nu}\equiv g_{\mu\nu}+V_\mu V_\nu\,,\tag{3·16a}$$

$$\omega_{\mu\nu}\equiv P_\mu{}^\alpha P_\nu{}^\beta(V_{\alpha;\beta}-V_{\beta;\alpha}),\tag{3·16b}$$

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$$\theta \equiv V^{\mu}{}_{;\mu}\,, \tag{3·16c}$$

$$\sigma_{\mu\nu} \equiv \frac{1}{2} P_{\mu}{}^{\alpha} P_{\nu}{}^{\beta} (V_{\alpha;\beta} + V_{\beta;\alpha}) - \frac{1}{n} \theta P_{\mu\nu}\,, \tag{3·16d}$$

$$a_{\mu} \equiv V_{\mu;\nu} V^{\nu}\,. \tag{3·16e}$$

Here $g_{\mu\nu}$ represents the metric tensor of a general spacetime. The quantities $\omega_{\mu\nu}$, θ, $\sigma_{\mu\nu}$ and a_{μ} are called the vorticity, expansion rate, shear and acceleration of the time-like unit vector field V^{μ}, respectively. Especially for the unit normal vector field to a family of space-like hypersurfaces, θ represents the volume expansion rate of the hypersurfaces along the normal vector.

Now for the unit normal vector field $\tilde{N}^{\mu}$ to constant time hypersurfaces, their components are given by

$$\tilde{N}_0 = -a(1+A\mathrm{Y}), \qquad \tilde{N}_i = 0\,, \tag{3·17}$$

$$\tilde{N}^0 = a^{-1}(1-A\mathrm{Y}), \qquad \tilde{N}^i = -B\mathrm{Y}^i\,. \tag{3·18}$$

Then with the help of the formulas for the perturbed Christoffel symbols given in Appendix D, the geometrical quantities defined in Eqs. (3·16) are calculated as

$$\tilde{\omega}_{\mu\nu} = 0\,, \tag{3·19a}$$

$$\tilde{\theta} = na^{-2}a'[1+\mathcal{K}_g \mathrm{Y}], \tag{3·19b}$$

$$\tilde{\sigma}_{00} = \tilde{\sigma}_{0j} = 0\,, \tag{3·19c}$$

$$\tilde{\sigma}_{ij} = ak\sigma_g \mathrm{Y}_{ij}\,, \tag{3·19d}$$

$$\tilde{a}_0 = 0\,, \tag{3·19e}$$

$$\tilde{a}_j = -kA\mathrm{Y}_j\,, \tag{3·19f}$$

where

$$\mathcal{K}_g \equiv -A + \frac{1}{n}\left(\frac{a'}{a}\right)^{-1} kB + \left(\frac{a'}{a}\right)^{-1} H'\,, \tag{3·20}$$

$$\sigma_g \equiv k^{-1} H_T' - B\,. \tag{3·21}$$

Equation (3·19a) reflects the hypersurface orthogonality of $\tilde{N}^{\mu}$. For the unperturbed background, $\tilde{\theta}_g$ coincides with the expansion rate of spatial volume per unit proper-time, $n\dot{a}/a$, and the shear $\tilde{\sigma}_{\mu\nu}$ vanishes due to the spatial isotropy. Hence $\mathcal{K}_g$ represents the amplitude of perturbation in the expansion rate and σ_g that of the shear. The variable A represents the amplitude of the acceleration by itself.

Now, from Eqs. (3·8), (3·20) and (3·21), $\mathcal{C}$ is related to $\mathcal{K}_g$ and σ_g as

$$\mathcal{C} = \mathcal{K}_g - \left(\frac{a'}{a}\right)^{-2} \left\{ a\left(\frac{a'}{a^2}\right)' - \frac{k^2}{n} \right\} \sigma_g\,. \tag{3·22}$$

Also, from Eqs. (3·6), (3·14) and (3·21), Φ is expressed in terms of $\mathcal{R}$ and σ_g as

$$\Phi = \mathcal{R} - k^{-1}(a'/a)\sigma_g\,. \tag{3·23}$$

Finally Ψ is expressed in terms of A and σ_g as

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$$\Psi = A - k^{-1}(a'/a)\sigma_g - k^{-1}\sigma_g' . \tag{3·24}$$

Similarly $\mathcal{A}$ and $\mathcal{B}$ are also expressed in terms of the geometrical quantities:

$$\mathcal{A} = A - a^{-1}[(a'/a)^{-1}a\mathcal{R}]' , \tag{3·25}$$

$$\mathcal{B} = k(a'/a)^{-1}\mathcal{R} - \sigma_g . \tag{3·26}$$

Hence $\mathcal{R}$, σ_g, A and $\mathcal{K}_g$ are not independent but related as

$$\mathcal{K}_g = -A - \frac{k}{n}\left(\frac{a'}{a}\right)^{-1}\sigma_g + \left(\frac{a'}{a}\right)^{-1}\mathcal{R}' . \tag{3·27}$$

In the Newtonian limit A is interpreted as the conventional gravitational potential. As will be shown in §4, a part of the Einstein equations reduces to an equation for Ψ which takes the same form as the Newtonian Poisson equation with Ψ playing a role of the gravitational potential. Then as expected from Eq. (3·24) this equation becomes most similar to the Poisson equation under the shear free condition $\sigma_g=0$, which will be explicitly shown in §III-1. Since for the gauge transformation (3·1) σ_g changes as

$$\bar{\sigma}_g = \sigma_g - kT , \tag{3·28}$$

the condition $\sigma_g=0$ specifies the time slicing of the perturbed spacetime. From the argument above this time slicing can be called "Newtonian slicing".

Equations (3·22)~(3·24) show that the gauge-invariant quantities constructed only from the perturbation variables of the metric have the most clear geometrical or physical meaning for the Newtonian slicing $\sigma_g=0$. Namely Φ, $\mathcal{C}$ and Ψ represent the amplitudes of perturbation in the intrinsic curvature of the space, in the expansion rate and in the gravitational potential, respectively, for this slicing.

Now let us proceed to the matter variables. Since $\tilde{u}^i/\tilde{u}^0 = dx^i/d\eta$, a gauge transformation of v is given as

$$\bar{v} = v + L' . \tag{3·29}$$

Since $\tilde{\rho}$ is a scalar quantity, $\tilde{\rho}$ transforms as

$$\bar{\tilde{\rho}}(\eta) = \tilde{\rho}(\eta - TY) \simeq \tilde{\rho}(\eta) - \rho' TY . \tag{3·30}$$

Hence using Eq. (0·7) we obtain

$$\bar{\delta} = \delta + n(1+w)(a'/a)T , \tag{3·31}$$

where

$$w \equiv p/\rho . \tag{3·32}$$

Similarly the amplitude of the isotropic pressure perturbation π_L transforms as

$$\bar{\pi}_L = \pi_L + \frac{c_s^2}{w} n(1+w)\frac{a'}{a}T , \tag{3·33}$$

where

$$c_s^2 \equiv \dot{p}/\dot{\rho} . \tag{3·34}$$

In order to find the transformation law of π_T, we consider the general transformation law

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of the energy-momentum tensor. From the same argument leading to Eq. (3·2) it follows that

$$\tilde{\tilde{T}}^{\mu}{}_{\nu}=\tilde{T}^{\mu}{}_{\nu}-T^{\alpha}{}_{\nu}\delta x^{\mu}{}_{,\alpha}+T^{\mu}{}_{\alpha}\delta x^{\alpha}{}_{,\nu}-T^{\mu}{}_{\nu,\alpha}\delta x^{\alpha}\,. \tag{3·35}$$

In particular for the spatial components this equation is written as

$$\begin{aligned}\tilde{\tilde{T}}^{i}{}_{j}&=\tilde{T}^{i}{}_{j}-T^{\alpha}{}_{j}\delta x^{i}{}_{,\alpha}+T^{i}{}_{\alpha}\delta x^{\alpha}{}_{,j}-T^{i}{}_{j,\alpha}\delta x^{\alpha}\\ &=\tilde{T}^{i}{}_{j}-p'\,T\mathrm{Y}\delta^{i}{}_{j}\,.\end{aligned} \tag{3·36}$$

This equation contains no term proportional to $\mathrm{Y}^{i}{}_{j}$. This means that π_T is gauge-invariant by itself,

$$\bar{\pi}_T=\pi_T\,. \tag{3·37}$$

There are two gauge-invariants which can be constructed only by the matter variables:

$$\Gamma\equiv\pi_L-\frac{c_s{}^2}{w}\delta\,, \tag{3·38}$$

$$\Pi\equiv\pi_T\,. \tag{3·39}$$

Since Γ vanishes for adiabatic perturbations $\delta p/\delta\rho=\dot{p}/\dot{\rho}$, Γ represents the amplitude of an entropy perturbation. The gauge-invariance of Γ also implies that the concept of adiabatic perturbation has the gauge-invariant meaning. In order to construct gauge-invariant quantities corresponding to v and δ, we must combine them with the geometrical quantities.

First let us consider the velocity variable v. Calculation of the geometrical quantities (3·16) for the perturbed $(n+1)$-velocity $\tilde{u}^{\mu}$ gives us insight into the structure of gauge-invariants associated with the velocity. Using the component expressions (2·10) and (2·11) and the formulas for the perturbed Christoffel symbols given in Appendix D, we obtain

$$\tilde{\omega}_{\mu\nu}=0\,, \tag{3·40a}$$

$$\tilde{\theta}=n(a'/a^2)(1+\mathcal{K}_m\mathrm{Y}), \tag{3·40b}$$

$$\tilde{\sigma}_{00}=\tilde{\sigma}_{0j}=0\,, \tag{3·40c}$$

$$\tilde{\sigma}_{ij}=ak\sigma_m\mathrm{Y}_{ij}\,, \tag{3·40d}$$

$$\tilde{a}_0=0\,, \tag{3·40e}$$

$$\tilde{a}_j=-kA_m\mathrm{Y}_j\,, \tag{3·40f}$$

where

$$\mathcal{K}_m\equiv-A+(a'/a)^{-1}(H_L{}'+(k/n)v), \tag{3·41}$$

$$\sigma_m\equiv k^{-1}H_T{}'-v\,, \tag{3·42}$$

$$A_m\equiv A-k^{-1}(a'/a)(v-B)-k^{-1}(v'-B'). \tag{3·43}$$

Among various coefficients appearing in these equations, the amplitude of the shear σ_m

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and the acceleration A_m are gauge-invariant by themselves. That is, if we define a gauge-invariant quantity by

$$V \equiv v - k^{-1} H_T' , \tag{3·44}$$

then σ_m and A_m are expressed as

$$\sigma_m = -V , \tag{3·45}$$

$$A_m = \Psi - k^{-1} a^{-1} (aV)' . \tag{3·46}$$

In contrast the amplitude of perturbation in the expansion rate of matter $\mathcal{K}_m$ is not gauge-invariant but transforms as

$$\bar{\mathcal{K}}_m = \mathcal{K}_m - (a'/a)^{-1} a (a'/a^2)' T . \tag{3·47}$$

Comparing this equation with Eq. (3·28), one finds that

$$\begin{aligned} \mathcal{C}_m &\equiv \mathcal{K}_m - (a'/a)^{-1} a (a'/a^2)' k^{-1} \sigma_g \\ &= -A + (a'/a)^{-1} (H_L' + (k/n) v) - (a'/a)^{-1} a (a'/a^2)' k^{-2} (H_T' - kB) \end{aligned} \tag{3·48}$$

is gauge-invariant. In fact $\mathcal{C}_m$ is expressed in terms of $\mathcal{C}$ and V as

$$\mathcal{C}_m = \mathcal{C} + (k/n)(a'/a)^{-1} V . \tag{3·49}$$

Since $\mathcal{C}_m$ and $\mathcal{C}$ are the gauge-invariant quantities corresponding to the expansion rates of matter and space, respectively, Eq. (3·49) shows that the difference of these two rates is equal to the divergence of the vector $V\mathrm{Y}_i$ except for the normalization factor.

These considerations strongly suggest that V is the most natural gauge-invariant representing the perturbation in velocity. The direct geometrical meaning of V is obviously the amplitude of shear of the material motion. As in the case of the geometrical gauge-invariant quantities, V has the most natural physical meaning in the Newtonian slicing $\sigma_g = 0$, in which $V = v - B$. Equation (3·18) shows that B represents the velocity of an observer moving along constant space-coordinates relative to the lines normal to constant time hypersurfaces. Hence V can be interpreted as representing the velocity of matter relative to the normal line observers. As noted above, furthermore, $a^{-1} (V\mathrm{Y}^i)_{|i}$ coincides with the difference between the expansion rates of matter and space in the Newtonian slicing. Therefore V represents the magnitude of the "proper velocity" of matter, namely the velocity of matter relative to the Newtonian space in this slicing.

In contrast to the velocity, there exists no unique natural definition of a gauge-invariant quantity corresponding to the density perturbation. Even if we limit ourselves to the simplest combinations, there are two choices:

$$\Delta_s \equiv \delta + n(1+w)(a'/a) k^{-1} \sigma_g \tag{3·50}$$

and

$$\begin{aligned} \Delta_g &\equiv \delta + n(1+w) \mathcal{R} \\ &= \Delta_s + n(1+w) \Phi , \end{aligned} \tag{3·51}$$

where Δ_s and Δ_g represent the density contrast on the Newtonian slicing and on the "flat" slicing, respectively. Here "flat" means that the perturbation of the scalar curvature of

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constant time hypersurfaces $\mathfrak{R}$ vanishes. As in the case of the geometrical quantities, any linear combination of V and Δ_s (or Δ_g) and their time-derivatives with coefficients of arbitrary functions of time is also gauge-invariant. One important such combination is Δ defined by

$$\Delta \equiv \Delta_s + n(1+w)(a'/a)k^{-1}V$$
$$= \delta + n(1+w)(a'/a)k^{-1}(v-B). \tag{3·52}$$

The variable Δ represents the density contrast in the slicing such that the material $(n+1)$-velocity is orthogonal to constant time hypersurfaces, as is seen from Eq. (2·11). As will be shown in the next section, the Einstein equations are written in the simplest form when one uses Δ and V as the fundamental variables.

(2) Vector perturbations

A vector type infinitesimal gauge transformation is expressed as

$$\bar{\eta} = \eta\,, \qquad \bar{x}^j = x^j + L^{(1)} Y^{(1)j}\,, \tag{3·53}$$

where $L^{(1)}$ is an arbitrary function of time. From Eqs. (2·20) and (3·2), the metric variables of a vector perturbation transform as

$$\bar{B}^{(1)} = B^{(1)} + L^{(1)\prime}\,, \tag{3·54a}$$

$$\bar{H}_T{}^{(1)} = H_T{}^{(1)} + kL^{(1)}\,. \tag{3·54b}$$

There exists only one gauge-invariant combination which is given by

$$\sigma_g{}^{(1)} \equiv k^{-1} H_T{}^{(1)\prime} - B^{(1)}\,. \tag{3·55}$$

Since the decomposition of $\tilde{N}_{\mu;\nu}$ for a vector perturbation yields

$$\tilde{\omega}_{\mu\nu} = \tilde{a}_\mu = \tilde{\sigma}_{00} = \tilde{\sigma}_{0j} = 0\,, \quad \tilde{\theta} = n(a'/a^2), \tag{3·56a}$$

$$\tilde{\sigma}_{ij} = ak\sigma_g{}^{(1)} Y^{(1)}{}_{ij}\,, \tag{3·56b}$$

$\sigma_g{}^{(1)}$ represents the amplitude of shear of the normal vector field $\tilde{N}^\mu$, which is gauge-invariant for a vector perturbation by itself. Note that only the shear is non-vanishing for vector perturbations.

The gauge transformation law of the matter variables is also easily obtained because there exists now no perturbation of the density and the isotropic pressure. Following the argument in (1), the matter variables transform as

$$\bar{v}^{(1)} = v^{(1)} + L^{(1)\prime}\,, \tag{3·57a}$$

$$\pi_T{}^{(1)} = \pi_T{}^{(1)}\,. \tag{3·57b}$$

The amplitude of an anisotropic stress perturbation is again gauge-invariant by itself:

$$\Pi^{(1)} \equiv \pi_T{}^{(1)}\,. \tag{3·58}$$

In contrast to a scalar perturbation the gauge transformation laws (3·54) and (3·57) infer two natural gauge-invariant combinations corresponding to a velocity perturbation:

$$V_s{}^{(1)} \equiv v^{(1)} - k^{-1} H_T{}^{(1)\prime}\,, \tag{3·59}$$

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$$V^{(1)} \equiv v^{(1)} - B^{(1)}$$
$$= V_s^{(1)} + \sigma_g^{(1)} . \tag{3·60}$$

The meaning of these quantities is found from the consideration on the decomposition of $\tilde{u}_{\mu;\nu}$. One obtains

$$\tilde{\omega}_{0j} = 0 , \tag{3·61a}$$
$$\tilde{\omega}_{ij} = a V^{(1)} (\mathrm{Y}^{(1)}{}_{i|j} - \mathrm{Y}^{(1)}{}_{j|i}), \tag{3·61b}$$
$$\tilde{\theta} = n(a'/a^2), \tag{3·61c}$$
$$\tilde{\sigma}_{00} = \tilde{\sigma}_{0j} = 0 , \tag{3·61d}$$
$$\tilde{\sigma}_{ij} = ak\sigma_m{}^{(1)} \mathrm{Y}^{(1)} , \tag{3·61e}$$
$$\tilde{a}_0 = 0 , \tag{3·61f}$$
$$\tilde{a}_j = V_s{}^{(1)\prime} \mathrm{Y}_j{}^{(1)} , \tag{3·61g}$$

where

$$\sigma_m{}^{(1)} = - V_s{}^{(1)} . \tag{3·62}$$

Thus $V_s{}^{(1)}$ and $V^{(1)}$ represent the amplitudes of the shear and the vorticity of the matter velocity field, respectively. The absence of the perturbation in the expansion rate of matter implies that the matter moves in an incompressible manner for vector perturbations. This is the reason why no perturbation in the density and the isotropic pressure appear in vector perturbations.

(3) Tensor perturbations

There exists no tensor type infinitesimal gauge transformation. Hence all the quantities associated with a tensor perturbation are gauge-invariant by themselves. Thus from Eqs. (2·24) there exist only two gauge-invariant quantities, $H_T{}^{(2)}$ and $\pi_T{}^{(2)}$, in this case. To be systematic, we introduce the notation $\Pi^{(2)}$ to represent the amplitude of an anisotropic stress perturbation instead of the original notation $\pi_T{}^{(2)}$:

$$\Pi^{(2)} \equiv \pi_T{}^{(2)} . \tag{3·63}$$

§ II-4. The Einstein equations for gauge-invariant variables

(1) Scalar perturbations

Since a gauge transformation is formally an infinitesimal coordinate transformation in the perturbed spacetime, the general covariance of the Einstein equations guarantees that the perturbation of the Einstein equations

$$\delta G^\mu{}_\nu = \chi^2 \delta T^\mu{}_\nu \tag{4·1}$$

can be written only in terms of gauge-invariant combinations of the original perturbation variables. From the expressions for $\delta G^\mu{}_\nu$ in terms of A, B, H_L and H_T given in Appendix D, it follows after a short calculation that Eq. (4·1) yields the following four independent equations written in terms of the gauge-invariant variables introduced in §3:

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$\delta G^0{}_0 = \varkappa^2 \delta T^0{}_0$:

$$n\left(\frac{a'}{a}\right)^2 \mathcal{A} - \frac{a'}{a} k \mathcal{B} = -\varkappa^2 \frac{a^2 \rho}{n-1} \Delta_g . \tag{4·2a}$$

$\delta G^0{}_j = \varkappa^2 \delta T^0{}_j$:

$$k \frac{a'}{a} \mathcal{A} - \left\{ K - \varkappa^2 \frac{a^2 h}{n-1} \right\} \mathcal{B} = \varkappa^2 \frac{a^2 h}{n-1} V . \tag{4·2b}$$

$\delta G^i{}_j = \varkappa^2 \delta T^i{}_j$:

The trace part;

$$\frac{a'}{a} \mathcal{A}' + \left\{ 2a\left(\frac{a'}{a^2}\right)' + n\left(\frac{a'}{a}\right)^2 \right\} \mathcal{A} = \varkappa^2 \frac{a^2 p}{n-1} \left(\Gamma + \frac{c_s{}^2}{w} \Delta_g \right) - \varkappa^2 \frac{1}{n} a^2 p \Pi . \tag{4·2c}$$

The traceless part;

$$\mathcal{A} + \frac{1}{k} a^{1-n} (a^{n-1} \mathcal{B})' = -\varkappa^2 \frac{a^2 p}{k^2} \Pi . \tag{4·2d}$$

Eliminating $\mathcal{A}$ from Eqs. (4·2a) and (4·2b), and using Eq. (3·6), we obtain an algebraic relation connecting Δ and Φ:

$$\varkappa^2 \rho \Delta = (n-1) a^{-2} (k^2 - nK) \Phi . \tag{4·3}$$

From the definition of Ψ, Eq. (3·7), Eq. (4·2d) can be written as

$$(n-2)\Phi + \Psi = -\varkappa^2 a^2 k^{-2} p \Pi . \tag{4·4}$$

With the aid of the formulas for the harmonic function Y given in Appendix C, this equation can be written as

$$\frac{1}{a^2}(\boldsymbol{\Delta} - nK)(\Psi \mathrm{Y}) = \varkappa^2 \frac{n-2}{n-1} \rho \Delta \mathrm{Y}$$
$$+ \varkappa^2 \frac{n(n-2)}{n-1} \boldsymbol{\Delta}^{-1s} \nabla_i{}^s \nabla_j (p \Pi \mathrm{Y}^{ij}). \tag{4·5}$$

Equation (4·5) has the same form as the Newtonian Poisson equation if the space curvature K and the anisotropic stress Π are neglected. This fact and the relation of Ψ and A given in §3 suggest that Ψ can be interpreted as a generalized gravitational potential.

With the aid of the equations for the time-derivatives of the cosmic scale factor a given in Appendix A, Eqs. (4·2c) and (4·2d) are written as

$$\mathcal{A}' + (a'/a)^{-1} \{ (2-n-nw) K + n(c_s{}^2 - w)(a'/a)^2 \} \mathcal{A} - c_s{}^2 k \mathcal{B}$$
$$= \varkappa^2 \left(\frac{a'}{a}\right)^{-1} a^2 p \left(\frac{\Gamma}{n-1} - \frac{\Pi}{n} \right), \tag{4·6a}$$

$$\mathcal{B}' + (n-1)(a'/a)\mathcal{B} + k\mathcal{A} = -k^{-1} \varkappa^2 a^2 p \Pi . \tag{4·6b}$$

These equations yield the time-evolution equation for $\mathcal{A}$ and $\mathcal{B}$ if Γ and Π are specified as functions of time or expressed in terms of $\mathcal{A}$ and $\mathcal{B}$. The gauge-invariant variables for

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the matter are determined by the algebraic relations,

$$x^2\rho\Delta=(n-1)\frac{a'}{a}\frac{k}{a^2}\left(1-\frac{nK}{k^2}\right)\mathcal{B}\,, \tag{4·6c}$$

$$x^2hV=(n-1)\frac{a'}{a}\frac{k}{a^2}\mathcal{A}+\left\{x^2h-(n-1)\frac{K}{a^2}\right\}\mathcal{B}\,. \tag{4·6d}$$

The set of equations (4·6) yields the Einstein equations for the gauge-invariant variables when the metric variables $\mathcal{A}$ and $\mathcal{B}$ are regarded as the fundamental variables.

In practical applications we are often interested rather in the evolution of the matter variables Δ and V. Hence it is desirable to write down the evolution equations directly in terms of Δ and V. Replacing $\mathcal{A}$ and $\mathcal{B}$ in Eqs. (4·6a) and (4·6b) by Δ and V with the aid of Eqs. (4·6c) and (4·6d), we obtain after a little cumbersome calculation the following equations:

$$\Delta'-nw\frac{a'}{a}\Delta=-\left(1-\frac{nK}{k^2}\right)(1+w)kV-(n-1)\left(1-\frac{nK}{k^2}\right)\frac{a'}{a}w\Pi\,, \tag{4·7a}$$

$$V'+\frac{a'}{a}V=-k\left[\frac{n-2}{n-1}x^2\frac{a^2\rho}{k^2-nK}-\frac{c_s{}^2}{1+w}\right]\Delta+k\frac{w}{1+w}\Gamma$$
$$-k\left[\frac{n-1}{n}\left(1-\frac{nK}{k^2}\right)\frac{1}{1+w}+x^2\frac{a^2\rho}{k^2}\right]w\Pi\,. \tag{4·7b}$$

Now the metric variables $\mathcal{A}$ and $\mathcal{B}$ are expressed as

$$\mathcal{A}=\left(\frac{a'}{a}\right)^{-2}\left\{K-x^2\frac{a^2h}{n-1}\right\}\frac{x^2a^2\rho}{(n-1)(k^2-nK)}\Delta+\frac{1}{n-1}\frac{1}{k}\left(\frac{a'}{a}\right)^{-1}x^2h\mathrm{V}\,, \tag{4·7c}$$

$$\mathcal{B}=\frac{1}{k}\frac{a'}{a}\frac{x^2a^2\rho}{(n-1)(k^2-nk)}\Delta\,. \tag{4·7d}$$

Note that Eqs. (4·7a) and (4·7b) can also be obtained directly from the perturbation of the equation of motion

$$\delta(\tilde{T}_\mu{}^\nu{}_{;\nu})=0\,. \tag{4·8}$$

Each term in Eqs. (4·7a) and (4·7b) allows simple physical interpretation except for those involving the space curvature and anisotropic stress: The second term on the left-hand side of Eq. (4·7a) represents the adiabatic change in the density contrast due to the cosmic expansion since the left-hand side is as a whole written as $(a^n\rho\Delta)'/a^n\rho$. The first term on the right-hand side represents the compression by the proper motion of matter since $kV\mathrm{Y}$ is written as ${}^s\nabla_i(V\mathrm{Y}^i)$. For the purpose of finding the physical interpretation of Eq. (4·7b), it is more convenient to write it as follows:

$$V'+\frac{a'}{a}V=k\left[\frac{c_s{}^2}{1+w}\Delta+\frac{w}{1+w}\Gamma\right]+k\Psi-k\frac{n-1}{n}\left(1-\frac{nK}{k^2}\right)\frac{w}{1+w}\Pi\,. \tag{4·7b$'$}$$

The second term on the left-hand side represents the adiabatic slowing-down of velocity due to the cosmic expansion. The slowing-down rate is exactly the same as that for a free particle. The first term on the right-hand side represents the force due to the pressure gradient and the secend term the gravitational force as it corresponds to the gradient of the generalized gravitational potential $\Psi\mathrm{Y}$.

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In the later applications a second-order form of the evolution equation is often used instead of the first-order system (4·7). Such a second-order form is obtained by eliminating V from Eqs. (4·7a) and (4·7b):

$$\Delta''-\{n(2w-c_s{}^2)-1\}\frac{a'}{a}\Delta'$$
$$+n\left[\left\{\frac{n}{2}w^2-(n+1)w-\frac{n-2}{2}+nc_s{}^2\right\}\left(\frac{a'}{a}\right)^2+\frac{nw^2+2-n}{2}K+\frac{k^2-nK}{n}c_s{}^2\right]\Delta=\mathscr{S}\,, \tag{4·9}$$

where

$$\mathscr{S}=-(k^2-nK)w\Gamma-(n-1)\left(1-\frac{nK}{k^2}\right)\frac{a'}{a}w\Pi'$$
$$+\left[\{n(w^2+c_s{}^2)-2w\}\left(\frac{a'}{a}\right)^2+w(nw+n-1)K+\frac{k^2-nK}{n}c_s{}^2\right]\left(1-\frac{nK}{k^2}\right)(n-1)\Pi\,. \tag{4·9a}$$

Equation (4·9) shows that an entropy perturbation and an anisotropic stress perturbation act as sources for density perturbations. In order to estimate their effects we must specify them as functions of time explicitly or express them in terms of Δ and V. In most of realistic situations, however, the appearance of Γ or Π is a consequence of some intrinsic structure of matter, especially of the multi-component nature of matter. For such matter Γ and Π are not expressible directly in terms of Δ and V, but given by density perturbations and/or velocity perturbations of the components. Hence the extension of the formalism to a multi-component system is necessary to discuss the generation and evolution of density perturbations which are not adiabatic. Such extension will be done in § 5.

(2) Vector perturbations

The derivation of the gauge-invariant equations for vector perturbations is much simpler than that for scalar perturbations because there are only three independent gauge-invariant variables. From the expression of $\delta G^\mu{}_\nu$ for a vector perturbation given in Appendix D, it follows that the Einstein equations reduce to the following two gauge-invariant equations:

$\delta G^0{}_j=\varkappa^2\delta T^0{}_j$:

$$\varkappa^2 hV^{(1)}=-\frac{k^2-(n-1)K}{2a^2}\sigma_g{}^{(1)}\,. \tag{4·10a}$$

$\delta G^i{}_j=\varkappa^2\delta T^i{}_j$:

$$\sigma_g{}^{(1)\prime}+(n-1)\frac{a'}{a}\sigma_g{}^{(1)}=\varkappa^2\frac{1}{k}a^2 p\Pi^{(1)}\,. \tag{4·10b}$$

Equation (4·10b) shows that $\Pi^{(1)}$ acts as a source for $\sigma^{(1)}{}_g$, namely, a vector type anisotropic stress perturbation produces an anisotropic expansion of space. This equation also shows that the anisotropy of space measured by

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$$|\tilde{\sigma}_{\mu\nu}\tilde{\sigma}^{\mu\nu}|^{1/2}=a^{-1}k|\sigma_g^{(1)}|[\mathrm{Y}_{ij}\mathrm{Y}^{ij}]^{1/2} \qquad (4\cdot11)$$

is damped in proportion to a^{-n}, hence it rapidly becomes negligible as the universe expands even if it is large in the early stage of the universe, provided that $\Pi^{(1)}=0$. This result is consistent with the result obtained by an exact analysis of spatially homogeneous anisotropic universe model of Bianchi-type I.

Equations (4·10) yield the following evolution equation for $V^{(1)}$:

$$V^{(1)\prime}+(1-nc_s{}^2)\frac{a'}{a}V^{(1)}=-\frac{k^2-(n-1)K}{2k}\frac{w}{1+w}\Pi^{(1)}. \qquad (4\cdot10\mathrm{c})$$

This equation implies that the vorticity of matter is generated by a vector type anisotropic stress perturbation. The structure of the left-hand side can be understood from the point of view of the angular momentum conservation since the left-hand side can be written as $[V\times(k^{-1}a)\times(\rho+p)(k^{-1}a)^n]'/(\rho+p)(k^{-1}a)^{n+1}$ in which the quantity in the square bracket represents the characteristic value of the angular momentum of matter contained in a region of proper size $k^{-1}a$.

From Eq. (4·10c) we can easily find the evolutionary behavior of the rotational motion of matter in the absence of the anisotropic stress perturbation. Let us measure the amplitude of this rotational motion by the dimensionless quantity

$$[\tilde{\omega}^{\mu\nu}\tilde{\omega}_{\mu\nu}]^{1/2}\Big/\frac{\dot{a}}{a}=\frac{k}{a}|V^{(1)}|[\mathrm{Z}_{ij}\mathrm{Z}^{ij}]^{1/2}\Big/\frac{\dot{a}}{a}$$

$$\propto\frac{k}{a}|V^{(1)}|\Big/\frac{\dot{a}}{a}, \qquad (4\cdot12)$$

where

$$\mathrm{Z}_{ij}\equiv k^{-1}(\mathrm{Y}^{(1)}_{i|j}-\mathrm{Y}^{(1)}_{j|i}). \qquad (4\cdot13)$$

The quantity (4·12) represents the ratio of the rotational velocity on a given scale to the cosmic expansion velocity on the same scale. In the case the cosmic matter satisfies the relativistic equation of state $p=\rho/n$ $(w=c_s{}^2=1/n)$ and there exists no anisotropic perturbation, it follows from Eq. (4·10c) that the rotational motion always grows as the universe expands,

$$|\tilde{\omega}^{\mu\nu}\tilde{\omega}_{\mu\nu}|^{1/2}/(\dot{a}/a)\propto a^{(n-1)/2}, \qquad (4\cdot14)$$

provided that the space curvature can be neglected, since in this case $\dot{a}/a\propto\rho^{1/2}\propto a^{-n(1+w)/2}$ from Eqs. (0·6) and (0·7).

(3) Tensor perturbations

For a tensor perturbation the Einstein equations reduce to a single gauge-invariant equation:

$\delta G^i{}_j=\chi^2\delta T^i{}_j$:

$$H_T{}^{(2)\prime\prime}+(n-1)(a'/a)H_T{}^{(2)\prime}+(k^2+2K)H_T{}^{(2)}=\chi^2a^2p\Pi^{(2)}. \qquad (4\cdot15)$$

Since $H_T{}^{(2)}$ corresponds to the divergenceless and traceless part of the metric tensor, it represents the amplitude of a gravitational wave. As is expected, the equation for $H_T{}^{(2)}$ (4·15) is of wave-equation type, and its evolution is completely determined by the tempo-

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ral behavior of the cosmic scale factor and does not depend directly on the material content of the universe except for possible contributions to the source term.

§ II-5. Extension to a multi-component system

Although the perturbation equations given in the last section have the general applicability in principle, they in their present form are suitable only for investigating the time-evolution of adiabatic perturbations in a universe dominated by a single fluid (by a single fluid, we mean a fluid described by an equation of state $p=p(\rho)$ effectively). In the history of the actual universe, however, there are times when the matter is more adequately described by a mixture of several fluid components (such as radiation, baryonic matter and neutrinos) and, in some cases, classical fields than by a single fluid. In such situations perturbations in the densities and velocities of individual components behave differently reflecting the difference in the dynamical properties, especially, the sound velocities. As a consequence the notion of adiabatic perturbation becomes inconsistent with the dynamics in the strict sense. So, it is preferable and worth while to reformulate the equations in the form directly applicable to a universe filled with a multi-component matter. As will be discussed in Part Two, this enables us to treat the interaction of adiabatic perturbations with the so-called isothermal ones, and also the time-evolution of density fluctuations when classical scalar fields are the essential ingredient of the cosmic matter.

In extending the formalism developed in §§2~4 to the case the matter consists of more than two components, there appear of course no new variables concerning the metric perturbation. As well, the gauge-invariant variables for the matter introduced so far are still meaningful in this case if they are regarded as representing, for example, the total energy density, the average velocity and so on. Furthermore since the Einstein equations contain only the variables describing the state of matter as a whole besides the metric variables, the gauge-invariant equations obtained in §4 are valid also in the multi-component case without any change, and the Einstein equations add no new equation. Hence what to be done in extending the formalism to a multi-component system is to define gauge-invariant variables representing the state of each component and decompose the gauge-invariant equation expressed in terms of $\varDelta$, V, $\varGamma$ and $\varPi$ into corresponding equations expressed in terms of the component-wise variables with the aid of the equation of motion for each component.

The concept of component used in this section is not so restrictive but means simply a part of matter which behaves uniformly in its dynamics. The only restrictive assumption made throughout is that the total energy-momentum tensor of matter is written as the direct sum of the energy-momentum tensor of each component which assumes the perfect fluid form in the unperturbed background. Hence in principle it is allowed that one component is actually composed of a lot of subcomponents further. We distinguish quantities pertaining to different components by the suffix of greek letters. When it may be confused with the spacetime indices, we distinguish it by enclosing it by round brackets.

In a multi-component system the energy-momentum tensor of each component $\tilde{T}_{(\alpha)}{}^{\mu}{}_{\nu}$ is not conserved independently and its divergence has a source term in general:

$$\tilde{T}_{(\alpha)}{}^{\nu}{}_{\mu;\nu}=\tilde{Q}_{(\alpha)\mu}\,. \tag{5·1}$$

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The source terms are restricted by the constraint obtained from the conservation law of the total energy-momentum tensor,

$$\tilde{T}^{\nu}{}_{\mu;\nu}=\sum_{\alpha}\tilde{T}_{(\alpha)}{}^{\nu}{}_{\mu;\nu}=0\,, \tag{5·2}$$

namely,

$$\sum_{\alpha}\tilde{Q}_{(\alpha)\mu}=0\,. \tag{5·3}$$

From the assumption, the energy-momentum tensor of α-component in the unperturbed background is written as

$$T_{(\alpha)\mu}{}^{\nu}=(\rho_{\alpha}+p_{\alpha})u_{\mu}u^{\nu}+p_{\alpha}\delta_{\mu}{}^{\nu}\,, \tag{5·4}$$

where

$$(u^{\mu})=(a^{-1},\,\mathbf{0}). \tag{5·5}$$

Hence the source term in the unperturbed background $Q_{(\alpha)\mu}$ is written as

$$(Q_{(\alpha)\mu})=(-aQ_{\alpha},\,\mathbf{0}), \tag{5·6}$$

and the equations of motion for α-component in the unperturbed background reduce to a single equation

$$\begin{aligned}\dot{\rho}_{\alpha}&=-n\frac{\dot{a}}{a}h_{\alpha}+Q_{\alpha}\\&=-n\frac{\dot{a}}{a}(1-q_{\alpha})h_{\alpha}\,,\end{aligned} \tag{5·7}$$

where

$$h_{\alpha}\equiv\rho_{\alpha}+p_{\alpha}\,, \tag{5·8}$$

$$q_{\alpha}\equiv Q_{\alpha}\Big/\left(n\frac{\dot{a}}{a}h_{\alpha}\right). \tag{5·9}$$

The relations of ρ_{α}, p_{α} and h_{α} to the quantities ρ, p and h are simply given by

$$\rho=\sum_{\alpha}\rho_{\alpha}\,, \tag{5·10a}$$

$$p=\sum_{\alpha}p_{\alpha}\,, \tag{5·10b}$$

$$h=\sum_{\alpha}h_{\alpha}\,. \tag{5·10c}$$

Supplemented by the constraint

$$\sum_{\alpha}Q_{\alpha}=0 \tag{5·11}$$

obtained from Eq. (5·3), the summation of Eq. (5·7) over α yields the total energy equation (0·7).

(1) Scalar perturbations

The expression for the perturbed energy-momentum tensor of each component in

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terms of gauge-dependent perturbation variables has exactly the same structure as that of the total matter. The $(n+1)$-velocity and the proper energy density of α-component, $\tilde{u}^{\mu}{}_{(\alpha)}$ and $\tilde{\rho}_{\alpha}$, are defined as the unit time-like eigenvector and the corresponding eigenvalue of $\tilde{T}_{(\alpha)}{}^{\mu}{}_{\nu}$:

$$\tilde{T}_{(\alpha)}{}^{\mu}{}_{\nu}\tilde{u}_{(\alpha)}{}^{\nu} = -\tilde{\rho}_{\alpha}\tilde{u}_{(\alpha)}{}^{\mu}\,, \tag{5·12}$$

and expressed in terms of gauge-dependent perturbation variables as

$$\tilde{u}_{(\alpha)}{}^{i}/\tilde{u}_{(\alpha)}{}^{0} = v_{\alpha}\mathrm{Y}^{i}\,, \tag{5·13}$$

$$\tilde{u}_{(\alpha)}{}^{0} = a^{-1}(1-A\mathrm{Y}), \tag{5·13a}$$

$$\tilde{u}_{(\alpha)i} = a(v_{\alpha}-B)\mathrm{Y}_{i}\,, \tag{5·14}$$

$$\tilde{u}_{(\alpha)0} = -a(1+A\mathrm{Y}), \tag{5·14a}$$

$$\tilde{\rho}_{\alpha} = \rho_{\alpha}(1+\delta_{\alpha}\mathrm{Y}). \tag{5·15}$$

These expressions are equivalent to

$$\tilde{T}_{(\alpha)}{}^{0}{}_{0} = -\rho_{\alpha}(1+\delta_{\alpha}\mathrm{Y}), \tag{5·16a}$$

$$\tilde{T}_{(\alpha)}{}^{0}{}_{j} = h_{\alpha}(v_{\alpha}-B)\mathrm{Y}_{j}\,, \tag{5·16b}$$

$$\tilde{T}_{(\alpha)}{}^{j}{}_{0} = -h_{\alpha}v_{\alpha}\mathrm{Y}^{j}\,. \tag{5·16c}$$

Similarly the expressions for the spatial components of $\tilde{T}_{(\alpha)}{}^{\mu}{}_{\nu}$ are obtained by simply attaching the suffix α to each variable in the corresponding expressions for $\tilde{T}^{\mu}{}_{\nu}$:

$$\tilde{T}_{(\alpha)}{}^{i}{}_{j} = p_{\alpha}[\delta^{i}{}_{j} + \pi_{L\alpha}\delta^{i}{}_{j} + \pi_{T\alpha}\mathrm{Y}^{i}{}_{j}]. \tag{5·16d}$$

Summation of Eqs. (5·16a)~(5·16d) over α and comparison with the corresponding equations for the total energy-momentum tensor, (2·14) and (2·17)~(2·19), yield the relation between the gauge-dependent perturbation variables for each component and those for the total matter:

$$\rho\delta = \sum_{\alpha}\rho_{\alpha}\delta_{\alpha}\,, \tag{5·17a}$$

$$hv = \sum_{\alpha}h_{\alpha}v_{\alpha}\,, \tag{5·17b}$$

$$p\pi_{L} = \sum_{\alpha}p_{\alpha}\pi_{L\alpha}\,, \tag{5·17c}$$

$$p\pi_{T} = \sum_{\alpha}p_{\alpha}\pi_{T\alpha}\,. \tag{5·17d}$$

The only difference from the previous case is the appearance of the new perturbation variables associated with the source term $\tilde{Q}_{(\alpha)\mu}$. For convenience we decompose it into two parts:

$$\tilde{Q}_{(\alpha)\mu} = \tilde{Q}_{\alpha}\tilde{u}_{\mu} + \tilde{f}_{(\alpha)\mu}\,, \tag{5·18}$$

where $\tilde{f}_{(\alpha)\mu}$ is required to be orthogonal to the average velocity $\tilde{u}^{\mu}$,

$$\tilde{u}^{\mu}\tilde{f}_{(\alpha)\mu} = 0\,. \tag{5·19}$$

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The quantities $\tilde{Q}_\alpha$ and $\tilde{f}_{(\alpha)\mu}$ represent the energy transfer rate and the momentum transfer rate to the α-component seen in the center of mass frame, respectively. Since $\tilde{u}_i$ and $\tilde{f}_{(\alpha)i}$ are both first-order quantities with respect to perturbation, it follows from Eq. (5·19) that

$$\tilde{f}_{(\alpha)0}=0\,. \tag{5·20}$$

Since $\tilde{Q}_\alpha$ and $\tilde{f}_{(\alpha)j}$ transform as a scalar and a vector under spatial coordinate transformations, respectively, they can be written as

$$\tilde{Q}_\alpha=Q_\alpha(1+\varepsilon_\alpha)\mathrm{Y}\,, \tag{5·21a}$$

$$\tilde{f}_{(\alpha)j}=(a'/a)h_\alpha f_\alpha \mathrm{Y}_j \tag{5·21b}$$

with two functions of time, ε_α and f_α. Combining these expressions with Eqs. (2·11), we obtain

$$\tilde{Q}_{(\alpha)0}=-aQ_\alpha[1+(A+\varepsilon_\alpha)\mathrm{Y}], \tag{5·22a}$$

$$\tilde{Q}_{(\alpha)j}=a[Q_\alpha(v-B)+(\dot{a}/a)h_\alpha f_\alpha]\mathrm{Y}_j\,. \tag{5·22b}$$

The constraint (5·3) is now expressed as

$$\sum_\alpha Q_\alpha\varepsilon_\alpha=0\,, \tag{5·23a}$$

$$\sum_\alpha h_\alpha f_\alpha=0\,. \tag{5·23b}$$

The gauge transformation properties of v_α, δ_α, $\pi_{L\alpha}$ and $\pi_{T\alpha}$ are the same as those of v, δ, π_L and π_T except for the modification arising from the difference of the energy equations:

$$\bar{v}_\alpha=v_\alpha+L'\,, \tag{5·24a}$$

$$\bar{\delta}_\alpha=\delta_\alpha+n(1+w_\alpha)(a'/a)(1-q_\alpha)T\,, \tag{5·24b}$$

$$\bar{\pi}_{L\alpha}=\pi_{L\alpha}+\frac{c_\alpha^2}{w_\alpha}n(1+w_\alpha)(1-q_\alpha)\frac{a'}{a}T\,, \tag{5·24c}$$

$$\bar{\pi}_{T\alpha}=\pi_{T\alpha}\,, \tag{5·24d}$$

where

$$w_\alpha\equiv p_\alpha/\rho_\alpha\,, \tag{5·25}$$

$$c_\alpha^2\equiv\dot{p}_\alpha/\dot{\rho}_\alpha\,. \tag{5·26}$$

Hence the gauge-invariant variables constructed from these gauge-dependent quantities are

$$V_\alpha\equiv v_\alpha-k^{-1}H_T'\,, \tag{5·27a}$$

$$\Delta_\alpha\equiv\delta_\alpha+n(1+w_\alpha)(1-q_\alpha)\frac{a'}{a}\frac{1}{k}(v_\alpha-B), \tag{5·27b}$$

$$\Gamma_\alpha\equiv\pi_{L\alpha}-\frac{c_\alpha^2}{w_\alpha}\delta_\alpha\,, \tag{5·27c}$$

$$\Pi_\alpha\equiv\pi_{T\alpha}\,. \tag{5·27d}$$

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As in the case of the total density perturbation there are lots of alternative definitions for the gauge-invariant density perturbation other than Δ_α:

$$\Delta_{s\alpha} \equiv \delta_\alpha + n(1+w_\alpha)(1-q_\alpha)\frac{a'}{a}\frac{1}{k}\sigma_g\,, \tag{5·28a}$$

$$\Delta_{g\alpha} \equiv \delta_\alpha + n(1+w_\alpha)(1-q_\alpha)\mathfrak{R}\,, \tag{5·28b}$$

$$\Delta_{c\alpha} \equiv \delta_\alpha + n(1+w_\alpha)(1-q_\alpha)\frac{a'}{a}\frac{1}{k}(v-B)\,. \tag{5·28c}$$

The relations among these various definitions are given as

$$\Delta_{g\alpha} = \Delta_{s\alpha} + n(1+w_\alpha)(1-q_\alpha)\Phi\,, \tag{5·29a}$$

$$\Delta_\alpha = \Delta_{s\alpha} + n(1+w_\alpha)(1-q_\alpha)\frac{a'}{a}\frac{1}{k}V_\alpha\,, \tag{5·29b}$$

$$\Delta_{c\alpha} = \Delta_{s\alpha} + n(1+w_\alpha)(1-q_\alpha)\frac{a'}{a}\frac{1}{k}V\,. \tag{5·29c}$$

Note that Δ_α is the density perturbation of the α-component relative to the hypersurface representing the rest frame of the α-component, whereas $\Delta_{c\alpha}$ is the density perturbation relative to the total matter rest frame. Hence though the former is adequate for the argument of the intrinsic perturbation of each component, the latter should be used when one compares density perturbations of different components. Equations (5·17a)~(5·17d) yield the relations among these gauge-invariant variables for each component and those for the total matter:

$$\begin{aligned}\rho\Delta &= \sum_\alpha \rho_\alpha \Delta_{c\alpha} \\ &= \sum_\alpha \rho_\alpha \Delta_\alpha + \frac{a}{k}\sum_\alpha Q_\alpha V_\alpha\,,\end{aligned} \tag{5·30a}$$

$$hV = \sum_\alpha h_\alpha V_\alpha\,, \tag{5·30b}$$

$$p\Gamma = p\Gamma_{\rm int} + p\Gamma_{\rm rel}\,, \tag{5·30c}$$

$$p\Pi = \sum_\alpha p_\alpha \Pi_\alpha\,, \tag{5·30d}$$

where

$$p\Gamma_{\rm int} \equiv \sum_\alpha p\Gamma_\alpha\,, \tag{5·31}$$

$$p\Gamma_{\rm rel} \equiv \sum_\alpha (c_\alpha^2 - c_s^2)\delta\rho_\alpha\,. \tag{5·32}$$

Since Γ and Γ_α are gauge-invariant, $\Gamma_{\rm rel}$ is gauge-invariant by itself. Its gauge-invariance can be directly seen by noting the expression for the sound velocity c_s^2:

$$\begin{aligned}c_s^2 &= \sum_\alpha c_\alpha^2 p_\alpha/\rho \\ &= \sum_\alpha \frac{h_\alpha}{h}(1-q_\alpha)c_\alpha^2\end{aligned}$$

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$$=\bar{c}_s^{\,2}-\sum_\alpha\frac{h_\alpha}{h}q_\alpha c_\alpha^{\,2}\,, \tag{5·33}$$

where

$$\bar{c}_s^{\,2}\equiv\sum_\alpha\frac{h_\alpha}{h}c_\alpha^{\,2}\,. \tag{5·34}$$

Hence eliminating δ_α from Eq. (5·32) with the help of Eq. (5·28c), we find

$$p\Gamma_{\rm rel}=\sum_\alpha(c_\alpha^{\,2}-c_s^{\,2})\rho_\alpha\Delta_{c\alpha}\,. \tag{5·35}$$

In order to see the physical meaning of $\Gamma_{\rm rel}$, it is more convenient to rewrite Eq. (5·35) further using the relation (5·33) again as

$$p\Gamma_{\rm rel}=\frac{1}{2}\sum_{\alpha,\beta}\frac{h_\alpha h_\beta}{h}(1-q_\alpha)(1-q_\beta)(c_\alpha^{\,2}-c_\beta^{\,2})\left[\frac{\Delta_{c\alpha}}{(1+w_\alpha)(1-q_\alpha)}-\frac{\Delta_{c\beta}}{(1+w_\beta)(1-q_\beta)}\right]$$

$$=\frac{1}{2}\sum_{\alpha,\beta}\frac{h_\alpha h_\beta}{h}(c_\alpha^{\,2}-c_\beta^{\,2})S_{\alpha\beta}+\frac{1}{2N}\frac{1}{1+w}\sum_{\alpha,\beta}\frac{1}{n}\left(\frac{\dot{a}}{a}\right)^{-1}(c_\beta^{\,2}-c_\alpha^{\,2})(Q_\beta-Q_\alpha), \tag{5·36}$$

where N is the number of components and

$$S_{\alpha\beta}\equiv\frac{\Delta_{c\alpha}}{1+w_\alpha}-\frac{\Delta_{c\beta}}{1+w_\beta}\,. \tag{5·37}$$

Equation (5·30c) supplemented by Eq. (5·35) or Eq. (5·36) is one of the most important equations in the gauge-invariant formalism for a multi-component system since the main aim of this formalism consists in finding the expression for the entropy perturbation in terms of the gauge-invariant variables pertaining to individual components. Especially it shows that the total entropy perturbation consists of two parts; a part coming from the intrinsic entropy perturbation of each component and a part coming from the difference of the dynamical behavior of components. One important interpretation of the quantity $S_{\alpha\beta}$, the main ingredient of the latter part, is obtained from the consideration on the following special case: Let one component be radiation by which the cosmological entropy is dominated and let the universe be in the radiative equilibrium. Then we can neglect the intrinsic entropy of the matter other than radiation and $S_{\alpha\beta}$ becomes

$$S_{\alpha r}=\frac{\delta n_\alpha}{n_\alpha}-\frac{\delta s}{s}=\frac{\delta(n_\alpha/s)}{(n_\alpha/s)}\,,\qquad(\alpha\neq r) \tag{5·38}$$

where the suffix r stands for radiation, s is the entropy density of radiation (or of the universe by assumption), and q_α is assumed to vanish. Cosmologically this is precisely what we call the isothermal perturbation. Thus $S_{\alpha\beta}$ turns out to be the relevant variable to describe the time-evolution of isothermal perturbations.

The gauge transformation properties of ε_α and f_α are obtained from that of $\tilde{Q}_{(\alpha)\mu}$ as an $(n+1)$-vector:

$$\bar{\tilde{Q}}_{(\alpha)\mu}(\eta,\,x^j)=\frac{\partial x^\nu}{\partial\bar{x}^\mu}\tilde{Q}_{(\alpha)\nu}(\eta-T{\rm Y},\,x^j-L{\rm Y}^j)$$

$$=\tilde{Q}_{(\alpha)\mu}+Q_{(\alpha)\nu}\delta x^\nu{}_{,\mu}-Q_{(\alpha)\mu,\nu}\delta x^\nu\,. \tag{5·39}$$

The result is

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$$\bar{\varepsilon}_a = \varepsilon_a - T\frac{Q_a'}{Q_a}, \tag{5·40a}$$

$$\bar{f}_a = f_a. \tag{5·40b}$$

Comparing these equations with Eqs. (3·3) and (5·24), one finds the following gauge-invariant combinations:

$$E_{ca} \equiv \varepsilon_a - \frac{Q_a'}{kQ_a}(v - B), \tag{5·41a}$$

$$F_{ca} \equiv f_a. \tag{5·41b}$$

As in the case of the density perturbation the following three other combinations are also possible as natural gauge-invariant representatives for ε_a:

$$\begin{aligned} E_a &\equiv \varepsilon_a - \frac{Q_a'}{kQ_a}(v_a - B) \\ &= E_{ca} - \frac{Q_a'}{kQ_a}(V_a - V), \end{aligned} \tag{5·42a}$$

$$E_{sa} \equiv \varepsilon_a - \frac{Q_a'}{kQ_a}\sigma_g, \tag{5·42b}$$

$$E_{ga} \equiv \varepsilon_a - \left(\frac{a'}{a}\right)^{-1}\frac{Q_a'}{Q_a}\mathcal{R}. \tag{5·42c}$$

Similarly we can find other simple gauge-invariant combinations representing the gauge-invariant momentum transfer rate. In particular the combination

$$F_a \equiv F_{ca} - nq_a(V_a - V) \tag{5·43}$$

is convenient for the later use. This quantity naturally arises if we use $\tilde{u}^\mu{}_{(a)}$ instead of $\tilde{u}^\mu$ in Eq. (5·18). The constraints (5·23) are written in terms of E_a and F_a as

$$\sum_a Q_a E_{ca} = \sum_a Q_a E_a + \frac{1}{k}\sum_a Q_a' V_a = 0, \tag{5·44a}$$

$$\sum_a h_a F_{ca} = 0. \tag{5·44b}$$

The gauge-invariant evolution equations for Δ_a, V_a and so on are obtained from the first-order part of the perturbed equation of motion for each component (5·1). Inserting the expressions for $T_{(a)}{}^\mu{}_\nu$, $\delta T_{(a)}{}^\mu{}_\nu$, $\Gamma^\mu{}_{\mu\lambda}$ and $\delta\Gamma^\mu{}_{\nu\lambda}$ into the general formula

$$\begin{aligned} \delta(\tilde{T}_{(a)}{}^\nu{}_{\mu;\nu}) &= \delta T_{(a)}{}^\nu{}_{\mu,\nu} + \delta\Gamma^\nu{}_{\nu\lambda}T_{(a)}{}^\lambda{}_\mu - \Gamma^\nu{}_{\nu\lambda}\delta T_{(a)}{}^\lambda{}_\mu \\ &\quad - \delta\Gamma^\lambda{}_{\mu\nu}T_{(a)}{}^\nu{}_\lambda - \Gamma^\lambda{}_{\mu\nu}\delta T_{(a)}{}^\nu{}_\lambda \\ &= 0, \end{aligned} \tag{5·45}$$

we find the following equations expressed in terms of the gauge-dependent perturbation variables:

$\delta(\tilde{T}_{(a)0}{}^\nu{}_{;\nu}) = \delta Q_{(a)0}$:

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$$(\rho_a\delta_a)'+n\frac{a'}{a}\rho_a\delta_a+h_a(kv_a+nH_L')+n\frac{a'}{a}p_a\pi_{La}=a(\epsilon_a-A)Q_a\,. \tag{5·46a}$$

$\delta(T_{(a)j}{}^{\nu}{}_{;\nu})=\delta Q_{(a)j}$:

$$[h_a(v_a-B)]'+(n+1)\frac{a'}{a}h_a(v_a-B)-kp_a\pi_{La}-kh_aA$$
$$+\frac{n-1}{n}\frac{k^2-nK}{k}p_a\pi_{Ta}=a\left[Q_a(v-B)+\frac{\dot a}{a}h_af_a\right]. \tag{5·46b}$$

From the definition of the gauge-invariant variables these equations are written as

$$(\rho_a\Delta_{ga})'+n\frac{a'}{a}\rho_a\Delta_{ga}+h_akV_a+n\frac{a'}{a}p_a\Gamma_a+n\frac{a'}{a}c_a{}^2\rho_a\Delta_{ga}$$
$$=aE_{sa}Q_a+a\mathcal{A}Q_a-a\left(\frac{a'}{a}\right)'Q_a'\Phi\,, \tag{5·47a}$$

$$[h_a(V_a-\mathcal{B})]'+(n+1)\frac{a'}{a}h_a(V_a-\mathcal{B})-kh_a\mathcal{A}-kp_a\left[\Gamma_a+\frac{c_a{}^2}{w_a}\Delta_{ga}\right]$$
$$+\frac{n-1}{n}\frac{k^2-nK}{k}p_a\Pi_a+a\mathcal{B}Q_a=a(F_{ca}+Q_aV). \tag{5·47b}$$

After a short calculation using Eqs. (3·7) and (5·7), Eq. (5·47b) is simplified as

$$V_a'+\frac{a'}{a}V_a=k\Psi+k\left(\frac{c_a{}^2}{1+w_a}\Delta_a+\frac{w_a}{1+w_a}\Gamma_a\right)$$
$$-\frac{n-1}{n}k\left(1-\frac{nK}{k^2}\right)\frac{w_a}{1+w_a}\Pi_a+\frac{a'}{a}F_a\,. \tag{5·48a}$$

In later applications we often find it more convenient to rewrite Eq. (5·47a) as the equation for Δ_a. It is easily done by expressing Δ_{ga} in terms of Δ_a and eliminating V_a' with the aid of Eq. (5·48a):

$$(\rho_a\Delta_a)'+n\frac{a'}{a}\rho_a\Delta_a=\frac{n}{n-1}\kappa^2ha^2h_a\frac{1}{k}(V-V_a)$$
$$-\left(1-\frac{nK}{k^2}\right)\left[h_akV_a+(n-1)\frac{a'}{a}p_a\Pi_a\right]+aQ_aE_a+\frac{a'}{a}h_aF_a\,. \tag{5·48b}$$

Equations (5·48) have the same structure as the corresponding equations for the total matter (4·7a) and (4·7b)′ except for the existence of the new source terms E_a, F_a and the term proportional to the difference of α-component velocity and the mean velocity. Among these new terms, E_a and F_a represent the perturbation in the energy transfer rate and the momentum transfer rate between the components due to interactions, respectively. They are expressed in terms of Δ_a and V_a if the interactions are specified. The term proportional to the velocity difference, namely the first term on the right-hand side of Eq. (5·48b), represents the change in the energy density of α-component due to additional work done upon the spacetime arising from the part of perturbation in the expansion rate of space provoked by the other components.

Although Eqs. (5·48) and Eq. (4·7) with Eqs. (5·30) and (5·36) yield a complete system of gauge-invariant dynamical equations for a multi-component system, the previ-

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ous argument on $\Gamma_{\rm rel}$ suggests that it is in some cases more convenient to write down the dynamical equations in terms of $S_{\alpha\beta}$ instead of Δ_α. First taking the difference of Eq. (4·7b)′ and Eq. (5·48a) and using the relation

$$\frac{\Delta_{c\alpha}}{1+w_\alpha}=\frac{\Delta}{1+w}+\sum_\gamma\frac{h_\gamma}{h}S_{\alpha\gamma}\,, \tag{5·49}$$

we obtain

$$\begin{aligned}(V_\alpha-V)'+\frac{a'}{a}(V_\alpha-V)&=k(c_\alpha^2-c_s^2)\frac{\rho\Delta}{h}+k\sum_\gamma\frac{h_\gamma}{h}c_\alpha^2S_{\alpha\gamma}\\&\quad-k\frac{p}{h}\Gamma_{\rm rel}+k\sum_\gamma\frac{h_\gamma}{h}\left[\Gamma_{\alpha\gamma}-\frac{n-1}{n}\left(1-\frac{nK}{k^2}\right)\Pi_{\alpha\gamma}\right]\\&\quad+\frac{a'}{a}F_\alpha+n\frac{a'}{a}c_\alpha^2(1-q_\alpha)(V_\alpha-V),\end{aligned} \tag{5·50}$$

where

$$\Gamma_{\alpha\beta}\equiv\frac{w_\alpha}{1+w_\alpha}\Gamma_\alpha-\frac{w_\beta}{1+w_\beta}\Gamma_\beta\,, \tag{5·51a}$$

$$\Pi_{\alpha\beta}\equiv\frac{w_\alpha}{1+w_\alpha}\Pi_\alpha-\frac{w_\beta}{1+w_\beta}\Pi_\beta\,. \tag{5·51b}$$

From Eqs. (5·48b) and (5·50) it follows after an elaborate calculation that

$$\begin{aligned}\left(\frac{\Delta_{c\alpha}}{1+w_\alpha}\right)'&=-kV_\alpha+n\frac{a}{k}KV+n\frac{a'}{a}\{(1-q_\alpha)c_s^2-q_\alpha(1+c_\alpha^2)\}\frac{\Delta}{1+w}\\&\quad+n\frac{a'}{a}\frac{w}{1+w}\Gamma_{\rm rel}-n\frac{a'}{a}q_\alpha(1+c_\alpha^2)\sum_\gamma\frac{h_\gamma}{h}S_{\gamma\alpha}\\&\quad-n\frac{a'}{a}\frac{w_\alpha}{1+w_\alpha}\Gamma_\alpha+n\frac{a'}{a}(1-q_\alpha)\frac{w}{1+w}\left[\Gamma_{\rm int}-\frac{n-1}{n}\left(1-\frac{nK}{k^2}\right)\Pi\right]\\&\quad+n\frac{a'}{a}q_\alpha E_{c\alpha}\,.\end{aligned} \tag{5·52}$$

Hence we obtain

$$\begin{aligned}&S_{\alpha\beta}'+n\frac{a'}{a}\frac{1}{2}\{q_\alpha(1+c_\alpha^2)+q_\beta(1+c_\beta^2)\}S_{\alpha\beta}\\&\quad+n\frac{a'}{a}[q_\alpha(1+c_\alpha^2)-q_\beta(1+c_\beta^2)]\sum_{\gamma\neq\alpha,\beta}\frac{h_\gamma}{h}\frac{1}{2}(S_{\alpha\gamma}+S_{\beta\gamma})\\&=-kV_{\alpha\beta}-n\frac{a'}{a}\Gamma_{\alpha\beta}+n\frac{a'}{a}E_{\alpha\beta}\\&\quad-n\frac{a'}{a}[(q_\alpha-q_\beta)c_s^2-q_\alpha(1+c_\alpha^2)+q_\beta(1+c_\beta^2)]\frac{\Delta}{1+w}\\&\quad-n\frac{a'}{a}(q_\alpha-q_\beta)\frac{w}{1+w}\left[\Gamma_{\rm int}-\frac{n-1}{n}\left(1-\frac{nK}{k^2}\right)\Pi\right],\end{aligned} \tag{5·53}$$

where

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$$V_{\alpha\beta} \equiv V_\alpha - V_\beta\,, \tag{5·54}$$

$$E_{\alpha\beta} \equiv q_\alpha E_\alpha - q_\beta E_\beta\,. \tag{5·55}$$

The structure of Eq. (5·53) suggests that it is more convenient to treat $V_{\alpha\beta}$ as the fundamental variable instead of V_α when we study the evolution of $S_{\alpha\beta}$. The dynamical equation for $V_{\alpha\beta}$ is easily obtained from Eq. (5·48). Using the relation

$$V_\alpha = V + \sum_\gamma \frac{h_\gamma}{h} V_{\alpha\gamma}\,, \tag{5·56}$$

we obtain

$$\begin{aligned}
&V'_{\alpha\beta} + \frac{a'}{a}V_{\alpha\beta} - \frac{n}{2}[c_\alpha{}^2 + c_\beta{}^2 - q_\alpha(1+c_\alpha{}^2) - q_\beta(1+c_\beta{}^2)]\frac{a'}{a}V_{\alpha\beta} \\
&\quad - n[c_\alpha{}^2 - c_\beta{}^2 - q_\alpha(1+c_\alpha{}^2) + q_\beta(1+c_\beta{}^2)]\frac{a'}{a}\sum_{\gamma\neq\alpha,\beta}\frac{h_\gamma}{h}\frac{1}{2}(V_{\alpha\gamma} + V_{\beta\gamma}) \\
&\quad = k(c_\alpha{}^2 - c_\beta{}^2)\frac{\Delta}{1+w} + k\frac{c_\alpha{}^2 + c_\beta{}^2}{2}S_{\alpha\beta} + k(c_\alpha{}^2 - c_\beta{}^2)\sum_\gamma \frac{h_\gamma}{h}\frac{1}{2}(S_{\alpha\gamma} + S_{\beta\gamma}) \\
&\quad\quad + \frac{a'}{a}F_{\alpha\beta} + k\left[\Gamma_{\alpha\beta} - \frac{n-1}{n}\left(1 - \frac{nK}{k^2}\right)\Pi_{\alpha\beta}\right],
\end{aligned} \tag{5·57}$$

where

$$F_{\alpha\beta} \equiv F_{c\alpha} - F_{c\beta}\,. \tag{5·58}$$

If we can neglect interactions among components, these equations become much simpler:

$$S'_{\alpha\beta} = -kV_{\alpha\beta} - n\frac{a'}{a}\Gamma_{\alpha\beta}\,, \tag{5·59}$$

$$\begin{aligned}
&V'_{\alpha\beta} + \left\{1 - \frac{n}{2}(c_\alpha{}^2 + c_\beta{}^2)\right\}\frac{a'}{a}V_{\alpha\beta} - n(c_\alpha{}^2 - c_\beta{}^2)\frac{a'}{a}\sum_{\gamma\neq\alpha,\beta}\frac{h_\gamma}{h}\frac{1}{2}(V_{\alpha\gamma} + V_{\beta\gamma}) \\
&\quad = k(c_\alpha{}^2 - c_\beta{}^2)\frac{\Delta}{1+w} + k\frac{c_\alpha{}^2 + c_\beta{}^2}{2}S_{\alpha\beta} \\
&\quad\quad + k(c_\alpha{}^2 - c_\beta{}^2)\sum_\gamma \frac{h_\gamma}{h}\frac{1}{2}(S_{\alpha\gamma} + S_{\beta\gamma}).
\end{aligned} \tag{5·60}$$

As a direct consequence of these equations we can prove that the concept of adiabatic perturbation is consistent with the dynamical equations only when all the components have the same sound velocity [Kodama (1983a)]. In fact if there are two components with different sound velocities (let them be α and β), the equation for Γ shows that both $\Gamma_{\rm int}$ and $S_{\alpha\beta}$ should vanish in general in order for the perturbation to be adiabatic. However, in that case, it follows from Eq. (5·59) that $V_{\alpha\beta}$ should vanish, which in turn implies that $(c_\alpha{}^2 - c_\beta{}^2)\Delta/(1+w)$ should vanish from Eq. (5·60). This leads to contradiction. The above consideration at the same time implies that isothermal perturbations (or entropy

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perturbations in general) and adiabatic perturbations (or total density perturbations in general) are coupled with each other and either one can be generated from the other in general. This problem will be discussed in detail in Chapter V.

(2) Vector perturbations

The extension of the gauge-invariant formalism for vector perturbations to a multi-component system is done in quite the same way as for scalar perturbations. The proper energy density and the $(n+1)$-velocity of each component are defined by Eq. (5·12) again. They are expressed in terms of gauge-dependent perturbation variables as

$$\tilde{u}_{(\alpha)}{}^0 = a^{-1}, \tag{5·61a}$$

$$\tilde{u}_{(\alpha)}{}^j = a^{-1} v_\alpha{}^{(1)} \mathrm{Y}^{(1)j}, \tag{5·61b}$$

$$\tilde{u}_{(\alpha)0} = -a, \tag{5·62a}$$

$$\tilde{u}_{(\alpha)j} = a(v_\alpha{}^{(1)} - B^{(1)}) \mathrm{Y}_j{}^{(1)}, \tag{5·62b}$$

$$\tilde{\rho}_\alpha = \rho_\alpha . \tag{5·63}$$

The perturbed energy-momentum tensor of α-component is expressed as

$$\tilde{T}_{(\alpha)}{}^0{}_0 = -\rho_\alpha, \tag{5·64a}$$

$$\tilde{T}_{(\alpha)}{}^0{}_j = h_\alpha (v_\alpha{}^{(1)} - B^{(1)}) \mathrm{Y}_j{}^{(1)}, \tag{5·64b}$$

$$\tilde{T}_{(\alpha)}{}^j{}_0 = -h_\alpha v_\alpha{}^{(1)} \mathrm{Y}_j{}^{(1)}, \tag{5·64c}$$

$$\tilde{T}_{(\alpha)}{}^i{}_j = p_\alpha [\delta^i{}_j + \pi_{T\alpha}{}^{(1)} \mathrm{Y}^{(1)i}{}_j]. \tag{5·64d}$$

Now the source term $\tilde{Q}_{(\alpha)\mu}$ is perturbed only in its spatial components:

$$\tilde{Q}_\alpha = Q_\alpha, \tag{5·65a}$$

$$\tilde{f}_{(\alpha)j} = (a'/a) h_\alpha f_\alpha{}^{(1)} \mathrm{Y}_j{}^{(1)}. \tag{5·65b}$$

The relations of the perturbed variables for each component to those for the total matter are given by

$$h v^{(1)} = \sum_\alpha h_\alpha v_\alpha{}^{(1)}, \tag{5·66a}$$

$$p \pi_T{}^{(1)} = \sum_\alpha p_\alpha \pi_{T\alpha}^{(1)}, \tag{5·66b}$$

and the constraint (5·3) is expressed as

$$\sum_\alpha h_\alpha f_\alpha{}^{(1)} = 0. \tag{5·67}$$

The gauge transformation properties of the perturbation variables are given by

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$$\bar{v}_a^{(1)}=v_a^{(1)}+L^{(1)\prime}, \tag{5·68a}$$

$$\bar{\pi}_{Ta}^{(1)}=\pi_{Ta}^{(1)}, \tag{5·68b}$$

$$\bar{f}_a^{(1)}=f_a^{(1)}. \tag{5·68c}$$

Following Eqs. (3·57)~(3·59), we define the gauge-invariant variables as

$$V_{sa}^{(1)}\equiv v_a^{(1)}-k^{-1}H_T^{(1)\prime}, \tag{5·69a}$$

$$V_a^{(1)}\equiv v_a^{(1)}-B^{(1)}$$

$$=V_{sa}^{(1)}+\sigma_g^{(1)}, \tag{5·69b}$$

$$\Pi_a^{(1)}\equiv\pi_{Ta}^{(1)}, \tag{5·69c}$$

$$F_a^{(1)}\equiv f_a^{(1)}. \tag{5·69d}$$

The relations (5·66) and (5·67) are written in terms of these variables as

$$hV_s^{(1)}=\sum_a h_a V_{sa}^{(1)}, \tag{5·70a}$$

$$hV^{(1)}=\sum_a h_a V_a^{(1)}, \tag{5·70b}$$

$$p\Pi^{(1)}=\sum_a p_a \Pi_a^{(1)}, \tag{5·70c}$$

$$\sum_a h_a F_a^{(1)}=0. \tag{5·71}$$

The Einstein equations yield one gauge-invariant equation of each component for a vector perturbation:

$\delta(T_{(a)j}{}^{\nu}{}_{;\nu})=\delta Q_{(a)j}$:

$$(h_a V_a^{(1)})'+(n+1)\frac{a'}{a}h_a V_a^{(1)}+\frac{1}{2k}\{k^2-(n-1)K\}p_a\Pi_a^{(1)}$$

$$=\frac{a'}{a}h_a(F_a^{(1)}+q_a V_a^{(1)}). \tag{5·72}$$

With the aid of Eq. (5·7) this equation can be written as

$$V_a^{(1)\prime}+\{1-nc_a^2(1-q_a)\}\frac{a'}{a}V_a^{(1)}$$

$$=-\frac{1}{2k}\{k^2-(n-1)K\}\frac{w_a}{1+w_a}\Pi_a^{(1)}+\frac{a'}{a}F_a^{(1)}. \tag{5·73}$$

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Equation (5·73) has the same structure as the corresponding equation for the total matter except for the appearance of q_α and the source term $F_\alpha^{(1)}$.

(3) Tensor perturbations

The extension to a multi-component system is trivial for tensor perturbations, since the matter is coupled to the metric perturbation only through the anisotropic stress perturbation. The gauge-invariant anisotropic stress perturbation of α-component is defined by

$$\Pi_\alpha^{(2)} \equiv \pi_{T\alpha}^{(2)}, \tag{5·74}$$

which obviously satisfies the relation

$$p\Pi^{(2)} = \sum_\alpha p_\alpha \Pi_\alpha^{(2)}. \tag{5·75}$$

Hence the gauge-invariant evolution equation for a tensor perturbation is the same as that given by Eq. (4·15) except that the relation (5·75) should be respected additionally.

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Chapter III

Relation to Gauge-Dependent Methods

§ III-1. Typical gauge conditions

In the actual application of the perturbation theory it often becomes necessary to fix the gauge in order to set the initial condition of the perturbation variables or to interpret the results obtained from the analysis of the evolution equation and compare them with observational data. For example, when one discusses the formation of galaxies, clusters of galaxies, and even larger scale structures of the universe in the light of the gravitational instability theory, one must follow the evolution of density irregularities from the initial linear-fluctuation stage to the late non-linear stage. In the early stage density irregularities generally have small amplitudes but their scales are much larger than the cosmic horizon size. Hence the relativistic linear perturbation theory is appropriate to study their evolution. In contrast, in the late stage, the density irregularities become very pronounced and the complete non-linear treatment taking account of various dissipation processes are necessary to study their evolution. However, their scales are generally much smaller than the horizon size, and the analysis based on Newtonian theory is more appropriate. Hence it becomes necessary to express the results of the analysis of the early stage in a gauge well suited for the later Newtonian treatment. Another example is the junction of perturbation variables at some singular space-like hypersurface which may appear associated with transition phenomena in the course of the cosmic evolution. In such cases it is obviously appropriate to work in the gauge such that the time-coordinate is constant on the singular surface.

There exists no gauge in which evolution equations of perturbations become simpler than the gauge-invariant equations. Hence it is most appropriate to work in the gauge-invariant formalism to study the temporal evolution of perturbations. Therefore what to be done in relating the gauge-invariant formalism to various gauge-dependent methods is to express the fundamental variables used in various gauge-dependent methods in terms of the gauge-invariant variables using their definitions and some of the gauge-invariant equations. Since this job is trivial for vector and tensor perturbations for which the gauge-invariant variables are much the same as the original gauge-dependent variables, we only consider scalar perturbations in this section.

In order to specify a gauge, we must impose two relations among the gauge-dependent variables; one for fixing the time-coordinate and one for the space-coordinates. The gauge condition for the time-coordinate, namely, the choice of time slicing of the perturbed spacetime, is given by imposing a constraint on one of the gauge-dependent variables whose change under the gauge transformation

$$\bar{\eta}=\eta+TY\,, \tag{1·1a}$$

$$\bar{x}^i=x^i+LY^i\,, \tag{1·1b}$$

is expressed only in terms of T. Typical examples of such variables are A, $v-B$, $\mathfrak{R}$, $\mathcal{K}_g$,

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σ_g and $\mathcal{K}_m$. In fact from Eqs. (II-3·3) and (II-3·28), these variables transform as

$$\bar{A}=A-T'-\frac{a'}{a}T\,, \qquad (1\cdot 2)$$

$$\bar{v}-\bar{B}=(v-B)-kT\,, \qquad (1\cdot 3)$$

$$\bar{\mathcal{R}}=\mathcal{R}-\frac{a'}{a}T\,, \qquad (1\cdot 4)$$

$$\bar{\mathcal{K}}_g=\mathcal{K}_g+\left(\frac{a'}{a}\right)^{-1}\left\{\left(\frac{a'}{a}\right)^2-\left(\frac{a'}{a}\right)'+\frac{k^2}{n}\right\}T\,, \qquad (1\cdot 5)$$

$$\bar{\sigma}_g=\sigma_g-kT\,, \qquad (1\cdot 6)$$

$$\bar{\mathcal{K}}_m=\mathcal{K}_m+\left(\frac{a'}{a}\right)^{-1}\left\{\left(\frac{a'}{a}\right)^2-\left(\frac{a'}{a}\right)'\right\}T\,. \qquad (1\cdot 7)$$

The simplest way to specify the time slicing is to require one of these quantities to vanish. Except for setting $A=0$, such condition completely eliminates the gauge ambiguity in δ as well as in the quantities listed above. For each time slicing the standard way to eliminate the spatial coordinate gauge freedom is to require a quantity whose gauge transformation involves L to vanish. Typical examples of such quantities are B, v, H_L and H_T. In the following we derive the equations relating the gauge-invariant variables and gauge-dependent ones for several typical gauge conditions.

(1) Proper-time slicing: $A=0$

The condition $A=0$ implies that the proper-time distance between two neighboring hypersurfaces along the normal vector coincides with the coordinate-time distance defining these hypersurfaces. As seen from Eq. (1·1) this condition does not completely specify the time slicing and leaves a gauge freedom parametrized by one arbitrary constant α,

$$T=\alpha a^{-1}\,. \qquad (1\cdot 8)$$

As a consequence an unphysical mode called a gauge mode appears in the density contrast δ, which is given by

$$c(1+w)\frac{a'}{a}\frac{1}{a}\,, \qquad (1\cdot 9)$$

where c is an arbitrary constant. To see this in more detail, let us express δ in terms of gauge-invariant variables. From the definitions of $\varDelta$ and V we have

$$\delta=\varDelta-n(1+w)\frac{a'}{a}V-n(1+w)\frac{a'}{a}\sigma_g\,. \qquad (1\cdot 10)$$

Since Eq. (II-3·7) is now written as

$$\sigma_g{}'+\frac{a'}{a}\sigma_g=-\varPsi\,, \qquad (1\cdot 11)$$

σ_g is expressed as

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$$\sigma_g = -\frac{1}{a}\int_{\eta_*}^{\eta} a\Psi d\eta - \frac{c_1}{a}, \tag{1·12}$$

where η_* denotes some reference time and c_1 is an integration constant. Hence with the aid of Eqs. (II-4·3) and (II-4·4), δ is expressed as

$$\delta = \Delta - n(1+w)\frac{a'}{a}\frac{1}{k}V - \frac{n(n-2)}{n-1}\frac{1+w}{k^2-nK}\frac{a'}{a}\frac{1}{a}\int_{\eta_*}^{\eta}\chi^2 a^3 \rho\Delta d\eta$$
$$-\frac{n(1+w)}{k^2}\frac{a'}{a}\frac{1}{a}\int_{\eta_*}^{\eta}\chi^2 a^3 p\Pi d\eta + n(1+w)\frac{a'}{a}\frac{c_1}{a}. \tag{1·13}$$

The last term represents the gauge mode, where c_1 transforms under the transformation (1·8) as

$$\bar{c}_1 = c_1 + \alpha. \tag{1·14}$$

Note that c_1 also changes its value when we change the reference time η_*.

(1a) Synchronous gauge: $A=B=0$

Among the gauge conditions belonging to the proper-time slicing, the most commonly used is the synchronous gauge [see, e.g., Weinberg (1972); Peebles (1980)], in which the space-coordinates are specified by the condition that the lines of constant space-coordinates are orthogonal to the constant time hypersurfaces. From Eq. (II-3·3b) one sees that the condition $B=0$ does not completely eliminate the gauge freedom of the space-coordinates either, but leaves a freedom given by

$$L = -ka\int\frac{d\eta}{a} + \beta, \tag{1·15}$$

where α is the constant appeared in Eq. (1·8) and β is an independent arbitrary constant. Since the synchronous gauge is quite frequently used, we discuss it in more detail than the other gauge conditions. Especially we derive the familiar perturbation equations in the synchronous gauge from the gauge-invariant equations. It reveals the correspondence between the perturbation equations in the two methods, though it is easier to derive them directly from the perturbed Einstein equations written in terms of the original gauge-dependent variables.

In the synchronous gauge the variables h_L, H_T, δ, v, Γ and Π are usually adopted as the fundamental variables where

$$h_L \equiv 2nH_L. \tag{1·16}$$

First we derive the perturbation equations written in terms of these variables. Substituting the expression (II-3·44) into Eq. (II-4·7b)′ and noting that Ψ is now expressed as

$$\Psi = -\frac{1}{k^2}\left(H_T'' + \frac{a'}{a}H_T'\right), \tag{1·17}$$

we obtain

$$v' + (1-nc_s^2)\frac{a'}{a}v = k\frac{c_s^2}{1+w}\delta + k\frac{w}{1+w}\left[\Gamma - \frac{n-1}{n}\left(1-\frac{nK}{k^2}\right)\Pi\right]. \tag{1·18}$$

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Similarly substituting Eqs. (II-3·52) and (II-3·44) to Eq. (II-4·7a), we obtain

$$\delta' + n(c_s{}^2 - w)\frac{a'}{a}\delta = (1+w)\Big[-k^2 + \frac{n}{n-1}(1+w)\varkappa^2\rho a^2\Big]\frac{v}{k}$$
$$+\Big(1-\frac{nK}{k^2}\Big)(1+w)H_T' - nw\frac{a'}{a}\Gamma\,. \tag{1·19}$$

In order to express H_T' by h_L, we use Eqs. (II-3·4) and (II-3·5) which are now written as

$$\mathcal{A} = -\Big(\frac{a'}{a}\Big)^{-1}\mathcal{R}' + \Big(\frac{a'}{a}\Big)^{-2}\Big\{\Big(\frac{a'}{a}\Big)' - \Big(\frac{a'}{a}\Big)^2\Big\}\mathcal{R}\,, \tag{1·20}$$

$$\mathcal{B} = k\Big(\frac{a'}{a}\Big)^{-1}\mathcal{R} - \frac{1}{k}H_T'\,, \tag{1·21}$$

where

$$\mathcal{R} = \frac{1}{2n}h_L + \frac{1}{n}H_T\,. \tag{1·22}$$

Then substituting Eqs. (1·19) and (1·20) into Eq. (II-4·2b), we obtain

$$\frac{1}{2}h_L' + \Big(1-\frac{nK}{k^2}\Big)H_T' = -\frac{n}{n-1}\varkappa^2 ha^2\frac{v}{k}\,. \tag{1·23}$$

Using this equation to eliminate H_T' from Eq. (1·19), we find

$$\delta' + n(c_s{}^2 - w)\frac{a'}{a}\delta = -(1+w)\Big(kv - \frac{1}{2}h_L'\Big) - nw\frac{a'}{a}\Gamma\,. \tag{1·24}$$

While, Eqs. (II-4·3) and (II-4·4) with Eq. (1·17) yield the time-evolution equation for H_T:

$$H_T'' + \frac{a'}{a}H_T' = \frac{n-2}{n-1}\frac{\varkappa^2\rho a^2}{1-nK/k^2}\Big[\delta + n(1+w)\frac{a'}{a}\frac{v}{k}\Big] + \varkappa^2 pa^2\Pi\,. \tag{1·25}$$

With the aid of Eqs. (1·23) and (1·18), Eq. (1·25) can be changed to the corresponding equation for h_L:

$$h_L'' + \frac{a'}{a}h_L' = -2\frac{n-2+nc_s{}^2}{n-1}\varkappa^2\rho a^2\delta - \frac{2n}{n-1}\varkappa^2 pa^2\Gamma\,. \tag{1·26}$$

The four variables δ, v, h_L and H_T are not dynamically independent. For example, it follows from Eqs. (1·21), (1·22), (II-3·6) and (II-4·3) that

$$H_T = \frac{n}{n-1}\frac{\varkappa^2\rho a^2}{k^2 - nK}\delta - \frac{1}{2}\frac{k^2}{k^2-nK}\frac{a}{a'}h_L' - \frac{1}{2}h_L\,. \tag{1·27}$$

Similarly h_L can be expressed in terms of δ, v, H_T and H_T'. Hence we can consider Eqs. (1·18), (1·24) and (1·26) as the fundamental perturbation equations in the synchronous gauge. Reflecting the existence of the residual gauge freedom parametrized by two arbitrary constants, these perturbation equations constitute a system of differential equations of fourth-order effectively. It is obvious that the analysis of the gauge-invariant equations which are of second-order is much easier than these fourth-order equations.

It is easy to find the expressions for the variables in the synchronous gauge in terms

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of the gauge-invariant variables. First note that Eq. (1·18) is written as

$$v'+\frac{a'}{a}v=\frac{c_s^2}{1+w}k\Delta+k\frac{w}{1+w}\left[\Gamma-\frac{n-1}{n}\left(1-\frac{nK}{k^2}\right)\Pi\right]. \tag{1·28}$$

Integrating this equation we obtain the expression for v:

$$v=\frac{k}{a}\int_{\eta_*}^{\eta}\frac{c_s^2}{1+w}a\Delta d\eta+\frac{k}{a}\int_{\eta_*}^{\eta}\frac{w}{1+w}a\left[\Gamma-\frac{n-1}{n}\left(1-\frac{nK}{k^2}\right)\Pi\right]d\eta-c_1\frac{k}{a}, \tag{1·29}$$

where c_1 is the same integration constant as appeared in Eq. (1·13). The expression for δ is obtained from the definition of Δ as

$$\delta=\Delta-n(1+w)\frac{a'}{a}\frac{1}{k}v, \tag{1·30}$$

which has been given already in Eq. (1·13), and the expression for H_T from the definition of V as

$$H_T=-k\int_{\eta_*}^{\eta}Vd\eta+k\int_{\eta_*}^{\eta}vd\eta+kc_2, \tag{1·31}$$

where c_2 is another integration constant. Finally from Eqs. (1·21) and (1·22) we obtain the expression for h_L:

$$h_L=\frac{2n}{n-1}\frac{\chi^2\rho a^2}{k^2-nK}\Delta+\frac{2n}{k}\frac{a'}{a}(v-V)-2H_T, \tag{1·32}$$

where H_T is given by Eq. (1·31). The appearance of the arbitrary constant c_2 is a result of the residual gauge freedom associated with space-coordinates. The constant c_2 transforms under the transformation (1·8) and (1·15) as

$$\bar{c}_2=c_2+\beta. \tag{1·33}$$

(1b) Comoving proper-time gauge: $A=v=0$

Another frequently used gauge condition on the space-coordinates is the comoving condition $v=0$. This condition restricts the residual gauge freedom to

$$L=\beta, \tag{1·34}$$

where β is an arbitrary constant. The expressions for the various gauge-dependent variables in terms of Δ, V, Γ and Π are given as follows. First by rewriting the definition of V we obtain

$$B=-V-\sigma_g, \tag{1·35}$$

$$H_T=-k\int_{\eta_*}^{\eta}Vd\eta+kc_3, \tag{1·36}$$

where σ_g is given by Eq. (1·12) and c_3 is an integration constant which transforms under the gauge transformation (1·34) as

$$\bar{c}_3=c_3+\beta. \tag{1·37}$$

Next it follows from Eq. (II-3·6) that

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$$\mathfrak{R}=\Phi+\frac{1}{k}\frac{a'}{a}\sigma_g\,. \tag{1·38}$$

Hence H_L is expressed as

$$H_L=\mathfrak{R}-\frac{1}{n}H_T=\frac{\chi^2\rho a^2}{(n-1)(k^2-nK)}\varDelta+\frac{1}{k}\frac{a'}{a}\sigma_g+\frac{k}{n}\int_{\eta_*}^{\eta}Vd\eta-\frac{kc_3}{n}\,. \tag{1·39}$$

(2) Velocity-orthogonal slicing: $v=B$

As noted in §II-3, $v-B$ represents the deviation of the matter velocity from the vector normal to the constant time hypersurfaces. Hence this slicing is such that the matter $(n+1)$-velocity is orthogonal to the constant time hypersurfaces. As is easily seen from Eq. (1·3) the condition $v=B$ completely eliminates the gauge freedom associated with the time slicing. Especially δ coincides with $\varDelta$ in this slicing:

$$\delta=\varDelta\,. \tag{1·40}$$

Hence the fundamental equation in this gauge is given by the same second-order differential equation as Eq. (II-4·9). From Eqs. (II-3·7), (II-3·43) and (II-4·7b), A is expressed as

$$A=-\frac{1}{1+w}[c_s{}^2\varDelta+w\Gamma]+\frac{n-1}{n}\left(1-\frac{nK}{k^2}\right)\frac{w}{1+w}\Pi\,. \tag{1·41}$$

(2a) Comoving time-orthogonal gauge: $v=B=0$

The residual gauge freedom in this gauge is also expressed by Eq. (1·34). The expressions for the gauge-dependent variables are obtained as follows. First from Eq. (II-3·44) it follows that

$$H_T'=-kV\,, \tag{1·42}$$

hence

$$H_T=-k\int_{\eta_*}^{\eta}Vd\eta+kc_3\,, \tag{1·43}$$

where c_3 is an integration constant and transforms following Eq. (1·37). Now Eq. (II-3·6) can be written as

$$\mathfrak{R}=\Phi-\frac{1}{k}\frac{a'}{a}V\,. \tag{1·44}$$

Hence H_L is expressed as

$$H_L=\frac{\chi^2\rho a^2}{(n-1)(k^2-nK)}\varDelta-\frac{1}{k}\frac{a'}{a}V+\frac{k}{n}\int_{\eta_*}^{\eta}Vd\eta-\frac{kc_3}{n}\,. \tag{1·45}$$

(2b) Velocity-orthogonal isotropic gauge: $v=B$, $H_T=0$

In this gauge there is no residual gauge freedom. Furthermore since v now coincides with V from the definition, the formalism in this gauge is closest to the gauge-invariant formalism. From Eq. (II-3·6) H_L is expressed as

$$\begin{aligned}H_L&=\Phi-\frac{1}{k}\frac{a'}{a}V\\&=\frac{\chi^2\rho a^2}{(n-1)(k^2-nK)}\varDelta-\frac{1}{k}\frac{a'}{a}V\,.\end{aligned} \tag{1·46}$$

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(3) Newtonian slicing: $\sigma_g=(1/k)H_T'-B=0$

From the meaning of σ_g the perturbation in the expansion rate is isotropic in this slicing. This slicing completely eliminates the gauge freedom T. From the definitions of Δ and V, δ is now expressed as

$$\delta=\Delta-n(1+w)\frac{a'}{a}\frac{1}{k}V\,, \tag{1·47}$$

and A coincides with Ψ,

$$A=\Psi=-\frac{n-2}{n-1}\frac{\chi^2\rho a^2}{k^2-nK}\Delta-\frac{\chi^2 p a^2}{k^2}\Pi\,. \tag{1·48}$$

In particular when $\Pi=0$ and $K=0$, Eq. (1·48) takes the same form as the harmonic expansion of the Poisson equation in Newtonian theory except that Δ contains V in addition to δ. The appearance of the velocity in the source term of the Poisson equation is a relativistic effect and its effect becomes negligible for perturbations on scales much smaller than the horizon size $a/\dot{a}$ and the corresponding matter velocity is smaller than the Hubble expansion velocity, since for such perturbations

$$\frac{a'}{a}\frac{V}{k}=\left(\frac{a}{k}\frac{\dot{a}}{a}\right)^2\frac{a}{k}\frac{kV}{a}\Big/\left(\frac{a}{k}\frac{\dot{a}}{a}\right)<\left[\frac{a}{k}\Big/\left(\frac{\dot{a}}{a}\right)\right]^2\ll 1\,. \tag{1·49}$$

(3a) Longitudinal gauge: $B=H_T'=0$

The residual gauge freedom in this gauge is expressed by Eq. (1·34), which is reflected in the appearance of an arbitrary constant c_3 in the expression for H_T:

$$H_T=kc_3\,. \tag{1·50}$$

The transformation law of c_3 is the same as Eq. (1·37). The expressions for the remaining variables are

$$v=V\,, \tag{1·51}$$

$$H_L=\Phi-\frac{1}{n}H_T=\frac{\chi^2\rho a^2}{(n-1)(k^2-nK)}\Delta-\frac{kc_3}{n}\,. \tag{1·52}$$

(3b) Comoving Newtonian gauge: $B=(1/k)H_T'$, $v=0$

The residual gauge freedom is the same as that of (3a). The expressions for H_T and H_L are given by Eqs. (1·43) and (1·45).

(4) Uniform Hubble slicing: $\mathcal{K}_g=-A+(a'/a)^{-1}H_L'+(1/n)(a'/a)^{-1}kB=0$

In this slicing the perturbation in the volume expansion rate of constant time hypersurfaces vanishes. There is no residual gauge freedom in the time-coordinate. The expressions for A and δ in this gauge are obtained as follows. First note that from the condition $\mathcal{K}_g=0$, $\mathcal{A}$ and $\mathcal{B}$ are written as

$$\mathcal{A}=-\left(\frac{a'}{a}\right)^{-1}\frac{k}{n}\sigma_g+\left(\frac{a'}{a}\right)^{-2}\left[\left(\frac{a'}{a}\right)'-\left(\frac{a'}{a}\right)^2\right]\mathfrak{R}\,, \tag{1·53}$$

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$$\mathcal{B}=\left(\frac{a'}{a}\right)\frac{1}{k}\mathcal{R}-\sigma_g\,. \tag{1·54}$$

Substituting these expressions into Eq. (II-4·2a), we obtain

$$\mathcal{R}=\frac{\chi^2\rho a^2}{(n-1)(k^2-nK)}\delta\,. \tag{1·55}$$

Hence noting that Δ is expressed as

$$\Delta=\delta+n(1+w)\frac{a'}{a}\frac{1}{k}V+n(1+w)\frac{a'}{a}\frac{1}{k}\sigma_g\,, \tag{1·56}$$

and using the relation of $\mathcal{B}$ and Δ, we obtain the following expressions:

$$\delta=\Delta-\left[1+\frac{n}{n-1}\frac{\chi^2 ha^2}{k^2-nK}\right]^{-1}n(1+w)\frac{a'}{a}\frac{1}{k}V\,, \tag{1·57}$$

$$\sigma_g=-\left[1+\frac{n}{n-1}\frac{\chi^2 ha^2}{k^2-nK}\right]^{-1}\frac{n}{n-1}\frac{\chi^2 ha^2}{k^2-nK}V\,. \tag{1·58}$$

The expression for A is obtained from $\mathcal{K}_g=0$ as

$$A=-\frac{1}{n}\left(\frac{a'}{a}\right)^{-1}k\sigma_g+\left(\frac{a'}{a}\right)^{-1}\mathcal{R}'\,, \tag{1·59}$$

where $\mathcal{R}$ is given by Eq. (1·55) with Eq. (1·57) and σ_g by Eq. (1·58).

(4a) Time-orthogonal uniform Hubble gauge: $B=\mathcal{K}_g=0$

The residual gauge freedom in this gauge is again given by Eq. (1·34). The expressions for the remaining gauge-dependent variables are given by

$$H_T=k\int_{\eta_*}^{\eta}\sigma_g d\eta+kc_3\,, \tag{1·60}$$

$$H_L=\mathcal{R}-\frac{1}{n}H_T\,, \tag{1·61}$$

$$v=V+\sigma_g\,. \tag{1·62}$$

(4b) Comoving uniform Hubble gauge: $v=\mathcal{K}_g=0$

The residual gauge freedom is the same as in (4a). The expression for B is given by

$$B=-\sigma_g+\frac{1}{k}H_T{}'=-V-\sigma_g\,. \tag{1·63}$$

The expressions for H_T and H_L are given by Eqs. (1·43) and (1·45), respectively.

§ III-2. Temporal behavior of density perturbations in simple cases

As noted in the paragraph following Eq. (II-4·9), it is useless to try to solve the perturbation equation exactly for general cases, since the variables Γ and Π are usually related to Δ or V, which can be determined only after one specifies rather detailed properties of the matter. However, the situation becomes very simple when the matter can be regarded as a single perfect fluid. In such cases one is allowed to regard Γ and

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Π simply as sources of (adiabatic) density perturbations which are independent of Δ and V. Furthermore, if the equation of state of the fluid is so simple that $w=c_s^2=$const holds and if the background spatial curvature can be neglected (i.e., $K=0$), Eq. (II-4·9) becomes exactly solvable.

In this section, we investigate the temporal behavior of density perturbations in such simple cases. First we rewrite Eq. (II-4·9) in the relevant form including the sources Γ and Π. Then we give the exact analytic solutions of it without sources and discuss their properties. We also consider the case when the background equation of state changes suddenly at a certain epoch and derive the junction condition on perturbation amplitudes. However, here we neither discuss the generation of density perturbations due to Γ and Π nor consider the case when the background equation of state changes gradually. These will be discussed in Chapter IV.

For the usual spatial dimension of $n=3$, the general solutions of the perturbation equation in the case $w=c_s^2=$const and $K=0$ were obtained by Sakai (1969) using a gauge-dependent method and their properties were discussed in detail by Bardeen (1980) in the gauge-invariant language. Since the analysis done by Bardeen is fairly complete, the following is essentially a repetition of his arguments but in a more compact form.

In order to solve Eq. (II-4·9), it is more convenient to regard $\rho a^n\Delta$ as the dynamical variable instead of Δ itself. Then assuming $K=0$, Eq. (II-4·9) is rewritten as

$$\begin{aligned}(\rho a^n\Delta)''+(1+nc_s^2)\frac{a'}{a}(\rho a^n\Delta)'+\left\{k^2c_s^2-\frac{n(n-2)}{2}(1+w)\left(\frac{a'}{a}\right)^2\right\}(\rho a^n\Delta)\\ =-k^2w\rho a^n\Gamma+(n-1)\left[\frac{k^2}{n}w+\{n(c_s^2+w^2)-2w\}\left(\frac{a'}{a}\right)^2\right]\rho a^n\Pi\\ -(n-1)w\frac{a'}{a}\rho a^n\Pi'\,. \qquad (2\cdot1)\end{aligned}$$

In the case $w=$const and $K=0$, the background equations have the well-known power-law solution given by

$$a=a_0\left(\frac{\eta}{\eta_0}\right)^\beta\,,\qquad \rho=\frac{n(n-1)}{2\chi^2}\beta^2\frac{1}{a^2\eta^2}\,, \qquad (2\cdot2\mathrm{a})$$

where

$$\beta=\frac{2}{nw+n-2}\,, \qquad (2\cdot2\mathrm{b})$$

and a_0 is the value of a at some reference time $\eta=\eta_0$. Now defining a new independent variable

$$x\equiv k\eta\,, \qquad (2\cdot3)$$

Eq. (2·1) takes the form

$$\left[\frac{d^2}{dx^2}+\frac{2-(n-3)\beta}{x}\frac{d}{dx}+c_s^2-(n-2)\frac{\beta(\beta+1)}{x^2}\right]f$$

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$$=wx^{\beta(n-2)-2}\left[-\Gamma+(n-1)\left\{\left(\frac{1}{n}+2(n-1)\frac{\beta^2}{x^2}\right)\Pi-\frac{\beta}{x}\frac{d}{dx}\Pi\right\}\right], \tag{2·4}$$

where f is defined by

$$f\equiv x^{\beta(n-2)-2}\Delta\propto\rho a^n\Delta\,. \tag{2·5}$$

Note that x represents roughly the ratio of the horizon size to the wavelength of perturbation, since $ka/a'=x/\beta$.

With $\Gamma=\Pi=0$, the general solution of Eq. (2·4) is expressed by Bessel functions. In terms of the spherical Bessel functions of order ν, it is

$$f=x^{\nu-\beta}\{C_0 j_\nu(c_s x)+D_0 n_\nu(c_s x)\}, \tag{2·6a}$$

where ν is given by

$$\nu=\frac{n-1}{2}\beta=\frac{n-1}{nw+n-2}, \tag{2·6b}$$

and C_0 and D_0 are arbitrary constants. For the usual case of $n=3$, we have $\nu=\beta$ and the solution (2·6a) reduces to the one obtained by Bardeen. Although the case $n=3$ is physically most important, let us proceed our analysis keeping n arbitrary so that dimension-independent properties of the solution may be revealed. Now, the gauge-invariant amplitudes Δ, Φ and V corresponding to the solution (2·6a) are determined by Eq. (2·5), Eqs. (II-4·3) and (II-4·7b) with $\Gamma=\Pi=K=0$. They are

$$\Delta=x^{2-\nu}\mathcal{Z}_\nu(c_s x), \tag{2·7a}$$

$$\Phi=\frac{n\beta^2}{2}x^{-\nu}\mathcal{Z}_\nu(c_s x), \tag{2·7b}$$

$$V=\frac{n\beta}{2}x^{1-\nu}\left\{\mathcal{Z}_\nu(c_s x)-\frac{c_s x}{\beta+1}\mathcal{Z}_{\nu-1}(c_s x)\right\}, \tag{2·7c}$$

where

$$\mathcal{Z}_\nu(c_s x)\equiv C_0 j_\nu(c_s x)+D_0 n_\nu(c_s x). \tag{2·7d}$$

Up to now, we have not imposed any particular condition on the value of $w(=c_s^2)$. However, if the matter is a usual fluid, the value of w should lie in the range $0\le w\le 1$ which ensures the sound velocity to be real and causal, $0\le c_s\le 1$. Although it is occasionally necessary to consider cases with $w<0$ (especially the case when $w\simeq -1$ which corresponds to a de Sitter-like universe), generally the assumption $c_s^2=w=\text{const}$ breaks down and a more intricate treatment is required in such cases. As an example of the case $w\simeq -1$, the perturbation of the so-called inflationary universe will be discussed in §§VI-2 and 3. Therefore, in the rest of this section, we assume that $0\le w\le 1$ unless otherwise stated. Then from Eqs. (2·2b) and (2·6b), β and ν take the values in the respective ranges of

$$\frac{1}{n-1}\le\beta\le\frac{2}{n-2}, \tag{2·8a}$$

$$\frac{1}{2}\le\nu\le\frac{n-1}{n-2}, \tag{2·8b}$$

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where the left equalities hold for $w=1$ and the right ones for $w=0$.

Let us first consider the behavior of the general solution at $c_s x \gg 1$. In this case the acoustic nature of the solution is apparent. However, one finds that the amplitudes of Δ and V are generally not constant with time; both of them change in proportion to $x^{1-\nu}$. Thus for $\nu<1$ $(w>1/n)$ they grow and for $\nu<1$ $(w<1/n)$ they decay. The critical case $\nu=1$ of constant amplitudes is realized for an isotropic relativistic fluid $(w=1/n)$. On the other hand, the amplitude of Φ is small by a factor of order x^{-2} relative to Δ and V. This is true as long as $x \gg 1$ even if $c_s x<1$, which consequently implies that Newtonian theory is applicable for perturbations with wavelength smaller than the horizon size. In fact, Φ corresponds to the Newtonian gravitational potential exactly since it represents the spatial curvature perturbation of the Newtonian hypersurface by definition [see Eq. (II-3·23)].

Contrary to the case $c_s x \gg 1$, the general solution has power-law behavior at $c_s x \ll 1$,

$$f \simeq Cx^{2\nu-\beta}+Dx^{-(\beta+1)}, \tag{2·9}$$

where, for convenience, the numerical factors in front of C_0 and D_0 in the original expression are absorbed into the new constants C and D defined by

$$C \equiv \frac{\sqrt{\pi}\, c_s{}^{\nu}}{2^{\nu+1}\Gamma(\nu+3/2)} C_0, \tag{2·10a}$$

$$D \equiv -\frac{2^{\nu}\Gamma(\nu+1/2)}{\sqrt{\pi}\, c_s{}^{\nu+1}} D_0. \tag{2·10b}$$

Correspondingly, the amplitudes Δ, Φ and V are given by

$$\Delta \simeq Cx^2+Dx^{-(2\nu-1)}, \tag{2·11a}$$

$$\Phi \simeq \frac{n}{2}\beta^2(C+Dx^{-(2\nu+1)}), \tag{2·11b}$$

$$V \simeq \frac{n}{2}\beta\left(-\frac{(n-2)\beta}{\beta+1}Cx+Dx^{-2\nu}\right). \tag{2·11c}$$

In terms of the scale factor given by Eq. (2·2a), the above solutions are expressed as

$$\Delta \simeq Cx_0{}^2\left(\frac{a}{a_0}\right)^{2/\beta}+Dx_0{}^{-(2\nu-1)}\left(\frac{a}{a_0}\right)^{-(n-1)+1/\beta}, \tag{2·12a}$$

$$\Phi \simeq \frac{n}{2}\beta^2\left(C+Dx_0{}^{-(2\nu+1)}\left(\frac{a}{a_0}\right)^{-(n-1)-1/\beta}\right), \tag{2·12b}$$

$$V \simeq \frac{n}{2}\beta\left(-\frac{(n-2)\beta}{\beta+1}Cx_0\left(\frac{a}{a_0}\right)^{1/\beta}+Dx_0{}^{-2\nu}\left(\frac{a}{a_0}\right)^{-(n-1)}\right), \tag{2·12c}$$

where $x_0=k\eta_0$. As we have seen in the previous section, the amplitudes of perturbation depend on the choice of hypersurfaces on which they are defined. As we shall see soon, this dependence becomes particularly strong at $x \ll 1$. However, by referring to the behavior of Δ which represents the density perturbation amplitude on velocity-orthogonal hypersurfaces, the first mode proportional to C is called the growing mode and the second one proportional to D the decaying mode in general.

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The growing mode is generally regarded as representing the gravitational instability (the usual Jeans instability). Although the notion of gravitational instability is rather unclear at $x<1$ due to the strong hypersurface-dependence of the perturbation amplitudes, it becomes meaningful at $x>1$ where the hypersurface-dependence disappears. On the other hand, Eqs. (2·11) are applicable up to $c_s x \simeq 1$. Therefore the notion of gravitational instability is relevant only for a fluid with $c_s \ll 1$ in the strict sense. Assuming $c_s \ll 1$, it is easy to find the critical wavenumber below which the gravitational instability sets in. By going back to Eq. (II-4·9) and neglecting all the terms proportional to w or c_s^2 except for the c_s^2 in front of k^2 and writing down the dispersion relation by assuming a'/a to be constant with time (which can be justified if the time interval under consideration is much smaller than the expansion time), the critical wavenumber is found to be simply given by setting the whole coefficient in front of Δ equal to zero,

$$\frac{k_c^2}{a^2}=\frac{n(n-2)}{2c_s^2}\left(\frac{a'}{a^2}\right)^2=\frac{(n-2)\chi^2\rho}{(n-1)c_s^2}\,. \tag{2·13}$$

This is the well-known Jeans wavenumber for a non-relativistic fluid. Note that this result may be obtained also from Eq. (2·1) or Eq. (2·4) by setting the coefficient in front of $\rho a^n \Delta$ or f in respective equations equal to zero, since the time-derivative of ρa^n is proportional to $w(\ll 1)$. However in the case when c_s^2 is not negligibly small, the above procedure becomes invalid and the definition of the critical wavenumber becomes ambiguous. This is a reflection of the fact that the notion of gravitational instability ceases to be relevant for a fluid with $c_s \lesssim 1$. Nevertheless, one may define a critical wavenumber somehow or other as a criterion for the gravitational instability of a given perturbation. The simplest possibility will be the one defined by setting the coefficient in front of $\rho a^n \Delta$ in Eq. (2·1) equal to zero. This yields

$$\frac{k_c^2}{a^2}=\frac{(n-2)\chi^2 h}{(n-1)c_s^2}\,. \tag{2·14}$$

From Eq. (2·4) the corresponding value of x is

$$x_c^2=\frac{(n-2)\beta(\beta+1)}{c_s^2}\,. \tag{2·15}$$

That this definition for the critically stable wavenumber is fairly reasonable can be understood by noting the behavior of the general solution (2·6a); x_c is approximately the value above which the spherical Bessel function has the oscillatory behavior and below which it has the power-law behavior. Thus x_c can be regarded as the definition of the sound horizon within which the perturbation oscillates acoustically.

Now, having seen the temporal behavior of the general solution, we come to the question that in what case the perturbation may be regarded as small when the wavelength exceeds the horizon size; in other words, among the gauge-invariant amplitudes including Δ, Φ and V, which actually represents the true amplitude of the perturbation at $x \ll 1$, if ever? For example, if it were Δ the linear perturbation theory would be valid as long as $C \ll x^{-2}$ and $D \ll x^{2\nu-1}$ even if Φ or V were large compared with unity. In order to settle this problem, we proceed as follows: We first find an appropriate minimal set of gauge-dependent quantities whose smallness guarantees the validity of the linear approximation. Then we find a gauge in which these quantities can be made as small as

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possible. This in turn leads to conditions to be satisfied by C and D, namely, gauge-invariant conditions for the validity of the linear perturbation approximation.

As for geometrical quantities, the candidates of the representative gauge-dependent quantities are the amplitudes of the intrinsic curvature perturbation $\mathfrak{R}$, the lapse function perturbation A, the shift vector perturbation B and the perturbation in the expansion rate $\mathcal{K}_g$, the shear $\tilde{\sigma}_{ij}$ and the acceleration $\tilde{a}_j$ of the normal vector field $\tilde{N}^\mu$. Among them the last two quantities should be normalized by the background expansion rate to represent their perturbation amplitudes. Therefore we define

$$\sigma_g{}^* \equiv \left(n\frac{\dot{a}}{a}\right)^{-1}(\tilde{\sigma}_{ij}\tilde{\sigma}^{ij})^{1/2}/(\mathrm{Y}_{ij}\mathrm{Y}^{ij})^{1/2} = \frac{x}{n\beta}\sigma_g\,, \tag{2·16a}$$

$$a_g \equiv \left(n\frac{\dot{a}}{a}\right)^{-1}(\tilde{a}_j\tilde{a}^j)^{1/2}/(\mathrm{Y}_j\mathrm{Y}^j)^{1/2} = \frac{x}{n\beta}A\,. \tag{2·16b}$$

We note also that the amplitude representing the intrinsic curvature perturbation is not really $\mathfrak{R}$ itself but the ratio of $k^2/a^2\mathfrak{R}$ to the characteristic background curvature scale $\sim(\dot{a}/a)^2$, similar to the cases of the shear and the acceleration. Hence, for example,

$$\mathfrak{R}^* \equiv \left(\frac{\dot{a}}{a}\right)^{-2}\frac{k^2}{a^2}\mathfrak{R} = \frac{x^2}{\beta^2}\mathfrak{R}\,, \tag{2·17}$$

represents the amplitude of the intrinsic curvature perturbation. However, $\mathfrak{R}$ does represent the amplitude of the metric perturbation as directly seen from Eq. (II-3·14). Hence not only $\mathfrak{R}^*$ but also $\mathfrak{R}$ should be small if the linear perturbation approximation should be valid. Here note that the condition $|\mathfrak{R}|\ll 1$ is sufficient to guarantee that both $|H_T|$ and $|H_L|$ are smaller than unity, because one finds it is always possible to choose a spatial gauge in which $|H_T|\ll 1$ by noting that H_T is affected only by purely spatial gauge transformations as seen from Eq. (II-3·3d). Then, because a_g and $\mathfrak{R}^*$ are much smaller than A and $\mathfrak{R}$, respectively, for $x\ll 1$, we do not have to worry about them and what to be respected are the amplitudes $\mathfrak{R}$, A, B, $\mathcal{K}_g$ and $\sigma_g{}^*$. Among these, all but B are independent of spatial gauge transformations. Then, noting the relation $B = dH_T/dx - n\beta x^{-1}\sigma_g{}^*$, we can choose a spatial gauge in which $dH_T/dx \simeq n\beta x^{-1}\sigma_g{}^*$, hence $H_T = O(\sigma_g{}^*)$ and $|B|\ll|\sigma_g{}^*|$ without affecting the values of the rest. Furthermore, since not all of $\mathfrak{R}$, A, $\mathcal{K}_g$ and $\sigma_g{}^*$ are independent of each other but by Eq. (II-3·27) they are related as

$$\mathcal{K}_g + A + \sigma_g{}^* - \frac{x}{\beta}\frac{d}{dx}\mathfrak{R} = 0\,, \tag{2·18}$$

if any three of them are small, the remaing one must also be small. Hence for convenience, we choose $\mathfrak{R}$, $\mathcal{K}_g$ and $\sigma_g{}^*$ as independent quantities. Finally, from the definitions of Φ and Ψ, Eqs. (II-3·23) and (II-3·24), respectively, and the Einstein equation (II-4·4) with $\Pi=0$, we obtain

$$\mathfrak{R} = \Phi + \frac{n\beta^2}{x^2}\sigma_g{}^*\,, \tag{2·19a}$$

$$\mathcal{K}_g = \left[(n-2) + \frac{x}{\beta}\frac{d}{dx}\right]\Phi - \frac{x^2 + n\beta(\beta+1)}{x^2}\sigma_g{}^*\,. \tag{2·19b}$$

These are the fundamental relations among the representative geometrical am-

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plitudes. It turns out that the amplitudes of perturbation in all components of the Christoffel symbols and the Riemann tensor can be made as small as of order $\mathcal{R}$, $\mathcal{K}_g$ and/or $\sigma_g{}^*$. This is seen from inspecting the expressions for $\delta\Gamma^\alpha{}_{\mu\nu}$ and $\delta R^\alpha{}_{\beta\mu\nu}$ given in Appendix D; one easily finds that their amplitudes, properly normalized by their background values ($\Gamma^\alpha{}_{\mu\nu}=O(a'/a)$ and $R^\alpha{}_{\beta\mu\nu}=O((a'/a)'))$, are at most of the order of the metric perturbation amplitudes A, B, H_L and/or H_T, to make which comparable to or smaller than $\mathcal{R}$,$\mathcal{K}_g$ and/or $\sigma_g{}^*$ in magnitude has been just shown to be possible.

As for the matter variables, the amplitudes of interest are the density perturbation δ, the velocity perturbation v and the perturbation in the expansion rate $\mathcal{K}_m$, the shear $\sigma_m{}^*$ and the acceleration a_m of the velocity field $\tilde{u}^\mu$, where the amplitudes $\sigma_m{}^*$ and a_m are the properly normalized ones similar to $\sigma_g{}^*$ and a_g defined by Eqs. (2·16). First, using the definitions of σ_m^*, $\sigma_g{}^*$ and $\varDelta$, the Einstein equation (II-4·3) with $K=0$ which relates $\varDelta$ to $\varPhi$, and Eqs. (2·19), δ and v are found to be expressed as

$$\delta=2\left(\mathcal{K}_g+\frac{x^2}{n\beta^2}\mathcal{R}\right), \tag{2·20a}$$

$$v=B-\frac{n\beta}{x}(\sigma_m{}^*-\sigma_g{}^*). \tag{2·20b}$$

Also, from the definitions of $\mathcal{K}_m$ and a_m, we obtain

$$\mathcal{K}_m=\mathcal{K}_g-(\sigma_m{}^*-\sigma_g{}^*), \tag{2·21a}$$

$$a_m=a_g-x^{1-\beta}\frac{d}{dx}[x^{\beta-1}(\sigma_m{}^*-\sigma_g{}^*)]. \tag{2·21b}$$

While, since $\sigma_m{}^*=-xV/n$ and V can be expressed in terms of $\varPhi$, hence of $\mathcal{R}$ and $\sigma_g{}^*$ through Eq. (2·19a), we obtain

$$\sigma_m{}^*=\frac{1}{n\beta^2(\beta+1)}x^{3-(n-2)\beta}\frac{d}{dx}(x^{(n-2)\beta}\mathcal{R})-\frac{1}{\beta+1}x^{3-(n-2)\beta}\frac{d}{dx}(x^{(n-2)\beta-2}\sigma_g{}^*), \tag{2·22}$$

and by noting the relation $2\nu=(n-1)\beta$, $\sigma_m{}^*-\sigma_g{}^*$ is expressed as

$$\sigma_m{}^*-\sigma_g{}^*=\frac{1}{n\beta^2(\beta+1)}x^{3-(n-2)\beta}\frac{d}{dx}(x^{(n-2)\beta}\mathcal{R})-\frac{1}{\beta+1}x^{2-2\nu}\frac{d}{dx}(x^{2\nu-1}\sigma_g{}^*). \tag{2·23}$$

Now, Eq. (2·20a) implies that δ is small if $\mathcal{K}_g$ and $\mathcal{R}$ are small and Eq. (2·22) that $\sigma_m{}^*$ is small if $\mathcal{R}$ and $\sigma_g{}^*$ are small. Then Eq. (2·21a) implies that $\mathcal{K}_m$ is also small. Therefore what we must be concerned with are the amplitudes v and a_m. By using Eq. (2·23), they are explicitly expressed in terms of the geometrical amplitudes as

$$v=B-\frac{1}{\beta(\beta+1)}x^{2-(n-2)\beta}\frac{d}{dx}(x^{(n-2)\beta}\mathcal{R})+\frac{n\beta}{\beta+1}x^{1-2\nu}\frac{d}{dx}(x^{2\nu-1}\sigma_g{}^*), \tag{2·24a}$$

$$\begin{aligned} a_m=a_g&-\frac{1}{n\beta^2(\beta+1)}x^{1-\beta}\frac{d}{dx}\left[x^{2-(n-3)\beta}\frac{d}{dx}(x^{(n-2)\beta}\mathcal{R})\right] \\ &+\frac{1}{\beta+1}x^{1-\beta}\frac{d}{dx}\left[x^{\beta+1-2\nu}\frac{d}{dx}(x^{2\nu-1}\sigma_g{}^*)\right]. \end{aligned} \tag{2·24b}$$

Among the terms on the right-hand side of Eq. (2·24a), the first term B can be made arbitrarily small by a purely spatial gauge transformation and the second term is of order

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$x|\mathcal{R}|\ll|\mathcal{R}|$. Hence only the last term which is formally of order $x^{-1}\sigma_g{}^*$ may invalidate the linear approximation even if $|\sigma_g{}^*|\ll 1$. The same is true for the terms on the right-hand side of Eq. (2·24b), since a_g is of order xA and the second term is of order $x\mathcal{R}$, but the last term is formally of order $x^{-1}\sigma_g{}^*$. However, as we shall discover soon, these problematic terms can be made arbitrarily small by choosing a suitable gauge while keeping $\mathcal{R}$, $\mathcal{K}_g$ and $\sigma_g{}^*$ small enough to ensure the geometrical perturbation to be of the linear order.

Let us now derive the condition under which the amplitudes $\mathcal{R}$, $\mathcal{K}_g$ and $\sigma_g{}^*$ are simultaneously made to take minimum values. Since the growing mode and the decaying mode are linearly independent we can argue on them separately. First, consider the growing mode. From Eq. (2·11b), the leading terms of the growing mode in Φ are given by

$$\Phi\simeq C_1+C_2x^2\,, \tag{2·25}$$

where C_2 is a constant of order $c_s{}^2C_1$ and $C_1=n\beta^2C/2$. Then from Eqs. (2·19) we can easily convince ourselves that $\mathcal{R}$ and $\mathcal{K}_g$ cannot be made to be of order C_2x^2 simultaneously for any choice of $\sigma_g{}^*$ of order C_2x^2. Therefore either $\mathcal{R}$ or $\mathcal{K}_g$ must be of order C_1 while $\sigma_g{}^*$ can be made of order C_2x^2 or less. Incidentally, the problematic terms of v and a_m become of order $|C_2x|\ll|C|$ and cease to be problematic. Thus, the linear theory is found to be valid independent of x provided $|C|\ll 1$.

Next, consider the decaying mode. The leading terms of Φ in this case are

$$\Phi\simeq D_1x^{-(2\nu+1)}+D_2x^{-(2\nu-1)}\,, \tag{2·26}$$

where D_2 is of order $c_s{}^2D_1$ and $D_1=n\beta^2D/2$. Inserting this expression into Eqs. (2·19) gives

$$\mathcal{R}\simeq D_1x^{-(2\nu+1)}+n\beta^2x^{-2}\sigma_g{}^*+D_2x^{-(2\nu-1)}\,, \tag{2·27a}$$

$$\mathcal{K}_g\simeq-\frac{1+\beta}{\beta}(D_1x^{-(2\nu+1)}+n\beta^2x^{-2}\sigma_g{}^*)+\frac{1-\beta}{\beta}D_2x^{-(2\nu-1)}\,. \tag{2·27b}$$

Thus one finds that if one chooses a gauge in which

$$\sigma_g{}^*=-\frac{D_1}{n\beta^2}\{x^{-(2\nu-1)}+O(x^{-(2\nu-3)})\}, \tag{2·28}$$

$\mathcal{R}$ and $\mathcal{K}_g$ are simultaneously made to be of order $D_1x^{-(2\nu-1)}$ which is of the same order in magnitude as $\sigma_g{}^*$. In such a gauge, the leading term of $\sigma_g{}^*$ just vanishes when inserted into each expression for v and a_m and the contribution of $\sigma_g{}^*$ to them becomes of order $D_1x^{-(2\nu-2)}\sim x\sigma_g{}^*$, hence does not invalidate the linear approximation provided $|\sigma_g{}^*|\ll 1$. Therefore, we conclude that the linear perturbation theory is valid as long as $|D|\ll x^{2\nu-1}$ for a given x.

It is worthwhile to note that in terms of the combination of gauge-invariant amplitudes

$$\Phi-\frac{1}{k}\frac{a'}{a}V=\Phi-\frac{\beta}{x}V=\frac{n\beta^2c_s}{2(\beta+1)}x^{1-\nu}\mathcal{Z}_{\nu-1}(c_sx)$$

$$\simeq\frac{n\beta^2}{2(\beta+1)}\left\{(2\nu+1)C+\frac{c_s{}^2}{2\nu+1}Dx^{1-2\nu}\right\}, \tag{2·29}$$

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the conditions on C and D can be united to a single condition,

$$|\Phi-\frac{1}{k}\frac{a'}{a}V|\ll 1\,, \tag{2·30}$$

provided $c_s{}^2$ is not too small compared with unity. From Eq. (1·44), we find this combination is just $\mathcal{R}$ measured on the velocity-orthogonal hypersurface $v=B$. This suggests that $\mathcal{R}_{v=B}\equiv\Phi-a'(ka)^{-1}V$ is the measure of the linear perturbation amplitude in general, not only for the case $w=c_s{}^2$. As we shall see in Chapter VI, this statement is supported by the fact that the combination $\Phi-\Upsilon$, where $\Upsilon\equiv a'(ka)^{-1}V$, does represent the amplitude of the perturbation in the inflationary cosmological models. However, as $c_s{}^2$ approaches zero, $\mathcal{R}_{v=B}$ ceases to be a good representative for the decaying mode amplitude. This is related to a peculiar property of a dust fluid, which will be discussed in more detail in §3, especially in connection with the method developed by Olson (1976).

To summarize, the linear perturbation theory remains to be valid if the amplitudes C and D of the general solution satisfy the condition

$$|C|\ll 1\,,\qquad |D|\ll x^{2\nu-1}\,. \tag{2·31}$$

It should be mentioned, however, that the above analysis is not perfectly complete. Strictly speaking, the linear perturbation theory is valid only if higher-order non-linear terms can be shown to be negligible compared with the linear terms. That is, the above conclusion is correct only if the convergence of the perturbation expansion considered here is sufficiently good. To examine whether this is the case or not requires a complicated analysis on non-linear terms for which neither harmonic expansion nor distinction of scalar, vector and tensor perturbations retains their significance. Since such an analysis is beyond the scope of the present article, we merely assume the good convergence of the perturbation expansion here and leave it as a problem to be settled in the future.

Finally, let us discuss the case when the background equation of state undergoes a discrete change at a certain epoch. We note that this situation is not so unrealistic as one might think, since it has become a common belief among cosmologists recently that there were several epochs of phase transitions in the early universe at each of which the background equation of state changed rather drastically within a short time compared with the expansion time. Therefore, as a first approximation, it is reasonable to assume a discrete change in the equation of state to see its effects on the behavior of density perturbations.

Suppose that at $a=a_*$, the equation of state changed suddenly from $w=w_-$ to $w=w_+$. Then Eq. (2·1) or (2·4) cannot apply to this epoch directly since the derivative of f has a discrete jump. Therefore we must go back to the set of first-order differential equations (II-4·7) for Δ and V. There, we easily notice that the natural junction is to make Δ and V continuous at $a=a_*$. Physically, this corresponds to the case when the relaxation time of the fluid to the thermal equilibrium is short compared with the timescale of the change in the equation of state so that the adiabaticity is maintained during the phase transition, which is consistent with the assumption $\Gamma=\Pi=0$ throughout the stage under considera-

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tion.[†)] Once this condition is kept in mind, we may return to the general solutions for Δ and V given in Eqs. (2·7) and make them continuous at $a=a_*$. In order to write down the continuity conditions for Δ and V, we must note that the continuity of the energy density requires βx^{-1} to be continuous, which simply means that the ratio of the proper wavelength of perturbation to the horizon size $\sim a'/ka$ cannot have a discrete jump. Thus at first we must have

$$\beta_-/x_-=\beta_+/x_+\,, \tag{2·32}$$

where and in what follows the minus index to any quantity indicates its value just before $a=a_*$ and the plus index just after $a=a_*$. Then the conditions for Δ and V to be continuous are found to be

$$\beta_-^{2-\nu_-}\mathcal{Z}_{\nu_-}(c_{s-}x_-)=\beta_+^{2-\nu_+}\mathcal{Z}_{\nu_+}(c_{s+}x_+), \tag{2·33a}$$

$$\frac{\beta_-^{2-\nu_-}}{\beta_-+1}(c_{s-}x_-)\mathcal{Z}_{\nu_--1}(c_{s-}x_-)=\frac{\beta_+^{2-\nu_+}}{\beta_++1}(c_{s+}x_+)\mathcal{Z}_{\nu_+-1}(c_{s+}x_+), \tag{2·33b}$$

where

$$\mathcal{Z}_{\nu_\pm}(c_{s\pm}x_\pm)=C_{0\pm}j_{\nu_\pm}(c_{s\pm}x_\pm)+D_{0\pm}n_{\nu_\pm}(c_{s\pm}x_\pm), \tag{2·33c}$$

as given by Eq. (2·7d).

Although one could derive the general relation among the amplitudes $C_{0\pm}$ and $D_{0\pm}$, it would be too complicated to extract any physically significant information out of it. While, in the case $x_+\ll 1$, the junction conditions become much more tractable and this is practically the most important case since perturbations with wavelength larger than the horizon size are generally of interest in the early universe. Therefore, let us concentrate our attention on the case $x_+\ll 1$. In this case, we may use Eqs. (2·11) to derive the junction conditions for Δ and V. Noting Eq. (2·32), we find

$$C_-x_-^{\,2}+D_-x_-^{-(2\nu_--1)}=C_+x_+^{\,2}+D_+x_+^{-(2\nu_+-1)}\,, \tag{2·34a}$$

$$-\frac{(n-2)\beta_-}{\beta_-+1}C_-x_-^{\,2}+D_-x_-^{-(2\nu_--1)}=-\frac{(n-2)\beta_+}{\beta_++1}C_+x_+^{\,2}+D_+x_+^{-(2\nu_+-1)}\,. \tag{2·34b}$$

Now, it is easy to write down the expressions for C_+ and D_+ in terms of C_- and D_-. First, subtracting both hand sides of Eq. (2·34a) from those of Eq. (2·34b) immediately yields

$$C_+=\left(\frac{\beta_-}{\beta_+}\right)^2\frac{(1+\beta_+)\{1-(n-3)\beta_-\}}{(1+\beta_-)\{1-(n-3)\beta_+\}}C_-\,. \tag{2·35}$$

Thus the growing mode amplitude after the phase transition is entirely determined by that before the phase transition and there is no chance of generating a growing mode out of a decaying mode. As for the decaying mode amplitude, after straightforward manipulation of Eq. (2·34a) with the help of Eqs. (2·32) and (2·35) we obtain

$$D_+x_+^{-(2\nu_+-1)}=D_-x_-^{-(2\nu_--1)}+C_-x_-^{\,2}\frac{(n-2)(\beta_--\beta_+)}{(\beta_-+1)\{1-(n-3)\beta_+\}}\,, \tag{2·36}$$

where we have intentionally kept the explicit $x_\pm$-dependence associated with the am-

[†)] Rigorously speaking, Γ is non-zero during the phase transition since $c_s^{\,2}\neq w$. Nevertheless the argument remains valid since $c_s^{\,2}\Delta+w\Gamma=w\Delta$ for adiabatic perturbations even if w varies with time, provided that $\delta p=w\delta\rho$ on the velocity-orthogonal hypersurfaces.

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plitudes $D_{\pm}$, since the actual magnitude of the decaying mode is not D itself but $Dx^{-(2\nu-1)}$ as was shown a while ago. From Eq. (2·36), we find that a decaying mode can be generated from a purely growing mode but the resultant amplitude is much smaller than that of the corresponding growing mode, since $|C_-|x_-^2 \ll |C_-| = O(C_+)$.

Formally, one could apply the junction conditions given above to the case of $\beta_- \simeq -1$, for which the resultant perturbation amplitude after the phase transition would become very large. However, as noted before, a more careful analysis is necessary for such a case in general (see §§VI-2 and 3). Hence we do not discuss it here.

§ III-3. Resolution of typical gauge-dependent ambiguities

Until recently, most of work on cosmological density perturbations has been done in the synchronous gauge. However, as we have seen in §1 of this chapter, the perturbation equations in this gauge are fourth-order differential equations, which consequently allow the appearance of two extra gauge modes in a general solution. Of these two modes, the one that arises from the residual spatial gauge freedom represented by Eq. (1·15) is quite harmless since it only gives rise to additional constant terms to the metric perturbation amplitudes H_T and h_L. On the other hand, the other mode that arises from the residual temporal gauge freedom represented by Eq. (1·8) gives rise to the appearance of the corresponding gauge mode in the density perturbation amplitude δ given by Eq. (1·9) which behaves as t^{-1} irrespective of the equation of state where t is the comoving proper-time of the background Friedmann universe. The presence of this mode has been a source of confusion; whether one could attach any physical significance to it or not, though it was recognized already by Lifshitz and Khalatnikov (1963) in their pioneering work that this mode is certainly unphysical. A main reason of this confusion seems to be due to the accidental cancellation of the leading term in the decaying mode of δ. As can be derived easily from Eqs. (1·29) and (1·30) with $\Gamma=\Pi=0$ and Δ given by Eq. (2·11a), δ in a synchronous gauge is at $x \ll 1$,

$$\begin{aligned}\delta &= \Delta - nc_s^2\beta x^{-(\beta+1)}\int_{x_*}^{x} x'^{\beta}\Delta(x')dx' + c_1 x^{-(\beta+1)} \\ &= \frac{3+(1-nc_s^2)\beta}{3+\beta}Cx^2 + \frac{n(w-c_s^2)\beta}{\beta+2-2\nu}Dx^{-(2\nu-1)} + c_1' x^{-(\beta+1)}, \end{aligned} \tag{3·1}$$

where the factor arising from the lower bound of the integral (a reference time) in the middle of the first line has been absorbed into the coefficient c_1' of the last gauge term in the second line. We readily find that the leading term of the decaying mode vanishes since $w=c_s^2$ by assumption. This implies that the actual leading term of the decaying mode is of order $Dc_s^2x^{3-2\nu}$ (see Eq. (2·26)). Hence in particular for a dust fluid with $c_s^2=0$, the decaying mode does not make its appearance in the expression for δ. It turns out, however, that for $w=c_s^2=0$ one has $\beta+1=2\nu-1=n/(n-2)$ and the unphysical gauge mode happens to be identical in its time-dependence with the physical decaying mode which would be present if there occurred no cancellation. Actually this is related to the fact that when $w=c_s^2=0$ one can choose a gauge in which the coordinates are both synchronous and comoving. In the comoving time-orthogonal gauge, we have $\delta=\Delta$ and the decaying mode contribution to δ has in fact the time-dependence $x^{-(2\nu-1)}\propto t^{-1}$ for $w=c_s^2=0$. Thus the synchronous gauge is a quite improper one to study the temporal

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evolution of perturbations.

Another source of confusion resides in that too much emphasis has been placed upon the power-law "growth rate" of density perturbations on scales larger than the horizon size in many of former investigations. At the same time, the amplitude of density perturbations in the synchronous gauge, for example, has often been misinterpreted as representing the true amplitude of perturbations. As we have shown in the previous section, the physical amplitudes are represented by the coefficient C for the growing mode and by $Dx^{-(2\nu-1)}$ for the decaying mode but not the amplitude δ itself even if one successfully removed unphysical gauge modes from δ. Actually, the term, "a density perturbation" which is frequently used to represent a scalar perturbation is somewhat improper and misleading when the wavelength exceeds the horizon size. In any case, it should be clear by now that the physical amplitude of perturbations on scales greater than the horizon does not grow at all but stays constant as to the growing mode while it dies off as to the decaying mode, provided $\nu>1/2$ $(w<1)$.

In an attempt to remove unphysical gauge modes from the perturbation equations, Hawking (1966) proposed a coordinate-independent method of perturbation analysis based upon the Bianchi identities and Olson (1976) corrected an error in Hawking's paper and gave the perturbation equation in a closed second-order differential form explicitly. Unfortunately, there is a defect in Olson's method that the perturbation equation reduces to a first-order one in the case $w=c_s^2=0$, thus turns out to lack one physical mode. In addition, the density perturbation amplitude implied by his method still involves a gauge mode which, however he claims, becomes physical in the case $w=c_s^2=0$. Actually, this is the typical phenomenon for a dust matter as we have just seen in discussions on the synchronous gauge. Therefore, it is worthwhile to clarify the essence of Olson's method in terms of our gauge-invariant language and to resolve the ambiguity in his method when $w=c_s^2=0$.

To be specific, let us consider the usual spatial dimension of $n=3$. Olson assumes the $K=0$ background and introduces a quantity S which is related to the curvature perturbation on a hypersurface orthogonal to the matter four velocity $\tilde{u}^\mu$ as

$$SY\equiv\frac{\delta^s R}{2x^2\rho}=\frac{2k^2}{x^2\rho a^2}\mathcal{R}Y\,, \tag{3·2}$$

where $\mathcal{R}$ is to be measured on the velocity-orthogonal hypersurface $B=v$. Then from Eq. (1·44), S is given by

$$S=\frac{2k^2}{x^2\rho a^2}\left(\Phi-\frac{a'}{ka}V\right). \tag{3·3}$$

In the case $w=c_s^2=$const, we may use the results given in the previous section and S reduces to

$$S=\frac{x^{2-2\beta}}{\beta+1}\frac{d}{dx}(x^{\beta+1}f)=\frac{c_s x^{3-\beta}}{\beta+1}\{C_0 j_{\beta-1}(c_s x)+D_0 n_{\beta-1}(c_s x)\}, \tag{3·4}$$

which is the solution obtained by Olson. However, this solution does not seem to degenerate into a single mode in the limit $c_s\to 0$ at a first glance. The crucial point is that

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taking the limit $c_s \to 0$ is a singular procedure and does not commute with differentiation of $x^{\beta+1}f$ with respect to x in the strict sense. In order to make the limit $c_s \to 0$ regular, one must use the renormalized amplitudes C and D defined by Eqs. (2·10) instead of C_0 and D_0. Then

$$S=\frac{x^{3-\beta}}{\beta+1}\left\{\frac{2^{\beta+1}\Gamma(\beta+3/2)}{\sqrt{\pi}}Cc_s^{-(\beta-1)}j_{\beta-1}(c_s x)-\frac{\sqrt{\pi}c_s^2 D}{2^\beta\Gamma(\beta+1/2)}c_s^\beta n_{\beta-1}(c_s x)\right\}. \tag{3·5}$$

Since $c_s^{-(\beta-1)}j_{\beta-1}(c_s x)$ and $c_s^\beta n_{\beta-1}(c_s x)$ are both regular and independent of c_s in the limit $c_s \to 0$, it is apparent now that the decaying mode is the one which ceases its contribution to S due to the presence of the additional factor c_s^2.

Now S is related to the density perturbation through the energy component of the Einstein equations as

$$\tilde{\rho}=\rho(1+\delta Y)=\frac{1}{3\chi^2}\tilde{\theta}^2(1+SY), \tag{3·6a}$$

where $\tilde{\theta}$ is given by Eq. (II-3·40b), namely,

$$\tilde{\theta}=3\frac{a'}{a^2}(1+\mathcal{K}_m Y). \tag{3·6b}$$

Hence we have

$$\delta=2\mathcal{K}_m+S\,. \tag{3·7}$$

Olson then chooses the proper-time comoving gauge ($A=v=0$) to express δ in terms of S. In this gauge Eq. (3·7) takes the form

$$\delta=2\left(\frac{a'}{a}\right)^{-1}H_L'+S\,, \tag{3·8}$$

while the energy equation for δ is

$$\delta'+3\frac{a'}{a}(c_s^2-w)\delta+3(1+w)H_L'=0\,. \tag{3·9}$$

Therefore in the case $w=c_s^2=\text{const}$, Eqs. (3·8) and (3·9) are combined to yield

$$\frac{1}{(1+\beta)\eta^\beta}(\eta^{\beta+1}\delta)'=S\,, \tag{3·10}$$

where the background solution (2·2a) has been used. Using the relation between the proper-time and the conformal time, $dt=ad\eta$, Eq. (3·10) can be rewritten as

$$\frac{d}{dt}(t\delta)=S\,, \tag{3·11}$$

which correctly reproduces the result of Olson. Thus the gauge mode associated with δ given by Eq. (3·11) is t^{-1}. In fact, the origin of this gauge mode is the same as in the case of synchronous gauge, since it is entirely due to the hypersurface condition $A=0$. Therefore by the same argument we can easily show that the same ambiguity as occurred in the sychronous gauge arises in the limit $w=c_s^2=0$.

There has also been a certain degree of ambiguity concerning the generation of

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density perturbations due to pressure inhomogeneities. However, we do not discuss it in this section since we considered only the homogeneous solutions of the perturbation equation so far. Careful consideration on several examples of inhomogeneous solutions generated by pressure or entropy perturbations will be given in Chapters IV and V.

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Chapter IV

General Analysis of Density Perturbations

As stated in §I-1, the presence of large-scale structures in the present universe implies that there were density perturbations on scales much larger than the Hubble horizon size (defined by $(\dot{a}/a)^{-1}$) in the early universe and one of the most important problems in cosmology is to clarify the origin and the structure of these primordial density perturbations [see e.g., Kodama (1982a)]. A natural step to approach this problem at first is to investigate whether large-scale density perturbations can be generated in the course of the cosmic evolution or not and, if so, to estimate their amplitudes.

In this chapter we investigate the generation of density perturbations on super-horizon scales under circumstances as general as possible. We limit our consideration to universes dominated by the matter with normal equations of state, namely with non-negative pressure, in which a scale comoving with the cosmic expansion increases more slowly than the Hubble horizon size. As an extreme case of contrast, the behavior of density perturbations in an inflationary universe, in which there appears a stage where the cosmic expansion rate stays nearly constant and a comoving scale increases much faster than the Hubble horizon size, will be discussed in Chapter VI.

Since the Hubble horizon size represents the characteristic scale on which local processes can causally influence each other, density perturbations on super-horizon scales cannot be generated directly by transporting energy over such scales. Then the only possibility is that they are generated as a secondary product of some other perturbations provoked spontaneously by transient phenomena in the early universe. As is easily seen from the time-evolution equation for the gauge-invariant density perturbation obtained in §II-4, such secondary generation of density perturbations can occur only from stress perturbations. This possibility has already been studied by Press and Vishniac (1980) and by Bardeen (1980) in the simple case that the equation of state of the background cosmic matter (which we abbreviate B-EOS from now on) does not change. Here we extend their analysis to the case that the B-EOS changes while the stress perturbations are working. As the completely general argument is impossible, we consider two special cases; one in which the change in B-EOS is small but its temporal behavior is not specified, and one in which the change in B-EOS is large but its temporal behavior is special. The fundamental method we adopt is to treat stress perturbations as source independent of the density perturbation in the gauge-invariant time-evolution equation for the density perturbation and estimate the amplitudes of density perturbations generated after the stress perturbations vanish. We only consider the case in which the universe has the spatial dimension $n=3$ throughout this chapter. We also neglect the background spatial curvature and put $K=0$.

§ IV-1. Basic equation for density perturbations in a universe with weak transient phenomena

In this and next two sections we investigate the generation of density perturbations from stress perturbations in the case that the change in B-EOS while the stress perturba-

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tions are provoked is small and the B-EOS is exactly the same before and after this transient epoch [Kodama (1983b, 1984)].

The fundamental equation for the analysis is Eq. (II-4·9). For the present purpose it is more convenient to use variable ζ which is proportional to the scale factor a as the time variable instead of η:

$$\frac{d\zeta}{d\eta}=kHl\zeta\,, \tag{1·1}$$

where

$$H\equiv\dot{a}/a\,, \tag{1·2}$$

$$l\equiv a/k\,. \tag{1·3}$$

Note that Hl represents the ratio of the reduced wavelength of a perturbation to the Hubble horizon size:

$$Hl=\frac{\text{reduced wavelength}}{\text{Hubble horizon size}}\,. \tag{1·4}$$

We normalize ζ by the condition that $\zeta=1$ when $Hl=1$, namely when the perturbation comes within the horizon, in this and next two sections. In terms of ζ Eq. (II-4·9) is rewritten as

$$\frac{d^2\Delta}{d\zeta^2}-\frac{\mu}{\zeta}\frac{d\Delta}{d\zeta}+\left[-\frac{2+\nu}{\zeta^2}+\frac{c_s{}^2}{f^2}\right]\Delta=\mathscr{S}\,, \tag{1·5}$$

where

$$\mathscr{S}=-\frac{w}{f^2}\left(\Gamma-\frac{2}{3}\Pi\right)+\frac{2}{\zeta^2}(3w^2+3c_s{}^2-2w)\Pi-\frac{2w}{\zeta}\frac{d\Pi}{d\zeta}\,, \tag{1·5a}$$

$$\mu\equiv-\frac{5}{2}(1-3w)+1-3c_s{}^2\,, \tag{1·5b}$$

$$\nu\equiv-\frac{1}{2}(1-3w)(7-3w)+3(1-3c_s{}^2), \tag{1·5c}$$

$$f\equiv Hl\zeta\,. \tag{1·5d}$$

From Eqs. (A·9a)′, (A·10)′ and (1·1), f is expressed in terms of w as

$$f=f_*\exp\left[\frac{1}{2}\int_{\zeta_*}^{\zeta}(1-3w)\frac{d\zeta}{\zeta}\right], \tag{1·6}$$

where $f_*=f(\zeta_*)$ and ζ_* is a value of ζ at some reference time.

Let us consider the following situation. First until some time t_1 the B-EOS is described by the simple relation $w=c_s{}^2=\text{const}\ (\equiv\gamma)$ and there exists no perturbation at all ($\Delta=\Gamma=\Pi=0$). Then at some time after t_1 stress perturbations are provoked by some mechanism, which at the same time makes w and/or $c_s{}^2$ deviate from γ. The stress perturbations and the deviation of w and $c_s{}^2$ from γ continue to exist for a finite time and then vanish by a time t_2. After t_2 the B-EOS is again given by $w=c_s{}^2=\gamma$.

Since we assume that the strength of the stress perturbations and the deviation of

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B-EOS are small ($|\Gamma|\ll 1$, $|\Pi|\ll 1$, $|w-\gamma|\ll 1$ and $|c_s{}^2-\gamma|\ll 1$), the amplitudes of generated density perturbations can be estimated by an iterative method. For that purpose we rewrite Eq. (1·5) as

$$\frac{d^2\varDelta}{d\zeta^2}-\frac{\mu_0}{\zeta}\frac{d\varDelta}{d\zeta}-\frac{2+\nu_0}{\zeta^2}\varDelta=\mathcal{S}+\frac{\delta\mu}{\zeta}\frac{d\varDelta}{d\zeta}+\frac{\delta\nu-c_s{}^2(Hl)^{-2}}{\zeta^2}\varDelta\,, \tag{1·7}$$

where

$$\mu=\mu_0+\delta\mu\,;\qquad \mu_0\equiv-\frac{3}{2}(1-3\gamma), \tag{1·7a}$$

$$\nu=\nu_0+\delta\nu\,;\qquad \nu_0\equiv-\frac{1}{2}(1-3\gamma)^2\,. \tag{1·7b}$$

Equation (1·7) can be transformed into an integral equation with the aid of the Green function for the differential operator on the left-hand side of Eq. (1·7). The Green function $G(\zeta,\zeta')$ is given by

$$G(\zeta,\zeta')=[U_+(\zeta)U_-(\zeta')-U_-(\zeta)U_+(\zeta')]W(\zeta')^{-1}\theta(\zeta-\zeta'), \tag{1·8}$$

where $\theta(\zeta)$ is the Heaviside function and

$$U_\pm(\zeta)=\zeta^{\beta_\pm};\quad \beta_+\equiv 1+3\gamma\,,\quad \beta_-\equiv-\frac{3}{2}(1-\gamma), \tag{1·9}$$

$$W(\zeta)\equiv U_-\frac{dU_+}{d\zeta}-U_+\frac{dU_-}{d\zeta}=\frac{3\gamma+5}{2}\zeta^{-3(1-3\gamma)/2}\,. \tag{1·10}$$

Applying the Green function (1·8) to Eq. (1·7), we obtain for $\zeta>\zeta_1$,

$$\varDelta(\zeta)=\varDelta_0(\zeta)+L*\varDelta(\zeta), \tag{1·11}$$

where

$$\varDelta_0(\zeta)=P_0(\zeta)U_+(\zeta)+Q_0(\zeta)U_-(\zeta); \tag{1·12}$$

$$P_0(\zeta)=\int_{\zeta_1}^{\zeta}\frac{d\zeta'}{W(\zeta')}U_-(\zeta')\mathcal{S}(\zeta'), \tag{1·13}$$

$$Q_0(\zeta)=-\int_{\zeta_1}^{\zeta}\frac{d\zeta'}{W(\zeta')}U_+(\zeta')\mathcal{S}(\zeta') \tag{1·14}$$

and

$$\begin{aligned}L*\varDelta(\zeta)=\;&U_+(\zeta)\int_{\zeta_1}^{\zeta}\frac{d\zeta'}{W(\zeta')}U_-(\zeta')\left[\frac{\delta\mu}{\zeta}\frac{d\varDelta}{\delta\zeta}+\frac{\delta\nu-c_s{}^2(Hl)^{-2}}{\zeta^2}\varDelta\right]\\&-U_-(\zeta)\int_{\zeta_1}^{\zeta}\frac{d\zeta'}{W(\zeta')}U_+(\zeta')\left[\frac{\delta\mu}{\zeta}\frac{d\varDelta}{d\zeta}+\frac{\delta\nu-c_s{}^2(Hl)^{-2}}{\zeta^2}\varDelta\right].\end{aligned} \tag{1·15}$$

Here and hereafter the indices 1 and 2 denote the values at $t=t_1$ and $t=t_2$, respectively. Since $\mathcal{S}$ vanishes after $t=t_2$, $P_0(\zeta)$ and $Q_0(\zeta)$ become constant for $t>t_2$. Hence in Eq. (1·12) the first term represents the pure growing mode and the second term the pure decaying mode for $t>t_2$.

Since we are interested in the generation of large-scale density perturbations in an

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early stage of the universe, Hl can be considered extremely large in the stage concerned $t_1 < t < t_2$. Hence in the discussion of the generation of density perturbations the terms proportional to $c_s{}^2(Hl)^{-2}$ in Eq. (1·15) can be neglected. Then Eq. (1·15) shows that the second term on the right-hand side of Eq. (1·11) is in general of a higher order than the first term with respect to the deviation of B-EOS. Hence we can estimate the amplitude of density perturbations generated by stress perturbations iteratively with respect to the deviation of B-EOS by Eq. (1·11).

§IV-2. The effect of isotropic stress perturbations

In this section we estimate the effect of the isotropic stress perturbations on the generation of density perturbations. First we evaluate the lowest-order term with respect to the deviation of B-EOS, Δ_0. What we are interested in is the amplitudes of density fluctuations at the time t_H when the fluctuations come within the horizon. Since $Hl=\zeta=1$ at $t=t_H$, $f(t_H)=1$ from Eq. (1·5d), so from Eq. (1·6) f is expressed as

$$f=\zeta^{(1-3\gamma)/2}\exp\left[\frac{3}{2}\int_\zeta^1 \delta w\frac{d\zeta'}{\zeta'}\right]. \tag{2·1}$$

Hence for the isotropic perturbation $\mathscr{S}=-w\Gamma/f^2$, $P_0(\zeta)$ and $Q_0(\zeta)$ are written as

$$P_0(\zeta)=\frac{2}{3\gamma+5}\int_{\zeta_1}^{\zeta}\frac{d\zeta'}{\zeta'}\Gamma_1(\zeta'), \tag{2·2}$$

$$Q_0(\zeta)=-\frac{2}{3\gamma+5}\int_{\zeta_1}^{\zeta}d\zeta'(\zeta')^{3(1+\gamma)/2}\Gamma_1(\zeta'), \tag{2·3}$$

where

$$\Gamma_1\equiv\zeta^{(1-3\gamma)}\mathscr{S}=-w\Gamma\exp\left[-3\int_\zeta^1\delta w\frac{d\zeta'}{\zeta'}\right]. \tag{2·4}$$

From these equations the following estimates are obtained:

$$|P_0(\zeta)|\leq\frac{2}{3\gamma+5}\|\Gamma_1\|\ln\frac{\zeta_2}{\zeta_1}, \tag{2·5}$$

$$|Q_0(\zeta)|\leq\left(\frac{2}{3\gamma+5}\right)^2\|\Gamma_1\|(\zeta\wedge\zeta_2)^{(3\gamma+5)/2}, \tag{2·6}$$

where $\|Z\|$ denotes the maximum value of $|Z|$ during $t_1<t<t_2$ and $\zeta\wedge\zeta_2$ represents the smaller one of either ζ or ζ_2. Especially it follows that

$$|\Delta_0(t_H)|\simeq O(\|\Gamma\|). \tag{2·7}$$

Equation (2·7) shows that the amplitude of density fluctuations produced by isotropic stress perturbations, when they come within the horizon, is of the same order as the strength of the original stress perturbations if the deviation of B-EOS is neglected. This confirms in a little more general way the conclusion obtained by Press and Vishniac (1980) who gave a delicate argument based on the synchronous gauge.

Now we estimate the higher-order contributions to $\Delta(\zeta)$ with respect to the deviation of B-EOS by solving the integral equation (1·11) iteratively. For that purpose we first

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perform partial integration and eliminate $d\Delta/d\zeta$ in Eq. (1·15). Then it follows that

$$L*\Delta(\zeta)=U_{+}(\zeta)\int_{\zeta_1}^{\zeta}d\zeta'\left(\sigma_1-\frac{2c_s^2(Hl)^{-2}}{3\gamma+5}\right)(\zeta')^{-2-3\gamma}\Delta(\zeta')$$

$$+U_{-}(\zeta)\int_{\zeta_1}^{\zeta}d\zeta'\left(\sigma_2+\frac{2c_s^2(Hl)^{-2}}{3\gamma+5}\right)(\zeta')^{(1-3\gamma)/2}\Delta(\zeta'), \tag{2·8}$$

where

$$\sigma_1\equiv\frac{2}{3\gamma+5}\left\{(1+3\gamma)\delta\mu+\delta\nu+\frac{45}{2}(1+w)(c_s^2-w)-9\chi_s(1+w)\right\}, \tag{2·9}$$

$$\sigma_2\equiv\frac{2}{3\gamma+5}\left\{\frac{3}{2}(1-\gamma)\delta\mu-\delta\nu-\frac{45}{2}(1+w)(c_s^2-w)+9\chi_s(1+w)\right\}, \tag{2·10}$$

in which

$$\chi_s\equiv\rho\frac{d^2p}{d\rho^2}. \tag{2·11}$$

Note that both σ_1 and σ_2 vanish outside the period $t_1<t<t_2$ and their magnitudes are of the same order as δw and δc_s^2.

Now the solution of Eq. (1·11) is expressed as the formal series

$$\Delta(\zeta)=\sum_{n=0}^{\infty}\Delta_n(\zeta);\quad \Delta_{n+1}(\zeta)=L*\Delta_n(\zeta). \tag{2·12}$$

As before let us express $\Delta_n(\zeta)$ as

$$\Delta_n(\zeta)=P_n(\zeta)U_{+}(\zeta)+Q_n(\zeta)U_{-}(\zeta). \tag{2·13}$$

Then $P_n(\zeta)$ and $Q_n(\zeta)$ satisfy the recurrence formula

$$P_{n+1}(\zeta)=\int_{\zeta_1}^{\zeta}d\zeta'\left(\sigma_1-\frac{2c_s^2(Hl)^{-2}}{3\gamma+5}\right)(\zeta')^{-1}[P_n(\zeta')+(\zeta')^{-(5+3\gamma)/2}Q_n(\zeta')], \tag{2·14}$$

$$Q_{n+1}(\zeta)=\int_{\zeta_1}^{\zeta}d\zeta'\left(\sigma_2+\frac{2c_s^2(Hl)^{-2}}{3\gamma+5}\right)(\zeta')^{3(1+\gamma)/2}[P_n(\zeta')+(\zeta')^{-(5+3\gamma)/2}Q_n(\zeta')]. \tag{2·15}$$

From Eqs. (2·5) and (2·6), $P_0(\zeta)$ and $Q_0(\zeta)$ satisfy the inequalities

$$|P_0(\zeta)|\leq P_0^*,$$

$$|Q_0(\zeta)|\leq Q_0^*(\zeta\wedge\zeta_2)^{(3\gamma+5)/2}, \tag{2·16}$$

where

$$P_0^*=\frac{2}{3\gamma+5}\|\Gamma_1\|\ln\frac{\zeta_2}{\zeta_1}, \tag{2·17}$$

$$Q_0^*=\left(\frac{2}{3\gamma+5}\right)^2\|\Gamma_1\|. \tag{2·18}$$

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Hence noting that from Eq. (2·1) Hl is expressed as

$$(Hl)^{-1}=\zeta/f=\zeta^{(1+3\gamma)/2} \qquad \text{for } \zeta>\zeta_2\,, \tag{2·19}$$

one obtains the following estimates for $P_1(\zeta)$ and $Q_1(\zeta)$:

$$|P_1(\zeta)|\leq(P_0{}^*+Q_0{}^*r_{P2}/r_P)r_P\,, \tag{2·20}$$

$$|Q_1(\zeta)|\leq(P_0{}^*+Q_0{}^*)r_Q(\zeta\wedge\zeta_2)^{(5+3\gamma)/2}, \tag{2·21}$$

where

$$r_P=\left\|\sigma_1-\frac{2c_s{}^2(Hl)^{-2}}{3\gamma+5}\right\|\ln\left(\frac{\zeta_2}{\zeta_1}\right)+\frac{2\gamma}{(3\gamma+5)(3\gamma+1)}\zeta^{1+3\gamma}\,, \tag{2·22}$$

$$r_Q=\frac{2}{3\gamma+5}\left\|\sigma_2+\frac{2c_s{}^2(Hl)^{-2}}{3\gamma+5}\right\|+\frac{4\gamma}{(3\gamma+5)(9\gamma+7)}\zeta^{1+3\gamma}\,, \tag{2·23}$$

and $r_{P2}=r_P(\zeta_2)$. Repeating this procedure one obtains

$$|P_n(\zeta)|\leq(P_0{}^*+Q_0{}^*r_{P2}/r_P)r_P(r_P+r_Q)^{n-1}\,, \tag{2·24}$$

$$|Q_n(\zeta)|\leq(P_0{}^*+Q_0{}^*)r_Q(r_P+r_Q)^{n-1}(\zeta\wedge\zeta_2)^{(5+3\gamma)/2} \tag{2·25}$$

for $n\geqq1$.

Equations (2·24) and (2·25) show that the formal series (2·12) converges absolutely if $\|\sigma_1\|\ln(\zeta_2/\zeta_1)<1/2$, $\|\sigma_2\|<1$, $0\leqq\gamma\leqq1/3$ and $\zeta\leqq1$. In this case the higher-order terms are estimated as

$$\begin{aligned}\left|\sum_{n=1}^{\infty}\Delta_n(\zeta)\right|&\leq\frac{1}{1-r_P-r_Q}(r_PP_0{}^*+r_QQ_0{}^*)U_+(\zeta)\left[1+\frac{r_Q(P_0{}^*+Q_0{}^*)}{r_PP_0{}^*+r_{P2}Q_0{}^*}\left(\frac{\zeta\wedge\zeta_2}{\zeta}\right)^{(3\gamma+5)/2}\right]\\&=O(\|\Gamma\|) \qquad \text{at } t=t_H\,.\end{aligned} \tag{2·26}$$

Since r_P and r_Q are of the same order and $\zeta_2\ll1$ in general, the second term in the square bracket can be neglected. Since $\Delta_0(\zeta)\sim P_0{}^*U_+(\zeta)$ for $\zeta\sim\zeta_H=1$ and $\zeta_2\ll1$, Eq. (2·26) shows that the higher-order terms are at most of the same order as $\Delta_0(\xi)$. Hence the higher-order contributions do not play an essential role in general. In fact, the zeroth-order terms in r_P and r_Q, namely the second terms on the right-hand sides of Eqs. (2·22) and (2·23), arise from the term $c_s{}^2(Hl)^{-2}\zeta^{-2}\Delta$ on the right-hand side of Eq. (1·7). Therefore the correction $\sum_{n=1}^{\infty}\Delta_n(\zeta)$ can be made of order genuinely higher than $\Delta_0(\zeta)$ with respect to the deviation of B-EOS if the fundamental solutions of the homogeneous version of Eq. (1·5) exact in the zeroth order,

$$\zeta^{(9\gamma-1)/4}J_\nu(2\gamma^{1/2}(1+3\gamma)^{-1}\zeta^{(1+3\gamma)/2})$$

and

$$\zeta^{(9\gamma-1)/4}N_\nu(2\gamma^{1/2}(1+3\gamma)^{-1}\zeta^{(1+3\gamma)/2}),$$

are used as $U_+(\zeta)$ and $U_-(\zeta)$ instead of ζ^{β_+} and ζ^{β_-}, where $\nu=(5+3\gamma)/2(1+3\gamma)$.

Since Γ is extremely small in general, Eqs. (2·17) and (2·26) imply that the isotropic stress perturbations which are not associated with a large change in B-EOS are not important in the discussion of the origin of the large-scale structures in the present universe. For example, let us consider an isotropic stress perturbation given rise to by

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statistical fluctuations of the distribution of some objects or components which have a B-EOS different from that of the rest of the cosmic matter. Let ρ_1 and c_1 be the energy density and the sound velocity of this component and ρ_2 and c_2 be those of the rest. Then if the average energy of each object is represented by E, the statistical fluctuation of ρ_1 on a scale L is given by

$$\delta\rho_1 \sim L^{-3}E(\rho_1 L^3/E)^{1/2}. \tag{2·27}$$

Since there exists essentially no fluctuation in the total energy density when the stress perturbation is provoked, Γ is expressed as

$$\begin{aligned}\Gamma &\simeq (c_1^2-c_2^2)\{(\rho_2+p_2)\delta\rho_1-(\rho_1+p_1)\delta\rho_2\}/p(\rho+p)\\ &\sim \frac{c_1^2-c_2^2}{w}\frac{(\rho_1 L^3/E)^{1/2}}{\rho L^3/E}\\ &\sim (\rho_1/\rho)^{1/2}(\rho L^3/E)^{-1/2}.\end{aligned} \tag{2·28}$$

Hence in the standard hot big-bang universe $\Delta_0(t_H)$ is given by

$$|\Delta(t_H)| \sim 1.4\times10^{-41}(\rho_1/\rho)^{1/2}(E/T)^{1/2}(M_B/10^{12}M_\odot)^{-1/2}(T_{BB}/2.7\mathrm{K})^{-3/2}(\Omega_0 h_0^2)^{1/2}, \tag{2·29}$$

where T is a typical temperature when the stress perturbation is working, M_B is the baryon mass contained in the perturbed scale, T_{BB} is the present photon-temperature of the universe, h_0 is the Hubble constant normalized by 100km/s/Mpc, and Ω_0 is the density parameter of the present universe. In general E cannot be greater than the energy contained within the horizon scale, which is expressed in the radiation-dominated stage as

$$E_H \simeq (T_{\mathrm{Pl}}/T)^2 T_{\mathrm{Pl}}, \tag{2·30}$$

where T_{Pl} is the Planck temperature $\sim 10^{19}$GeV. Hence from Eq. (2·29) it follows

$$|\Delta(t_H)| \sim 1.4\times10^{-41}(\rho_1/\rho)^{1/2}(T/T_{\mathrm{Pl}})^{-3/2}(M_B/10^{12}M_\odot)^{-1/2}. \tag{2·31}$$

This shows that isotropic pressure perturbations associated with transient phenomena which occurred while $T \gtrsim 10\mathrm{kev} \simeq 10^8\mathrm{K}$ produce density fluctuations with amplitude much smaller than $O(10^{-3})$ required to produce the present large-scale structure of the universe.

§ IV-3. The effect of anisotropic stress perturbations

In this section we study the effect of anisotropic stress perturbations. The contribution of an anisotropic stress perturbation to Δ_0 is obtained by putting $\mathscr{S}=2\zeta^{-2}\{3(1+w)c_s^2-2w\}\Pi-2w\zeta^{-1}d\Pi/d\zeta$ in Eqs. (1·13) and (1·14) (since the contribution from the term $(2/3)\,wf^{-2}\Pi$ is exactly the same as that from an isotropic stress perturbation, we neglect it in this section). Partial integrations yield

$$P_0=\int_{\zeta_1}^{\zeta}\frac{d\zeta'}{W(\zeta')}U_-(\zeta')(\zeta')^{-2}6w(2w-c_s^2-w)\Pi, \tag{3·1}$$

$$Q_0=\int_{\zeta_1}^{\zeta}\frac{d\zeta'}{W(\zeta')}U_+(\zeta')(\zeta')^{-2}6w\left(2w-c_s^2-\frac{1}{2}w+\frac{5}{6}\right)\Pi. \tag{3·2}$$

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Note that the growing-mode component of $\Delta_0(\zeta)$, the first term in Eq. (1·12), is of first-order in the deviation of B-EOS, $\delta w=w-\gamma$ and $\delta c_s{}^2=c_s{}^2-\gamma$, in contrast to the case of isotropic perturbations. Relacing the terms containing w, $c_s{}^2$ and Π by their average values and performing the integration, we obtain the following estimate of $\Delta_0(\zeta)$:

$$\Delta_0(\zeta)\simeq\frac{12\gamma}{(3\gamma+1)(3\gamma+5)}\langle(2w-\gamma-c_s{}^2)\Pi\rangle\frac{U_+(\zeta)}{U_+(\zeta_1)}$$

$$-\frac{4\gamma}{3(1-\gamma)}\langle\Pi\rangle\frac{U_-(\zeta)}{U_-(\zeta_2)}, \tag{3·3}$$

where $\langle Q\rangle$ denotes the average of Q during t_1 and t_2. Equation (3·3) apparently tempts us to conclude that anisotropic perturbations produce growing density perturbations with amplitudes $\sim\delta w\Pi$ just after the perturbations vanish, hence are much more effective than isotropic stress perturbations in generating density perturbations. Unfortunately, however, this conclusion is not correct due to correction term in Eq. (1·11), $L*\Delta(\zeta)$; it is now also of first-order with respect to the deviation of B-EOS. To get the correct conclusion we must add the contribution from $L*\Delta(\zeta)$ to Δ_0.

For that purpose we improve the series $\{P_n\}$ to a series of genuinely increasing order with respect to the deviation of B-EOS, not spoiled by the terms of $O(c_s{}^2(Hl)^{-2})$, in the perturbative expansion (2·12). This can be achieved by bringing back the term $c_s{}^2(Hl)^{-2}\zeta^{-2}\Delta$ on the right-hand side of Eq. (1·7) to the left-hand side of it and writing the corresponding elementary solutions of the new differential operator which reduce to ζ^{β_+} and ζ^{β_-} in the limit $Hl\to\infty$ as U_+ and U_-, respectively. Then among the coefficients of $U_+(\zeta)$ in Eq. (2·12), P_0 and P_1 are the only terms of first-order with respect to the deviation of B-EOS, and what we have to do is to see whether they cancel or not for an anisotropic stress perturbation.

With the above choice of U_+ and U_-, P_0 and Q_0 are expressed as

$$P_0(\zeta)=\int_{\zeta_1}^{\zeta}d\zeta'\frac{U_-}{(\zeta')^2W}2w[3(2w-c_s{}^2-\gamma)+\tilde{\beta}_--\beta_-]\Pi\,, \tag{3·4}$$

$$Q_0(\zeta)=\int_{\zeta_1}^{\zeta}d\zeta'\frac{U_+}{(\zeta')^2W}2w[3(2w-c_s{}^2-\gamma)+\tilde{\beta}_+-\beta_-]\Pi\,, \tag{3·5}$$

where

$$W\equiv U_-\frac{dU_+}{d\zeta}-U_+\frac{dU_-}{d\zeta}=\frac{3\gamma+5}{2}\zeta^{-3(1-3\gamma)/2}, \tag{3·6}$$

$$\tilde{\beta}_\pm\equiv\frac{\zeta}{U_\pm}\frac{dU_\pm}{d\zeta}, \tag{3·7}$$

and the recurrence formula for P_n and Q_n is given by

$$P_{n+1}=\int_{\zeta_1}^{\zeta}d\zeta'\frac{U_-}{(\zeta')^2W}(P_nU_++Q_nU_-)$$

$$\times\left[\delta\nu-\delta(c_s{}^2(Hl)^{-2})-\left(\frac{1-9w}{2}+\beta_-\right)\delta\mu-\zeta\frac{d\delta\mu}{d\zeta}\right], \tag{3·8}$$

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$$Q_{n+1}=\int_{\zeta_1}^{\zeta} d\zeta' \frac{U_+}{(\zeta')^2 W}(P_n U_+ + Q_n U_-)$$
$$\times\left[\delta\nu-\delta(c_s^2(Hl)^{-2})-\left(\frac{1-9w}{2}+\beta_+\right)\delta\mu-\zeta\frac{d\delta\mu}{d\zeta}\right]. \tag{3·9}$$

In particular the dominant part of P_1 with respect to the deviation of B-EOS is expressed as

$$P_1=-\int_{\zeta_1}^{\zeta} d\zeta' \frac{U_+}{(\zeta')^2 W}2w[\tilde{\beta}_+-\tilde{\beta}_-]\Pi\int_{\zeta'}^{\zeta} d\zeta'' \frac{\sigma_3}{(\zeta'')^2 W}U_-^2$$
$$+\text{higher-order terms}, \tag{3·10}$$

where

$$\sigma_3\equiv-(3+9\gamma)(w-r)+\frac{1}{2}(21+27\gamma)(c_s^2-\gamma)$$
$$+3\zeta\frac{dc_s^2}{d\zeta}-\delta(c_s^2(Hl)^{-2})+\text{higher-order terms.} \tag{3·11}$$

From Eqs. (3·4) and (3·10) it follows that if the equality

$$\frac{6w-3c_s^2-3\gamma+\tilde{\beta}_--\beta_-}{\tilde{\beta}_+-\tilde{\beta}_-}\frac{U_-}{U_+}=\int_{\zeta}^{\zeta_2} d\zeta' \frac{\sigma_3}{(\zeta')^2 W}U_-^2 \tag{3·12}$$

holds for $\zeta_1<\zeta<\zeta_2$, then P_0 and P_1 cancel each other up to the first order in the deviation of B-EOS for an arbitrary Π. If one writes the right-hand side of Eq. (3·12) as X, this condition is equivalent to the equality

$$\frac{dX}{d\zeta}=-\frac{\sigma_3}{\zeta^2 W}U_-^2.$$

The explicit calculation yields

$$\frac{dX}{d\zeta}\Big/\left(-\frac{U_-^2}{\zeta^2 W}\right)=-(3+9\gamma)(w-\gamma)+\frac{1}{2}(21+27\gamma)(c_s^2-\gamma)+\delta(\beta_-)^2$$
$$+3(1+\gamma)\delta\beta_-+3\zeta\frac{dc_s^2}{d\zeta}+\delta(c_s^2(Hl)^{-2})+\text{higher-order terms}, \tag{3·13}$$

which coincides with σ_3 given by Eq. (3·11) except for the extremely small terms of $O((Hl)^{-2})$. Therefore, surprisingly enough, the first-order terms of P_0 and P_1 cancel each other exactly. This means that the amplitudes of density perturbations generated from anisotropic stress perturbations, when they come within the horizon, are of the same order as the original stress perturbations at least up to the first order with respect to the deviation of B-EOS.

§ IV-4. Behavior of density perturbations in the radiation-dust universe

In this and next sections we study the generation and evolution of density perturbations in a universe in which the background matter is composed of radiation and pressure-free particles (dust), which we call the radiation-dust universe. Except for exotic stages

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in which some kinds of exotic matter such as coherent classical fields (e.g., the Higgs fields or the axion field in the cosmological models based on grand unified theories) dominate the cosmic expansion, the cosmic matter is well described by such a mixture in almost all stages of the realistic cosmological model, though the definite content of "dust" may change from one stage to another. Hence the study of the above problem is of great importance. First in this section we examine the behavior of adiabatic density perturbations in the radiation-dust universe. The generation of density perturbations will be discussed in the next section.

The adiabatic perturbation in the radiation-dust universe is described by Eq. (1·5) with $\mathscr{S}=0$:

$$\frac{d^2\Delta}{d\zeta^2}-\frac{\mu}{\zeta}\frac{d\Delta}{d\zeta}-\frac{2+\nu}{\zeta^2}\Delta=0\,, \tag{4·1}$$

where we have omitted the term $c_s{}^2/f^2$ because we only consider perturbations with scale much larger than the Hubble horizon size. In this and next sections we normalize ζ as $\zeta=1$ at the equal time $t=t_e$, that is when the energy densities of radiation and dust coincide. Then from the assumption on the composition of the background cosmic matter, the energy density and pressure of the unperturbed universe are given by

$$\rho=\frac{1}{2}\zeta^{-4}+\frac{1}{2}\zeta^{-3}\,, \tag{4·2}$$

$$p=\frac{1}{6}\zeta^{-4}\,. \tag{4·3}$$

From these equations it immediately follows that

$$w\equiv p/\rho=\frac{1}{3z}\,, \tag{4·4}$$

$$c_s{}^2\equiv\dot{p}/\dot{\rho}=\frac{4}{3(1+3z)}\,, \tag{4·5}$$

where

$$z\equiv 1+\zeta\,. \tag{4·6}$$

From Eqs. (1·5b), (1·5c), (4·4) and (4·5), Eq. (4·1) is written in terms of the new variable z as

$$\frac{d^2\Delta}{dz^2}+\left(\frac{5}{2}\frac{1}{z}-\frac{3}{3z+1}\right)\frac{d\Delta}{dz}+\left[-\frac{2}{(z-1)^2}+\frac{3}{4}\frac{1}{z-1}+\frac{1}{2z^2}-\frac{3}{z}+\frac{27}{4}\frac{1}{3z+1}\right]\Delta=0\,. \tag{4·7}$$

To find the general solution of this equation, let us change the unknown function Δ to u defined by

$$\Delta=u/z^{1/2}(z-1). \tag{4·8}$$

Then after a little calculation we find

$$\frac{d^2u}{dz^2}+\left(\frac{3}{2z}-\frac{2}{z-1}-\frac{3}{3z+1}\right)\frac{du}{dz}=0\,. \tag{4·9}$$

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The general solution of this equation is

$$u(z)=c_1X(z)+c_2\,, \tag{4·10}$$

where

$$X(z)=z^{-1/2}\left(z^3-\frac{25}{9}z^2+\frac{5}{3}z-\frac{5}{3}\right), \tag{4·11}$$

and c_1 and c_2 are arbitrary integration-constants. Hence the general solution of Eq. (4·7) is given by

$$\Delta(z)=AU_G(z)+BU_D(z), \tag{4·12}$$

where

$$U_D(z)=z^{-1/2}(z-1)^{-1}\,, \tag{4·13}$$

$$U_G(z)=(X(z)+x)U_D(z) \tag{4·14}$$

with A and B being arbitrary constants and x being some constant determined below [cf. Chernin (1966); Nariai, Tomita and Kato (1967)]. For $\zeta\gg1(z\gg1)$, which corresponds to the dust-dominated stage, U_D and U_G behave as

$$U_D\sim z^{-3/2}\sim\zeta^{-3/2}\,, \tag{4·15}$$

$$U_G\sim z\sim\zeta\,. \tag{4·16}$$

Hence U_D and U_G represent the decaying mode and the growing mode in the dust-dominated stage, respectively. On the other hand in the radiation-dominated stage $\zeta\ll1(z\sim1)$, their asymptotic behavior is

$$U_D=\frac{1}{\zeta}-\frac{1}{2}+\frac{3}{8}\zeta-\frac{5}{16}\zeta^2+O(\zeta^3), \tag{4·17}$$

$$U_G=-\frac{16}{9}\frac{1}{\zeta}+\frac{8}{9}-\frac{2}{3}\zeta+\frac{5}{3}\zeta^2+x\left(\frac{1}{\zeta}-\frac{1}{2}+\frac{3}{8}\zeta-\frac{5}{16}\zeta^2\right)+O(\zeta^3). \tag{4·18}$$

From this asymptotic behavior we find that the choice of the constant x as

$$x=16/9 \tag{4·19}$$

makes U_G purely growing in the radiation-dominated stage:

$$U_G=\frac{10}{9}\zeta^2+O(\zeta^3). \tag{4·20}$$

In §III-2 we have studied the behavior of density perturbations when the B-EOS changes discretely. There is one problem in comparing the result obtained there with the case when the B-EOS of the universe changes smoothly from one stage to another. In particular in the purely radiation-dominated or the purely dust-dominated stage, the behavior of the density perturbation is quite simple and there are natural definitions of the growing mode and the decaying mode. In the present case, however, in the limit $\zeta\to\infty$ there remains always a freedom of adding a constant multiple of the decaying mode in defining the growing mode. Similarly in the limit $\zeta\to0$ there is a freedom of adding a constant

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multiple of the growing mode in defining the decaying mode. This arbitrariness makes it difficult to make correspondence between the modes in the purely radiation- or dust-dominated universe and those in the universe treated here.

In order to resolve this difficulty we consider the universe which is purely radiation-dominated during $\zeta<\zeta_1$ with an arbitrary ζ_1 and changes discretely at $\zeta=\zeta_1$ to the radiation-dust universe. Then by taking the limit $\zeta_1\to 0$ we can find the natural correspondence between the modes in the two universes. As shown in §III-2 (see Eq. (III-2·12a)) the general solution for the density perturbation in the purely radiation-dominated stage is

$$\Delta=CU_+ + DU_- , \tag{4·21}$$

where

$$U_+=\zeta^2 , \tag{4·22}$$

$$U_-=\zeta^{-1} . \tag{4·23}$$

We connect this solution to the general solution in the radiation-dust dominated stage (4·12) at $\zeta=\zeta_1$ so that Δ and V are both continuous there as done in §III-2. Then solving the junction condition

$$\Delta_b=\Delta_a , \tag{4·24a}$$

$$\frac{1}{1+w_b}\left(\frac{d\Delta}{d\zeta}\right)_b-\frac{3w_b}{1+w_b}\frac{\Delta_b}{\zeta_1}=\frac{1}{1+w_a}\left(\frac{d\Delta}{d\zeta}\right)_a-\frac{3w_a}{1+w_a}\frac{\Delta_a}{\zeta_1} , \tag{4·24b}$$

where the suffixes b and a denote the values of quantities in the purely radiation-dominated stage just before ζ_1 and in the radiation-dust dominated stage just after it, respectively, we find the following relation between (A, B) and (C, D):

$$AU_G(\zeta_1)=\frac{9}{10}K(z_1)\frac{X(z_1)+16/9}{X(z_1)+x}CU_+(\zeta_1), \tag{4·25a}$$

$$BU_D(\zeta_1)=\frac{9}{10}\left\{\frac{10}{9}-K(z_1)+\frac{K(z_1)}{X(z_1)+16/9}\left(x-\frac{16}{9}\right)\right\}CU_+(\zeta_1)+DU_-(\zeta_-), \tag{4·25b}$$

where

$$K(z)=(z-1)^{-3}\left(z^3-\frac{25}{9}z^2+\frac{5}{3}z-\frac{5}{3}+\frac{16}{9}z^{1/2}\right). \tag{4·26}$$

Here note that

$$K(z)\to\begin{cases}10/9 ; & z\to 1 ,\\ 1 ; & z\to\infty .\end{cases} \tag{4·27}$$

In Eqs. (4·25) we left x unspecified. These equations show that the decaying mode in the radiation-dust dominated stage, U_D, exactly corresponds to the one in the purely radiation-dominated stage in the limit $\zeta\to 0$ only when $x=16/9$. Hence the choice of x to make U_G purely growing in the $\zeta\to 0$ limit turned out to be the most natural choice in the viewpoint of the decaying mode correspondence. With this choice Eqs. (4·25) become

$$AU_G(\zeta_1)=\frac{9}{10}K(z_1)CU_+(\zeta_1), \tag{4·28a}$$

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$$BU_D(\zeta_1)=\left(1-\frac{9}{10}K(z_1)\right)CU_+(\zeta_1)+DU_-(\zeta_1). \tag{4·28b}$$

One important implication obtained from these equations is that the decaying mode in the radiation-dominated stage (C=0) evolves to the pure decaying mode in the dust-dominated stage of the radiation-dust universe irrespective of the value of x. This result completely coincides with the result for the case of discrete change in B-EOS discussed in §III-2. Of course, since a slightly generic situation is considered now, the matching relation of the amplitudes is different from the one in §III-2. However, the difference is at most within a factor of $O(1)$. The matching relations (III-2·35) and (III-2·36) with $n=3$, $\beta_-=1$ and $\beta_+=2$ are obtained from Eqs. (4·28) by replacing $K(z)$ by $K(\infty)=1$.

§ IV-5. Generation of density perturbations in the radiation-dust universe

In this section we investigate the generation of density perturbations from stress perturbations in the radiation-dust universe. Since the evolution equation of the density perturbation without the source term (coming from stress perturbations) can be explicitly solved in this universe as was shown in §4, we can study this problem without appealing to the iterative method as in §§1~3. Hence, though the situation considered may be a little bit special, the following investigation is expected to yield some insight into the problem of the generation of density perturbations in the cases that the B-EOS changes largely while stress perturbations are working; such cases could not be discussed in the iterative treatment of §§1~3.

We consider the same situation as in §1. There exists no density perturbation before some time t_1 ($\Delta=V=0$) and then a stress perturbation is provoked during a finite time interval $t_1<t<t_2$. The universe is assumed to be described by the radiation-dust universe at least until t_2. However, as for the behavior of the universe after t_2 we consider two cases; the case the universe remains radiation-dust dominated throughout and the case the universe undergoes a discrete change from the radiation-dust dominated stage to the pure radiation-dominated stage at some time $t_*(>t_2)$.

First let us consider the isotropic stress perturbation. By using the same notation as in §4, the fundamental equation is written as

$$\mathcal{D}\Delta=-f^{-2}w\Gamma\,, \tag{5·1}$$

where $\mathcal{D}$ is the differential operator on the left-hand side of Eq. (4·7). With the aid of the Green function

$$G(z,\,z')=[U_G(z)U_D(z')-U_D(z)U_G(z')]W(z')^{-1}\theta(z-z'), \tag{5·2}$$

where

$$W(z)\equiv U_D\frac{dU_G}{dz}-U_G\frac{dU_D}{dz}=\frac{5}{6}z^{-5/2}(3z+1), \tag{5·3}$$

Eq. (5·1) can be easily solved:

$$\Delta(z)=A(z)U_G(z)+B(z)U_D(z), \tag{5·4}$$

where

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$$A(z)=-\int_{z_1}^{z}\frac{w(z')}{f(z')^2}\frac{U_D(z')}{W(z')}\Gamma(z')dz'\,, \tag{5·5a}$$

$$B(z)=\int_{z_1}^{z}\frac{w(z')}{f(z')^2}\frac{U_D(z')}{W(z')}\Gamma(z')dz' \tag{5·5b}$$

for $z>z_1$. Since Γ vanishes for $z>z_2$, $A(z)$ and $B(z)$ become constant for $z>z_2$, which we write as A and B, respectively. From Eqs. (1·6) and (4·4) f is now given by

$$f=(z/z_2)^{1/2}f_2\,. \tag{5·6}$$

Hence the constants A and B are expressed as

$$A=-\frac{5}{6}\alpha\int_{z_1}^{z_2}\frac{\Gamma(z)}{(3z+1)(z-1)}dz\,, \tag{5·7a}$$

$$B=\frac{5}{6}\alpha\int_{z_1}^{z_2}\frac{(X(z)+16/9)\Gamma(z)}{(3z+1)(z-1)}dz\,, \tag{5·7b}$$

where

$$\alpha\equiv z_2/3f_2^{\,2}\,. \tag{5·8}$$

To estimate A and B, we must specify the ζ-dependence of Γ. We consider the three cases: $\Gamma=\text{const}$, $\Gamma=\gamma(z-1)$ and $\Gamma=\gamma(z-1)^n$ $(n\geqq 1)$, during $z_1<z<z_2$ for all the cases.

(i) $\Gamma=\text{const}$

In this case the explicit integration yields

$$A=-\frac{5}{24}\alpha\Gamma\left(\ln\frac{z_2-1}{z_1-1}-\ln\frac{3z_2+1}{3z_1+1}\right), \tag{5·9a}$$

$$B=\frac{5}{6}\alpha\Gamma\left[\frac{2}{9}z^{3/2}-\frac{32}{27}z^{1/2}+\frac{40}{27\sqrt{3}}\tan^{-1}(\sqrt{3}z)+\frac{4}{9}\ln\frac{(\sqrt{z}+1)^2}{(3z+1)^3}\right]_{z_1}^{z_2}. \tag{5·9b}$$

In this form we cannot yet see whether the amplitude of the generated density perturbation is large or not. What we have to do is to estimate the amplitude of the perturbation when it enters the particle horizon (we denote that time by t_H). First let us assume that the universe remains in the radiation-dust dominated stage until the time t_H. Then from Eq. (5·6), $f/z^{1/2}$ remains constant until $t=t_H$. Since $f=Hl\zeta=\zeta_H$ at $t=t_H$, α is written as

$$\alpha=z_H/3\zeta_H^{\,2}\simeq 1/3\zeta_H \qquad \text{for } \zeta_H\gg 1\,. \tag{5·10}$$

In order to estimate $\Delta(t_H)$, let us consider two limiting cases. First in the limit $\zeta_1<\zeta_2\ll 1$,

$$A\simeq-\frac{5}{24}\alpha\Gamma\ln\frac{z_2-1}{z_1-1}\,, \tag{5·11a}$$

$$B\simeq\frac{5}{72}\alpha\Gamma[(z_2-1)^3-(z-1)^3], \tag{5·11b}$$

hence if $\zeta_H\gg 1$, we obtain the estimate

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$$\Delta(t_H)\simeq -\frac{5}{72}\Gamma\ln\frac{\zeta_2}{\zeta_1}+\frac{5}{216}\Gamma\zeta_2^{1/2}\left(\frac{\zeta_2}{\zeta_H}\right)^{5/2}\left[1-\left(\frac{\zeta_1}{\zeta_2}\right)^3\right]. \tag{5·12}$$

The first term comes from the growing mode and the second term from the decaying mode; the latter is negligibly small compared with the former. Equation (5·12) shows that the amplitude of the generated density perturbation when it comes within the horizon, is of the order of the strength of the original stress perturbation.

Next we consider the limit $\zeta_1 \ll 1 \ll \zeta_2$. In this case A and B are approximately given by

$$A\simeq -\frac{5}{24}a\Gamma\left[\ln\frac{1}{z_1-1}+\ln\frac{4}{3}-\frac{4}{3}\frac{1}{z_2}\right], \tag{5·13a}$$

$$B\simeq\frac{5}{24}a\Gamma z_2^{3/2}. \tag{5·13b}$$

Hence we obtain

$$\Delta(t_H)\simeq -\frac{5}{72}\Gamma\left[-\ln\zeta_1+\ln\frac{4}{3}-\frac{4}{3}\frac{1}{\zeta_2}\right]+\frac{5}{81}\Gamma\frac{1}{\zeta_H}\left(\frac{\zeta_2}{\zeta_H}\right)^{3/2}. \tag{5·14}$$

Thus the amplitude of $\Delta(t_H)$ is again of the order of Γ. From the comparison of Eqs. (5·12) and (5·14) we find that the amplitude increases with ζ_2 while $\zeta_2<1$ but it saturates when ζ_2 becomes greater than 1. This means that the generation of density perturbations during the matter-dominated stage is negligible if $\Gamma(\zeta)$ is constant. The main reason of this is that the decrease in w at the matter-dominated stage depresses the effect of Γ because Γ acts as the source of density perturbations in the combination $w\Gamma$.

Now we estimate the amplitude of the generated density perturbation in the case the universe suddenly changes from the radiation-dust dominated stage to the purely radiation-dominated stage at $t=t_*(>t_2$ and $<t_H)$. The relation between the amplitudes of the growing mode and the decaying mode in the two stages is again given by Eqs. (4·28) and (4·29) with the replacement $z_1\to z_*$, since the junction condition (4·24) is unchanged under the exchange of the two stage before and after the junction time. Now assuming that $t_H\gg t_*$, a is given by

$$a=z_*/3f_*^2=z_*/3\zeta_H^2, \tag{5·15}$$

since f is constant in the purely radiation-dominated stage and coincides with ζ_H at $t=t_H$. Hence in the limit $\zeta_1<\zeta_2\ll 1$, $\Delta(t_H)$ is estimated as

$$\Delta(t_H)\simeq -\frac{5}{72}\Gamma\ln\frac{\zeta_2}{\zeta_1}. \tag{5·16}$$

Thus again $\Delta(t_H)$ is of order Γ and reconfirm the result obtained in §2 by the iterative argument. On the other hand in the limit $\zeta_1\ll 1\ll\zeta_2$ we obtain

$$\Delta(t_H)\simeq -\frac{5}{72}\Gamma\left[-\ln\zeta_1+\ln\frac{4}{3}-\frac{4}{3}\frac{1}{\zeta_2}\right]. \tag{5·17}$$

Hence the amplitude is exactly the same as that in the case the universe remains radiation-dust dominated throughout. This result may look strange considering the fact that the growth of density perturbations is slower in the dust dominated stage than in the

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radiation-dominated stage if expressed in terms of the cosmic scale factor; $\Delta \propto \zeta^2$ in the radiation-dominated stage and $\Delta \propto \zeta$ in the dust-dominated stage for the growing mode. This seeming contradiction is resolved if we note that the Hubble horizon size H^{-1} increases more slowly in the radiation-dominated stage than in the dust-dominated one if expressed in terms of the cosmic scale factor; the density perturbations come within the horizon earlier in the case the universe evolves into the radiation-dominated stage than in the case it remains radiation-dust dominated.

(ii) $\Gamma=\gamma(z-1)$

In this case A is given by

$$A=-\frac{5}{18}\alpha\gamma\ln\frac{3z_2+1}{3z_1+1}. \tag{5·18}$$

Since the contribution of the decaying mode is negligibly small as shown in (i), we do not consider it from now on. Further since $\Delta(t_H)$ does not depend on whether the universe undergoes the transition to the purely radiation-dominated stage or not as shown in case (i), we only consider the case in which the universe remains radiation-dust dominated throughout.

In the limit $\zeta_1<\zeta_2\ll 1$. A is approximately given by

$$A\simeq-\frac{5}{9}\alpha\gamma(z_2-z_1)\simeq-\frac{5}{9}\alpha\Gamma(z_2). \tag{5·19}$$

Hence the amplitude of the generated density perturbation is of the same order as that in case (i) except for the absence of the logarithmic factor. On the other hand in the limit $\zeta_1\ll 1\ll\zeta_2$, A is given by

$$A\simeq-\frac{5}{18}\alpha\gamma\ln\left(\frac{3}{4}\zeta_H\right)=-\frac{5\gamma}{54\zeta_H}\ln\left(\frac{3}{4}\zeta_H\right). \tag{5·20}$$

Hence

$$\Delta(t_H)\simeq-\frac{5}{54}\gamma\ln\left(\frac{3}{4}\zeta_H\right)=-\frac{5}{54}\Gamma(z_2)\frac{1}{\zeta_2}\ln\left(\frac{3}{4}\zeta_H\right). \tag{5·21}$$

Now the resultant amplitude is depressed by the factor $1/\zeta_2$ compared with case (i) except for the unimportant logarithmic enhancement factor. This is easily understood by noting that Γ is ineffective in the radiation-dominated stage due to the form of $\Gamma(\propto\zeta)$ and that the effect of Γ in the dust-dominated stage is depressed by w as explained previously.

(iii) $\Gamma=\gamma(z-1)^n$ $(n\geqq 2)$

In this case A is estimated as

$$A\simeq\begin{cases}-\dfrac{5}{24n}\alpha\gamma[(z_2-1)^n-(z_1-1)^n]\simeq-\dfrac{5}{24n}\alpha\Gamma(z_2) & \text{for } \zeta_1<\zeta_2\ll 1,\\[2ex] -\dfrac{5}{18(n-1)}\alpha\gamma {z_2}^{n-1} & \text{for } \zeta_1\ll 1\ll\zeta_2.\end{cases} \tag{5·22}$$

The result in the limit $\zeta_1<\zeta_2\ll 1$ is the same as in (i) and (ii). In the limit $\zeta_1\ll 1\ll\zeta_2$, $\Delta(t_H)$ is approximately given by

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$$\Delta(t_H) \simeq -\frac{5}{18(n-1)} \Gamma(z_2) \frac{1}{\zeta_2} . \tag{5·23}$$

Hence the result in this limit is also the same as that for the corresponding limit in case (ii).

To summarize, as in the case of a small change in B-EOS, the amplitude of the density perturbations generated from isotropic stress perturbations is, when they enter the horizon, of the same order as the strength of the original stress perturbations at most in the radiation-dust universe even if the stress perturbations continue to work while the universe changes from radiation-dominated to dust-dominated ones. In addition the density perturbations generated after the universe becomes dust-dominated is negligibly smaller than that generated at the radiation-dominated stage, unless the stress perturbations are rather enhanced in the dust-dominated stage.

So far we have considered only the generation from isotropic stress perturbations. The effect of anisotropic stress perturbations can be estimated in the same way. Of the source terms arising from the anisotropic stress perturbation in Eq. (1·2), the term $(2/3)wf^{-2}\Pi$ acts exactly in the same way as the one from the isotropic stress perturbation $-wf^{-2}\Gamma$. Hence the difference of the effect of the anisotropic stress perturbation comes from the part

$$\begin{aligned} \mathscr{S}^* &= 2\{3(w^2+c_s{}^2)-2w\}\frac{\Pi}{\zeta^2} - 2\frac{w}{\zeta}\frac{d\Pi}{d\zeta} \\ &= 2\left(\frac{1}{3z^2}+\frac{4}{1+3z}-\frac{2}{3z}\right)\frac{\Pi}{(z-1)^2} - \frac{2}{3}\frac{1}{z(z-1)}\frac{d\Pi}{dz} . \end{aligned} \tag{5·24}$$

Replacing the right-hand side of Eq. (5·1) by $\mathscr{S}^*$. We find that the amplitude of the generated growing mode is given by

$$A = -2\int_{z_1}^{z_2} dz \frac{U_D}{W}\left(\frac{1}{3z^2}+\frac{4}{1+3z}-\frac{2}{3z}\right)\frac{\Pi}{(z-1)^2} + \frac{2}{3}\int_{z_1}^{z_2} dz \frac{U_D}{W}\frac{1}{z(z-1)}\frac{d\Pi}{dz} . \tag{5·25}$$

By partial integration we find that the right-hand side of Eq. (5·25) exactly vanishes. Thus no density perturbation is generated from the source term $\mathscr{S}^*$ in the radiation-dust universe irrespective of the stage it works. This result and the result in §3, if combined, strongly suggest that isotropic and anisotropic stress perturbations act in the same way in generating density perturbations even in cases when the B-EOS undergoes a more general change while the stress perturbations are working.

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Chapter V

Perturbations in the Baryon-Photon System

§ V-1. Basic equations

According to the standard model of the universe, the cosmic temperature was once high enough for statistical equilibrium among photons, nucleons, electrons and neutrinos. Further, the net baryon number is assumed to be positive with the baryon-photon ratio of $10^{-8} \lesssim n_B/n_r \lesssim 10^{-10}$ and neutrinos are assumed to be non-degenerate [see, e.g., Sato, Matsuda and Takeda (1971); Peebles (1971)].

As the temperature decreases neutrinos first decouples from the thermal equilibrium and then nucleons. Then at temperatures $100\text{keV} \gtrsim T \gtrsim 10\text{keV}$, light elements such as deuterium and helium are synthesized. After this era up to the epoch of hydrogen recombination at $T \sim 0.35\text{eV}$ ($\sim 4000\text{K}$), we can regard the cosmic matter to be composed of photons, baryons in the form of nuclei, electrons and neutrinos with their relative abundances kept essentially constant. Among them the important species for discussion of cosmological perturbations are photons and baryons.

Photons are important since radiation dominates the cosmic energy density during most of the stage under consideration and the pressure throughout the stage. Baryons are important since they are the ones that are responsible for the observed large-scale structures.

On the contrary, provided neutrinos are massless, they play no significant role in regard to the formation of structures in the universe, though their contribution to the energy density of the universe is non-negligible before the universe becomes baryon-dominated. Although it has been suggested recently that neutrinos are massive with mass of order 10eV which might have played an important role in the formation of the observed large-scale structures [Sato and Takahara (1980, 1981); Bond, Efstathiou and Silk (1980)], we assume they are massless for simplicity. As for electrons, it is apparent that their contribution to the cosmic energy density can be neglected. However, they are important in the sense that they provide the coupling between photons and baryons through Thomson scattering.

Hence the stage of the universe with the temperature range $1\text{keV} \gtrsim T \gtrsim 0.35\text{eV}$ can be described by the baryon-photon system with interaction provided by Thomson scattering of photons by electrons. Furthermore, because of the small neutron-fraction of baryons, $n_N/n_B \sim 0.1$, expected in the standard model of the universe, we may neglect it in a first approximation and regard baryons as totally protons. Under these assumptions the energy density and the pressure of the universe are given by

$$\rho = \rho_r + \rho_m \quad ; \quad \rho_r = 4\sigma_0 T^4, \quad \rho_m = n_B M, \tag{1·1a}$$

$$p = p_r + p_m \quad ; \quad p_r = \frac{1}{3}\rho_r, \qquad p_m = (1 + x_e) n_B T, \tag{1·1b}$$

where σ_0 is the Stephan-Boltzmann constant, M is the proton mass, $x_e \equiv n_e/n_B$ is the

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fractional hydrogen ionization with n_e being the number density of free electrons. In the above equations, the pressure of the matter (baryons and electrons) is included since it plays an important role in determining the behavior of matter density perturbations, though its value is negligibly smaller than the radiation pressure in the contribution to the total pressure. The gauge-invariant analysis of cosmological perturbations at this stage of the universe provides a good example of applications of the formalism developed in §II-5 for a multi-component system because of the simplicity and the reality of it.

Before investigating the detailed behavior of perturbations in this system, it is necessay to give some arguments on the nature of the system especially concerning the mutual interactions among photons, protons and electrons. In general the matter temperature may deviate from the radiation temperature because of an incomplete thermal equilibrium. Further the matter temperature may not be well-defined since the proton and electron temperatures may take values different from each other. In what follows we examine whether these deviations from the thermal equilibrium can be neglected or not, which eventually yields the proper form of interactions to be taken into account. We express the temperature of photons by T, that of the matter by T_m, of protons by T_p and of electrons by T_e.

First consider the case when T_m deviates from T. In such a case, the net energy transfer between photons and baryons will be non-vanishing. The energy transfer rate due to the temperature difference can be calculated by evaluating the collision integral (E·39) given in Appendix E, assuming the Maxwell distribution with temperature T_m for electrons and Planck distribution with temperature T for photons. After a rather lengthy calculation, we obtain

$$C[f]=\frac{1}{4\pi}\frac{n_e\sigma_T T^2}{m}\frac{T_m-T}{T}\left(x^3\frac{d^2}{dx^2}+4x^2\frac{d}{dx}\right)f(x), \tag{1·2}$$

where $x=q/T$ with q being the photon energy, σ_T is the Thomson cross section, m is the electron mass, $f(x)=[4\pi^3(e^x-1)]^{-1}$ and the inequality $(T_m-T)/T\ll 1$ has been assumed. Then the energy transfer rate per unite volume Q from photons to electrons is given by [Weymann (1965)]

$$\begin{aligned} Q&=-4\pi\int C[f]q^2dq \\ &=\frac{4\rho_r n_e\sigma_T}{m}(T-T_m). \end{aligned} \tag{1·3}$$

Assuming that equipartition of energy between protons and electrons is rapidly established, the equation for the internal energy of the matter yields

$$\begin{aligned} \dot{T}_m+2\frac{\dot{a}}{a}T_m&=\frac{2}{3(1+x_e)n_B}Q \\ &=\frac{8x_e\rho_r\sigma_T}{3(1+x_e)m}(T-T_m). \end{aligned} \tag{1·4}$$

On the other hand, the equation for the radiation temperature is

$$\dot{T}+\frac{\dot{a}}{a}T=-\frac{T}{4\rho_r}Q$$

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$$= -\frac{x_e n_B \sigma_T T}{m}(T - T_m). \tag{1·5}$$

Explicit evaluation of the coefficients appearing on the right-hand sides of these equations reads

$$\tau_{\rm th} \equiv \frac{3(1+x_e)m}{8x_e\rho_r\sigma_T} \simeq 1.1\times10^5 \frac{1+x_e}{x_e}\left(\frac{T}{1\text{eV}}\right)^{-4}\text{sec}\,, \tag{1·6a}$$

$$\tau_{\rm rad} \equiv \frac{m}{x_e n_B \sigma_T T} = \frac{2}{1+x_e}\left(\frac{s}{n_B}\right)\tau_{\rm th}\,, \tag{1·6b}$$

where s/n_B is the entropy per baryon which is approximately equal to the inverse of the baryon-photon ratio $10^8 \lesssim (n_B/n_r)^{-1} \lesssim 10^{10}$. Since the adiabatic damping of the matter temperature is severer than that of the radiation temperature, $\tau_{\rm th}$ represents the timescale on which T_m would adjust to T if the cosmic expansion timescale $\tau_{\rm ex} \equiv (\dot{a}/a)^{-1} = H^{-1}$ were much greater than $\tau_{\rm th}$. Under such a situation we have

$$\frac{T-T_m}{\tau_{\rm rad}} = \frac{\tau_{\rm th}}{\tau_{\rm rad}}\frac{T}{\tau_{\rm ex}} = \frac{1+x_e}{2}\left(\frac{n_B}{s}\right)\frac{T}{\tau_{\rm ex}} \ll \frac{T}{\tau_{\rm ex}}\,. \tag{1·7}$$

Therefore we may neglect the energy loss of photons and assume $T_m = T$. Now, according to the standard cosmological scenario, $\tau_{\rm ex}$ is given by

$$\tau_{\rm ex} \simeq 3.4\times10^{12}\left(\frac{T}{1\text{eV}}\right)^{-2}\left[1+1.25\times10^9\left(\frac{s}{n_B}\right)^{-1}\left(\frac{T}{1\text{eV}}\right)^{-1}\right]^{-1/2}\text{sec}$$

$$\simeq \begin{cases} 3.4\times10^{12}\left(\dfrac{T}{1\text{eV}}\right)^{-2}\text{sec}\,; & T \gg 1\text{eV}\,, \\ 9.6\times10^7\left(\dfrac{s}{n_B}\right)^{1/2}\left(\dfrac{T}{1\text{eV}}\right)^{-3/2}\text{sec}\,; & T \ll 1\text{eV}\,, \end{cases} \tag{1·8}$$

assuming s/n_B is more or less close to 10^9. Then comparing $\tau_{\rm th}$ with $\tau_{\rm ex}$ one finds that the condition $\tau_{\rm th} \ll \tau_{\rm ex}$ is satisfied if

$$T \gtrsim 7\times10^{-2}\left(\frac{s}{n_B}\right)^{-1/5}\left(\frac{x_e}{1+x_e}\right)^{-2/5}\text{eV}\,. \tag{1·9}$$

Thus if x_e were always close to unity, the critical temperature below which $T_m = T$ no longer holds would be about 10^{-3}eV. In reality, however, rapid recombination of electrons to protons at $T \sim 0.35$eV decreases x_e considerably below that temperature and x_e levels off at a value $\sim 10^{-4}$ below $T \sim 0.1$eV [Sato, Matsuda and Takeda (1971)]. Incidentally, Eq. (1·9) implies that this leveling-off temperature provides the actual lower limit of the temperature range for $\tau_{\rm th} \ll \tau_{\rm ex}$ approximately.

The validity of Eq. (1·4) for the matter temperature is restricted by the condition that the timescale $\tau_{\rm ep}$ for equipartition of energy between protons and electrons be less than $\tau_{\rm th}$. Therefore let us next clarify in what stage $\tau_{\rm ep} \ll \tau_{\rm th}$ is satisfied. However, even if $\tau_{\rm ep} > \tau_{\rm th}$, we may still assume $T_p = T_e = T_m$ when seen on the expansion timescale if $\tau_{\rm ep} \ll \tau_{\rm ex}$ is satisfied. This implies that for perturbations which do not oscillate many times within the timescale $\tau_{\rm ex}$, the assumption $T_p = T_e = T_m$ is valid provided $\tau_{\rm ep} \ll \tau_{\rm ex}$. Thus the condition $\tau_{\rm ep} \ll \tau_{\rm ex}$ is often sufficient to justify the assumption $T_p = T_e = T_m$. The time-

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scale $\tau_{\rm ep}$ can be roughly estimated as follows. Let $\boldsymbol{v}$ be the electron velocity and $\boldsymbol{V}$ the proton velocity. In the proton rest frame the electron momentum is $m(\boldsymbol{v}-\boldsymbol{V})$ so that the momentum flux coming from this direction is $n_e|\boldsymbol{v}-\boldsymbol{V}|m(\boldsymbol{v}-\boldsymbol{V})/4\pi \simeq n_e v m(\boldsymbol{v}-\boldsymbol{V})/4\pi$, where $v=|\boldsymbol{v}|$ and we have assumed $v \gg |\boldsymbol{V}|$. Thus the momentum transferred to the proton per unit time is

$$\frac{d\boldsymbol{P}}{dt}=\frac{1}{4\pi}\int n_e v m(\boldsymbol{v}-\boldsymbol{V})\sigma_c d\Omega = -n_e\sigma_c v m \boldsymbol{V}\,, \tag{1·10}$$

where the integral is over all directions of $\boldsymbol{v}$ and σ_c is the effective Coulomb cross section for momentum transfer [see, e.g., Ichimaru (1973)],

$$\sigma_c \simeq \frac{3}{2v^4}\sigma_T \ln\Lambda\,;\quad \Lambda=\frac{3}{2}\left(\frac{T^3}{\pi\alpha_{\rm em}^3 n_B(1+x_e)}\right)^{1/2} \tag{1·11}$$

with $\alpha_{\rm em}$ being the electromagnetic fine structure constant. Then the average energy loss rate per proton is

$$\begin{aligned}\left\langle -\boldsymbol{V}\cdot\frac{d\boldsymbol{P}}{dt}\right\rangle &= n_e\sigma_c v m\langle V^2\rangle \\ &= 3\frac{m}{M}n_e\sigma_c v T_p\,. \end{aligned}\tag{1·12}$$

In the thermal equilibrium this energy loss is compensated by the thermal energy of electrons. Therefore the equation for the proton temperature is given by

$$\dot{T}_p + 2\frac{\dot{a}}{a}T_p = \frac{1}{\tau_{\rm ep}}(T_e - T_p), \tag{1·13a}$$

where

$$\tau_{\rm ep} \equiv \frac{\sqrt{3}M}{x_e n_B m\sigma_T \ln\Lambda}\left(\frac{T}{m}\right)^{3/2}. \tag{1·13b}$$

Comparing this with Eq. (1·6a) one finds that $\tau_{\rm ep} \ll \tau_{\rm th}$ is satisfied only for $T \lesssim 10$eV. Hence the energy equipartition is not exactly established at $T \gtrsim 10$eV. Nevertheless, we find the inequality $\tau_{\rm ep} \ll \tau_{\rm ex}$ is still well satisfied. Thus

$$\frac{T_e - T_p}{T_p} \simeq \frac{2\tau_{\rm ep}}{\tau_{\rm ex}} \ll 1\,, \tag{1·14}$$

when seen on the timescale $\tau_{\rm ex}$. As we shall see in the following sections, all modes of perturbations which are of physical interests at $T \gtrsim 10$eV are not oscillatory within the timescale $\tau_{\rm ex}$. Hence we may assume $T_m = T_p = T_e = T$ above $T \sim 0.1$eV up to the temperature at which electrons become relativistic ($T \gtrsim 10^2$keV). This implies that the matter temperature plays no dynamical role by itself and it is not necessary to consider perturbations of the matter temperature separately at this stage.[†)] However this does not imply that we can neglect the matter pressure completely. As mentioned before, the matter pressure actually plays an important role in the form of sound velocity c_m when one considers the evolution of matter density perturbations with wavelength comparable

[†)] Except for the stage just after recombination. For detail, see the discussion in § 4.

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to or less than the matter sound horizon $\lambda \lesssim c_m t$.

Keeping the above arguments in mind, now we can write down the basic equations for evolution of perturbations in the baryon-photon system. The background equations of motion are

$$\dot{\rho}_r + 4\frac{\dot{a}}{a}\rho_r = 0\,, \tag{1·15a}$$

$$\dot{\rho}_m + 3\frac{\dot{a}}{a}\rho_m = 0\,, \tag{1·15b}$$

with the background Friedmann equation

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2} = \frac{\kappa^2}{3}(\rho_r + \rho_m). \tag{1·16}$$

The relevant equations of motion for scalar, vector and tensor perturbations are derived in Appendix E, taking account of the baryon-photon interaction through Thomson scattering of photons by electrons.

(1) Scalar perturbations

Comparing the general equations of motion for α-component given in Eqs. (II-5·46) with Eqs. (E·46) and (E·47), we readily find that the variable f_α representing the momentum transfer rate to each component is given by

$$f_r = \frac{\tau_{\rm ex}}{\tau_c}(v_m - v_r) = R_c(v_m - v_r), \tag{1·17a}$$

$$f_m = \frac{\tau_{\rm ex}}{\tau_{\rm drag}}(v_r - v_m) = \frac{4\rho_r}{3\rho_m}R_c(v_r - v_m), \tag{1·17b}$$

for $\alpha = r$ and m, with τ_c, $\tau_{\rm drag}$ and R_c defined by

$$\tau_c \equiv \frac{1}{n_e\sigma_T}\,, \tag{1·17c}$$

$$\tau_{\rm drag} \equiv \frac{3\rho_m}{4\rho_r}\tau_c\,, \tag{1·17d}$$

$$R_c \equiv \left(\frac{\dot{a}}{a}\right)^{-1} n_e\sigma_T = \frac{1}{H\tau_c}\,. \tag{1·17e}$$

Here, τ_c is the mean collision time of photons with electrons, $\tau_{\rm drag}$ is the timescale on which baryons would adjust their motion to the motion of photons by the Thomson dragging and R_c is the ratio of the horizon size to the mean free path for photons colliding with electrons. Inserting numerical values, we find

$$R_c = 7.8\times10^{12} x_e \left(\frac{s}{n_B}\right)^{-1}\left(\frac{T}{1\,\mathrm{eV}}\right)\left[1 + 1.25\times10^9\left(\frac{s}{n_B}\right)^{-1}\left(\frac{T}{1\,\mathrm{eV}}\right)^{-1}\right]^{-1/2}. \tag{1·18}$$

Hence before decoupling ($T \gtrsim 0.35$eV, $x_e \simeq 1$), R_c is much greater than unity, provided $s/n_B \sim 10^9$. While, the background and perturbed energy transfer rate are both zero by assumption. Thus from Eqs. (II-5·41b) and (II-5·43) we have $F_\alpha = F_{c\alpha} = f_\alpha$ ($\alpha = r, m$), that is, the momentum transfer rate is invariant under a first-order change of the frame in

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which it is observed. The equations for gauge-invariant variables are from Eqs. (II-5·48),

$$(\rho_r \Delta_r)' + 3\frac{a'}{a}\rho_r \Delta_r = 2\chi^2 h a^2 \rho_r \frac{1}{k}(V - V_r)$$

$$-\left(1-\frac{3K}{k^2}\right)\left(\frac{4}{3}\rho_r k V_r + \frac{2}{3}\frac{a'}{a}\rho_r \Pi_r\right) + \frac{4}{3}\frac{a'}{a}\rho_r R_c (V_m - V_r), \quad (1\cdot 19\text{a})$$

$$V_r' + \frac{a'}{a}V_r = k\Psi + \frac{1}{4}k\Delta_r - \frac{1}{6}k\left(1-\frac{3K}{k^2}\right)\Pi_r + \frac{a'}{a}R_c(V_m - V_r), \quad (1\cdot 19\text{b})$$

$$(\rho_m \Delta_m)' + 3\frac{a'}{a}\rho_m \Delta_m = \frac{3}{2}\chi^2 h a^2 \rho_m \frac{1}{k}(V - V_m)$$

$$-\left(1-\frac{3K}{k^2}\right)\rho_m k V_m + \frac{a'}{a}\rho_m R_c (V_r - V_m), \quad (1\cdot 19\text{c})$$

$$V_m' + \frac{a'}{a}V_m = k\Psi + k c_m^2 \Delta_m + \frac{a'}{a}\frac{4\rho_r}{3\rho_m}R_c(V_r - V_m), \quad (1\cdot 19\text{d})$$

and from Eq. (E·46c),

$$\Pi_r' - \frac{8}{5}kV_r = -\frac{a'}{a}R_c \Pi_r . \quad (1\cdot 19\text{e})$$

As we have emphasized in §II-5, when one considers the coupling between isothermal perturbations and adiabatic perturbations, it is more convenient to rewrite the equations in terms of $S_{\alpha\beta}$ and $V_{\alpha\beta}$. From Eqs. (II-5·53) and (II-5·57), they are in the present case given by

$$S_{mr}' = -kV_{mr}, \quad (1\cdot 20\text{a})$$

$$V_{mr}' + \frac{a'}{a}\left\{\frac{4}{3}\frac{\rho_r}{h}(1-3c_m^2) + \frac{h}{\rho_m}R_c\right\}V_{mr}$$

$$= k\left(c_m^2 - \frac{1}{3}\right)\frac{\rho\Delta}{h} + k\frac{\frac{1}{3}\rho_m + \frac{4}{3}c_m^2\rho_r}{h}S_{mr} + \frac{1}{6}k\left(1-\frac{3K}{k^2}\right)\Pi_r , \quad (1\cdot 20\text{b})$$

where $\rho = \rho_r + \rho_m$ and $h = \rho_r + p_r + \rho_m$. Further, Eqs. (1·20a) and (1·20b) can be combined into the single second-order equation for S_{mr}:

$$S_{mr}'' + \frac{a'}{a}\left\{\frac{4\rho_r}{3h}(1-3c_m^2) + \frac{h}{\rho_m}R_c\right\}S_{mr}' + k^2\frac{\rho_m + 4c_m^2\rho_r}{3h}S_{mr}$$

$$= k^2\frac{\rho}{3h}(1-3c_m^2)\Delta - \frac{1}{6}k^2\left(1-\frac{3K}{k^2}\right)\Pi_r . \quad (1\cdot 21)$$

Since $S_{mr} = \delta(n_B/s)/(n_B/s)$ from Eq. (II-5·38), this is the equation for isothermal perturbations. It shows how isothermal perturbations are coupled with adiabatic perturbations. On the other hand, the equation for total energy density perturbations has been given by Eq. (II-4·9). Writing it in the form appropriate for the present case, we have

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$$(\rho a^3\varDelta)''+(1+3c_s{}^2)\frac{a'}{a}(\rho a^3\varDelta)'+\left\{(k^2-3K)c_s{}^2-\frac{\chi^2}{2}ha^2\right\}(\rho a^3\varDelta)$$
$$=\left(1-\frac{3K}{k^2}\right)\left[\frac{4\rho_r\rho_m a^3}{9h}k^2(1-3c_m{}^2)S_{mr}\right.$$
$$\left.+\frac{2}{9}\{(k^2+3(1+3c_s{}^2)K)+\chi^2(\rho_r-3c_s{}^2\rho)a^2\}\rho_r a^3\varPi_r-\frac{2}{3}a'(\rho_r a^2\varPi_r)'\right], \tag{1·22}$$

where c_s is the total sound velocity defined by Eq. (II-5·33). Explicitly it is given by

$$c_s{}^2=\frac{4\rho_r+9c_m{}^2\rho_m}{3(4\rho_r+3\rho_m)}. \tag{1·23}$$

We note that although perturbations in the total energy density are frequently termed as "adiabatic", this terminology is somewhat misleading. Rather, though not yet exactly correct, it would be more relevant to regard $\varDelta_r$ or $\varDelta_{cr}$ as representing adiabatic perturbations. However, since the initial situation we are concerned with is radiation-dominated to a high degree, we have $\varDelta_r\simeq\varDelta_{cr}\simeq\varDelta$ initially and regarding $\varDelta$ as the amplitude of an adiabatic perturbation is justified. Of course, the final amplitude of $\varDelta$ after integrating the set of Eqs. (1·21) and (1·22) would not represent the amplitude of the adiabatic perturbation but one should properly separate the contribution of S_{mr} from $\varDelta$. Given solutions of S_{mr} and $\varDelta$, this can be easily done by noting the relation

$$\rho\varDelta=\frac{3}{4}h\varDelta_{cr}+\rho_m S_{mr}. \tag{1·24}$$

The above argument suggests that sometimes it would be more convenient to regard $\varDelta_{cr}$ and S_{mr} as the dynamical variables from the beginning. Then replacing $\varDelta$ in Eq. (1·21) by the expression (1·24), we find

$$S''_{mr}+\frac{a'}{a}\left\{\frac{4\rho_r}{3h}(1-3c_m{}^2)+\frac{h}{\rho_m}R_c\right\}S'_{mr}+c_m{}^2k^2S_{mr}$$
$$=\frac{1}{4}k^2(1-3c_m{}^2)\varDelta_{cr}-\frac{1}{6}k^2\left(1-\frac{3K}{k^2}\right)\varPi_r. \tag{1·25}$$

This clearly shows that isothermal perturbations can have oscillatory behavior only on scales smaller than the matter sound horizon. Also inserting the expression (1·24) into Eq. (1·22) and eliminating S''_{mr} with the help of Eq. (1·25) give

$$(ha^3\varDelta_{cr})''+(1+3c_s{}^2)\frac{a'}{a}(ha^3\varDelta_{cr})'+\left(\frac{1}{3}k^2-\frac{\chi^2}{2}ha^2\right)(ha^3\varDelta_{cr})$$
$$=\frac{4}{3}\rho_m a^3\left\{\frac{\chi^2}{2}ha^2S_{mr}-\frac{a'}{a}\left(1+3c_m{}^2-\frac{h}{\rho_m}R_c\right)S'_{mr}\right\}$$
$$+\frac{1}{6}k^2ha^3\varPi_r+\frac{2}{9}\chi^2(\rho_r-3c_s{}^2\rho)a^2\frac{4}{3}\rho_r a^3\varPi_r-\frac{2}{3}a'\left(\frac{4}{3}\rho_r a^2\varPi_r\right)'. \tag{1·26}$$

Finally, we rewrite Eq. (1·19e) for $\varPi_r$ in the relevant form by expressing V_r in terms of V and V_{mr} and eliminating them with the help of Eqs. (II-4·7a) and (1·20a). The result

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is

$$\Pi_r' + \frac{a'}{a}\left(\frac{16\rho_r}{15h} + R_c\right)\Pi_r = -\frac{8}{5}\left\{\frac{k^2(\rho a^3 \Delta)'}{(k^2-3k)ha^3} - \frac{\rho_m}{h}S'_{mr}\right\}, \tag{1·27}$$

or in terms of Δ_{cr} and S_{mr}, it takes the form

$$\Pi_r' + \frac{a'}{a}\left(\frac{16\rho_r}{15h} + R_c\right)\Pi_r = -\frac{6}{5}\left\{\frac{k^2(ha^3\Delta_{cr})'}{(k^2-3K)ha^3} + \frac{4K}{k^2-3K}\frac{\rho_m}{h}S'_{mr}\right\}. \tag{1·28}$$

Thus either the set of Eqs. (1·21), (1·22) and (1·27) or that of Eqs. (1·25), (1·26) and (1·28) can be used for investigating the time-evolution of scalar perturbations in the baryon-photon system.

(2) Vector perturbations

From Eqs. (E·48) of Appendix E, we readily obtain

$$V_r^{(1)\prime} + \frac{1}{8}k\left(1-\frac{2K}{k^2}\right)\Pi_r^{(1)} = \frac{a'}{a}R_c(V_m^{(1)} - V_r^{(1)}), \tag{1·29a}$$

$$V_m^{(1)\prime} + \frac{a'}{a}(1-3c_m^2)V_m^{(1)} = \frac{4\rho_r}{3\rho_m}\frac{a'}{a}R_c(V_r^{(1)} - V_m^{(1)}), \tag{1·29b}$$

$$\Pi_r^{(1)\prime} + \frac{a'}{a}R_c\Pi_r^{(1)} = \frac{8}{5}k(V_r^{(1)} - \sigma_g^{(1)}). \tag{1·29c}$$

Since $\sigma_g^{(1)}$ is algebraically related to $V^{(1)}$ through Eq. (II-4·10a) and $V^{(1)}$ is expressed in terms of $V_r^{(1)}$ and $V_m^{(1)}$ by Eq. (II-5·70b), Eqs. (1·29) form already a complete set of equations describing the time-evolution of vector perturbations. Alternatively, we may rewrite Eqs. (1·29) in terms of $V^{(1)}$, $\Pi_r^{(1)}$ and $V_{mr}^{(1)} \equiv V_m^{(1)} - V_r^{(1)}$. The resulting equations are

$$(ha^4V^{(1)})' = -\frac{1}{6}k\left(1-\frac{2K}{k^2}\right)\rho_r a^4 \Pi_r^{(1)}, \tag{1·30a}$$

$$\Pi_r^{(1)\prime} + \frac{a'}{a}R_c\Pi_r^{(1)} = \frac{8}{5}kV^{(1)}\left(1+\frac{2x^2ha^2}{k^2-2K}\right) - \frac{8}{5}k\frac{h_m}{h}V_{mr}^{(1)}, \tag{1·30b}$$

$$V_{mr}^{(1)\prime} + \frac{a'}{a}\left\{\frac{4\rho_r}{3h}(1-3c_m^2) + \frac{h}{\rho_m}R_c\right\}V_{mr}^{(1)} = \frac{1}{8}k\left(1-\frac{2K}{k^2}\right)\Pi_r^{(1)} - \frac{a'}{a}(1-3c_m^2)V^{(1)}. \tag{1·30c}$$

The first two equations can be combined into the single second-order equation for $V^{(1)}$:

$$(ha^4V^{(1)})'' + \frac{a'}{a}R_c(ha^4V^{(1)})' + \frac{4\rho_r}{15h}(k^2-2K+2x^2ha^2)(ha^4V^{(1)}) = \frac{4\rho_m\rho_r a^4}{15h}(k^2-2K)V_{mr}^{(1)}. \tag{1·31}$$

This shows how the vorticity is affected by the relative motion of radiation and matter. It also shows that gravity can never enhance the vorticity, the fact which is apparently expected.

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(3) Tensor perturbations

As for tensor perturbations, Eq. (E·49) for $\Pi_T^{(2)}$ is the only non-trivial equation of motion. Therefore Eq. (II-4·15) with the identification $\Pi^{(2)}=\Pi_T^{(2)}$ and Eq. (E·49) describe the time-evolution of tensor perturbations completely. Namely,

$$H_T^{(2)\prime\prime}+2\frac{a'}{a}H_T^{(2)\prime}+(k^2+2K)H_T^{(2)}=\chi^2\frac{\rho_T a^2}{3}\Pi_T^{(2)}, \tag{1·32a}$$

$$\Pi_T^{(2)\prime}+\frac{a'}{a}R_c\Pi_T^{(2)}=-\frac{8}{5}H_T^{(2)\prime}. \tag{1·32b}$$

In the rest of this chapter, we shall not be concerned with tensor perturbations any more. However, here we mention one important implication of Eqs. (1·32). In the case $R_c\gg ka/a'$, $\Pi_T^{(2)\prime}$ in Eq. (1·32b) can be neglected (which corresponds to the usual viscous fluid approximation). Under this approximation Eq. (1·31a) takes the form

$$H_T^{(2)\prime\prime}+2\frac{a'}{a}\left(1+\frac{4}{5}\frac{\chi^2\rho_T a^2}{\chi^2\rho a^2-3K}\frac{1}{R_c}\right)H_T^{(2)\prime}+(k^2+2K)H_T^{(2)}=0. \tag{1·33}$$

This shows how the viscosity ($\propto R_c^{-1}$) affects the propagation of gravitational waves. More definitely, further assuming $ka/a'\gg 1$ enables us to solve Eq. (1·33) by the WKB method. The solution is

$$H_T^{(2)}\propto\frac{1}{a}\exp\left[\int^{\eta}\left(\pm ik-\frac{4\rho_T}{5\rho}\frac{a'}{aR_c}\right)d\eta\right], \tag{1·34}$$

where we have put $K=0$ for simplicity. Thus the presence of viscosity leads to additional damping of gravitational waves, besides the usual adiabatic damping due to cosmological expansion. This effect was first observed by Hawking (1966). The characteristic decay time is given by

$$\tau_{\text{decay}}\simeq\frac{5\rho}{4\rho_T}R_c\tau_{\text{ex}}=\frac{5\rho}{4\rho_T}n_e\sigma_T\tau_{\text{ex}}^2, \tag{1·35}$$

irrespective of the wave number k.

§ V-2. Perturbations on super-horizon scales

In this section, we investigate the behavior of perturbations on scales greater than the horizon, i.e., we assume $a'/ka\gg 1$. For such large scales, it is apparent that the matter sound velocity plays no essential role. Hence we may put $c_m^2=0$ and the system becomes equivalent to the radiation-dust universe considered in §§ IV-4 and 5 as a whole. The difference is that we take account of the interaction between photons and baryons explicitly in the present case.

(1) Scalar perturbations

For convenience, we choose the set of Eqs. (1·25), (1·26) and (1·28) as the fundamental equations. Then introducing a new set of variables X, Y and Z by

$$X\equiv ha^3\Delta_{cr}, \tag{2·1a}$$

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$$Y \equiv \rho_m a^3 S_{mr}\,, \tag{2·1b}$$

$$Z \equiv h a^3 \Pi_r\,, \tag{2·1c}$$

and putting $c_m{}^2=0$, these equations take the form

$$\begin{aligned} X''+(1+3c_s{}^2)\frac{a'}{a}X'+\Big(\frac{1}{3}k^2-\frac{\varkappa^2}{2}ha^2\Big)X &\\ =\frac{4}{3}\Big\{\frac{\varkappa^2}{2}ha^2Y-\frac{a'}{a}\Big(1-\frac{h}{\rho_m}R_c\Big)Y'\Big\} &\\ +\frac{2}{9}\Big\{k^2+\varkappa^2(3\rho_r+\rho_m-6c_s{}^2\rho)a^2\frac{h_r}{h}\Big\}Z-\frac{2}{3}\frac{h_r}{h}\frac{a'}{a}Z'\,, & \end{aligned}\tag{2·2a}$$

$$Y''+\Big(\frac{h_r}{h}+\frac{h}{\rho_m}R_c\Big)\frac{a'}{a}Y'=k^2\frac{\rho_m}{4h}\Big(X-\frac{2}{3}Z\Big), \tag{2·2b}$$

$$Z'+\Big(\frac{9h_r}{5h}+R_c\Big)\frac{a'}{a}Z=-\frac{6}{5}X'\,, \tag{2·2c}$$

where we have set $K=0$ for simplicity.

First let us consider the homogeneous versions of Eqs. (2·2) by setting all the right-hand sides of them equal to zero. As for Eq. (2·2a), the left-hand side of it has exactly the same structure as the equation considered in §IV-4 in the limit $a'/ka \gg 1$. Therefore the solution is given by

$$X \simeq \begin{cases} A\zeta+B\zeta^{-2}\,; & \zeta \ll 1\,,\\ \frac{3}{4}\Big(\frac{9}{16}A\zeta+B\zeta^{-3/2}\Big); & \zeta \gg 1\,, \end{cases} \tag{2·3}$$

where ζ is the cosmic scale factor normalized to unity at epoch $\rho_r=\rho_m$. Because of their simple structures, the homogeneous solutions of Eqs. (2·2b) and (2·2c) can be obtained by direct integration. In terms of the variable ζ, they are

$$Y=Y_0+Y_1\int_{\zeta_*}^{\zeta}\frac{d\zeta_1}{\sqrt{1+\zeta_1}}\exp\Big[-\int_{\zeta_*}^{\zeta_1}F(\zeta_2)\frac{d\zeta_2}{\zeta_2}\Big] \tag{2·4}$$

and

$$Z=Z_0\exp\Big[-\int_{\zeta_*}^{\zeta}G(\zeta_1)\frac{d\zeta_1}{\zeta_1}\Big], \tag{2·5}$$

where $F(\zeta)$ and $G(\zeta)$ are defined by

$$F(\zeta)\equiv\frac{h}{\rho_m}R_c(\zeta)+\frac{h_r}{h}=\frac{\zeta+4/3}{\zeta}R_c(\zeta)+\frac{1}{1+3\zeta/4}\,, \tag{2·6a}$$

$$G(\zeta)\equiv R_c(\zeta)+\frac{9h_r}{5h}=R_c(\zeta)+\frac{9}{5(1+3\zeta/4)}, \tag{2·6b}$$

and $\zeta_*=\zeta(\eta_*)$ with η_* being an arbitrary reference time. As evaluated in Eq. (1·18), the quantity R_c, which is the ratio of the horizon size to the photon mean free path, is much greater than unity at the cosmological stage of our interest. Hence we have $F\gg1$ and $G \gg 1$. For the solution Y given by Eq. (2·4), this implies that the second term of it settles

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down to a constant value immediately after it is provoked. Thus the isothermal perturbation has essentially one mode which is constant with time. This is because the matter is so tightly coupled with radiation that it cannot move relative to radiation and inhomogeneities in the matter density relative to radiation density are virtually frozen. Note that Eq. (2·4) is valid even on scales smaller than the horizon size down to the matter sound horizon. Therefore, the above consideration on the isothermal mode remains true on almost all scales of interest, provided $R_c \gg 1$. For the solution (2·5) of Z, it vanishes immediately after it is provoked. Thus there exists no independent mode with respect to the anisotropic stress perturbation. That is, any instantaneous perturbation in the anisotropic stress cannot persist independently of the state of the other variables.

Now let us consider the coupling among the variables X, Y and Z. The effect of the presence of Y or Z on X has been discussed in detail in §IV-5. In the case Y is provoked at certain epoch ζ_*, the value of it, hence of S_{mr} stays constant as we have shown above. On the other hand, from Eq. (II-5·36) we have

$$\Gamma = \Gamma_{\rm rel} = -\frac{\zeta}{1+3\zeta/4} S_{mr} . \tag{2·7}$$

Therefore assuming $\zeta_* \ll 1$, Γ increases as ζ initially and becomes constant at $\zeta \gg 1$. Then from the results obtained in § IV-5, cases (i) and (ii), we find the generated amplitude of the adiabatic density perturbation when it enters the horizon is of order S_{mr}. As for Z, since it works only within the interval of order $\Delta\zeta \sim \zeta_*/R_c(\zeta_*) \ll \zeta_*$, the result given by Eq. (IV-5·16) with the identifications $\Gamma = -2/3\Pi_\gamma(\zeta_*)$ and $\ln(\zeta_2/\zeta_1) \simeq \Delta\zeta/\zeta_*$ is relevant, which reads

$$\Delta(t_H) \simeq \frac{10}{216} \frac{\Pi_\gamma(\zeta_*)}{R_c(\zeta_*)} . \tag{2·8}$$

Hence the generated amplitude is suppressed by the factor $R_c(\zeta_*) \gg 1$ compared with the case of S_{mr}. This implies that the anisotropic stress can never be an effective source of density perturbations in the stage under consideration ($T \lesssim 10$ keV).

It now remains to be considered the case when X is present from the beginning but Y and Z are not. First consider Eq. (2·2c) for Z. This can be readily integrated by using the homogeneous solution (2·5). The result is

$$Z(\zeta) = -\frac{6}{5} \int_{\zeta_0}^{\zeta} \exp\left[-\int_{\zeta_1}^{\zeta} G(\zeta_2) \frac{d\zeta_2}{\zeta_2} \right] \frac{d}{d\zeta_1} X(\zeta_1) d\zeta_1 , \tag{2·9}$$

where ζ_0 is the epoch at which the initial condition for X is specified. Since $G \sim R_c \gg 1$ while X is relatively slowly varying according to Eq. (2·3), Eq. (2·9) is simplified to give

$$Z(\zeta) \simeq -\frac{6}{5} \frac{\zeta}{R_c(\zeta)} \frac{d}{d\zeta} X(\zeta). \tag{2·10}$$

This is just the anisotropic stress one would obtain under the viscous fluid approximation, which is valid if the wavelength is much greater than the photon mean free path. That this condition is satisfied in the present case should be apparent since $R_c a'/ka \gg a'/ka \gg 1$. Now consider Eq. (2·2b) for Y. Rewriting it in terms of the independent variable ζ and integrating it twice, we obtain

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$$Y(\zeta)=\frac{1}{2\gamma_{\mathrm{eq}}^2}\int_{\zeta_0}^{\zeta}\int_{\zeta_0}^{\zeta_1}\exp\left[-\int_{\zeta_2}^{\zeta_1}F(\zeta_3)\frac{d\zeta_3}{\zeta_3}\right]S(\zeta_2)\frac{d\zeta_2}{\sqrt{1+\zeta_2}}\frac{d\zeta_1}{\sqrt{1+\zeta_1}}, \tag{2·11}$$

where

$$S(\zeta)\equiv\frac{\rho_m}{h}\left[X(\zeta)-\frac{2}{3}Z(\zeta)\right]=\frac{\zeta}{\zeta+4/3}\left[X(\zeta)-\frac{2}{3}Z(\zeta)\right], \tag{2·12}$$

and $\gamma_{\mathrm{eq}}=(a'/ka)_{\zeta=1}$ is the ratio of the reduced proper wavelength to horizon size at $\rho_r=\rho_m$. Under the same approximation used to derive Eq. (2·10) for $Z(\zeta)$, the solution $Y(\zeta)$ given by Eq. (2·11) reduces to

$$Y(\zeta)\simeq\frac{1}{2\gamma_{\mathrm{eq}}^2}\int_{\zeta_0}^{\zeta}\frac{\zeta_1{}^3}{(\zeta_1+1)(\zeta_1+4/3)^2}\frac{X(\zeta_1)}{R_c(\zeta_1)}d\zeta_1, \tag{2·13}$$

where the fact that $Z(\zeta)\ll X(\zeta)$ implied by Eqs. (2·3) and (2·10) has been used to neglect the contribution of $Z(\zeta)$ to $S(\zeta)$.

Thus we are left with an interesting possibility that isothermal perturbations may be generated from adiabatic perturbations. In order to investigate this possibility further, consider the case when X has only the growing mode initially at $\zeta\ll1$. Then from Eq. (2·3) we have

$$X\simeq\begin{cases}A\zeta & ;\quad \zeta\ll1,\\ \left(\frac{3}{4}\right)^3A\zeta & ;\quad \zeta\gg1.\end{cases} \tag{2·14}$$

While from Eq. (1·17c) or (1·18), R_c is expressed in terms of ζ as

$$R_c=\frac{\sqrt{2}}{(1+\zeta)^{1/2}\zeta}R_{\mathrm{eq}}, \tag{2·15}$$

where R_{eq} is the value of R_c at $\zeta=1$ and we have assumed $x_e=1$ throughout the stage under consideration. Inserting Eqs. (2·14) and (2·15) into Eq. (2·13) gives

$$Y(\zeta)\simeq\begin{cases}\dfrac{A}{12\sqrt{2}\gamma_{\mathrm{eq}}^2R_{\mathrm{eq}}}\left(\dfrac{3}{4}\right)^2\zeta^6; & \zeta\ll1,\\ \dfrac{A}{56\sqrt{2}\gamma_{\mathrm{eq}}^2R_{\mathrm{eq}}}\left(\dfrac{3}{4}\right)^2\zeta^{7/2}; & \zeta\gg1.\end{cases} \tag{2·16}$$

This leads to the expression for S_{mr} in terms of $\varDelta_{cr}$, R_c and γ as

$$S_{mr}(\zeta)\simeq\begin{cases}\dfrac{\varDelta_{cr}}{32R_c\gamma^2}\zeta^3 & ;\quad \zeta\ll1,\\ \dfrac{\varDelta_{cr}}{84R_c\gamma^2} & ;\quad \zeta\gg1,\end{cases} \tag{2·17}$$

where $\gamma\equiv a'/ka$. Thus when the wavelength comes into the horizon ($\gamma\simeq1$) we find

$$S_{mr}(\zeta_H)\simeq\begin{cases}\dfrac{\varDelta_{cr}(\zeta_H)}{32R_c(\zeta_H)}\zeta_H{}^3 & ;\quad \zeta_H\ll1,\\ \dfrac{\varDelta_{cr}(\zeta_H)}{84R_c(\zeta_H)} & ;\quad \zeta_H\gg1,\end{cases} \tag{2·18}$$

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where ζ_H is the value of ζ at $\gamma=1$. This implies the generated amplitude of the isothermal perturbation is depressed by a factor of order $1/R_c(\zeta_H)$ compared with the amplitude of the adiabatic perturbation, where $R_c(\zeta_H)\gg 1$ by assumption.

Although practically it was found to be totally negligible, it is conceptually interesting if isothermal perturbations could ever be generated from adiabatic ones. In this respect, we note that the amplitude in Eq. (2·17) is not really what we call isothermal. As we have discussed before, a real isothermal perturbation is characterized by its constancy with time, i.e., the mode carried by S_{mr} intrinsically. On the contrary, the amplitude given by Eq. (2·17) is always associated with an adiabatic perturbation, hence it is intrinsically not isothermal. In fact, as we shall see in the next section, this induced perturbation in S_{mr} is the origin of damped adiabatic oscillation when the wavelength comes into the horizon. However, since the distinction of the isothermal mode from the (two) adiabatic modes is not really possible on scales greater than the horizon size, if S_{mr} and Δ_{cr} have non-negligible influence on each other, we cannot deny the possibility that some portion of the amplitude generated in S_{mr} according to Eq. (2·17) turns into the intrinsic isothermal mode when the perturbation appears on the horizon. For example, consider a fictitious situation in which one is allowed to control the value of R_c as one wishes. Then let R_c be such that it is large but finite up to an epoch $\zeta=\zeta_1$ and it is infinite (i.e., the strong coupling limit) thereafter. Under such a situation, Eq. (2·13) implies $Y(\zeta)$, hence $S_{mr}(\zeta)$ becomes constant with time at $\zeta>\zeta_1$. Thus the purely isothermal mode is generated. Physically, this is interpreted as follows: Initially, the matter is tightly coupled with photons and they move together, but finiteness of R_c allows the gradual diffusion of photons away from the matter and S_{mr} is generated consequently. Then after $\zeta=\zeta_1$, the infinitely strong coupling given between the matter and radiation forces the relative density contrast imprinted on S_{mr} to be frozen and the purely isothermal mode is left behind.

In conclusion, adiabatic perturbations can be generated from isothermal ones but the generated amplitude when they appear on the horizon is of the same order as the initial amplitude of isothermal perturbations. Anisotropic stress perturbations can never be an effective source for adiabatic perturbations. Adiabatic perturbations existing from the beginning induce isothermal perturbations but the resulting amplitude is by no means appreciable nor it is intrinsically isothermal. Whether the mechanism of generating purely isothermal perturbations from adiabatic ones discussed here has some relevance in the actual history of the universe is a future problem.

(2) Vector perturbations

Basic equations are those given by Eqs. (1·30). However, for convenience, we use Eqs. (1·30c) and (1·31) to investigate the behavior of vector perturbations, where $\Pi_r^{(1)}$ appearing in the former equation should be eliminated with the help of Eq. (1·30a) so that these equations form a closed system for the variables $V^{(1)}$ and $V_{mr}^{(1)}$. Defining the variables $X^{(1)}$ and $Y^{(1)}$ by

$$X^{(1)}\equiv ha^4 V^{(1)}, \tag{2·19a}$$

$$Y^{(1)}\equiv h_r a^4 V_{mr}^{(1)}, \tag{2·19b}$$

they take the form

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$$X^{(1)\prime\prime}+R_c\frac{a'}{a}X^{(1)\prime}+\frac{1}{5}\left(\frac{h_r}{h}k^2+8\frac{\rho_r}{\rho}\left(\frac{a'}{a}\right)^2\right)X^{(1)}=\frac{\rho_m}{5h}k^2Y^{(1)}, \tag{2·20a}$$

$$Y^{(1)\prime}+\left(\frac{h_r}{h}+\frac{h}{\rho_m}R_c\right)\frac{a'}{a}Y^{(1)}=-X^{(1)\prime}-\frac{h_r}{h}\frac{a'}{a}X^{(1)}, \tag{2·20b}$$

where we have put $c_m{}^2=K=0$ as before. We see that the left-hand side of Eq. (2·20b) has exactly the same form as that of Eq. (2·2b) with Y' identified as $Y^{(1)}$. Therefore we readily obtain

$$Y^{(1)}(\zeta)=-\int_{\zeta_0}^{\zeta}\exp\left[-\int_{\zeta_1}^{\zeta}F(\zeta_2)\frac{d\zeta_2}{\zeta_2}\right]S^{(1)}(\zeta_1)\frac{d\zeta_1}{\zeta_1}, \tag{2·21}$$

where

$$S^{(1)}(\zeta)\equiv\zeta\frac{d}{d\zeta}X^{(1)}(\zeta)+\frac{1}{1+3\zeta/4}X^{(1)}(\zeta), \tag{2·22}$$

and $F(\zeta)$ has been defined by Eq. (2·6a). In the limit $R_c\gg1$, Eq. (2·21) reduces to

$$Y^{(1)}(\zeta)\simeq-\frac{\zeta}{(\zeta+4/3)R_c(\zeta)}S^{(1)}(\zeta). \tag{2·23}$$

When this is inserted into Eq. (2·20a), we find that the contribution of $Y^{(1)}$ to this equation is negligible if $(a'/ak)R_c\gg1$, which is just the condition for the viscous fluid approximation. Hence changing the independent variable from η to ζ, Eq. (2·20a) under the assumptions $R_c\gg1$ and $(a'/ak)R_c\gg1$ takes the form

$$\left[\frac{d^2}{d\zeta^2}+\frac{R_c(\zeta)}{\zeta}\frac{d}{d\zeta}+\frac{W(\zeta)}{\zeta^2}\right]X^{(1)}(\zeta)=0, \tag{2·24}$$

where

$$W(\zeta)\equiv\frac{1}{5}\left(\frac{h_rk^2a'^2}{ha^2}+\frac{8\rho_r}{\rho}\right)=\frac{1}{5}\left\{\frac{1}{\gamma^2(1+3\zeta/4)}+\frac{8}{1+\zeta}\right\}. \tag{2·25}$$

Furthermore, with the same assumptions empolyed so far, one can put Eq. (2·24) into the factorized form,

$$\left[\frac{d}{d\zeta}+\frac{R_c(\zeta)}{\zeta}\right]\left[\frac{d}{d\zeta}+\frac{W(\zeta)}{R_c(\zeta)\zeta}\right]X^{(1)}(\zeta)=0. \tag{2·26}$$

Note that in reducing the original equations into Eq. (2·26), it has not been necessary to impose the condition $ka/a'\ll1$. Therefore Eq. (2·26) and consequently the following analysis is valid even on scales smaller than the horizon size, provided $ka/a'\ll R_c$ and $R_c\gg1$.

Because of its factorized form, the general solution to Eq. (2·26) can be easily obtained, yielding

$$X^{(1)}(\zeta)=X_0\exp\left[-\int_{\zeta_0}^{\zeta}\frac{W(\zeta_1)}{R_c(\zeta_1)}\frac{d\zeta_1}{\zeta_1}\right]$$
$$+X_1\int_{\zeta_0}^{\zeta}\exp\left[-\int_{\zeta_1}^{\zeta}\frac{W(\zeta_2)}{R_c(\zeta_2)}\frac{d\zeta_2}{\zeta_2}\right]\exp\left[-\int_{\zeta_0}^{\zeta_1}R_c(\zeta_3)\frac{d\zeta_3}{\zeta_3}\right]d\zeta_1. \tag{2·27}$$

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Now due to the assumption $R_c \gg 1$, the integrand of the second term is non-vanishing only within the initial interval $\Delta\zeta_0 \sim \zeta_0/R_c(\zeta_0) \ll 1$. For this interval we have

$$\frac{W(\zeta_0)}{R_c(\zeta_0)}\frac{\Delta\zeta_0}{\zeta_0} \sim \frac{W(\zeta_0)}{R_c^2(\zeta_0)} \ll 1\,, \tag{2·28}$$

where the last double-inequality follows from the assumptions $R_c \gg 1$ and $R_c \gg ka/a'$. Hence for $\zeta \gg \zeta_0$, the second term is simply given by

$$\sim \frac{\zeta_0}{R_c(\zeta_0)} X_1 \exp\left[-\int_{\zeta_0}^{\zeta} \frac{W(\zeta_1)}{R_c(\zeta_1)}\frac{d\zeta_1}{\zeta_1}\right], \tag{2·29}$$

which is equivalent to the first term. Therefore, there is essentially a unique mode associated with Eq. (2·26).

As it is immediately clear, this mode is a decaying mode with the characteristic decay time given by

$$\tau_{\text{decay}} \simeq \frac{R_c}{W}\tau_{\text{ex}}\,. \tag{2·30}$$

Since $X^{(1)}$ represents the vorticity of a fluid element within unit comoving volume, the above result implies that the photon viscosity induces damping of the vorticity. From the definitions of R_c and W, Eqs. (1·17c) and (2·25), respectively, τ_{decay} is more specifically expressed as

$$\tau_{\text{decay}} \simeq \begin{cases} \dfrac{5\rho}{8\rho_r} n_e\sigma_T \tau_{\text{ex}}^2 & ; \quad \lambda/2\pi \gg \tau_{\text{ex}}\,, \\ \dfrac{5h}{h_r} n_e\sigma_T \left(\dfrac{\lambda}{2\pi}\right)^2 & ; \quad (n_e\sigma_T)^{-1} \ll \lambda/2\pi \ll \tau_{\text{ex}}\,, \end{cases} \tag{2·31}$$

where $\lambda = 2\pi a/k$ is the proper wavelength of a perturbation. Thus the damping is severe for wavelengths which satisfy $\tau_{\text{decay}} < \tau_{\text{ex}}$, that is,

$$\lambda < \lambda_d^{(1)} \equiv 2\pi\sqrt{\frac{h_r}{5h}\frac{\tau_{\text{ex}}}{n_e\sigma_T}}\,. \tag{2·32}$$

As one will find in the next section, this critical wavelength $\lambda_d^{(1)}$ for the viscous damping of vorticity is essentially the same as that for the damping of adiabatic perturbations.

§ V-3. Perturbations on sub-horizon scales

For perturbations on scales smaller than the horizon size, we need not worry about the gauge freedom associated with perturbations and the variables Δ and V, for example, may be interpreted directly as the density perturbation amplitude and the velocity perturbation amplitude, respectively. Further, we may confine our attention to the behavior of perturbations on timescales much shorter than the expansion time τ_{ex}. That is, we can assume the background to be static and use the technique of dispersion relation by putting all the perturbation amplitudes proportional to $e^{i\omega t}$.

(1) Scalar perturbations

Although we have chosen the set of Eqs. (1·25), (1·26) and (1·28) as the basic

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equations in the previous section, here we choose the set of Eqs. (1·21), (1·22) and (1·27) instead, since the quantities related to the matter and those related to the radiation appear in the symmetric way in those equations, which is a rather convenient property for the following analysis. Neglecting the effect of cosmic expansion, they take the form

$$\ddot{\Delta}+(\omega_s{}^2-\omega_J{}^2)\Delta=\frac{h_r h_m}{\rho h}(\omega_r{}^2-\omega_m{}^2)S_{mr}+\frac{h}{2\rho}\omega_r{}^2\Pi_r\,, \tag{3·1a}$$

$$\ddot{S}_{mr}+\frac{h}{h_m}\omega_c\dot{S}_{mr}+\left(\frac{h_m}{h}\omega_r{}^2+\frac{h_r}{h}\omega_m{}^2\right)S_{mr}=\frac{\rho}{h}(\omega_r{}^2-\omega_m{}^2)\Delta-\frac{1}{2}\omega_r{}^2\Pi_r\,, \tag{3·1b}$$

$$\dot{\Pi}_r+\omega_c\Pi_r=\frac{8}{5}\left(\frac{h_m}{h}\dot{S}_{mr}-\frac{\rho}{h}\dot{\Delta}\right), \tag{3·1c}$$

where

$$\omega_r{}^2\equiv c_r{}^2\frac{k^2}{a^2}\,,\qquad \omega_m{}^2\equiv c_m{}^2\frac{k^2}{a^2}\,,$$

$$\omega_s{}^2\equiv c_s{}^2\frac{k^2}{a^2}=\frac{h_r}{h}\omega_r{}^2+\frac{h_m}{h}\omega_m{}^2\,,$$

$$\omega_J{}^2\equiv\frac{\chi^2}{2}(\rho+p),\quad \omega_c\equiv\frac{\dot{a}}{a}R_c=n_e\sigma_T\,, \tag{3·2}$$

and the spatial curvature K has been neglected, which is surely valid for $k/a\gg\dot{a}/a$. Note that although $\omega_J{}^2$ is of order $(\dot{a}/a)^2$ if the background equation were used, this term is retained since it is due to the gravity of the perturbed density rather than the effect of expansion.

It is now straightforward to derive the dispersion relation for the system of Eqs. (3·1) by setting Δ, S_{mr} and Π_r proportional to $e^{i\omega t}$. First from Eq. (3·1c) we obtain

$$\Pi_r=2\alpha\left(\frac{h_m}{h}S_{mr}-\frac{\rho}{h}\Delta\right), \tag{3·3a}$$

where α is defined by

$$\alpha\equiv\frac{4}{5}\frac{i\omega}{\omega_c+i\omega}\,. \tag{3·3b}$$

Inserting Eq. (3·3a) into Eqs. (3·1a) and (3·1b) readily yields

$$(-\omega^2+\tilde{\omega}_s{}^2-\omega_J{}^2)\Delta=\frac{h_r h_m}{\rho h}(\tilde{\omega}_r{}^2-\omega_m{}^2)S_{mr}\,, \tag{3·4a}$$

$$\left(-\omega^2+\frac{h_m}{h}\tilde{\omega}_r{}^2+\frac{h_r}{h}\omega_m{}^2+i\frac{h}{h_m}\omega_c\omega\right)S_{mr}=\frac{\rho}{h}(\tilde{\omega}_r{}^2-\omega_m{}^2)\Delta\,, \tag{3·4b}$$

where $\tilde{\omega}_r{}^2$ and $\tilde{\omega}_s{}^2$ are defined by

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$$\tilde{\omega}_r^2 \equiv (1+\alpha)\omega_r^2 , \tag{3·5a}$$

$$\tilde{\omega}_s^2 \equiv \frac{h_r}{h}\tilde{\omega}_r^2 + \frac{h_m}{h}\omega_m^2 . \tag{3·5b}$$

Then combining Eqs. (3·4a) and (3·4b) and arranging the result in order, we find

$$(\omega^2-\hat{\omega}_r^2)(\omega^2-\hat{\omega}_m^2)=(i\omega\hat{\omega}_c+\omega_J^2)(\omega^2-\hat{\omega}_s^2), \tag{3·6}$$

where

$$\hat{\omega}_r^2 \equiv \tilde{\omega}_r^2 - \omega_J^2 , \quad \hat{\omega}_m^2 \equiv \omega_m^2 - \omega_J^2 ,$$

$$\hat{\omega}_s^2 \equiv \tilde{\omega}_s^2 - \omega_J^2 , \quad \hat{\omega}_c \equiv \frac{h}{h_m}\omega_c . \tag{3·7}$$

This is the equation from which the dispersion relation $\omega=\omega(k)$ is to be derived.

It is, however, neither an easy task nor a sensible attempt to look for general solutions of Eq. (3·6). Instead, it is physically more significant to consider several limiting cases in which simple analytic formulas for ω can be derived. The cases we consider are (i) $\omega_c=0$, (ii) $\omega_c^{-1}=0$ and (iii) $\omega_J=0$.

(i) $\omega_c=0$

Consider first the simplest case of decoupled matter and radiation. In this case, although α defined by Eq. (3·3b) is non-vanishing, it is not relevant to use this value of α. This is because the derivation of the equation for Π_r given in Appendix E is based on the assumption that the directional moments of the photon distribution function higher than the second order can be neglected. However when photons are collisionless, the higher-order moments come into play inevitably. In order to close the system, it is then more reasonable to assume $\Pi_r=0$ in this case. Hence we put $\alpha=0$. Under this assumption, Eq. (3·6) can be easily solved to give

$$\omega^2=\frac{1}{2}\left\{\omega_r^2+\omega_m^2-\omega_J^2\pm\sqrt{(\omega_r^2-\omega_m^2-\omega_J^2)^2+4\frac{h_m}{h}\omega_J^2(\omega_r^2-\omega_m^2)}\right\}. \tag{3·8}$$

Note that the term inside the square root is always positive since $\omega_r^2-\omega_m^2>0$[cf. Grischuk and Zel'dovich (1981)]. In the limit $\omega_r^2 \gg \omega_m^2$, ω_J^2 it reduces to

$$\omega^2 \simeq \begin{cases} \omega_r^2-\dfrac{h_r}{h}\omega_J^2 , & (3·9a) \\[2ex] \omega_m^2-\dfrac{h_m}{h}\omega_J^2 . & (3·9b) \end{cases}$$

The upper solution gives real ω which is essentially determined by the photon sound velocity. On the other hand the lower solution gives pure imaginary ω if $\omega_m^2<(h_m/h)\omega_J^2$, i.e., if

$$\lambda^2>\lambda_J^{(m)2}\equiv\frac{2(2\pi)^2c_m^2}{\chi^2\rho_m} \quad ; \qquad \lambda=\frac{2\pi a}{k} . \tag{3·10}$$

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This is the well-known formula for the gravitational instability of the matter with $\lambda_J^{(m)}$ known as the Jeans length. One finds the qualitative features of the solutions are the same even if $\omega_J^2 \gtrsim \omega_r^2$ since one of the solutions ω^2 of Eq. (3·8) is always positive and of order ω_r^2. This implies that there is no gravitational instability associated with photons when they are decoupled from the matter.

The fact that the upper solution in Eq. (3·9) corresponds to a perturbation in the radiation density and the lower one to that in the matter density can be easily seen by inserting Eq. (3·9) into Eq. (3·4a) and deriving the relation between the amplitudes δ_m and δ_r. The result is

$$\delta_m \simeq -\frac{\rho_r}{h}\frac{\omega_J^2+\omega_m^2}{\omega_r^2}\delta_r \ll \delta_r \qquad \text{for } \omega^2 \simeq \omega_r^2 - \frac{h_r}{h}\omega_J^2\,, \tag{3·11a}$$

$$\delta_r \simeq \frac{4}{3}\frac{\omega_J^2}{\omega_r^2}\delta_m \ll \delta_m \qquad \text{for } \omega^2 \simeq \omega_m^2 - \frac{h_m}{h}\omega_J^2\,, \tag{3·11b}$$

where the relations $\Delta = (\rho_r\delta_r + \rho_m\delta_m)/\rho$ and $S_{mr} = \delta_m - 3\delta_r/4$ have been used.

(ii) $\omega_c^{-1} = 0$

This is the strong coupling limit at which the matter and radiation behave together as a single fluid. In this limit, Eq. (3·6) simplifies to

$$\omega(\omega^2 - \omega_s^2 + \omega_J^2) = 0\,. \tag{3·12}$$

Hence we have $\omega = 0$ or $\omega^2 = \omega_s^2 - \omega_J^2$. The former solution corresponds to the isothermal mode and the latter to the (two) adiabatic modes. As we have seen in the previous section, their essential features are the same even on scales greater than the horizon size. The Jeans length for the adiabatic modes is given by

$$\lambda_J^{(a)2} = \frac{2(2\pi)^2 c_s^2}{\chi^2(\rho + p)}\,, \tag{3·13}$$

which is again a well-known result.

From Eqs. (3·4a) we readily find

$$\delta_r \simeq -\frac{\rho_m}{\rho_r}\frac{\omega_m^2 - \omega_J^2}{\omega_r^2}S_{mr} \qquad \text{for } \omega = 0\,, \tag{3·14a}$$

$$S_{mr} = 0 \qquad \text{for } \omega^2 = \omega_s^2 - \omega_J^2\,, \tag{3·14b}$$

where the fact that $\omega_r^2 \gg \omega_J^2$ for wavelengths smaller than the horizon size has been used in deriving Eq. (3·14a). From this equation the isothermal nature of the $\omega = 0$ mode is clear. Also, the adiabatic nature of the $\omega^2 = \omega_s^2 - \omega_J^2$ mode is trivially apparent from Eq. (3·14b).

(iii) $\omega_J^2 = 0$

This is the case when the gravity can be neglected. Then Eq. (3·6) reduces to

$$(\omega^2 - \tilde{\omega}_r^2)(\omega^2 - \omega_m^2) = i\omega\tilde{\omega}_c(\omega^2 - \tilde{\omega}_s^2). \tag{3·15}$$

Since qualitative features of the resultant dispersion relations change greatly when the relative magnitude of ω_c to ω_r or ω_m is varied, we consider the cases $\omega_c \gg \omega_r$, $\omega_c \ll \omega_m$ and $\omega_m \ll \omega_c \ll \omega_r$ separately.

First consider the case $\omega_c \gg \omega_r$. In this case there are essentially four solutions which

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are given by

$$\omega \simeq i\frac{h}{h_m}\omega_c\,, \tag{3·16a}$$

$$\omega \simeq i\frac{h_m}{h_r}\frac{\omega_m^2}{\omega_c}\,, \tag{3·16b}$$

$$\omega \simeq \pm\omega_s + i\gamma_s\,, \tag{3·16c}$$

where

$$\gamma_s \equiv \frac{h_m^2\omega_r^2}{2h^2\omega_c}\left(1+\frac{4h_rh}{5h_m^2}\right). \tag{3·16d}$$

The first solution is a rapidly decaying mode whose existence may be neglected from the beginning. The second one is, on the contrary, a very slowly decaying mode which can be virtually regarded as a time-independent mode. Actually these two modes correspond to the two isothermal modes found in the previous section; only the latter has been found to be relevant. The other two solutions (3·16c) represent the adiabatic modes. They have the positive imaginary part which leads to the viscous damping of adiabatic perturbations [Silk (1968); Sato (1971); Weinberg (1971)]. The characteristic wavelength $\lambda_d^{(a)}$ below which an adiabatic perturbation is damped severely can be found by multiplying γ_s by the expansion time $\tau_{\rm ex}$ and setting the result equal to unity. This gives

$$\lambda_d^{(a)} = 2\pi\sqrt{\frac{h_m^2}{6h^2}\left(1+\frac{4h_rh}{5h_m^2}\right)\frac{\tau_{\rm ex}}{\omega_c}}\,, \tag{3·17}$$

which is approximately the same as $\lambda_d^{(1)}$ for the damping of vorticity found in the previous section.

From Eq. (3·4a) the amplitudes δ_r and S_{mr} are found to be related as

$$\delta_r \simeq -\frac{\rho_m}{\rho_r}\frac{\omega_m^2}{\omega_r^2}S_{mr} \qquad \text{for } \omega \simeq i\frac{h_m}{h_r}\frac{\omega_m^2}{\omega_c}\,, \tag{3·18a}$$

$$S_{mr} \simeq \mp i\left(\frac{3}{4}\right)^2\frac{\rho_m}{\rho_r}\frac{\omega_s}{\omega_c}\delta_r \qquad \text{for } \omega \simeq \pm\omega_s + i\gamma_s\,. \tag{3·18b}$$

The former relation is essentially the same as Eq. (3·14a) of the case $\omega_c^{-1}=0$ and exhibits the isothermal nature of the mode. The latter corresponds to Eq. (3·14b) but here ω_c is assumed to be finite though large. This appearance of small but finite S_{mr} with its phase completely anti-correlated to δ_r causes the damping of adiabatic perturbations. As we have mentioned in the previous section the induced perturbation in S_{mr} on large scales, Eq. (2·17), essentially corresponds to S_{mr} of Eq. (3·18b). Hence as we have concluded there, the induced perturbation in S_{mr} would never turn into a real isothermal perturbation unless there were a certain degree of mode-mixing, which could occur only on scales greater than the horizon size where distinction of the modes from each other is rather ambiguous.

Next, we consider the case $\omega_c \ll \omega_m$. As we have discussed in (i), the case $\omega_c=0$, it is more reasonable to assume $\alpha=0$ than to use α given by Eq. (3·3b) when the coupling between the matter and radiation is weak. Therefore let us also set $\alpha=0$ in the present case. Then it is not difficult to solve Eq. (3·15) for ω. The solutions are

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$$\omega \simeq \pm \omega_r + i\frac{1}{2}\omega_c\,, \tag{3·19a}$$

$$\omega \simeq \pm \omega_m + i\frac{h_r}{2h_m}\omega_c\,. \tag{3·19b}$$

These correspond to the solutions given in Eqs. (3·9) if $\omega_c=0$ but $\omega_J{}^2 \neq 0$. Comparing the imaginary parts of these solutions with the relevant interaction timescales appearing in the original equations (1·19b) and (1·19d), we find that perturbations in the density of each component (the radiation and matter) decay within the timescale of momentum transfer to each component.

As before, the relations between the relevant perturbation amplitudes can be derived from Eq. (3·4a). One finds

$$\delta_m \simeq \mp i\frac{\rho_r}{\rho_m}\frac{\omega_c}{\omega_r}\delta_r \ll \delta_r \qquad \text{for} \quad \omega \simeq \pm\omega_r + i\frac{1}{2}\omega_c\,, \tag{3·20a}$$

$$\delta_r \simeq \pm i\frac{4\omega_m\omega_c}{3\omega_r{}^2}\delta_m \ll \delta_m \qquad \text{for} \quad \omega \simeq \pm\omega_m + i\frac{h_r}{2h_m}\omega_c\,. \tag{3·20b}$$

Again, these relations clearly show that the modes of Eq. (3·19a) represent a perturbation in the radiation density and those of Eq. (3·19b) represent that in the matter density.

Finally, we consider the case $\omega_m \ll \omega_c \ll \omega_r$. In this case, a careful inspection of Eq. (3·15) reveals that the solution (3·16b) which has been derived by assuming $\omega_c \gg \omega_r$ is still valid and the solutions (3·19a) derived by assuming $\omega_c \ll \omega_m$ are also valid if $\alpha \ll \omega_c/\omega_r$. The remaining fourth solution is found to be proportional to ω_c similar to the solution (3·16a). Thus we have

$$\omega \simeq i\frac{h_r}{h_m}\omega_c\,, \tag{3·21a}$$

$$\omega \simeq i\frac{h_m}{h_r}\frac{\omega_m{}^2}{\omega_c}\,, \tag{3·21b}$$

$$\omega \simeq \pm\omega_r + i\frac{1}{2}\omega_c\,. \tag{3·21c}$$

Although the inequality $\alpha \ll \omega_c/\omega_r$ may not hold in reality, we still expect $\alpha \ll 1$ since the radiation can be regarded as decoupled from the matter if $\omega_c/\omega_r \ll 1$ and the correction due to α should be correspondingly small. Then Eq. (3·21c) should be qualitatively correct at least.

The physical interpretations of the solutions (3·21b) and (3·21c) are the same as before. The former represents the usual isothermal mode and the latter the two modes of perturbation in the radiation density. There exists no adiabatic mode in this range of ω_c since the matter and radiation are not sufficiently bound together. The mode (3·21a) is the rapidly decaying isothermal mode. The relation between the amplitudes δ_r and S_{mr} for this mode is given by

$$\delta_r \simeq -\left(\frac{4}{3}\right)^2 \frac{\rho_r}{\rho_m}\frac{\omega_c{}^2}{\omega_r{}^2}S_{mr}\,. \tag{3·22}$$

Note that Eq. (3·21b) implies that there also exists a characteristic wavelength $\lambda_d{}^{(i)}$ for the isothermal mode below which the damping of it is appreciable, similar to $\lambda_d{}^{(a)}$ for

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the adiabatic modes. This wavelength is given by

$$\lambda_d^{(i)} = 2\pi c_m \sqrt{\frac{h_m}{h_r} \frac{\tau_{\rm ex}}{\omega_c}} . \tag{3·23}$$

It is worthwhile to mention that the ratio $\lambda_d^{(i)}/\lambda_d^{(a)}$ is approximately equal to the ratio of the sound velocities, c_m/c_r, if the energy density of matter and that of radiation are of the same order of magnitude.

(2) Vector perturbations

Neglecting the effect of cosmic expansion and setting $K=0$, the basic equations (1·30) reduce to

$$\dot{V}^{(1)} = -\frac{h_r}{8h}\omega_0 \Pi_r^{(1)} , \tag{3·24a}$$

$$\dot{\Pi}_r^{(1)} + \omega_c \Pi_r^{(1)} = \frac{8}{5}\omega_0 \left(V^{(1)} - \frac{h_m}{h} V_{mr}^{(1)} \right), \tag{3·24b}$$

$$\dot{V}_{mr}^{(1)} + \frac{h}{h_m}\omega_c V_{mr}^{(1)} = \frac{1}{8}\omega_0 \Pi_r^{(1)} , \tag{3·24c}$$

where

$$\omega_0 \equiv k/a , \tag{3·24d}$$

and ω_c has been defined in Eqs. (3·2). We find that these equations do not involve the matter sound velocity. Hence the analysis given in the previous section is still valid and the same conclusions should be drawn here, provided $\omega_c \gg \omega_0$ and $\omega_c \tau_{\rm ex} \gg 1$. However, for completeness, let us investigate Eqs. (3·24) in terms of dispersion relations.

Assuming $V^{(1)}$, $V_{mr}^{(1)}$ and $\Pi_r^{(1)}$ to be proportional to $e^{i\omega t}$, Eqs. (3·24) become

$$i\omega V^{(1)} = -\frac{h_r}{8h}\omega_0 \Pi_r^{(1)} , \tag{3·25a}$$

$$(i\omega + \omega_c)\Pi_r^{(1)} = \frac{8}{5}\omega_0 \left(V^{(1)} - \frac{h_m}{h} V_{mr}^{(1)} \right), \tag{3·25b}$$

$$\left(i\omega + \frac{h}{h_m}\omega_c \right) V_{mr}^{(1)} = \frac{1}{8}\omega_0 \Pi_r^{(1)} . \tag{3·25c}$$

Combining these equations, we obtain

$$\omega^3 - i\left(2 + \frac{h_r}{h_m}\right)\omega_c \omega^2 - \left\{ \frac{1}{5}\omega_0^2 + \left(1 + \frac{h_r}{h_m}\right)\omega_c^2 \right\} \omega + i\frac{h_r}{5h_m}\omega_0^2 \omega_c = 0 . \tag{3·26}$$

Although this can be analytically solved for ω, the solutions are not quite meaningful when $\omega_c \ll \omega_0$, since Eq. (3·25b) is no longer valid under such a situation as already discussed previously. Therefore it is sufficient to consider the case when $\omega_c \gg \omega_0$. In this case, the solutions of Eq. (3·26) are expressed in the simple form as

$$\omega \simeq i\frac{h}{h_m}\omega_c , \tag{3·27a}$$

$$\omega \simeq i\omega_c , \tag{3·27b}$$

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$$\omega \simeq i\frac{h_r}{5h}\frac{\omega_0^2}{\omega_c}. \tag{3·27c}$$

The first two solutions are the rapidly decaying modes which have appeared in the kernel of Eq. (2·21) and that in the second term of Eq. (2·27), respectively. Also the last solution (3·27c) gives the viscous damping rate of vorticity which has already been obtained in the previous section. The critical wavelength below which the damping is severe has been given by Eq. (2·32).

The physical meaning of each mode given by Eqs. (3·27) can be clarified by deriving the relation between the amplitudes $V^{(1)}$, $V_{mr}^{(1)}$ and $\Pi_r^{(1)}$. Inserting the expression for each mode into Eqs. (3·25) in sequence, we obtain

$$V^{(1)} \simeq \frac{h_m^3}{5h^3}\frac{\omega_0^2}{\omega_c^2}V_{mr}^{(1)} \qquad \text{for} \quad \omega \simeq i\frac{h}{h_m}\omega_c\,, \tag{3·28a}$$

$$V^{(1)} \simeq -\frac{h_r^2}{hh_m}V_{mr}^{(1)} \simeq \frac{h_r}{8h}\frac{\omega_0}{\omega_c}\Pi_r^{(1)} \qquad \text{for} \quad \omega \simeq i\omega_c\,, \tag{3·28b}$$

$$V_{mr}^{(1)} \simeq \frac{h_m}{5h}\frac{\omega_0^2}{\omega_c^2}V^{(1)} \qquad \text{for} \quad \omega \simeq i\frac{h_r}{5h}\frac{\omega_0^2}{\omega_c}. \tag{3·28c}$$

Thus the first mode can be termed the "isovortical" mode. The second mode represents the anisotropic stress perturbation and the third mode the usual vorticity perturbation for a viscous fluid. The fact that the decay rates of the first and second modes are large is simply explained by the strong coupling between the matter and radiation, which does not allow $V_{mr}^{(1)}$ or $\Pi_r^{(1)}$ to deviate greatly from zero.

Finally, let us comment on the case when $\omega_c \ll \omega_0$. Although Eq. (3·25b) fails to be valid in this case, Eqs. (3·25a) and (3·25c) would be still valid. Then if $\Pi_r^{(1)}$ could be assumed to be vanishing, one would obtain

$$\omega = 0 \qquad \text{for} \ V^{(1)} \neq 0\,, \tag{3·29a}$$

$$\omega = i\frac{h}{h_m}\omega_c \qquad \text{for} \ V_{mr}^{(1)} \neq 0\,. \tag{3·29b}$$

Thus the vorticity would be conserved while isovortical perturbations would decay within a timescale of order $(h_m/h)\omega_c^{-1}$.

§ V-4. Summary and implications

Let us put together the results obtained in the last two sections and discuss their implications by following the time-evolution of density perturbations with given comoving wavelengths. Since implications are quite straightforward in the cases of vector and tensor perturbations, we concentrate our attention on the case of scalar perturbations in this section.

For convenience, we define the following characteristic wavelengths (or length-scales) which play important roles in determining the behavior of density perturbations, some of which have been introduced already in §3:

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$$\lambda_H \equiv \frac{2\pi}{H} = 2\pi\tau_{\rm ex}\,, \tag{4·1a}$$

$$\lambda_J^{(a)} \equiv c_s\sqrt{\frac{2\rho}{3h}}\lambda_H\,, \tag{4·1b}$$

$$\lambda_J^{(m)} \equiv c_m\sqrt{\frac{2\rho}{3\rho_m}}\lambda_H\,, \tag{4·1c}$$

$$\lambda_C \equiv \frac{2\pi}{\omega_C} = 2\pi\tau_C\,, \tag{4·1d}$$

$$\lambda_C^{(r)} \equiv c_r\lambda_C\,, \tag{4·1e}$$

$$\lambda_C^{(m)} \equiv c_m\lambda_C\,, \tag{4·1f}$$

$$\lambda_d^{(a)} \equiv \sqrt{\frac{\rho_m^2}{6h^2}\left(1+\frac{4h_r h}{5\rho_m^2}\right)\frac{\lambda_C H}{2\pi}}\lambda_H\,, \tag{4·1g}$$

$$\lambda_d^{(i)} \equiv c_m\sqrt{\frac{\rho_m}{h_r}\frac{\lambda_C H}{2\pi}}\lambda_H = c_m\sqrt{\frac{\tau_{\rm drag}}{\tau_{\rm ex}}}\lambda_H\,. \tag{4·1h}$$

The variations of these wavelengths as compared with fixed comoving scales in a typical baryon-photon dominated universe are depicted in Fig. 1 as functions of the cosmic temperature. Most of the lines representing these wavelengths terminate at the recombination time since they cease to be well-defined at $T < T_{\rm rec} \sim 0.35$eV. We denote their terminal values at $T = T_{\rm rec}$ by attaching the subscript "rec" to each of them. The meaning and role of the line indicated by $\lambda_d^{(T)}$, which appears on the lower temperature side of the $T = T_{\rm rec}$ line in the figure, will be explained in time.

Let us discuss the behavior of density perturbations before and after the recombination time separately. However, before we go into details, the following comments are in order. As we have stressed several times, the behavior of adiabatic density perturbations on super-horizon scales depends crucially on the choice of time slicing and a special care is needed for interpreting the physical nature of such perturbations. Actually, as we have shown in § III-2, the amplitude of an adiabatic perturbation at $\lambda > \lambda_H$ should be regarded as constant with time for the "growing" mode. Nevertheless, if this fact is kept in mind, it is convenient to represent a density perturbation by the amplitude Δ simply because its behavior shows no characteristic change across the horizon; the characteristic change occurs only when the perturbation appears on the sound horizon (Jeans radius), which helps to simplify our discussion. On the other hand, an isothermal perturbation is quite uniquely represented by S_{mr} irrespective of the choice of time slicing. Hence, in the following, when we talk about the behavior of adiabatic and isothermal density perturbations we mean that of Δ and S_{mr}, respectively.

First we consider the stage before recombination. Adiabatic perturbations would grow as a^2 in the radiation-dominated stage and as a in the matter-dominated stage up to the epoch when it comes within the Jean radius $\lambda_J^{(a)}$. After it enters the Jean radius it begins to oscillate acoustically with frequency $\omega = 2\pi c_s/\lambda$. Hence perturbations with $\lambda_{\rm rec} > \lambda_{J,\rm rec}^{(a)}$ would never experience the oscillation but continue to grow, where $\lambda_{\rm rec}$ is the wavelength of a perturbation at $T = T_{\rm rec}$. For those with $\lambda_{\rm rec} < \lambda_{J,\rm rec}^{(a)}$, they would eventu-

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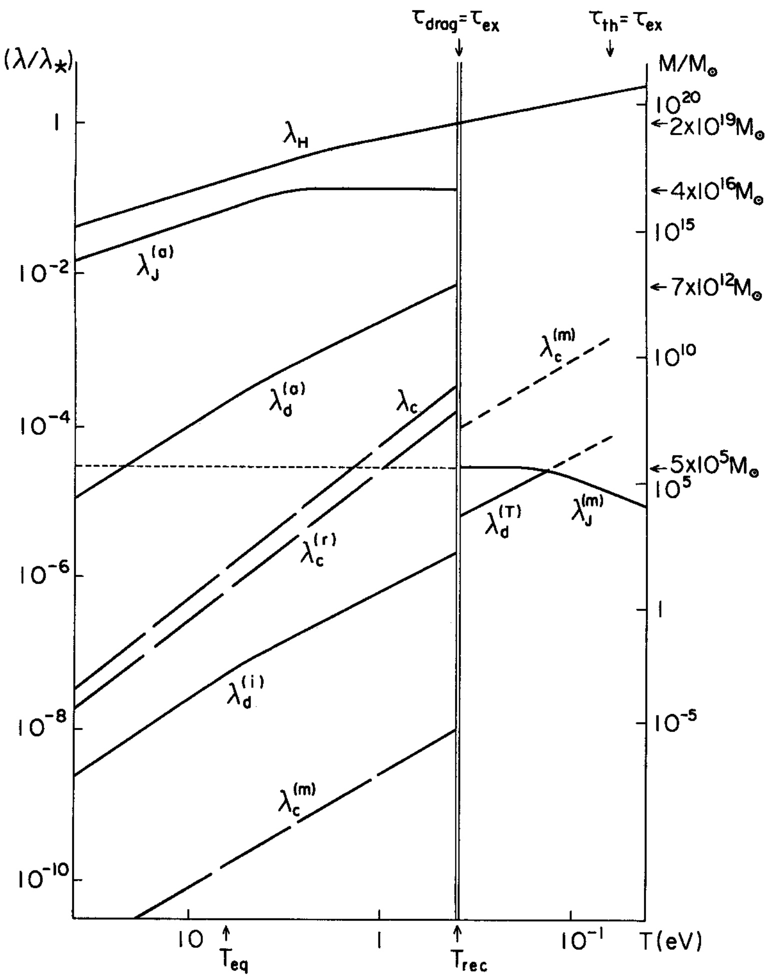

Fig. 1. Variations of comoving scales corresponding to the characteristic wavelengths (see Eqs. (4·1) and (4·3) for their definitions) in the baryon-photon dominated universe of $\Omega_0 h^2=0.64(s/n_B\simeq 2\times 10^8)$ with respect to the cosmic temperature. For simplicity, the recombination is assumed to be instantaneous at $T=T_{rec}(\simeq 0.35\text{ eV})$ and the ionization fraction x_e is assumed to be $x_e=1$ at $T>T_{rec}$ and $x_e=10^{-4}$ at $T<T_{rec}$. The left vertical coordinate is the comoving wavelength normalized by that of the horizon at $T=T_{rec}(\lambda_*\equiv(a/a_{rec})\lambda_{H,rec})$ and the right one is the corresponding baryonic mass-scale.

ally come inside the radius $\lambda_J{}^{(a)}$. Then the gravity can be neglected for $\lambda<\lambda_J{}^{(a)}$ and the analysis of §3, case (iii) applies. Note that the notion of adiabatic perturbation is well-defined only for $\lambda>\lambda_c{}^{(r)}$. For such a wevelength, the frequency is given by Eq. (3·16c) and the perturbations would be severely damped if $\lambda<\lambda_d{}^{(a)}$. Thus only those with $\lambda_{rec}>\lambda_{d,rec}^{(a)}$ would survive until the recombination time. Although no perturbation can be adiabatic on scales $\lambda<\lambda_c{}^{(r)}$, perturbations in the radiation density are naturally related to adiabatic perturbations on scales $\lambda>\lambda_c{}^{(r)}$. Hence let us also consider their behavior here. Since $\lambda_d{}^{(a)}\gg\lambda_c{}^{(r)}$, no adiabatic perturbation would develop into a radiation density perturbation with $\lambda<\lambda_c{}^{(r)}$. Further, even if a perturbation were provoked at $\lambda<\lambda_c{}^{(r)}$ by some mechanism it would decay almost instantaneously since the characteristic decay time is $\omega_c{}^{-1}\ll\tau_{ex}$ as given by Eq. (3·19a) or (3·21c).

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On the other hand, an isothermal perturbation would never become oscillatory nor its amplitude would grow. However, the other properties are essentially the same as those of adiabatic perturbations. Namely, (1) isothermal perturbations would be damped if $\lambda < \lambda_d^{(i)}$ [Eq. (3·16a) or (3·21b)], (2) the notion of isothermal perturbation is ambiguous for $\lambda < \lambda_c^{(m)}$ and (3) perturbations in the matter density with $\lambda < \lambda_c^{(m)}$, which are naturally related to isothermal perturbations, would decay within the timescale $\tau_{\rm drag} \equiv (3\rho_m / 4\rho_r)\omega_c^{-1} \ll \tau_{\rm ex}$ [Eq. (3·19b)] if they could ever be provoked.

Next we consider the stage after recombination. At recombination the ionization fraction x_e decreases considerably. Its final value is expected to be about 10^{-4}. Then both τ_c and $\tau_{\rm drag}$ exceed $\tau_{\rm ex}$ so that photons and baryons are essentially decoupled from each other and neither the notion of adiabatic perturbation nor that of isothermal one is well-defined any more. Hence a perturbation which was originally adiabatic would turn into two independent perturbations of radiation and matter and a perturbation originally isothermal would turn into that of matter.

As mentioned in the previous section, perturbations in the radiation density at this stage can be analysed correctly only if the collisionless nature of photons is taken into account. In addition, they would never contribute to the formation of cosmic structures. Hence we shall not consider them here but only comment on a well-known feature of them whose detail can be found in literature. That is the famous free streaming effect which smears out perturbations in the distribution of relativistic collisionless particles on scales smaller than the Hubble horizon size [Stewart (1972)]. Thus no perturbations in the radiation density would remain within the horizon in the end.

As for perturbations in the matter density, they would start to grow as a if $\lambda > \lambda_J^{(m)}$ which is essentially the same as the behavior of Δ in a dust-dominated universe. Hence originally adiabatic perturbations would form structures of the universe on comoving scales greater than $\lambda_{d,\rm rec}^{(a)}$ at $T = T_{\rm rec}$. The mass-scale corresponding to $\lambda_{d,\rm rec}^{(a)}$ is numerically given by

$$M_{d,\rm rec}^{(a)} \simeq 4.3 \times 10^{12} (\Omega_0 h^2)^{-5/4} M_\odot \tag{4·2}$$

for $\Omega_0 h^2 \gtrsim 0.1$ ($s/n_B \lesssim 1.3 \times 10^9$), where Ω_0 is the density parameter of the present universe and h is the Hubble constant in the unit of 100 km s^{-1}Mpc^{-1}. On the other hand, originally isothermal perturbations would be responsible for the formation of smaller scale structures. Since $\lambda_J^{(m)} \ll \lambda_c^{(m)}$ at $T < T_{\rm rec}$ as shown in Fig. 1, one would expect that Eq. (3·19b) applies for comoving wavelenghts of range $\lambda_{d,\rm rec}^{(i)} < \lambda_{\rm rec} < \lambda_{J,\rm rec}^{(m)}$. Then, since $\tau_{\rm drag} \gg \tau_{\rm ex}$ after recombination, one would conclude that the oscillation would never be damped effectively and the wavelength of a perturbation would eventually exceed $\lambda_J^{(m)}$ and begins to grow. Thus one would be tempted to conclude that the smallest possible comoving scale is determined by $\lambda_{d,\rm rec}^{(i)}$. However, it turns out that this conclusion is incorrect but the lower limit is determined by $\lambda_{J,\rm rec}^{(m)}$ actually.

The above incorrect conclusion is simply due to neglection of the matter temperature T_m as a dynamical variable in our analysis. As we have seen in §1, although the Thomson dragging is ineffective at $T < T_{\rm rec}$, the thermal coupling is still effective until $T \sim 0.1$eV when $\tau_{\rm th}$ exceeds $\tau_{\rm ex}$. Therefore the matter temperature would still keep up with the radiation temperature T for a while after recombination. However since a perturbation in the matter density oscillates with frequency $\omega = 2\pi c_m / \lambda$, the matter temperature is

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forced to oscillate with the same frequency, while the radiation temperature must be uniform on the present scale of concern. This induces the thermal drag of the matter similar to the Thomson drag [Carr and Rees (1984)]. A simple analysis of perturbations by dispersion relation which takes account of a perturbation in T_m yields a characteristic scale below which matter density perturbations are strongly damped. It is given by

$$\lambda_d^{(T)}=\sqrt{\frac{(1+x_e)m}{2M}}\lambda_d^{(i)}, \tag{4·3}$$

where the factor in the square root is just the ratio of $\tau_{\rm th}$ to $\tau_{\rm drag}$,

$$\tau_{\rm th}=\frac{(1+x_e)m}{2M}\tau_{\rm drag}, \tag{4·4}$$

as naturally expected.

Now from Eqs. (4·1c), (4·1h) and (4·3) one finds

$$\lambda_d^{(T)}=\sqrt{\frac{3\rho_m}{2\rho}\frac{\tau_{\rm th}}{\tau_{\rm ex}}}\lambda_J^{(m)}. \tag{4·5}$$

Hence in the matter-dominated stage where $\rho\simeq\rho_m$, $\lambda_d^{(T)}$ eventually catches up with $\lambda_J^{(m)}$, which occurs at the epoch characterized by $\tau_{\rm th}=(2\rho/3\rho_m)\tau_{\rm ex}(<\tau_{\rm ex})$ and the value of $\lambda_J^{(m)}$ at this epoch determines the shortest comoving wavelength of perturbations which may exist. On the other hand if $\tau_{\rm th}<\tau_{\rm ex}$ one may assume $T_m=T$. Then from Eq. (4·1c) one easily finds $\lambda_J^{(m)}\propto a$ at such a stage. Thus $\lambda_J^{(m)}/a=\lambda_{J,{\rm rec}}^{(m)}/a_{\rm rec}$ at the epoch $\lambda_d^{(T)}=\lambda_J^{(m)}$. This is the reason why the smallest possible comoving scale of perturbations is determined by $\lambda_{J,{\rm rec}}^{(m)}$. In Fig. 1, a typical behavior of $\lambda_d^{(T)}$ and $\lambda_J^{(m)}$ is shown, which is in accordance with the above discussion. Numerically, the scale $\lambda_{J,{\rm rec}}^{(m)}$ corresponds to the mass given by

$$M_{J,{\rm rec}}^{(m)}\simeq 4\times10^5(\Omega_0h^2)^{-1/2}M_\odot\,. \tag{4·6}$$

To summarize, adiabatic perturbations could have been responsible for the formation of cosmic structures on scales $M\gtrsim M_{d,{\rm rec}}^{(a)}$[Eq. (4·2)] while isothermal ones for that on scales $M\gtrsim M_{J,{\rm rec}}^{(m)}$[Eq. (4·6)]. It is worthwhile to mention that $M_{d,{\rm rec}}^{(a)}$ is more or less close to the mass of a giant galaxy or a cluster of galaxies and $M_{J,{\rm rec}}^{(m)}$ to the mass of a globular cluster. Another mass-scale which we have not mentioned explicitly but may be of some significance is the one that corresponds to $\lambda_{J,{\rm rec}}^{(a)}$. It is given by

$$M_{J,{\rm rec}}^{(a)}\simeq 1.8\times10^{16}(\Omega_0h^2)^{-2}M_\odot \tag{4·7}$$

for $\Omega_0h^2\gtrsim0.1$. Note that this is close to the mass of a supercluster of galaxies.

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Chapter VI

Perturbations in Classical Scalar Field Dominated Systems

In some cosmological models, especially in models based on grand unified gauge theories (GUTs) which are actively studied recently [see, e.g., Langacker (1981)], there appear stages during which classical fields dominate the cosmic expansion [see, e.g., Turner, Wilczek and Zee (1983); Stecker and Shafi (1983); Sato (1984)]. Since classical fields behave quite differently from fluids, it is interesting and important to investigate the behavior of perturbations in a system containing classical fields. In this chapter, we consider perturbations in classical scalar field dominated systems.

§ VI-1. Formulation

In this section we give the fundamental equations describing perturbations in a system composed of a classical scalar field and some species of fluid. Such equations are easily obtained by specifying the general formalism developed in §II-5 to the system concerned [Sasaki (1983a, c)].

Let $\tilde{\phi}=(\tilde{\phi}_A)$ be a multi-component classical scalar field. If the direct interactions with other matter are neglected, its dynamics in a curved spacetime with metric $\tilde{g}_{\mu\nu}$ is determined by the Lagrangian

$$\mathcal{L}=-(1/2)\sqrt{-\tilde{g}}\,[\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\cdot\partial_\nu\tilde{\phi}+\xi\tilde{R}\tilde{\phi}^2+2U(\tilde{\phi})], \tag{1·1}$$

where ξ is a constant, $\partial_\mu\phi\cdot\partial_\nu\phi\equiv\sum_A\partial_\mu\phi_A\partial_\nu\phi_A$ and $U(\tilde{\phi})$ is assumed to be independent of $\tilde{g}_{\mu\nu}$. This Lagrangian yields the field equation

$$\tilde{\Box}\tilde{\phi}-\xi\tilde{R}\tilde{\phi}-U_\phi(\tilde{\phi})=0\,, \tag{1·2}$$

where

$$\tilde{\Box}\tilde{\phi}\equiv(-\tilde{g})^{-1/2}\partial_\mu((-\tilde{g})^{1/2}\tilde{g}^{\mu\nu}\partial_\nu\tilde{\phi}) \tag{1·3}$$

and

$$(U_\phi)_A\equiv\partial U/\partial\phi_A\,, \tag{1·4}$$

and the energy-momentum tensor

$$\begin{aligned}\tilde{T}^\mu_{(\phi)\nu}=&\tilde{\nabla}^\mu\tilde{\phi}\,\tilde{\nabla}_\nu\tilde{\phi}-\frac{1}{2}[\tilde{\nabla}^\lambda\tilde{\phi}\,\tilde{\nabla}_\lambda\tilde{\phi}+2\tilde{U}]\delta^\mu{}_\nu\\&+\xi[\tilde{G}^\mu{}_\nu\tilde{\phi}^2-\tilde{\nabla}^\mu\tilde{\nabla}_\nu(\tilde{\phi}^2)+\delta^\mu{}_\nu\tilde{\Box}(\tilde{\phi}^2)].\end{aligned} \tag{1·5}$$

Equation (1·2) guarantees the conservation of $\tilde{T}_{(\phi)\mu\nu}$:

$$\tilde{\nabla}^\mu\tilde{T}_{(\phi)\mu\nu}=0\,. \tag{1·6}$$

In actual situations it often occurs that there are some interactions of ϕ field with other matter. Such interactions can be phenomenologically expressed by adding a source

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term to the right-hand side of Eq. (1·2):

$$\tilde{\Box}\tilde{\phi}-\xi\tilde{R}\tilde{\phi}-U_\phi(\tilde{\phi})=\tilde{S}\,, \tag{1·7}$$

where $\tilde{S}$ depends on matter variables as well as on $\tilde{\phi}$ in general. In the case that such a source term exists, $\tilde{T}_{(\phi)\mu\nu}$ is no longer conserved by itself. From Eq. (1·7) it follows that

$$\tilde{\nabla}^\mu\tilde{T}_{(\phi)\mu\nu}=\tilde{Q}_{(\phi)\nu}\,, \tag{1·8}$$

where

$$\tilde{Q}_{(\phi)\mu}=\tilde{S}\cdot\partial_\mu\tilde{\phi}\,. \tag{1·9}$$

The perturbed field $\tilde{\phi}$ consists of the unperturbed part ϕ and a perturbation $\delta\phi$:

$$\tilde{\phi}=\phi+\delta\phi\,. \tag{1·10}$$

From the assumption of homogeneity ϕ depends only on the cosmic time and follows the equation

$$\phi''+(n-1)(a'/a)\phi'+\xi a^2R\phi+a^2U_\phi=-a^2S\,. \tag{1·11}$$

From Eq. (1·5) the unperturbed energy density ρ_ϕ and the unperturbed pressure p_ϕ are given by

$$\rho_\phi\equiv-T_{(\phi)}{}^0{}_0=(1/2)a^{-2}(\phi')^2+U+\xi a^{-2}[-a^2G^0{}_0\phi^2+n(a'/a)(\phi^2)'], \tag{1·12}$$

$$\begin{aligned}p_\phi\equiv(1/n)T_{(\phi)}{}^j{}_j&=(1/2)a^{-2}(\phi')^2-U\\&\quad+\xi a^{-2}[(1/n)a^2G^j{}_j\phi^2-(\phi^2)''-(n-2)(a'/a)(\phi^2)'].\end{aligned} \tag{1·13}$$

In particular,

$$\begin{aligned}h_\phi&\equiv\rho_\phi+p_\phi\\&=a^{-2}(\phi')^2+\xi a^{-2}[a^2\{(1/n)G^j{}_j-G^0{}_0\}\phi^2-(\phi^2)''+2(a'/a)(\phi^2)'].\end{aligned} \tag{1·14}$$

Since from Eqs. (1·9) and (II-5·6), the background energy transfer rate Q_ϕ is expressed as

$$Q_\phi=-a^{-1}S\cdot\phi'\,, \tag{1·15}$$

the unperturbed part of Eq. (1·8) yields the following energy equation:

$$\rho_\phi'=-n(a'/a)h_\phi(1-q_\phi), \tag{1·16}$$

where

$$q_\phi=-(1/n)(a/a')h_\phi^{-1}S\cdot\phi'\,. \tag{1·17}$$

In order to find the "sound velocity" c_ϕ, we need the expression for p_ϕ'. With the aid of Eq. (1·11) and its time derivative, Eq. (1·13) yields

$$\begin{aligned}p_\phi'=&-a^{-2}\phi'\cdot\left(n\frac{a'}{a}\phi'+2a^2U_\phi+a^2S\right)+\xi a^{-2}\Big[a^2\left(\frac{1}{n}G^j{}_j{}'+2\xi R'+2\xi R\frac{a'}{a}\right)\phi^2\\&+\left\{\frac{a^2}{n}G^j{}_j+\left(\frac{a'}{a}\right)'-(n-1)\left(\frac{a'}{a}\right)^2+\frac{7}{2}\xi a^2R\right\}(\phi^2)'+4(n-1)\frac{a'}{a}(\phi')^2\end{aligned}$$

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$$+6a^2U_\phi\cdot\phi'+2a^2\frac{a'}{a}U_\phi\cdot\phi+2a^2\phi\cdot U_{\phi\phi}\cdot\phi'+2a^2S'\cdot\phi+6a^2S\cdot\phi'+2a^2\frac{a'}{a}S\cdot\phi\Big]. \qquad (1\cdot18)$$

The ratio of p_ϕ' and ρ_ϕ' gives c_ϕ^2. We do not write the explicit expression for it here because it is too complicated. What to be noted here is that c_ϕ^2 not only changes its sign in general but also becomes infinite when ρ_ϕ' vanishes. For example, in the case $\xi=Q_\phi=0$ and ϕ behaves oscillatorily, $\rho_\phi'=0$ at $\phi'=0$. Thus a well-defined concept of sound velocity does not exist for classical scalar fields. This fact is one of the reason why the density perturbation behaves quite differently from the usual case when classical fields play important dynamical roles [Sasaki (1984a, b)].

Since $\tilde{\phi}$ transforms as a scalar under the gauge transformation, $\delta\phi$ is of the scalar type and can be written as:

$$\delta\phi(\eta,\ \boldsymbol{x})=\chi(\eta)\mathrm{Y}(\boldsymbol{x}). \qquad (1\cdot19)$$

Under the infinitesimal gauge transformation

$$\bar{\eta}=\eta+T(\eta)\mathrm{Y}(\boldsymbol{x}), \qquad \bar{x}^j=x^j+L(\eta)\mathrm{Y}^j(\boldsymbol{x}), \qquad (1\cdot20)$$

$\tilde{\phi}$ transforms as

$$\bar{\tilde{\phi}}=\tilde{\phi}-\phi' T\mathrm{Y}\,, \qquad (1\cdot21)$$

hence χ transforms as

$$\bar{\chi}=\chi-\phi'\mathrm{T}\,. \qquad (1\cdot22)$$

Comparison of this transformation law with Eq. (II-3·3) suggests us to define the gauge-invariant amplitude for the ϕ field perturbation as

$$X\equiv\chi-k^{-1}\sigma_g\phi'\,. \qquad (1\cdot23)$$

In order to write down the gauge-invariant perturbation equations for a system containing ϕ field, we first find the expressions for $\varDelta_\phi$, V_ϕ, $\varGamma_\phi$, $\varPi_\phi$, E_ϕ and F_ϕ in terms of X. These expressions are derived from the expression for the energy-momentum tensor of ϕ field in terms of χ and the perturbation variables for the metric. The explicit expressions for $\delta T^\mu{}_\nu$ are given in Appendix F. From the expression for $\delta T^0{}_0$ we obtain the following expression for $\varDelta_{s\phi}$:

$$\begin{aligned}\rho_\phi\varDelta_{s\phi}&\equiv\rho_\phi\delta_\phi+n(1-q_\phi)h_\phi(\mathcal{R}-\varPhi)\\
&=a^{-2}[\phi'\cdot X'+a^2U_\phi\cdot X-(\phi')^2\varPsi]\\
&\quad+\xi na^{-2}\Big[2\frac{a'}{a}(\phi\cdot X)'+\Big\{\Big(\frac{a'}{a}\Big)^2+K+\frac{2k^2}{n}\Big\}\phi\cdot X\\
&\quad-\Big\{(n-1)\Big(\frac{a'}{a}\Big)^2\phi^2+\frac{a'}{a}(\phi^2)'\Big\}\mathcal{A}-\frac{a'}{a}(\phi^2)'\varPsi\\
&\quad-\Big\{(n-1)\frac{a'}{a}\Big[\Big(\frac{a'}{a}\Big)^2-\Big(\frac{a'}{a}\Big)'+K-\frac{k^2}{n}\Big]\phi^2+\Big[\Big(\frac{a'}{a}\Big)^2-\Big(\frac{a'}{a}\Big)'\Big](\phi^2)'\Big\}\frac{1}{k}\mathcal{B}\Big].\end{aligned} \qquad (1\cdot24)$$

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The expressions for the other gauge-invariant density perturbation variables, Δ_ϕ, $\Delta_{g\phi}$ and $\Delta_{c\phi}$ can be obtained from the relations (II-5·29). From the expression for $\delta T^j{}_0 = -h_\phi v_\phi Y^j$ the gauge-invariant velocity V_ϕ is given by

$$\begin{aligned} h_\phi V_\phi &\equiv h_\phi v_\phi - h_\phi k^{-1} H_T' \\ &= a^{-2}k\phi'\cdot X + \xi a^{-2}k\Big[-2(\phi\cdot X)' + 2\frac{a'}{a}\phi\cdot X + (\phi^2)'\Psi \\ &\quad + (n-1)\frac{a'}{a}\phi^2\mathcal{A} + (n-1)\Big\{\Big(\frac{a'}{a}\Big)^2 - \Big(\frac{a'}{a}\Big)'\Big\}\phi^2\frac{1}{k}\mathcal{B}\Big]. \end{aligned} \tag{1·25}$$

From the isotropic part of $\delta T^i{}_j$, $p_\phi \pi_{\phi L}$, the gauge-invariant entropy perturbation Γ_ϕ is expressed as

$$p_\phi \Gamma_\phi = (1-c_\phi{}^2)\rho_\phi \Delta_{s\phi} - 2U_\phi\cdot X + \xi p_\phi{}^\xi\Gamma_\phi\,, \tag{1·26}$$

where

$$\begin{aligned} a^2\,{}^\xi\Gamma_\phi &= -2(\phi\cdot X)'' - 4(n-1)\frac{a'}{a}(\phi\cdot X)' \\ &\quad - \Big[2(n-1)\Big(\frac{a'}{a}\Big)' + (n^2-2n+2)\Big\{\Big(\frac{a'}{a}\Big)^2 + K\Big\} + \frac{2n-1}{n}k^2\Big]\phi\cdot X \\ &\quad + \Big\{n\frac{a'}{a}(\phi^2)' - \frac{n-1}{n}k^2\phi^2\Big\}\Psi \\ &\quad + \Big\{(\phi^2)' + (n-1)\frac{a'}{a}\phi^2\Big\}\mathcal{A}' + \Big\{2(\phi^2)'' + (3n-4)\frac{a'}{a}(\phi^2)'\Big\}\mathcal{A} \\ &\quad + 2(n-1)\Big\{\Big(\frac{a'}{a}\Big)' + (n-1)\Big(\frac{a'}{a}\Big)^2\Big\}\phi^2\mathcal{A} + (\phi^2)'\frac{1}{k}\mathcal{B}'' \\ &\quad + \Big\{2(\phi^2)'' + (n-2)\frac{a'}{a}(\phi^2)'\Big\}\frac{1}{k}\mathcal{B}' \\ &\quad + \Big[n(n-1)(1-c_s{}^2)\frac{a'}{a}\Big\{\Big(\frac{a'}{a}\Big)^2 - \Big(\frac{a'}{a}\Big)' + K\Big\}\phi^2 - \frac{(n-1)(2n-1)}{n}\frac{a'}{a}k^2\phi^2 \\ &\quad + 2\frac{a'}{a}(\phi^2)'' + \Big\{(3n-4)\Big(\frac{a'}{a}\Big)^2 - 2(n-1)\Big(\frac{a'}{a}\Big)'\Big\}(\phi^2)'\Big]\frac{1}{k}\mathcal{B}\,. \end{aligned} \tag{1·27}$$

In contrast to the entropy perturbation, the anisotropic perturbation does not vanish only for $\xi\neq 0$ case. In fact, from the expression for the anisotropic part of $\delta T^i{}_j$, $p_\phi\pi_{\phi T}$, it is expressed as

$$p_\phi \Pi = -\xi k^2 a^{-2}[2\phi\cdot X + \{(n-2)\Phi + \Psi\}\phi^2]. \tag{1·28}$$

The existence of the anisotropic stress perturbation is an interesting feature of a non-minimally coupled scalar field.

The remaining perturbation variables are E_ϕ and F_ϕ. The definition of $\tilde{Q}_\phi$ and $\tilde{f}_{(\phi)\mu}$ are in the present case written as

$$\tilde{Q}_\phi \tilde{u}_\mu + \tilde{f}_{(\phi)\mu} = \tilde{S}\cdot\partial_\mu\tilde{\phi}\,. \tag{1·29}$$

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From this equation it follows that

$$\tilde{Q}_\phi = -\tilde{u}^\mu \tilde{S}\cdot\partial_\mu\tilde{\phi}\,, \tag{1·30}$$

$$\tilde{f}_{(\phi)j} = \tilde{S}\cdot\partial_j\tilde{\phi} - \tilde{Q}_\phi\tilde{u}_j\,. \tag{1·31}$$

First from Eq. (1·30) the perturbation of $\tilde{Q}_\phi$ is written as

$$\begin{aligned}\delta Q_\phi &= -(\delta u^0)S\cdot\phi' - u^0\phi'\cdot\delta S - u^0 S\cdot\delta\phi' \\ &= a^{-1}[-\phi'\cdot\delta S + (AS\cdot\phi' - S\cdot X' - k^{-1}S\cdot\phi'\sigma_g' - k^{-1}S\cdot\phi''\sigma_g)\mathrm{Y}].\end{aligned} \tag{1·32}$$

Since $\tilde{S}$ is a scalar quantity, the quantity DS defined by

$$DS\,\mathrm{Y} \equiv \delta S - k^{-1}\sigma_g S'\mathrm{Y} \tag{1·33}$$

is gauge-invariant. With this definition, the gauge-invariant quantity corresponding to ε_ϕ, E_ϕ, is written as

$$Q_\phi E_\phi = -a^{-1}S\cdot X' - a^{-1}\phi'\cdot DS - Q_\phi\Psi - k^{-1}Q_\phi' V_\phi\,. \tag{1·34}$$

As shown in §II-5, $\delta f_{(\phi)\mu}$ is itself gauge-invariant. From Eq. (1·31) it follows that

$$h_\phi F_\phi = -k(a'/a)^{-1}[S\cdot X + ak^{-1}Q_\phi V_\phi]. \tag{1·35}$$

As seen from Eqs. (1·24) and (1·25), X cannot be expressed only in terms of Δ_ϕ, V_ϕ and the gauge-invariant metric quantities when ϕ is a multi-component field. Hence the evolution equations of Δ_ϕ and V_ϕ do not form a closed system for such a case. With future application to such cases in mind, here we give the equation for X directly derived from the field equation (1·7). This equation is also necessary when one studies cases in which the time-derivative of the ϕ field background energy density, ρ_ϕ', vanishes and the perturbation amplitude V_ϕ is ill-defined. We do not give the detail of the derivation, but simply write down the result. The required formulas are that for the perturbation of $\tilde{\Box}\tilde{\phi}$ given in Appendix F, those for various geometrical quantities and the definitions of corresponding gauge-invariant variables. The resultant gauge-invariant perturbation equation for X is

$$\begin{aligned}&X'' + (n-1)\frac{a'}{a}X' + (k^2 + a^2U_{\phi\phi} + \xi a^2 R)X \\ &\quad= -a^{-2}(DS + 2S\Psi) + \phi'(\Psi - n\Phi)' - 2a^2U_\phi\Psi \\ &\qquad + 2n\xi\left[\frac{a^2\chi^2}{n-1}(p\Gamma + c_s{}^2\rho\Delta_s) + \left(-\frac{k^2}{n} + \frac{a^2\chi^2 h}{n-1}\right)\Phi - \frac{a^2}{n}R\Psi\right]\phi\,.\end{aligned} \tag{1·36}$$

The important feature of this equation is the appearance of the source terms depending on the gauge-invariant geometrical variables Φ and Ψ. This feature is the crucial difference between the equation obtained by the naive treatment neglecting the effect of the perturbation of gravitational field and that obtained by the correct argument [cf. Guth and Pi (1982)].

Now we write down the gauge-invariant equations for Δ_ϕ and V_ϕ. From now on we assume that the classical field has only one component. In contrast to the case of a multi-component field, X, hence Γ_ϕ and Π_ϕ can be expressed in terms of Δ_ϕ and V_ϕ for a

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single-component field. Though the expressions can be derived in principle irrespective of the value of the coupling parameter ξ, they become quite lengthy and complicated for a non-minimally coupled field $\xi \neq 0$. Hence we shall consider only the minimally coupled field, $\xi=0$, in the following. In this special case the expressions for various quantities become very simple. For example, the expressions for Δ_ϕ and V_ϕ in terms of X are written as

$$\rho_\phi \Delta_\phi = \rho_\phi \Delta_{s\phi} + nh_\phi(1-q_\phi)k^{-1}(a'/a)V_\phi$$
$$= a^{-2}[X'\phi' + \{-\phi'' + (a'/a)\phi'\}X - \phi'^2 \Psi], \tag{1·37a}$$

$$h_\phi V_\phi = a^{-2} k\phi' X\,. \tag{1·37b}$$

Furthermore noting the equation

$$n(a'/a)(1-c_\phi{}^2)h_\phi(1-q_\phi)V_\phi = -(\rho_\phi{}' - p_\phi{}')V_\phi = -2U_\phi\phi' V_\phi = -2kU_\phi X\,, \tag{1·38}$$

we find that Γ_ϕ is expressed as

$$p_\phi \Gamma_\phi = (1-c_\phi{}^2)[\rho_\phi \Delta_{s\phi} + nk^{-1}(a'/a)h_\phi(1-q_\phi)V_\phi]$$
$$= (1-c_\phi{}^2)\rho_\phi \Delta_\phi\,. \tag{1·39}$$

Finally from Eq. (1·15), (1·35) and (1·37b) it follows that

$$F_\phi = 0\,. \tag{1·40}$$

Substituting the equation for Γ_ϕ into Eq. (II-5·48a) with $F_\phi = \Pi_\phi = 0$, we obtain

$$V'_\phi + \frac{a'}{a}V_\phi = k\Psi + k\frac{\Delta_\phi}{1+w_\phi}\,. \tag{1·41}$$

This equation can also be derived from Eqs. (1·37) by eliminating X from them with the aid of the field equation for ϕ. Note that there is no term containing Q_ϕ explicitly. Of course if we use $\Delta_{s\phi}$ instead of Δ_ϕ, such a term appears. Contrarily, Q_ϕ appears explicitly in the equation for Δ_ϕ:

$$(\rho_\phi \Delta_\phi)' + n\frac{a'}{a}\rho_\phi \Delta_\phi = -k\left(1 - \frac{nK}{k^2}\right)h_\phi V_\phi$$
$$+ \frac{n^2}{2}(1+w)h_\phi\left\{\left(\frac{a'}{a}\right)^2 + K\right\}\frac{1}{k}(V - V_\phi) + aQ_\phi E_\phi\,. \tag{1·42}$$

In order to express E_ϕ in terms of Δ_ϕ and V_ϕ, we must specify S. For definiteness we assume the form

$$\tilde{S} = \gamma\,\mathrm{sgn}(\phi')[-\tilde{g}^{\mu\nu}\partial_\mu\tilde{\phi}\partial_\nu\tilde{\phi}]^{\alpha/2}|\tilde{\phi}|^\beta\,, \tag{1·43}$$

where α, β and γ are constants, and $\mathrm{sgn}(\phi')$ denotes the sign of ϕ'; $\mathrm{sgn}(\phi')$ is well-defined only when $\partial_\mu\tilde{\phi}$ is a time-like vector, which we assume. In the unperturbed background S and Q_ϕ reduce to

$$S = \gamma a^{-\alpha}\phi'|\phi'|^{\alpha-1}|\phi|^\beta\,, \tag{1·44}$$

$$Q_\phi = -\gamma a^{-\alpha-1}|\phi'|^{\alpha+1}|\phi|^\beta\,. \tag{1·45}$$

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Hence $\tilde{S}$ of the form (1·43) represents some kind of dissipation process for the field whose energy is transformed to other forms of energy. The perturbation of $\tilde{S}$ is given by

$$\delta S=\gamma a^{-\alpha}|\phi'|^{\alpha-1}|\phi|^{\beta-2}\phi[\beta\phi'\chi+\alpha\phi\chi'-\alpha\phi\phi' A]Y\ . \tag{1·46}$$

Hence the gauge-invariant quantity DS is expressed in terms of X as

$$DS=\gamma a^{-\alpha}|\phi'|^{\alpha-1}|\phi|^{\beta-2}\phi[\beta\phi' X+\alpha\phi X'-\alpha\phi\phi'\Psi]. \tag{1·47}$$

Substituting this equation and Eqs. (1·44), (1·45) and (1·37b) into Eq. (1·34), we find

$$Q_\phi E_\phi=n(\alpha+1)(\dot{a}/a)q_\phi\rho_\phi\Delta_\phi\ , \tag{1·48}$$

where

$$q_\phi=-(1/n)\gamma(a'/a)^{-1}a^{-\alpha+2}|\phi'|^{\alpha-1}|\phi|^\beta\ . \tag{1·49}$$

Hence Eq. (1·42) are written as

$$(\rho_\phi\Delta_\phi)'+n\frac{a'}{a}\{1-(\alpha+1)q_\phi\}\rho_\phi\Delta_\phi$$
$$=\frac{n^2}{2}(1+w)h_\phi\left\{\left(\frac{a'}{a}\right)^2+K\right\}\frac{1}{k}(V-V_\phi)-k\left(1-\frac{nK}{k^2}\right)h_\phi V_\phi\ . \tag{1·50}$$

So far we have concentrated on the variables associated with the classical field. To proceed further we must specify the content of matter other than ϕ field. For simplicity we restrict ourselves to the case in which the matter is composed of radiation and pressure-free particles (dust), which are referred to by the indices r and d, respectively. Then various background quantities are written as

$$\rho=\rho_\phi+\rho_r+\rho_d\ , \tag{1·51}$$

$$p=p_\phi+(1/n)p_r\ , \tag{1·52}$$

$$h=\dot{\phi}^2+(1+1/n)\rho_r+\rho_d\ , \tag{1·53}$$

$$w_r=c_r{}^2=1/n\ , \tag{1·54}$$

$$w_d=c_d{}^2=0\ . \tag{1·55}$$

In addition we assume that the isotropic and anisotropic stress perturbations intrinsic to each component, Γ_α and Π_α, vanish except for Γ_ϕ. We also assume $K=0$. First we consider the equations for the variables of the total matter, Δ and V. For later application it is more convenient to use the variables Φ and Υ defined by

$$\Upsilon\equiv ak^{-1}HV\ , \tag{1·56}$$

where H denotes the expansion rate of the universe $\dot{a}/a$. In the following we use the variable ζ proportional to the cosmic scale factor as the time variable instead of the conformal time η (the normalization of ζ is left arbitrary throughout this chapter unless otherwise stated). Then with the aid of the equation

$$\frac{dH}{da}=-\frac{n}{2}(1+w)\frac{H}{a}\ , \tag{1·57}$$

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the equation for Δ, (II-4·7a), is written as

$$\frac{d\Phi}{d\zeta}+(n-2)\frac{\Phi}{\zeta}=-\frac{n}{2}(1+w)\frac{\Upsilon}{\zeta}. \tag{1·58a}$$

Similarly the equation for V, (II-4·7b), is written as

$$\frac{d\Upsilon}{d\zeta}+\frac{1}{2}n(1+w)\frac{\Upsilon}{\zeta}=-(n-2)\frac{\Phi}{\zeta}+\frac{1}{\zeta}\frac{1}{h}[c_s{}^2\rho\Delta+p\Gamma]. \tag{1·58b}$$

From Eqs. (II-5·30a) and (II-5·30c), $\rho\Delta$ and $p\Gamma$ are written as

$$\rho\Delta=\rho_\phi\Delta_{c\phi}+\rho_r\Delta_{cr}+\rho_d\Delta_{cd}, \tag{1·59}$$

$$p\Gamma=p_\phi\Gamma_\phi+p\Gamma_{\text{rel}}, \tag{1·60}$$

where

$$\begin{aligned}p\Gamma_{\text{rel}}&=\sum_a(c_a{}^2-c_s{}^2)\rho_a\Delta_{ca}\\&=c_\phi{}^2\rho_\phi\Delta_{c\phi}+c_r{}^2\rho_r\Delta_{cr}+c_d{}^2\rho_d\Delta_{cd}-c_s{}^2\rho\Delta.\end{aligned} \tag{1·61}$$

Hence together with Eq. (1·39) and the relation between Δ_{ca} and Δ_a these expressions yield

$$\begin{aligned}c_s{}^2\rho\Delta+p\Gamma&=\rho_\phi\Delta_\phi+(1/n)\rho_r\Delta_r\\&\quad+ak^{-1}Hh_r(1-q_r)(V-V_r)+nak^{-1}Hc_\phi{}^2h_\phi(1-q_\phi)(V-V_\phi),\end{aligned} \tag{1·62}$$

which is to be inserted into the relevant term of Eq. (1·58b) if necessary.

Now we write down the dynamical equations for variables pertaining to each component. For later convenience we introduce the new variables Φ_a and Υ_a ($a=\phi$, r and d) defined by

$$(n-1)k^2\Phi_a\equiv\chi^2a^2\rho_a\Delta_a, \tag{1·63a}$$

$$h\Upsilon_a\equiv k^{-1}aHh_aV_a. \tag{1·63b}$$

Corresponding to Eqs. (II-5·30a) and (II-5·30b), Φ and Υ are expressed in terms of these quantities as

$$\Phi=\sum_a\Phi_a+(n^2/2)(aH/k)^2(1+w)\sum_a q_a\Upsilon_a, \tag{1·64a}$$

$$\Upsilon=\sum_a\Upsilon_a. \tag{1·64b}$$

Then Eqs. (1·41) and (1·50) are written as

$$\begin{aligned}&\frac{d\Upsilon_\phi}{d\zeta}+n\left\{\frac{1+w}{2}+c_\phi{}^2-c_s{}^2-q_\phi(1+c_\phi{}^2)\right\}\frac{\Upsilon_\phi}{\zeta}\\&\qquad=-(n-2)\frac{h_\phi}{h}\frac{\Phi}{\zeta}+\frac{2}{n}\frac{k^2}{(1+w)a^2H^2}\frac{\Phi_\phi}{\zeta},\end{aligned} \tag{1·65a}$$

$$\frac{d\Phi_\phi}{d\zeta}+\{n-2-n(1+\alpha)q_\phi\}\frac{\Phi_\phi}{\zeta}$$

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$$=\frac{n^3}{4}(1+w)^2\frac{k^2}{a^2H^2}\frac{1}{\zeta}\left(\frac{h_\phi}{h}\Upsilon-\Upsilon_\phi\right)-\frac{n}{2}(1+w)\frac{\Upsilon_\phi}{\zeta}, \qquad (1\cdot 65\text{b})$$

where the form of $\tilde{S}$ given by Eq. (1·44) has been used. The corresponding equations for radiation and dust can be easily obtained from Eqs. (II-5·48a) and (II-5·48b):

$$\frac{d\Upsilon_r}{d\zeta}+n\left(\frac{1+w}{2}+\frac{1}{n}-c_s{}^2-\frac{n+1}{n}q_r\right)\frac{\Upsilon_r}{\zeta}$$

$$=-(n-2)\frac{h_r}{h}\frac{\Phi}{\zeta}+\frac{2}{n^2}\frac{k^2}{(1+w)a^2H^2}\frac{\Phi_r}{\zeta}+\frac{aH}{k}\frac{h_r}{h}F_r, \qquad (1\cdot 66\text{a})$$

$$\frac{d\Phi_r}{d\zeta}+(n-2)\frac{\Phi_r}{\zeta}=\frac{n^3}{4}(1+w)^2\frac{k^2}{a^2H^2}\frac{1}{\zeta}\left(\frac{h_r}{h}\Upsilon-\Upsilon_r\right)$$

$$-\frac{n}{2}(1+w)\frac{\Upsilon_r}{\zeta}+\frac{n^2}{2}\frac{a^2H^2}{k^2}\frac{h_r}{\rho}q_r\frac{E_r}{\zeta}-\frac{n}{2}\frac{a^2H^2}{k^2}\frac{h_r}{\rho}\frac{F_r}{\zeta}, \qquad (1\cdot 66\text{b})$$

$$\frac{d\Upsilon_d}{d\zeta}+n\left(\frac{1+w}{2}-c_s{}^2-q_d\right)\frac{\Upsilon_d}{\zeta}=-(n-2)\frac{\rho_d}{h}\frac{\Phi}{\zeta}+\frac{aH}{k}\frac{\rho_d}{h}\frac{F_d}{\zeta}, \qquad (1\cdot 67\text{a})$$

$$\frac{d\Phi_d}{d\zeta}+(n-2)\frac{\Phi_d}{\zeta}=\frac{n^3}{4}(1+w)^2\frac{k^2}{a^2H^2}\frac{1}{\zeta}\left(\frac{\rho_d}{h}\Upsilon-\Upsilon_d\right)$$

$$-\frac{n}{2}(1+w)\frac{\Upsilon_d}{\zeta}+\frac{n^2}{2}\frac{a^2H^2}{k^2}\frac{\rho_d}{\rho}q_d\frac{E_d}{\zeta}-\frac{n}{2}\frac{a^2H^2}{k^2}\frac{\rho_d}{\rho}\frac{F_d}{\zeta}. \qquad (1\cdot 67\text{b})$$

Here from Eq. (II-5·44) E_r, E_d, F_r and F_d satisfy the relations

$$h_rF_r+\rho_dF_d=-n(k/aH)h(q_r\Upsilon_r+q_d\Upsilon_d), \qquad (1\cdot 68\text{a})$$

$$h_rq_rE_r+\rho_dq_dE_d=-(2/n)(\alpha+1)(k/aH)^2q_\phi\rho_\phi\Phi_\phi$$

$$-(1/n)H^{-2}(Q_\phi\Upsilon_\phi+Q_r\Upsilon_r+Q_d\Upsilon_d). \qquad (1\cdot 68\text{b})$$

§ VI-2. Simple decoupled systems in the inflationary universe

One of the most important cases in which a classical scalar field plays an essential role is the inflationary universe model [Sato (1981); Guth (1981)]. In the inflationary universe scenario one assumes the existence of a scalar field whose energy density changes very slowly during a certain stage of the early universe. At this stage the energy density of the field dominates the cosmic expansion since the energy density of the ordinary matter quickly decreases in proportion to some inverse power of the cosmic scale factor. Because the scalar field energy density stays nearly constant with time, the universe expands exponentially, namely experiences "inflation" during this stage as is easily seen from the Einstein equations. This stage terminates when the energy of the scalar field is transformed to the ordinary matter by some dissipation mechanism [Albrecht, Steinhardt, Turner and Wilczek (1982)]. There are several different versions of the inflationary universe scenario [for a review, see, e.g., Sasaki (1983b); Sato (1984)]. The most promising version among them is the one called the new inflationary universe scenario [Linde (1982); Albrecht and Steinhardt (1982)]. Therefore we shall mainly consider this scenario in the following.

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Let us review the general idea of the new inflationary universe scenario with special attention paid to the behavior of perturbation [see, e.g., Linde (1983); Turner (1983)]. The evolution of the universe around the inflationary stage is divided into six regimes:

(i) Primordial Friedmann regime: $\phi=0$ and $\rho_r \gg \rho_\phi$
(ii) Supercooling regime: $\phi=0$ and $\rho_r \ll \rho_\phi$
(iii) Quantum or thermal transition: $\phi=0 \rightarrow \phi \neq 0$
(iv) New inflationary regime: $\phi \neq 0$ and $h \ll \rho$
(v) Reheating: $\rho_\phi \rightarrow \rho_r$
(vi) Final Friedmann regime: $\phi \simeq \phi_{\min}$ and $p_r \gg \rho_\phi$

It is generally assumed that the universe started from a very hot state so that the effective potential of the scalar field had only one minimum $\phi=0$ $(U(0)>0)$. In this primordial regime (i), the universe is well described by a Friedmann model and the perturbations of the scalar field and radiation are decoupled as will be shown later. As the universe cooled down, there appeared a new minimum besides $\phi=0$ and it soon became an absolute minimum $(\phi=\phi_{\min})$. However, in the new inflationary scenario, the potential has a barrier near $\phi=0$ and the transition to this absolute minimum does not proceed too rapidly. In order for sufficient inflation to be achieved after ϕ passes through the barrier, the scalar field is stuck at $\phi=0$ by this potential barrier at least until the radiation temperature falls below H. In this supercooling regime (ii), $\rho_r(\propto T^4)$ soon becomes much smaller than $\rho_\phi(=U(0)\sim\text{const})$ and the universe undergoes inflation.

After the supercooling regime has lasted for some time, the scalar field develops a non-vanishing expectation value by quantum or thermal effects. Since this regime (iii) does not allow the classical description of the scalar field, it is out of scope of the present analysis. This introduces ambiguity into the junction of the perturbations before and after the regime (iii). In what follows we simply assume that the perturbation of radiation is smoothly matched across this transient regime.

It is expected that after the stage (iii) there appears a regime in which the scalar field can be treated as a classical field, which is nearly uniform over a scale of the Hubble horizon size at the beginning. This stage is the so-called new inflationary regime; the universe continues to expand exponentially due to the slow decrease in the potential energy. Since the perturbations of the scalar field and radiation are coupled gravitationally, the preexisting perturbation of radiation may induce the perturbation of the scalar field which eventually turns into the density perturbations in the final regime (vi).

As the time proceeds in the new inflationary regime, the background value of the scalar field ϕ increases and finally comes close to the absolute minimum point of the potential, where the potential has a deep dip. Then the field begins rapid oscillations and its energy is converted to radiation through direct interactions between the scalar field and other fields [Abbott, Farhi and Wise(1982); Hosoya and Sakagami(1984); Morikawa and Sasaki(1984)]. During this reheating regime (v) a perturbation of the scalar field is transferred to a perturbation of radiation through the direct interactions as well as the gravitational interaction. If this reheating process is completed within a time not too longer than the cosmic expansion time, the universe again becomes radiation-dominated with sufficiently high temperature.

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The inflationary stage is different from the ordinary Friedmann stage especially in the following two points. First the comoving scale increases rapidly whereas the Hubble horizon size $1/H$ is nearly constant. As a consequence the wavelength of a perturbation comes out of the Hubble horizon size, as opposed to the Friedmann case. Second the energy density is dominated by the classical field whose dynamical properties are quite different from ordinary matter. Therefore the general idea on the behavior of perturbations obtained from the study of Friedmann universes cannot be applied to the inflationary universe. In this section we examine the behavior of density perturbations in the inflationary universe on the bases of the formalism given in §1 [cf. Frieman and Will (1982); Bardeen, Steinhardt and Turner (1983); Frieman and Turner (1984)].

In order to do such investigation without going into details of elementary particle theory, e.g., grand unified theories(GUTs), on which the inflationary universe scenario is based on, we must make some assumptions for simplification. First we assume that the scalar field playing the central role is described by a single-component classical field coupled minimally with the gravitational field, i.e., $\xi=0$. Second we assume that the space curvature K can be neglected, $K=0$. Though the scalar field managing the inflation has many components in realistic models [Moss (1983); Breit, Gupta and Zacks (1983); Sato and Kodama (1984)], we believe that we can find the essential feature of the perturbation in the inflationary universe with a single-component scalar field as far as the behavior of density perturbations is concerned. The assumption on the spatial curvature is much more natural since the effect of the spatial curvature becomes less and less important as the inflation proceeds.

Concerning the perturbation in the inflationary universe, what we are most interested in is the relation between the amplitude of perturbation long after the reheating of the universe and that of perturbation at or before the inflationary stage. In this respect we first of all look for a quantity which represents the perturbed state of the inflationary stage and determines the eventual amplitude of density perturbations. Assuming that the universe remains radiation-dominated throughout $(w=1/n)$ after the reheating, the temporal behavior of the gauge-invariant quantities is from §III-2 given as

$$\Delta=\zeta^{(n-1)/2}[Aj_1(c_*\zeta^{(n-1)/2})+Bn_1(c_*\zeta^{(n-1)/2})], \tag{2·1a}$$

$$V=-\frac{n}{n+1}\zeta^{-(n-3)/2}\left(\frac{d\Delta}{d\zeta}-\frac{\Delta}{\zeta}\right), \tag{2·1b}$$

where $\zeta(\propto a)$ has been normalized to $\zeta=1$ at $k/aH=1$, and

$$c_*\equiv\frac{1}{\sqrt{n}}\frac{2}{n-1}. \tag{2·2}$$

As stated in §III-2, Δ begins to oscillate when ζ becomes greater than unity, namely, when the perturbation comes within the Hubble horizon, and its amplitude approaches a constant:

$$\Delta\simeq-c_*^{-1}[A\cos(c_*\zeta^{(n-1)/2})+B\sin(c_*\zeta^{(n-1)/2})]. \tag{2·3}$$

Hence the eventual amplitude of the density perturbation, $\|\Delta\|_H$, can be written as

$$\|\Delta\|_H\simeq c_*^{-1}[A^2+B^2]^{1/2}. \tag{2·4}$$

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We find it is convenient to express the constants A and B in terms of the values of Υ and Φ at some early epoch $\zeta=\zeta_*(\ll 1)$, denoted by Υ_* and Φ_*, respectively. Noting the relation

$$\Phi=(n/2)(aH/k)^2\Delta=(n/2)\zeta^{-(n-1)}\Delta\,, \tag{2·5a}$$

$$\Upsilon=(aH/k)V=\zeta^{-(n-1)/2}V\,, \tag{2·5b}$$

we find after a short calculation

$$A=c_*^2\Big[\frac{1}{\sqrt{n}}\{(n+1)\Upsilon_*+(n-3)\Phi_*\}\zeta_*^{n-1}n_1(c_*\zeta_*^{(n-1)/2})$$
$$+\frac{2}{n}\Phi_*\zeta_*^{3(n-1)/2}n_1'(c_*\zeta_*^{(n-1)/2})\Big], \tag{2·6a}$$

$$B=-c_*^2\Big[\frac{1}{\sqrt{n}}\{(n+1)\Upsilon_*+(n-3)\Phi_*\}\zeta_*^{n-1}j_1(c_*\zeta_*^{(n-1)/2})$$
$$+\frac{2}{n}\Phi_*\zeta_*^{3(n-1)/2}j_1'(c_*\zeta_*^{(n-1)/2})\Big]. \tag{2·6b}$$

In the limit $\zeta_*\to 0$ these equations reduce to

$$A=-\frac{n+1}{\sqrt{n}}(\Upsilon_*-\Phi_*)+O(\zeta_*^{(n-1)/2}), \tag{2·7a}$$

$$B=O(\zeta_*^{3(n-1)/2}). \tag{2·7b}$$

Thus we are led to the simple relation

$$\|\Delta\|_H\simeq\frac{n^2-1}{2}|\Upsilon_*-\Phi_*|. \tag{2·8}$$

Now we proceed to the analysis of perturbations in the inflationary stage. Since the equations describing a multi-component system containing a classical scalar field are quite complicated as seen from Eqs. (1·65)~(1·67), the general analysis can be performed only in rather restricted cases. Simplest cases are those in which the cosmic matter is composed only of a scalar field or only one component is perturbed in a multi-component system. In this section, as a first step, we study such simple cases. The analysis of a more complicated case will be done in the next section.

Since perturbations pertaining to different components are in general coupled gravitationally, some further restriction is required in order that the assumption of the single-component perturbation is consistent with the evolution equations. Of course this problem does not occur in the purely classical field dominated case, namely, in the case the cosmic matter is composed only of a classical field. However, if we study the behavior of perturbations of matter other than the scalar field in the inflationary stage, we are inevitably involved with a multi-component system. Fortunately there are several important simple multi-component systems in which the assumption of the single-component perturbation is consistent with the evolution equations. Namely, the cases such that the system is composed of two components, one is the scalar field and the other is an ordinary fluid, and the unperturbed scalar field ϕ stays at a meta-stable local minimum of the

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potential $U(\phi)$ (the corresponding value of ϕ is taken to be zero and $U(0)>0$ is assumed). In such cases, the perturbation of the scalar field is completely decoupled from the perturbation of the other component as is easily seen from Eqs. (1·24), (1·25) and (1·36), provided that the direct interaction between the components is absent. In the following first we study the two of such cases: One is the case the other component is dust and the other is the case it is radiation. Then we consider the system of a classical field alone to extract the essential nature of the perturbation of a classical scalar field in the inflationary stage.

Now let us consider the case in which ϕ stays at the local minimum $\phi=0$ of the potential U. As stated above the perturbation of the scalar field is completely decoupled from the perturbations of gravitational field and the other matter component. From Eq. (1·36) the gauge-invariant amplitude of the perturbation of ϕ field is subject to the equation

$$\ddot{X}+n\frac{\dot{a}}{a}\dot{X}+\left(\frac{k^2}{a^2}+m^2\right)X=0\,. \tag{2·9}$$

After change of the time variable from t to ζ, this equation is written as

$$\frac{d^2X}{d\zeta^2}+\frac{1}{\zeta}\left\{n+1-\frac{n(1+w)}{2}\right\}\frac{dX}{d\zeta}+\left(\frac{k^2}{a^2H^2}+\frac{m^2}{H^2}\right)\frac{1}{\zeta^2}=0\,, \tag{2·10}$$

where the normalization of ζ is left arbitrary again. Since $\phi=0$ throughout, the energy density of the component other than the scalar field soon becomes negligible and $1+w$ approaches zero and H becomes constant. Hence except for an early short period, the solution of Eq. (2·10) is given by

$$X=\zeta^{-n/2}[AJ_\lambda(\zeta^{-1})+BJ_{-\lambda}(\zeta^{-1})], \tag{2·11}$$

where

$$\lambda=[(\frac{n}{2})^2-(\frac{m}{H})^2]^{1/2}\,, \tag{2·11a}$$

and ζ has been normalized to $\zeta=1$ at $aH/k=1$ once again. From this equation the following asymptotic behavior of X is obtained:

$$X\sim\begin{cases}\dfrac{2}{\pi}\zeta^{-(n-1)/2}\left[A\cos\left(\zeta^{-1}-\dfrac{2\lambda+1}{4}\pi\right)+B\cos\left(\zeta^{-1}-\dfrac{1-2\lambda}{4}\pi\right)\right] & \text{for } \zeta\ll1\,,\\ \dfrac{A}{\Gamma(1+\lambda)}\zeta^{-\lambda-n/2}+\dfrac{B}{\Gamma(1-\lambda)}\zeta^{\lambda-n/2} & \text{for } \zeta\gg1\,.\end{cases} \tag{2·12}$$

Thus $\delta\phi=\chi Y=XY$ rapidly decreases except for the case m/H is nearly zero.

Contrarily the perturbation of the material component is coupled with the perturbation of metric. For the case the matter is dust, the perturbation equation is rather simple. Since $h=\rho_d$, $c_s=0$, $\Phi=\Phi_d$ and $\Upsilon=\Upsilon_d$ now, Eqs. (1·67) are written as

$$\frac{d\Upsilon}{d\zeta}+n\frac{1+w}{2}\frac{\Upsilon}{\zeta}=-(n-2)\frac{\Phi}{\zeta}\,, \tag{2·13a}$$

$$\frac{d\Phi}{d\zeta}+(n-2)\frac{\Phi}{\zeta}=-\frac{n}{2}(1+w)\frac{\Upsilon}{\zeta}\,. \tag{2·13b}$$

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Especially the subtraction of Eq. (2·13b) from Eq. (2·13a) yields a simple equation

$$\frac{d}{d\zeta}(\varUpsilon-\varPhi)=0\,. \tag{2·14}$$

Hence

$$\varUpsilon-\varPhi=\varUpsilon_0-\varPhi_0=\text{const}\,, \tag{2·15}$$

where the suffix 0 of $\varUpsilon$ and $\varPhi$ denotes the values of them at some reference time ζ_0.

Eliminating $\varUpsilon$ from Eq. (2·13b) with the aid of Eq. (2·15) we obtain the first-order differential equation for $\varPhi$

$$\frac{d\varPhi}{d\zeta}+\left\{n-2+\frac{n}{2}(1+w)\right\}\frac{\varPhi}{\zeta}=\frac{n}{2}(1+w)\frac{\varPhi_0-\varUpsilon_0}{\zeta}\,, \tag{2·16}$$

which is integrated to yield

$$\varPhi=(\zeta/\zeta_0)^{2-n}(H/H_0)\varPhi_0+\frac{n}{2}(\varPhi_0-\varUpsilon_0)\frac{H}{\zeta^{n-2}}\int_{\zeta_0}^{\zeta}\frac{1+w_1}{H_1}\zeta_1{}^{n-2}\frac{d\zeta_1}{\zeta_1}\,, \tag{2·17}$$

where the suffix 1 of a quantity denotes the value of it at $\zeta=\zeta_1$. Since $1+w=\rho_d/\rho$ decreases approximately in proportion to ζ^{-n}, the integral on the right-hand side of Eq. (2·17) converges for $\zeta\to\infty$. Hence for $\zeta\gg\zeta_0$, $\varPhi$ decreases as

$$\varPhi\simeq C(H/H_0)(\zeta/\zeta_0)^{2-n}\,, \tag{2·18}$$

where

$$C=\varPhi_0+\frac{n}{2}(\varPhi_0-\varUpsilon_0)\int_{\zeta_0}^{\infty}(1+w_1)\frac{H_0}{H_1}\left(\frac{\zeta_1}{\zeta_0}\right)^{n-2}\frac{d\zeta_1}{\zeta_1}\,, \tag{2·19}$$

and from Eq. (2·15) $\varUpsilon$ approaches a constant

$$\varUpsilon\to\varUpsilon_0-\varPhi_0\,. \tag{2·20}$$

If we translate this behavior to that of $\varDelta_d$ and $\varPhi_d$, we obtain

$$\varDelta_d=\frac{(n-1)k^2\varPhi_d}{x^2a^2\rho_d}\longrightarrow\frac{n-1}{x^2}\left(\frac{k}{a_0}\right)^2\frac{C}{\rho_{d0}}\frac{H}{H_0}\simeq\varDelta_{d0}\,, \tag{2·21a}$$

$$V_d=\frac{k}{aH}\varUpsilon_d\longrightarrow\frac{k}{aH}(\varUpsilon_0-\varPhi_0). \tag{2·21b}$$

Therefore the density contrast of dust stays nearly constant throughout the inflationary stage, while its velocity perturbation is damped adiabatically. This behavior is quite similar to that of the so-called isothermal mode in the radiation-baryon universe (see Chapter V). Note that the Hubble horizon size has no influence on the behavior of dust perturbation as is expected [Eqs. (2·13) are invariant under a scale change in ζ].

The behavior of the perturbation of radiation in the inflationary stage is different from the dust case due to the finite sound velocity. From Eqs. (1·66) the evolution equations for radiation perturbation are given by

$$\frac{d\varUpsilon}{d\zeta}+\frac{n}{2}(1+w)\frac{\varUpsilon}{\zeta}=\left[-(n-2)+\frac{2}{n^2}\frac{1}{(1+w)l^2H^2}\right]\frac{\varPhi}{\zeta}\,, \tag{2·22a}$$

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$$\frac{d\Phi}{d\zeta}+(n-2)\frac{\Phi}{\zeta}=-\frac{n}{2}(1+w)\frac{\Upsilon}{\zeta}, \tag{2·22b}$$

where

$$l\equiv a/k \tag{2·23}$$

is the reduced wavelength. Now $\Upsilon-\Phi$ is not constant but the ζ-derivative of it is given by

$$\frac{d}{d\zeta}(\Upsilon-\Phi)=\frac{2}{n^2}\frac{1}{(1+w)H^2l^2}\frac{\Phi}{\zeta}. \tag{2·24}$$

Hence Eqs. (2·22) are essentially second-order. Elimination of Υ from Eqs. (2·22) yields

$$\frac{d^2\Phi}{d\zeta^2}+\left\{2n-\frac{n}{2}(1+w)\right\}\frac{1}{\zeta}\frac{d\Phi}{d\zeta}$$
$$+\left\{(n+1)(n-2)-n(n-2)(1+w)+\left(\frac{H_k}{H}\right)^2\frac{1}{\zeta^2}\right\}\frac{\Phi}{\zeta^2}=0, \tag{2·25}$$

where we have normalized $\zeta=1$ at $lH=1/\sqrt{n}$ and H_k is the value of H at the same time. Except for a short period in the very early phase of inflation, $1+w=(n+1)\rho_r/n\rho$ is negligibly smaller than unity and H is constant with good accuracy. Hence we neglect this term. Under this approximation the solution of Eq. (2·25) is given by

$$\Phi=\zeta^{-n}[A(\zeta^2\sin\zeta^{-1}-\zeta\cos\zeta^{-1})+B(\zeta^2\cos\zeta^{-1}+\zeta\sin\zeta^{-1})]. \tag{2·25a}$$

Substituting this expression into Eq. (2·22b) we obtain

$$\Upsilon=\frac{2}{n}\frac{\zeta^{-n}}{(1+w)}(A\sin\zeta^{-1}+B\cos\zeta^{-1}). \tag{2·25b}$$

While the wavelength of perturbation is smaller than the Hubble horizon size ($\zeta\ll1$), Φ shows the damped oscillatory behavior

$$\Phi\simeq\zeta^{1-n}(-A\cos\zeta^{-1}+B\sin\zeta^{-1}), \tag{2·26a}$$

but Υ grows oscillatorily

$$\Upsilon\simeq\frac{2}{n+1}\left(\frac{\rho}{\rho_r}\right)_k\zeta(A\sin\zeta^{-1}+B\cos\zeta^{-1}), \tag{2·26b}$$

where $(\rho_r/\rho)_k$ denotes the value of ρ_r/ρ at $\zeta=1$. An interesting feature of the behavior of perturbation is that the ratio of the amplitudes of Φ and Υ, $\|\Phi\|/\|\Upsilon\|$, is independent of A and B and determined only by $(\rho_r/\rho)_k$ and ζ:

$$\frac{\|\Phi\|}{\|\Upsilon\|}\simeq\frac{n+1}{2}\left(\frac{\rho_r}{\rho}\right)_k\zeta^{-n}. \tag{2·27}$$

On the other hand when the wavelength of a perturbation exceeds the Hubble horizon size ($\zeta\gg1$), Φ decreases monotonically

$$\Phi\simeq\zeta^{2-n}\left[\frac{1}{3\zeta^3}A+B\right], \tag{2·28a}$$

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and Υ grows monotonically

$$\Upsilon \simeq \frac{2}{n+1}\left(\frac{\rho}{\rho_r}\right)_k (A+B\zeta). \tag{2·28b}$$

For sufficiently large ζ, the ratio of Φ and Υ becomes independent of A and B again:

$$\frac{\|\Phi\|}{\|\Upsilon\|}=\frac{n+1}{2}\left(\frac{\rho_r}{\rho}\right)_k \zeta^{1-n}. \tag{2·29}$$

Comparing Eqs. (2·28) with Eq. (2·8), one may expect that perturbations in the radiation energy density which makes negligible contribution to the total energy density produce very large density perturbations after reheating. However, this is not correct. As seen from Eq. (2·25b), the growth of Υ results from the rapid decrease in $1+w$. In realistic inflationary models, there is always a small amount of non-decreasing contribution to $1+w$ from the scalar field. Hence when the radiation energy density becomes smaller than this contribution, $1+w$ stops decreasing. As a result Υ begins to decrease, as will be shown in the next section. In addition there is one subtle point in applying this result to a realistic situation. In the exponentially expanding phase the cosmic expansion is too rapid for particles to interact among themselves within one expansion time. The collisionless nature in general means the breakdown of the fluid approximation and the free streaming damping [Stewart (1972)] may occur for perturbations with scale smaller than the Hubble horizon size. Since the scale corresponding to the present size of the universe is much smaller than the Hubble horizon size at this phase, this may imply that the perturbations existing before the inflationary stage is smoothed out and have no influence on the later stage as far as the scales relevant to the presently observed part of the universe is concerned.

The behavior of Φ and Υ found above can be easily translated to that of Δ_r and V_r. For $\zeta \ll 1$, Δ_r and V_r oscillate with constant amplitudes:

$$\Delta_r=\frac{(n-1)k^2}{\chi^2 a^2}\Phi \simeq 2\left(\frac{\rho}{\rho_r}\right)_k(-A\cos\zeta^{-1}+B\sin\zeta^{-1}), \tag{2·30a}$$

$$V_r=\frac{\Upsilon}{Hl}\simeq\frac{\sqrt{n}}{n+2}\left(\frac{\rho}{\rho_r}\right)_k(A\sin\zeta^{-1}+B\cos\zeta^{-1}). \tag{2·30b}$$

On the other hand for $\zeta \gg 1$, Δ_r begins to grow monotonically

$$\Delta_r \simeq 2\left(\frac{\rho}{\rho_r}\right)_k\left(A\frac{1}{3\zeta^2}+B\zeta\right), \tag{2·31a}$$

while V_r approaches a constant

$$V_r \simeq \frac{2\sqrt{n}}{n+1}\left(\frac{\rho}{\rho_r}\right)_k\left(A\frac{1}{\zeta}+B\right). \tag{2·31b}$$

Note that this behavior of Δ_r does not mean the growth of Δ because Δ is depressed by the factor ρ_r/ρ which decreases in proportion to $\zeta^{-(n+1)}$.

Finally we consider the perturbation in the purely classical field dominated case. The time-evolution equations in this case are from Eqs. (1·65)

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$$\frac{d\Upsilon}{d\zeta}+\frac{n}{2}(1+w)\frac{\Upsilon}{\zeta}=\left[-(n-2)+\frac{2}{n}\frac{1}{(1+w)l^2H^2}\right]\frac{\Phi}{\zeta}, \tag{2·32a}$$

$$\frac{d\Phi}{d\zeta}+(n-2)\frac{\Phi}{\zeta}=-\frac{n}{2}(1+w)\frac{\Upsilon}{\zeta}. \tag{2·32b}$$

Note that in the present case $1+w$ is simply given by

$$1+w=\dot{\phi}^2/\rho\,, \tag{2·33}$$

which vanishes for $\dot{\phi}=0$. Hence Eq. (2·32a) is singular at $\dot{\phi}=0$. However, as seen from Eq. (1·36) which is not singular anywhere, it does not mean that the perturbation behaves abnormally around $\dot{\phi}=0$, but simply means that the quantity Υ or V loses meaning at $\dot{\phi}=0$ [see Eq. (1·25)]. In reality $\dot{\phi}$ vanishes in the two stages; at the beginning of the inflationary stage and in the reheating phase in which ϕ oscillates around the absolute minimum of the potential. Around the time when the scalar field begins to deviate from the meta-stable state ($\phi=0$), its quantum or thermal perturbations are very large and it cannot be treated as a coherent classical field. Also, in the reheating phase the radiation generated from the scalar field cannot be neglected. Hence the present simplification assumption fails at the stages mentioned above. Therefore we limit our consideration to the stage in which ϕ changes monotonically. Then we can use Eqs. (2·32) safely. Some aspects of the scalar field perturbation in the oscillatory phase will be discussed in §4. Though $1+w$ does not vanish at such a monotonic stage, its value should be much smaller than unity in general in order that sufficient inflation occurs. In fact during the monotonic stage, we can neglect $\ddot{\phi}$ term in Eq. (1·11), hence

$$\dot{\phi}\sim-U_\phi/H\,. \tag{2·34}$$

On the other hand sufficient inflation is possible only for

$$H\phi/\dot{\phi}\gg 1\,. \tag{2·35}$$

Hence it follows that

$$\dot{\phi}^2\ll-\phi U_\phi\lesssim U\,. \tag{2·36}$$

In order to see the temporal behavior of Υ and Φ, we first combine Eqs. (2·32) to the second-order differential equation for Φ:

$$\frac{d^2\Phi}{d\zeta^2}+\left\{n-1+n(c_\phi{}^2-w)+\frac{n}{2}(1+w)\right\}\frac{1}{\zeta}\frac{d\Phi}{d\zeta}$$
$$+\left\{n(n-2)(c_\phi{}^2-w)+\frac{1}{l^2H^2}\right\}\frac{\Phi}{\zeta^2}=0\,. \tag{2·37}$$

As a first approximation we assume $1+w=$const. Then from the equation

$$\frac{dw}{d\zeta}=-n(c_\phi{}^2-w)(1+w)\frac{H}{\zeta}, \tag{2·38}$$

we find $c_\phi{}^2=w$ under this approximation. Further from Eq. (1·57) ζ-dependence of H^2 is

$$H^2\propto\zeta^{n(1+w)}. \tag{2·39}$$

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Hence Eq. (2·37) is approximately written as

$$\frac{d^2\Phi}{d\zeta^2}+\left\{n-1+\frac{n}{2}(1+w)\right\}\frac{1}{\zeta}\frac{d\Phi}{d\zeta}+\frac{\Phi}{\zeta^{4+n(1+w)}}=0\,, \tag{2·40}$$

where ζ has been normalized to $\zeta=1$ at $lH=1$.

The general solution of this equation is written in terms of the Bessel functions as

$$\Phi=\zeta^{\alpha}[AJ_{\nu}(\beta\zeta^{-\gamma})+BJ_{-\nu}(\beta\zeta^{-\gamma})], \tag{2·41}$$

where[†]

$$\alpha\equiv-\frac{n-2}{2}-\frac{n}{4}(1+w), \tag{2·41a}$$

$$\beta\equiv 1/|\gamma|, \tag{2·41b}$$

$$\gamma\equiv-1+\frac{n}{2}(1+w), \tag{2·41c}$$

$$\nu\equiv|\alpha/\gamma|. \tag{2·41d}$$

This solution coincides with Eq. (III-2·7b) if expressed in terms of ζ instead of η, as it should do. The expression for Υ may be obtained from Eq. (2·32b). The asymptotic behavior of this solution for $\zeta\ll 1$ is

$$\Phi\simeq\sqrt{\frac{2}{\pi\beta}}\zeta^{-(n-3)/2-n(1+w)/2}\times\left[A\cos\left(\beta\zeta^{-\gamma}-\frac{1+2\nu}{4}\pi\right)+B\cos\left(\beta\zeta^{-\gamma}-\frac{1-2\nu}{4}\pi\right)\right], \tag{2·42a}$$

$$\Upsilon\simeq-\frac{2}{n(1+w)}\sqrt{\frac{2}{\pi\beta}}\zeta^{-(n-1)/2}\times\left[A\sin\left(\beta\zeta^{-\gamma}-\frac{1+2\nu}{4}\pi\right)+B\sin\left(\beta\zeta^{-\gamma}-\frac{1-2\nu}{4}\pi\right)\right]. \tag{2·42b}$$

These expressions show that for $n>3$ Φ and Υ are damped oscillatorily while the perturbation wavelength is smaller than the Hubble horizon size. However, for the realistic dimension $n=3$, Φ stays nearly constant since $1+w\ll 1$. Similar to the perturbation of radiation discussed previously, the ratio of the amplitudes of Φ and Υ for $\zeta\ll 1$ is independent of the coefficients A and B:

$$\frac{\|\Upsilon\|}{\|\Phi\|}\simeq\frac{2}{n}\frac{\zeta^{-1+n(1+w)/2}}{1+w}\,. \tag{2·43}$$

In particular for $\zeta\sim 1$, namely, when the wavelength is equal to the Hubble horizon size,

$$\|\Phi\|_k\sim\frac{n}{2}(1+w)\|\Upsilon\|_k\,, \tag{2·44}$$

[†] Because of the limited number of symbols available, the notation used in this section is essentially isolated; that is, except for the symbols for basic variables introduced in §1 and those listed in Appendix G, symbols such as α, β and γ are used to denote quantities different from those denoted by the same symbols in §1. Similarly, the notation used in the next section will also be isolated.

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where the suffix k denotes the value at $\zeta=1$. Hence $\|\Phi\|_k$ is negligibly smaller than $\|\Upsilon\|_k$.

On the other hand, for $\zeta\gg 1$, Φ and Υ are approximately given as

$$\Phi\simeq\frac{1}{\Gamma(1-\nu)}\left(\frac{\beta}{2}\right)^{-\nu}B\,, \tag{2·45a}$$

$$\Upsilon\simeq-\frac{2}{n}\frac{1}{1+w}\frac{n-2+\alpha+\nu}{\Gamma(1-\nu)}\left(\frac{\beta}{2}\right)^{-\nu}B\,, \tag{2·45b}$$

which are both constant with time. Since $\alpha+\nu=O(1+w)\ll 1$, the ratio of the amplitudes of Φ and Υ are now

$$\frac{\|\Upsilon\|}{\|\Phi\|}\simeq\frac{2(n-2)}{n}\frac{1}{1+w}\,. \tag{2·46}$$

Hence the relation (2·44) is preserved for $\zeta>1$.

It has so far been assumed that $1+w$ is nearly constant. We can check, at least for $\zeta\gg 1$, whether the result obtained above depends essentially on this assumption or not, by transforming Eqs. (2·32) into integral equations and making a perturbation analysis with respect to $1+w(\ll 1)$. After a short calculation, Eqs. (2·32) are transformed into the integral equations

$$\Phi=(\zeta/\zeta_0)^{2-n}\Phi_0-\frac{n}{2}\zeta^{2-n}\int_{\zeta_0}^{\zeta}(1+w_1)\zeta_1^{\,n-3}\Upsilon_1 d\zeta_1\,, \tag{2·47a}$$

$$\Upsilon=(H/H_0)\Upsilon_0-\frac{2}{n}\int_{\zeta_0}^{\zeta}\frac{H}{H_1}\left[\frac{n(n-2)}{2}-\left(\frac{H_k}{H_1}\right)^2\frac{1}{1+w_1}\frac{1}{\zeta_1^{\,2}}\right]\Phi_1\frac{d\zeta_1}{\zeta_1}\,. \tag{2·47b}$$

Substituting Eq. (2·47b) into Eq. (2·47a) and performing partial integration, we obtain

$$\Phi=(\zeta/\zeta_0)^{2-n}\Phi_0-\frac{n}{2}F(\zeta)\Upsilon_0+\int_{\zeta_0}^{\zeta}K(\zeta,\,\zeta_1)\Phi_1\frac{d\zeta_1}{\zeta_1}\,, \tag{2·48}$$

where

$$F(\zeta)=\zeta^{2-n}\int_{\zeta_0}^{\zeta}\frac{H_1}{H_0}(1+w_1)\zeta_1^{\,n-3}d\zeta_1\,, \tag{2·48a}$$

$$K(\zeta,\,\zeta_1)=\zeta^{2-n}\left[\frac{n(n-2)}{2}-\left(\frac{H_k}{H_1}\right)^2\frac{\zeta_1^{-2}}{1+w_1}\right]\int_{\zeta_1}^{\zeta}\frac{H_2}{H_1}(1+w_2)\zeta_2^{\,n-3}d\zeta_2\,. \tag{2·48b}$$

The formal solution of Eq. (2·48) is given by

$$\Phi(\zeta)=A(\zeta)+\sum_{n=1}^{\infty}\int_{\zeta_0}^{\zeta}K_n(\zeta,\,\zeta_1)A(\zeta_1)\frac{d\zeta_1}{\zeta_1}\,, \tag{2·49}$$

where

$$A(\zeta)\equiv(\zeta/\zeta_0)^{2-n}\Phi_0-\frac{n}{2}F(\zeta)\Upsilon_0\,, \tag{2·49a}$$

$$K_n(\zeta,\,\zeta_1)=\int_{\zeta_1}^{\zeta}K_{n-1}(\zeta,\,\zeta_2)K(\zeta_2,\,\zeta_1)\frac{d\zeta_2}{\zeta_2}\,;\quad K_1=K\,. \tag{2·49b}$$

Provided that $|1+w|\ll 1$ and $\zeta_0\gg 1$, one can prove that the above formal series converges

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uniformly for $\|1+w\|\ln(\zeta/\zeta_0)\ll 1$ and yields the solution of Eq. (2·48). Up to the lowest order with respect to $1+w$ and ζ_0^{-2} in Eq. (2·49), we obtain

$$\Phi(\zeta)\simeq\Phi_0\left[\frac{n}{2(n-2)}(1+w)-\frac{1}{n(n-2)}\zeta_0^{-2}\right]-\frac{n}{2(n-2)}(1+w)\Upsilon_0\,, \qquad (2\cdot 50\mathrm{a})$$

$$\Upsilon(\zeta)\simeq\Upsilon_0\left[1-\frac{n}{2(n-2)}(1+w)+\int_{\zeta_0}^{\zeta}\left(\frac{H_k}{H_1}\right)^2\frac{1}{1+w_1}\frac{F_1}{\zeta_1^2}\frac{d\zeta_1}{\zeta_1}\right]-\left[1-\frac{2}{n^2}\frac{1}{(1+w)\zeta_0^2}\right]\Phi_0\,. \qquad (2\cdot 50\mathrm{b})$$

Equations (2·50) show that the relation (2·46) holds for $\zeta_0\gg 1$ even if w is not constant, as far as $|1+w|\ll 1$.

To summarize, the behavior of perturbation at the inflationary stage for the purely classical field dominated case is such that Φ and Υ are damped oscillatorily while the reduced wavelength of a perturbation l is smaller than the Hubble horizon size $1/H$, and approaches a constant when l becomes larger than $1/H$. The ratio of the amplitudes of Υ to Φ is approximately given by $(1+w)^{-1}(Hl)^{-1}$ for $Hl<1$ and $(1+w)^{-1}$ for $Hl>1$, irrespective of their initial values.

§ VI-3. Radiation-scalar field coupled systems in the inflationary universe

In §2 we have studied the behavior of perturbations for cases in which only one component is perturbed. In this section we extend the analysis to the case the cosmic matter is composed of radiation and a scalar field with monotonically changing ϕ, hence the perturbations of radiation and field are coupled gravitationally. For definiteness we only consider the new inflationary scenario.

From the story of the inflationary universe explained in §2 one sees that there are two regimes where the interaction between the perturbations of radiation and the scalar field becomes important; one is the regime (iv) where the perturbation of the scalar field is induced from that of radiation and the other is the regime (v) where the inverse process occurs. An important point here is that the nature of the interaction is different in these two regimes; in the regime (iv) the gravitational interaction is the dominant part, while direct non-gravitational interactions play an important role in the regime (v). Since the analysis of the latter regime is extremely complicated, we leave it to future study and we only consider the former regime in this paper [cf., however, Den and Tomita (1984)].

The fundamental equations describing the behavior of perturbations in the regime (iv) are obtained from Eqs. (1·65) and (1·66) by putting $q_\phi=q_r=E_r=F_r=0$, $\Phi=\Phi_r+\Phi_\phi$ and $\Upsilon=\Upsilon_r+\Upsilon_\phi$:

$$\frac{d\Upsilon_\phi}{d\zeta}+n\left\{\frac{1+w}{2}+c_\phi^2-c_s^2\right\}\frac{\Upsilon_\phi}{\zeta}=\left[-(n-2)\frac{h_\phi}{h}+\frac{2}{n}\frac{1}{1+w}\frac{1}{H^2l^2}\right]\frac{\Phi_\phi}{\zeta}$$
$$-(n-2)\frac{h_\phi}{h}\frac{\Phi_r}{\zeta}\,, \qquad (3\cdot 1\mathrm{a})$$

$$\frac{d\Phi_\phi}{d\zeta}+(n-2)\frac{\Phi_\phi}{\zeta}=-\frac{n}{2}(1+w)\left[1+\frac{n^2}{2}H^2l^2\frac{h_r}{\rho}\right]\frac{\Upsilon_\phi}{\zeta}+\frac{n^3}{4}(1+w)\frac{h_\phi}{\rho}H^2l^2\frac{\Upsilon_r}{\zeta}\,, \qquad (3\cdot 1\mathrm{b})$$

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$$\frac{d\Upsilon_r}{d\zeta}+n\left(\frac{1+w}{2}+\frac{1}{n}-c_s^2\right)\frac{\Upsilon_r}{\zeta}=\left[-(n-2)\frac{h_r}{h}+\frac{2}{n^2}\frac{1}{1+w}\frac{1}{H^2l^2}\right]\frac{\Phi_r}{\zeta}$$

$$-(n-2)\frac{h_r}{h}\frac{\Phi_\phi}{\zeta}, \tag{3·2a}$$

$$\frac{d\Phi_r}{d\zeta}+(n-2)\frac{\Phi_r}{\zeta}=-\frac{n}{2}(1+w)\left[1+\frac{n^2}{2}H^2l^2\frac{h_\phi}{\rho}\right]\frac{\Upsilon_r}{\zeta}+\frac{n^3}{4}(1+w)\frac{h_r}{\rho}H^2l^2\frac{\Upsilon_\phi}{\zeta}. \tag{3·2b}$$

In order to study the behavior of perturbations, it is more convenient to put these into the second-order equations for Φ_ϕ and Φ_r:

$$\mathscr{D}_\phi\Phi_\phi=\zeta^{-2}\mathscr{S}_\phi, \tag{3·3}$$

where

$$\mathscr{D}_\phi=\frac{d^2}{d\zeta^2}+\left\{n-1+n\left(c_\phi^2+1-\frac{1+w}{2}\right)+(n-1)\frac{n^2H^2l^2h_r/2\rho}{1+n^2H^2l^2h_r/2\rho}\right\}\frac{1}{\zeta}\frac{d}{d\zeta}$$

$$+\frac{1}{\zeta^2}\left[n(n-2)(c_\phi^2-w)+(n-1)(n-2)\frac{n^2H^2l^2h_r/2\rho}{1+n^2H^2l^2h_r/2\rho}+\frac{1}{H^2l^2}\right.$$

$$\left.+n(n-1)\frac{h_r}{\rho}\right], \tag{3·3a}$$

$$\mathscr{S}_\phi=-\frac{1}{2}n(n-1)(1+w)\frac{n^2H^2l^2h_\phi/2\rho}{1+n^2H^2l^2h_r/2\rho}\Upsilon_r+\frac{n^2}{2}\frac{h_\phi}{\rho}\Phi_r \tag{3·3b}$$

and

$$\mathscr{D}_r\Phi_r=\zeta^{-2}\mathscr{S}_r, \tag{3·4}$$

where

$$\mathscr{D}_r=\frac{d^2}{d\zeta^2}+\left[2n-\frac{n}{2}(1+w)-\{2-n(1+c_\phi^2)\}\frac{n^2H^2l^2h_\phi/2\rho}{1+n^2H^2l^2h_\phi/2\rho}\right]\frac{1}{\zeta}\frac{d}{d\zeta}$$

$$+\left[(n+1)(n-2)-n(n-2)(1+w)-(n-2)\{2-n(1+c_\phi^2)\}\frac{n^2H^2l^2h_\phi/2\rho}{1+n^2H^2l^2h_\phi/2\rho}\right.$$

$$\left.+\frac{1}{2}n(n-1)\frac{h_\phi}{\rho}+\frac{1}{n}\frac{1}{H^2l^2}\right]\frac{1}{\zeta^2}, \tag{3·4a}$$

$$\mathscr{S}_r=\frac{n}{2}\{2-n(1+c_\phi^2)\}(1+w)\frac{n^2H^2l^2h_r/2\rho}{1+n^2H^2l^2h_\phi/2\rho}\Upsilon_\phi+n(n-1)\frac{h_r}{\rho}\Phi_\phi. \tag{3·4b}$$

Since it is very difficult to solve the complete system (3·3)~(3·4) by analytic methods, we first solve Eq. (3·4) with $\mathscr{S}_r=0$ and then solve Eq. (3·3) by substituting the solutions of Φ_r and Υ_r so obtained into $\mathscr{S}_\phi$. By this method we estimate the amplitude of the scalar field perturbation at the end of the regime (iv) induced from the perturbation of radiation existing at the beginning of this regime. Further we assume that $\rho_r \ll \dot{\phi}^2$ and w is nearly constant, hence $w\simeq c_\phi^2$. Then the temporal behavior of Hl is given by

$$Hl\propto\zeta^{-n(1+w)/2}. \tag{3·5}$$

In the following we adopt the normalization of ζ such that $\zeta=1$ at $Hl=1$.

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The general behavior of solutions of the equation

$$\mathcal{D}_r \Phi_r = 0 \tag{3·6}$$

is different for $n^2(1+w)l^2H^2/2 \ll 1$ and $n^2(1+w)l^2H^2/2 \gg 1$. First consider the case $n^2(1+w)l^2H^2/2 \ll 1 (\zeta \ll (1+w)^{-1/2})$. In this case we can solve Eq. (3·6) iteratively by rewriting it as

$$\frac{d^2\Phi_r}{d\zeta^2} + \frac{2n}{\zeta}\frac{d\Phi_r}{d\zeta} + \left\{(n+1)(n-2) + \frac{n}{2}(1+w) + \frac{1}{n}\zeta^{-2+n(1+w)}\right\}\frac{\Phi_r}{\zeta^2}$$
$$= \left[\frac{n}{2}(1+w) + \{2-n(1+w)\}\frac{n^2(1+w)H^2l^2/2}{1+n^2(1+w)H^2l^2/2}\right]\left\{\frac{1}{\zeta}\frac{d\Phi_r}{d\zeta} + \frac{n-2}{\zeta^2}\Phi_r\right\}. \tag{3·7}$$

This equation is equivalent to the integral equation

$$\Phi_r = AU_+ + BU_- + \int_{\zeta_0}^{\zeta}\frac{d\zeta_1}{\zeta_1^2}\frac{U_+U_{-1} - U_-U_{+1}}{W_1}$$
$$\times\left[\frac{n}{2}(1+w) + \{2-n(1+w)\}\frac{n^2(1+w)H^2l^2/2}{1+n^2(1+w)H^2l^2/2}\right]_1\left\{\zeta\frac{d\Phi_r}{d\zeta} + (n-2)\Phi_r\right\}_1, \tag{3·8}$$

where[†)]

$$U_\pm(\zeta) = \zeta^{-(2n-1)/2} J_{\pm\nu}(\beta\zeta^{-\gamma}), \tag{3·8a}$$

$$W \equiv \frac{dU_+}{d\zeta}U_- - \frac{dU_-}{d\zeta}U_+ = -2\frac{\gamma}{\pi}\sin(\nu\pi)\zeta^{-2n}, \tag{3·8b}$$

$$\beta = 1/\gamma\sqrt{n}, \tag{3·8c}$$

$$\gamma = 1 - n(1+w)/2, \tag{3·8d}$$

$$\nu = \gamma^{-1}[9/4 - n(1+w)/2]^{1/2}. \tag{3·8e}$$

Hence in the lowest order with respect to $1+w$ as well as to $(1+w)(lH)^2$,

$$\Phi_r \simeq A[(1+f_{-+})U_+ - f_{++}U_-] + B[(1-f_{+-})U_- + f_{--}U_+], \tag{3·9a}$$

$$\zeta\frac{d\Phi_r}{d\zeta} + (n-2)\Phi_r = A[(1+f_{-+})u_+ - f_{++}u_-] + B[(1-f_{+-})u_- + f_{--}u_+], \tag{3·9b}$$

where

$$u_\pm = \left[\zeta\frac{d}{d\zeta} + (n-2)\right]U_\pm, \tag{3·10}$$

$$f_{ij} = \int_{\zeta_0}^{\zeta}\frac{d\zeta_1}{\zeta_1^2}\frac{U_{i1}}{W_1}\frac{n}{2}(1+w_1)\{1+2n\zeta_1^{2\gamma}\}u_{j1}. \quad (i, j = +, -) \tag{3·11}$$

From the asymptotic behavior of the Bessel functions the asymptotic behavior of $U_\pm$ and

[†)] See the footnote below Eq. (2·41).

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$u_\pm$ is given as follows:

For $\zeta \ll 1$,

$$U_\pm \simeq \sqrt{2/\pi\beta}\,\zeta^{-n+1-n(1+w)/4}\cos[\beta\zeta^{-\gamma}-(1\pm 2\nu)\pi/2], \tag{3·12a}$$

$$u_\pm \simeq \sqrt{2/\pi\beta n}\,\zeta^{-n+1+n(1+w)/4}\sin[\beta\zeta^{-\gamma}-(1\pm 2\nu)\pi/2]. \tag{3·12b}$$

For $1 \ll \zeta \ll (1+w)^{-1/2}$,

$$U_+ \simeq (\beta/2)^\nu \Gamma(1+\nu)^{-1}\zeta^{-n+1/2-\gamma\nu} \simeq (4/3\sqrt{\pi})(2\sqrt{n})^{-3/2}\zeta^{1-n}, \tag{3·13a}$$

$$U_- \simeq (\beta/2)^{-\nu}\Gamma(1-\nu)^{-1}\zeta^{-n+1/2+\gamma\nu} \simeq (1/2\sqrt{\pi})(2\sqrt{n})^{3/2}\zeta^{2-n}, \tag{3·13b}$$

$$\begin{aligned} u_+ &\simeq -(3/2+\gamma\nu)(\beta/2)^\nu\Gamma(1+\nu)^{-1}\zeta^{-n+1/2-\gamma\nu} \\ &\simeq -(4/\sqrt{\pi})(2\sqrt{n})^{-3/2}\zeta^{1-n}, \end{aligned} \tag{3·13c}$$

$$\begin{aligned} u_- &\simeq (-3/2+\gamma\nu)(\beta/2)^\nu\Gamma(1-\nu)^{-1}\zeta^{-n+1/2+\gamma\nu} \\ &\simeq -(1/12\sqrt{\pi})(2\sqrt{n})^{3/2}n(1+w)\zeta^{2-n}. \end{aligned} \tag{3·13d}$$

Hence f_{ij} is estimated as follows:

For $\zeta \ll 1$,

$$f_{ij} = O(1+w). \tag{3·14}$$

For $\zeta > \zeta_0 \gg 1$,

$$f_{-+} \simeq -(1/2)n^2(1+w)(\zeta^{2\gamma}-\zeta_0^{2\gamma})+f_{-+}(\zeta_0), \tag{3·15a}$$

$$f_{++} \simeq (1/3)\sqrt{n}(1+w)(\zeta_0^{2\gamma-3}-\zeta^{2\gamma-3})+f_{++}(\zeta_0), \tag{3·15b}$$

$$f_{+-} \simeq (1/36)n^3(1+w)^2(\zeta^{2\gamma}-\zeta_0^{2\gamma})+f_{+-}(\zeta_0), \tag{3·15c}$$

$$f_{--} \simeq -(1/30)n^4\sqrt{n}(1+w)^2(\zeta^{3+2\gamma}-\zeta_0^{3+2\gamma})+f_{--}(\zeta_0). \tag{3·15d}$$

Next consider the case $n^2(1+w)l^2H^2/2 \gg 1\,(\zeta \gg (1+w)^{-1/2})$. In this case we rewrite Eq. (3·6) as

$$\begin{aligned} &\frac{d^2\Phi_r}{d\zeta^2}+2(n-1)\frac{1}{\zeta}\frac{d\Phi_r}{d\zeta}+\left\{(n-1)(n-2)+\frac{n}{2}(1+w)+\frac{1}{n}\zeta^{-2+n(1+w)}\right\}\Phi_r \\ &=\left\{\frac{n}{2}(1+w)-\frac{2+n^3(1+w)^2H^2l^2/2}{1+n^2(1+w)H^2l^2/2}\right\}\left\{\frac{1}{\zeta}\frac{d\Phi_r}{d\zeta}+\frac{\Phi_r}{\zeta^2}\right\}. \end{aligned} \tag{3·16}$$

The corresponding integral equation is

$$\begin{aligned} \Phi_r = \hat{A}\hat{U}_+ + \hat{B}\hat{U}_- + \int_{\zeta_0}^{\zeta}\frac{d\zeta_1}{\zeta_1^2}\frac{\hat{U}_+\hat{U}_{-1}-\hat{U}_-\hat{U}_{+1}}{\hat{W}_1} \\ \times\left[\frac{n}{2}(1+w)-\frac{2+n^3(1+w)^2H^2l^2/2}{1+n^2(1+w)H^2l^2/2}\right]_1\left\{\zeta\frac{d\Phi_r}{d\zeta}+(n-2)\Phi_r\right\}_1, \end{aligned} \tag{3·17}$$

where

$$\hat{U}_\pm(\zeta) = \zeta^{-(2n-3)/2}J_{\pm\hat{\nu}}(\beta\zeta^{-\gamma}), \tag{3·17a}$$

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$$\hat{W} \equiv \frac{d\hat{U}_+}{d\zeta}\hat{U}_- - \frac{d\hat{U}_-}{d\zeta}\hat{U}_+ = -2\frac{\gamma}{\pi}\sin(\hat{\nu}\pi)\zeta^{-2n}, \tag{3·17b}$$

$$\hat{\nu} = \gamma^{-1}[1/4 - n(1+w)/2]^{1/2}, \tag{3·17c}$$

and β and γ are the constants given in Eqs. (3·8c) and (3·8d). Hence in the lowest order with respect to $1+w$ as well as to $1/(1+w)(Hl)^2$,

$$\Phi_r \simeq \hat{A}[(1+\hat{f}_{-+})\hat{U}_+ - \hat{f}_{++}\hat{U}_-] + \hat{B}[(1-\hat{f}_{+-})\hat{U}_- + \hat{f}_{--}\hat{U}_+], \tag{3·18a}$$

$$\zeta\frac{d\Phi_r}{d\zeta} + (n-2)\Phi_r = \hat{A}[(1+\hat{f}_{-+})\hat{u}_+ - \hat{f}_{++}\hat{u}_-] + \hat{B}[(1-\hat{f}_{+-})\hat{u}_- + \hat{f}_{--}\hat{u}_+], \tag{3·18b}$$

where

$$\hat{u}_\pm = \left[\zeta\frac{d}{d\zeta} + (n-2)\right]\hat{U}_\pm, \tag{3·19}$$

$$\hat{f}_{ij} = -\int_{\zeta_0}^{\zeta}\frac{d\zeta_1}{\zeta_1^2}\frac{\hat{U}_{i1}}{\hat{W}_1}\left[\frac{n}{2}(1+w) + \frac{1}{n^2}\frac{1}{(1+w)H^2l^2}\right]_1 \hat{u}_{j1}. \quad (i,\ j=+,\ -) \tag{3·20}$$

In the range of ζ concerned now, $\hat{U}_\pm$ and $\hat{u}_\pm$ are approximately expressed as

$$\hat{U}_- \simeq 2(2\pi\sqrt{n})^{1/2}\zeta^{-n+2-n(1+w)/2}, \tag{3·21a}$$

$$\hat{U}_+ \simeq (2\sqrt{n}/\pi)^{1/2}\zeta^{-n+1+n(1+w)/2}, \tag{3·21b}$$

$$\hat{u}_- \simeq -n(2\pi\sqrt{n})^{-1/2}(1+w)\zeta^{-n+2-n(1+w)/2}, \tag{3·21c}$$

$$\hat{u}_+ \simeq \hat{U}_+. \tag{3·21d}$$

Therefore

$$\hat{f}_{-+} \simeq (n/2)(1+w)\ln(\zeta/\zeta_0) + n^{-2}(1+w)^{-1}(\zeta_0^{2\gamma} - \zeta^{2\gamma}), \tag{3·22a}$$

$$\hat{f}_{++} \simeq (n\sqrt{n}/2)(1+w)(\zeta_0^{1-2\gamma} - \zeta^{1-2\gamma}) + (1/3n\sqrt{n})(1+w)^{-1}(\zeta_0^{-3} - \zeta^{-3}), \tag{3·22b}$$

$$\hat{f}_{+-} \simeq -(n^2/4)(1+w)\ln(\zeta/\zeta_0) - (1/2n)(\zeta_0^{-2\gamma} - \zeta^{-2\gamma}), \tag{3·22c}$$

$$\hat{f}_{--} \simeq -(n\sqrt{n}/4)(1+w)^2(\zeta_0^{2\gamma-1} - \zeta^{2\gamma-1}) + (1/n\sqrt{n})(\zeta_0^{-1} - \zeta^{-1}). \tag{3·22d}$$

Substituting the asymptotic behavior of $U_\pm$, $u_\pm$, $\hat{U}_\pm$ and $\hat{u}_\pm$ into Eqs. (3·9a), (3·9b), (3·18a) and (3·18b) and noting the relation between Υ_r and Φ_r obtained from Eq. (3·2b) with $\Upsilon_\phi = 0$, that is

$$\Upsilon_r = -\frac{2}{n}\frac{1}{1+w}\left[1 + \frac{n^2}{2}(1+w)\zeta^{2\gamma}\right]^{-1}\left[\zeta\frac{d\Phi_r}{d\zeta} + (n-2)\Phi_r\right], \tag{3·23}$$

we obtain the following estimate of Φ_r and Υ_r:

For $\zeta \ll 1$,

$$\Phi_r \simeq \left(2\frac{\sqrt{n}}{\pi}\right)^{1/2}\zeta^{1-n}\left[A\cos\left(\beta\zeta^{-\gamma} - \frac{1+2\nu}{4}\pi\right) + B\cos\left(\beta\zeta^{-\gamma} - \frac{1-2\nu}{4}\pi\right)\right], \tag{3·24a}$$

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$$\zeta\frac{d\Phi_r}{d\zeta}+(n-2)\Phi_r\simeq\left(\frac{2}{\pi\sqrt{n}}\right)^{1/2}\zeta^{-n}\left[A\sin\left(\beta\zeta^{-\gamma}-\frac{1+2\nu}{4}\pi\right)+B\sin\left(B\zeta^{-\gamma}-\frac{1-2\nu}{4}\pi\right)\right], \tag{3·24b}$$

$$\Upsilon_r\simeq-\frac{2}{n}\left(\frac{2}{\pi\sqrt{n}}\right)^{1/2}\frac{1}{1+w}\zeta^{-n}\left[A\sin\left(\beta\zeta^{-\gamma}-\frac{1+2\nu}{4}\pi\right)+B\sin\left(\beta\zeta^{-\gamma}-\frac{1-2\nu}{4}\pi\right)\right]. \tag{3·24c}$$

For $1\ll\zeta\ll(1+w)^{-1/2}$,

$$\Phi_r\simeq A\left[\frac{4}{3\sqrt{\pi}}\left(\frac{1}{2\sqrt{n}}\right)^{3/2}\frac{1+f_{-+}}{\zeta^3}+\frac{(2\sqrt{n})^{3/2}}{2\sqrt{\pi}}f_{++}\right]\zeta^{2-n}$$
$$+B\left[-\frac{(2\sqrt{n})^{3/2}}{2\sqrt{\pi}}(1-f_{+-})+\frac{4}{3\sqrt{\pi}}\left(\frac{1}{2\sqrt{n}}\right)^{3/2}\frac{f_{--}}{\zeta^3}\right]\zeta^{2-n}, \tag{3·25a}$$

$$\zeta\frac{d\Phi_r}{d\zeta}+(n-2)\Phi_r\simeq A\left[-\frac{4}{\sqrt{\pi}}\left(\frac{1}{2\sqrt{n}}\right)^{3/2}\frac{1+f_{-+}}{\zeta^3}+\frac{n(2\sqrt{n})^{3/2}}{12\sqrt{\pi}}(1+w)f_{++}\right]\zeta^{2-n}$$
$$+(1+w)B\left[-\frac{n(2\sqrt{n})^{3/2}}{12\sqrt{\pi}}(1-f_{+-})-\frac{4}{\sqrt{\pi}}\left(\frac{1}{2\sqrt{n}}\right)^{3/2}\frac{f_{--}}{(1+w)\zeta^3}\right]\zeta^{2-n}, \tag{3·25b}$$

$$\Upsilon_r\simeq A\left[\frac{8}{n\sqrt{\pi}}\left(\frac{1}{2\sqrt{n}}\right)^{3/2}\frac{1+f_{-+}}{(1+w)\zeta^3}-\frac{(2\sqrt{n})^{3/2}}{6\sqrt{\pi}}f_{++}\right]\zeta^{2-n}$$
$$+B\left[\frac{(2\sqrt{n})^{3/2}}{6\sqrt{\pi}}(1-f_{+-})+\frac{8}{n\sqrt{\pi}}\left(\frac{1}{2\sqrt{n}}\right)^{3/2}\frac{f_{--}}{(1+w)\zeta^3}\right]\zeta^{2-n}. \tag{3·25c}$$

For $\zeta\gg(1+w)^{-1/2}$,

$$\Phi_r\simeq\hat{A}\left[\frac{(2\sqrt{n})^{1/2}}{\sqrt{\pi}}\frac{1+\hat{f}_{-+}}{\zeta}-\frac{2}{\sqrt{\pi}}\left(\frac{1}{2\sqrt{n}}\right)^{1/2}\hat{f}_{++}\right]\zeta^{2-n}$$
$$+\hat{B}\left[\frac{2}{\sqrt{\pi}}\left(\frac{1}{2\sqrt{n}}\right)^{1/2}(1-\hat{f}_{+-})+\frac{(2\sqrt{n})^{1/2}}{\sqrt{\pi}}\frac{\hat{f}_{--}}{\zeta}\right]\zeta^{2-n}, \tag{3·26a}$$

$$\zeta\frac{d\Phi_r}{d\zeta}+(n-2)\Phi_r\simeq\hat{A}\left[\left(\frac{2\sqrt{n}}{\pi}\right)^{1/2}\frac{1+\hat{f}_{-+}}{\zeta}+\frac{n}{\sqrt{\pi}}\left(\frac{1}{2\sqrt{n}}\right)^{1/2}(1+w)\hat{f}_{++}\right]\zeta^{2-n}$$
$$+\hat{B}\left[-\frac{n}{\sqrt{\pi}}\left(\frac{1}{2\sqrt{n}}\right)^{1/2}(1+w)(1-\hat{f}_{+-})+\frac{(2\sqrt{n})^{1/2}}{\sqrt{\pi}}\frac{\hat{f}_{--}}{\zeta}\right]\zeta^{2-n}, \tag{3·26b}$$

$$\Upsilon_r\simeq-\frac{\hat{A}}{(1+w)^2}\left[\frac{2(2\sqrt{n})^{1/2}}{n^3\sqrt{\pi}}\frac{1+\hat{f}_{-+}}{\zeta}+\frac{2}{n^2\sqrt{\pi}}\left(\frac{1}{2\sqrt{n}}\right)^{1/2}(1+w)\hat{f}_{++}\right]\zeta^{-n}$$
$$+\frac{\hat{B}}{1+w}\left[\frac{2}{n^2\sqrt{\pi}}\left(\frac{1}{2\sqrt{n}}\right)^{1/2}(1-\hat{f}_{+-})-\frac{2(2\sqrt{n})^{1/2}}{n^3\sqrt{\pi}}\frac{\hat{f}_{--}}{(1+w)\zeta}\right]\zeta^{-n}. \tag{3·26c}$$

These equations show that the perturbation of radiation is rapidly damped in the new inflationary stage (iv), provided that $\rho_r\ll\dot{\phi}^2$.

Substituting Eqs. (3·24a)~(3·26c) into Eq. (3·3b) we can estimate the source term for

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the perturbation of the scalar field:

For $\zeta \ll 1$,

$$\mathscr{S}_\phi \simeq \frac{n^2}{2}\left(\frac{2}{\pi\sqrt{n}}\right)^{1/2}(1+w)\Big[(n-2)\zeta^\gamma\Big\{A\sin\Big(\beta\zeta^{-\gamma}-\frac{1+2\nu}{4}\pi\Big)+B\sin\Big(\beta\zeta^{-\gamma}-\frac{1-2\nu}{4}\pi\Big)\Big\}$$
$$+\sqrt{n}\Big\{A\cos\Big(\beta\zeta^{-\gamma}-\frac{1+2\nu}{4}\pi\Big)+B\cos\Big(\beta\zeta^{-\gamma}-\frac{1-2\nu}{4}\pi\Big)\Big\}\Big]\zeta^{1-n-(n/4)(1+w)}. \tag{3·27a}$$

For $1 \ll \zeta \ll (1+w)^{-1/2}$,

$$\mathscr{S}_\phi \simeq \frac{n}{8}\left(\frac{2\sqrt{n}}{\pi}\right)^{1/2}(1+w)\Big[A\Big\{-4(n-1)\frac{1+f_{-+}}{\zeta}+4n\sqrt{n}f_{++}\Big\}$$
$$+B\Big\{-4n\sqrt{n}(1-f_{+-})-\frac{n-1}{4}\frac{f_{--}}{\zeta}\Big\}\Big]\zeta^{2-n}. \tag{3·27b}$$

For $(1+w)^{-1/2} \ll \zeta$,

$$\mathscr{S}_\phi \simeq \frac{1}{2}\left(\frac{1}{2\pi\sqrt{n}}\right)^{1/2}(1+w)\Big[\hat{A}\Big\{2\sqrt{n}(n-1)\frac{1+\hat{f}_{-+}}{(1+w)\zeta}-n(n+1)\hat{f}_{++}\Big\}$$
$$+\hat{B}\Big\{n(n+1)(1-\hat{f}_{+-})+2\sqrt{n}(n-1)\frac{\hat{f}_{--}}{(1+w)\zeta}\Big\}\Big]\zeta^{2-n}. \tag{3·27c}$$

Comparing the values of $\mathscr{S}_\phi$ at $\zeta \sim (1+w)^{-1/2}$ obtained from Eq. (3·27b) and Eq. (3·27c), we find the magnitude relation between A, B and $\hat{A}$, $\hat{B}$:

$$\hat{A}=O[(1+w)A \quad \text{or} \quad (1+w)^{1/2}B],$$
$$\hat{B}=O[(1+w)^{1/2}A \quad \text{or} \quad B]. \tag{3·28}$$

Now we estimate the amplitude of the perturbation of the scalar field induced from Φ_r and Υ_r. Since we assume $(1+w)\approx$const, the two independent solutions of the equation $\mathscr{D}_\phi U_\phi=0$ are from Eq. (2·41),

$$U_{\phi\pm}=\zeta^{-(n-2)/2-n(1+w)/4}J_{\pm\nu'}(\beta'\zeta^{-\gamma}), \tag{3·29}$$

where γ is the constant given by Eq. (3·8d) and

$$\beta'=1/\gamma\,, \tag{3·30}$$

$$\nu'=\Big[\frac{n-2}{2}+\frac{n}{4}(1+w)\Big]/\gamma\,. \tag{3·31}$$

Hence the general solution of Eq. (3·3) is expressed as

$$\Phi_\phi=(C+F(\zeta))U_{\phi+}+(D-G(\zeta))U_{\phi-}\,, \tag{3·32}$$

where

$$F(\zeta)=\int_{\zeta_0}^{\zeta}\frac{d\zeta_1}{\zeta_1{}^2}\frac{U_{\phi-1}}{W_{\phi 1}}\mathscr{S}_{\phi 1}\,, \tag{3·32a}$$

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$$G(\zeta)=\int_{\zeta_0}^{\zeta}\frac{d\zeta_1}{\zeta_1^2}\frac{U_{\phi+1}}{W_{\phi 1}}\mathscr{S}_{\phi 1}\,, \tag{3·32b}$$

$$W_\phi \equiv \frac{dU_{\phi+}}{d\zeta}U_{\phi-}-\frac{dU_{\phi-}}{d\zeta}U_{\phi+}$$
$$=(2\gamma/\pi)\sin(\nu'\pi)\zeta^{1-n-n(1+w)/2}\,. \tag{3·32c}$$

Substituting Eq. (3·32) into Eq. (3·1b), we obtain

$$\Upsilon_\phi \simeq -(2/n)(1+w)^{-1}[(C+F)u_{\phi+}+(D-G)u_{\phi-}]+(n^2/2)(1+w)\zeta^{2\gamma}\Upsilon_r\,, \tag{3·33}$$

where

$$u_{\phi\pm}=\left[\zeta\frac{d}{d\zeta}+(n-2)\right]U_{\phi\pm}\,. \tag{3·33a}$$

The constants C and D should be determined by the condition

$$\Phi_\phi(\zeta_0)=\Upsilon_\phi(\zeta_0)=0\,, \tag{3·34}$$

which yields

$$C=(n^3\pi/8\gamma)(1+w)^2\zeta_0^{(n+2)/2-n(1+w)/4}J_{\nu'}(\beta'\zeta_0^{-\gamma})\Upsilon_{r0}\,, \tag{3·35a}$$

$$D=-(n^3\pi/8\gamma)(1+w)^2\zeta_0^{(n+2)/2-n(1+w)/4}J_{-\nu'}(\beta'\zeta_0^{-\gamma})\Upsilon_{r0}\,. \tag{3·35b}$$

For $\zeta \gg 1$, $U_{\phi\pm}$ and $u_{\phi\pm}$ have the asymptotic behavior

$$U_{\phi+}\simeq 2^{-\gamma\nu'}\Gamma(1+\nu')^{-1}\zeta^{2-n-n(1+w)/2}\,, \tag{3·36a}$$

$$U_{\phi-}\simeq 2^{\gamma\nu'}\Gamma(1-\nu')^{-1}\,, \tag{3·36b}$$

$$u_{\phi+}\simeq -(n/2)(1+w)2^{-\gamma\nu'}\Gamma(1+\nu')^{-1}\zeta^{2-n-n(1+w)/2}\,, \tag{3·36c}$$

$$u_{\phi-}\simeq (n-2)2^{\gamma\nu'}\Gamma(1-\nu')^{-1}\,. \tag{3·36d}$$

Since $\mathscr{S}_\phi$ decreases approximately in proportion to ζ^{2-n}, this behavior guarantees that both $F(\zeta)$ and $G(\zeta)$ approach constants as $\zeta\to\infty$. Hence, noting that $\zeta^{2\gamma}\Upsilon_r\to 0$ as $\zeta\to\infty$ from Eq. (3·26c), we find that both Φ_ϕ and Υ_ϕ level off to constant values for sufficiently large ζ:

$$\Phi_\phi\to\Phi_\infty\equiv\frac{2^{\gamma\nu'}}{\Gamma(1-\nu')}[D-G(\infty)], \tag{3·37a}$$

$$\Upsilon_\phi\to\Upsilon_\infty\equiv-\frac{2(n-2)}{n}\frac{1}{1+w}\frac{2^{\gamma\nu'}}{\Gamma(1-\nu')}[D-G(\infty)]. \tag{3·37b}$$

Note that the relation between $\|\Phi\|$ and $\|\Upsilon\|$ given in Eq. (2·46) holds also for Φ_∞ and Υ_∞.

Substituting Eqs. (3·27a)~(3·27c) into the definition of $G(\zeta)$ and noting that the coefficients of A and B in the square brackets of Eqs. (3·27) are of order unity, we find that $G(\infty)=O(A,(1+w)B)$, while D tends to zero as $\zeta_0^{(n+3)/2-n(1+w)/2}$ in the limit $\zeta_0\to 0$. Hence we obtain the estimate for the final amplitudes of the scalar field perturbation, Φ_∞ and Υ_∞:

$$\Upsilon_\infty=-\frac{2(n-2)}{n}\frac{1}{1+w}\Phi_\infty$$

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$$=O(A \text{ or } B). \tag{3·38}$$

We mention that the dominant contribution of $\mathscr{S}_\phi$ to $G(\infty)$ comes from the range $\zeta \leq 1$.

Equation (3·38) means that the amplitude of the scalar field perturbation induced from the radiation perturbation is of the order of Φ_r, namely the curvature perturbation, at the time when the wavelength of the perturbation coincides with the Hubble horizon size. Since the amplitude of Φ_r decreases monotonically in proportion to a^{1-n} while the perturbation scale is smaller than the Hubble horizon size both in the regimes (ii) and (iv), A and B are expected to be very small in general. Now unless the anomalous enhancement of the perturbation occurs in the reheating regime (v), the amplitude of density perturbations, when they enter the Hubble horizon again in the regime (vi), is at most of the order of $|\Upsilon_\infty - \Phi_\infty|$. Hence we conclude that the perturbation of radiation existing before the inflationary stage has negligible influence on the present structure of the universe even if it is not erased away by free streaming damping, provided that the new inflationary regime lasts long enough. In other words we must seek for sources of perturbations either in the new inflationary stage itself, such as quantum fluctuations [Hawking (1982); Guth and Pi (1982); Starobinskii (1982); Hawking and Moss (1983); Vilenkin (1983)], or in the Friedmann stage after reheating in order to account for the large-scale structures of the present universe [see, however, Sasaki and Kodama (1982); Kodama, Sasaki and Sato (1982); Kodama, Sato and Sasaki (1983) in the case of the original inflationary universe scenario].

§ VI-4. Comments on peculiar properties of scalar field perturbations

In the previous sections we have studied the behavior of perturbation in a stage dominated by a classical field for the case the background classical field changes monotonically. It is shown that the behavior of perturbations in such a case is rather simple. However, this does not mean that its behavior is easily understood. For example, the perturbation of a classical field shows the oscillatory behavior as if its sound velocity were equal to the velocity of light, though its actual value is not so.

In actual situations it sometimes occurs that a classical field behaves not monotonically but oscillatorily while its energy density dominates the cosmic expansion. One example is the early stage of the reheating phase in the inflationary universe scenario explained in §2. Another important example is the axion field dominated universe [Turner, Wilczek and Zee (1983); Stecker and Shafi (1983); Fukugita and Yoshimura (1983); Yoshimura (1983)]. In the latter example the peculiarity of the perturbation of a coherent classical scalar field becomes more vivid [Sasaki (1984a, b)].

In this section we point out the problematic aspect of the perturbation of a classical scalar field in the oscillatory stage. We only consider a minimally coupled single-component real field in the usual four-dimensional spacetime and assume that its potential is purely quadratic:

$$U(\phi)=\frac{1}{2}m^2\phi^2. \tag{4·1}$$

Further for simplicity we assume that the cosmic matter is composed only of the scalar field.

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First we examine the behavior of the unperturbed field ϕ. From Eq. (1·11), on the above assumptions ϕ is subject to the equation

$$\ddot{\phi}+3H\dot{\phi}+m^2\phi=0\,, \tag{4·2}$$

where

$$H\equiv\frac{\dot{a}}{a}=\frac{\chi}{\sqrt{3}}\left[\frac{1}{2}\dot{\phi}^2+\frac{1}{2}m^2\phi^2\right]^{1/2}. \tag{4·3}$$

This equation cannot be solved exactly, but it is easily checked that its solution can be approximately expressed for $mt\gg1$ as

$$\phi\simeq\frac{\sqrt{2}}{\chi}\frac{1}{mt}\sin(mt+c), \tag{4·4a}$$

$$H\simeq\frac{2}{3}\frac{1}{t}, \tag{4·4b}$$

where c is an arbitrary constant.

Equation (4·4b) shows that the cosmic scale factor a increases in proportion to $t^{2/3}$ and the energy density decreases in proportion to a^{-3}. The reason of this dust-like behavior is easily understood by looking at the structure of the energy density and the pressure

$$\rho=\frac{1}{2}\dot{\phi}^2+\frac{1}{2}m^2\phi^2\,, \tag{4·5a}$$

$$p=\frac{1}{2}\dot{\phi}^2-\frac{1}{2}m^2\phi^2\,. \tag{4·5b}$$

Since ϕ behaves oscillatorily, p vanishes if averaged over a period longer than $1/m$.

This apparently suggests that the perturbation of a classical field also behaves like dust. However, the situation is not so simple. From Eq. (3·3) with $h_r=\mathscr{S}_\phi=0$, the gauge-invariant amplitude Φ obeys the equation

$$\ddot{\Phi}+(4+3c_s{}^2)H\dot{\Phi}+[l^{-2}+3(c_s{}^2-w)H^2]\Phi=0\,. \tag{4·6}$$

This equation should be compared with the equation for Φ in the purely dust-dominated universe:

$$\ddot{\Phi}+4H\dot{\Phi}=0\,. \tag{4·7}$$

There are several important differences between Eqs. (4·6) and (4·7).

First the sound velocity c_s and the pressure $p=w\rho$ appear in Eq. (4·6) though not in Eq. (4·7). Especially the appearance of $c_s{}^2$ introduces a very intricate problem. Since $c_s{}^2$ is expressed from Eqs. (1·16) and (1·18) as

$$c_s{}^2=1+\frac{2m^2\phi}{3H\dot{\phi}}\,, \tag{4·8}$$

it becomes infinite when $\dot{\phi}=0$. Hence Eq. (4·6) is singular. Of course this does not mean that the perturbation exhibits a singular behavior. This can be seen as follows. Noting the definition of Υ_ϕ [Eq. (1·63b)] and the expression for V_ϕ in terms of X[Eq. (1·37b)],

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Eq. (1·65b) is now written as

$$\dot{\Phi}=-H\Phi-\frac{1}{2}\kappa^2\dot{\phi}X\,. \tag{4·9a}$$

This equation and the equation obtained from Eq. (1·36) by rewriting it in the first-order form

$$\dot{X}=Y\,, \tag{4·9b}$$

$$\dot{Y}=-3HY-[l^{-2}+m^2-2\kappa^2\dot{\phi}^2]X+2[3H\dot{\phi}+m^2\phi]\Phi\,, \tag{4·9c}$$

form a system of regular first-order differential equations for X, Y and Φ. These quantities are constrained by the equation obtained from Eq. (1·63a) by rewriting it with the aid of Eq. (1·37a):

$$(3H\dot{\phi}+m^2\phi)X+\dot{\phi}Y+[\dot{\phi}^2-2\kappa^{-2}l^{-2}]\Phi=0\,, \tag{4·10}$$

which is consistent with the evolution equations (4·9). Thus for any initial data of X, Y and Φ satisfying Eq. (4·10), their subsequent temporal evolution is regularly determined by Eqs. (4·9) and there appears no singularity. The apparent singularity in Eq. (4·6) appears through the reduction of the constrained system to the second-order differential equation. From the inspection of Eq. (4·10) one easily finds that there exists no linear combination of X, Y and Φ in terms of which the reduced second-order equation becomes regular.

From this argument it is clear that we must match the solution at points where $\dot{\phi}=0$ so that it is smooth across these points, if we determine the time-evolution of Φ by Eq. (4·6). This smoothness condition requires that Φ and $\dot{\Phi}$ are related by

$$\dot{\Phi}=-H\Phi\,, \tag{4·11}$$

at the singular points. The existence of this periodic constraint is expected to make the solution of Eq. (4·6) behave quite differently from that of Eq. (4·7).

The second difference is the appearance of the wavelength in Eq. (4·6). Comparing it with Eq. (3·4) with $h_\phi=\mathscr{S}_r=0$, one finds that Eq. (4·6) is quite similar to the perturbation equation for a fluid with $c_s{}^2=1$. It has been shown in §2 that this similarity is not just an apparent one but reflects the similarity in dynamics at least for the scalar field perturbation in the inflationary stage; namely, the scalar field perturbation shows the oscillatory behavior when its wavelength is shorter than the Hubble horizon size. Thus it is expected that the solution of Eq. (4·6) also shows the oscillatory behavior for sufficiently short wavelengths.

Finally the coefficients of Eq. (4·6) oscillate with a period much shorter than the cosmic expansion time-scale if $mt\gg1$. Several authors argued that in a time-scale much larger than $1/m$, the temporal behavior of Φ will be approximately given by the solution of the equation obtained from Eq. (4·6) by replacing $c_s{}^2$ and w with their average values, that is by eliminating them [Turner, Wilczek and Zee (1983)]. However, this is not correct. The most famous counter-example against such argument is the Mathieu equation. The Mathieu equation

$$\frac{d^2u}{dz^2}+[A-2B\cos(2z)]u=0\,, \tag{4·12}$$

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has a growing (unstable) solution even if $A>0$, though the equation obtained from Eq. (4·12) by replacing its coefficient with the averaged value has none if $A>0$. This type of parametric enhancement is expected to occur also in Eq. (4·6). In fact we have found by a numerical calculation that Eq. (4·6) has an oscillatorily growing solution at least for sufficiently large Hl.

Though these arguments are rather qualitative, they are sufficient to show that the behavior of the perturbation in the oscillatory coherent scalar field dominated universe is quite different from that in the purely dust-dominated universe. Of course we must analyse the constrained system (4·9) and (4·10) in order to answer the question; then how does the perturbation behave in detail? This problem is left to future work.

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Chapter VII

Conclusion

In this article we have developed the theory of cosmological perturbations referring only to gauge-invariant amplitudes of perturbations, that is quantities representing the perturbation independent of the way of embedding a fictitious unperturbed background universe into a perturbed real universe. We believe we have succeeded in showing how powerful the gauge-invariant formalism is not only in resolving and disentangling the conceptual problems but also in investigating the generation and evolution of perturbations in specific problems.

The fundamental idea of the formalism developed in this paper owes to Bardeen (1980). However, we have succeeded in adding quite a few new developments. First we investigated the geometrical meaning of the gauge-invariant variables in more detail. Further on the basis of this investigation we clarified the relation between the gauge-invariant formalism and the conventional gauge-dependent methods. In particular we have succeeded in finding out a gauge-invariant quantity characterizing the genuine amplitude of the perturbation and discussing the validity limit of the linear perturbation theory more clearly than Bardeen. Second the part concerned with the extension to a multi-component system is completely original. Since the multi-component nature of cosmic matter has recently become more and more important especially in connection with the galaxy formation problem, we expect that this extension will play important roles in the future study.

We have given several examples of application of the formalism to the problem of generation and evolution of cosmological perturbations. Although these examples are not complete, we can draw conclusions from them to some extent on the fundamental problem of cosmological perturbations, namely, the origin of the perturbations as the seed for the large-scale inhomogeneous structure of the present universe. First, the result of Chapter IV clearly shows that weak transient phenomena in a nearly Friedmann universe cannot generate perturbations of enough amplitude to account for the present structure of the universe. Hence if the seed perturbation was generated in the course of the cosmic evolution, we must attribute it to some strong transient phenomena, probably of very exotic nature, associated with a great deviation of the cosmic expansion law from the Friedmann one. The investigation of perturbations in the inflationary universe in Chapter VI was included to obtain some insight into such cases. Though it was limited to some simple cases, the result presented there indicates that we must appeal to some exotic non-classical phenomena, such as the freezing of zero-point oscillations of quantum fields, in order for a sufficient perturbation to be generated. Second, the analysis of perturbations in the interacting baryon-photon system in Chapter V, where the gauge-invariant formalism extended to a multi-component system is fully utilized, clarified the nature of the so-called isothermal mode. Several people argue that the seed perturbation must be of isothermal origin based on the severe observational limit on the anisotropy of 3K microwave background radiation. However, as shown in Chapter V, the isothermal mode does not grow in a strong coupling regime, and the generation of it from the

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adiabatic perturbation while the coupling weakens gradually is quite ineffective. Hence appealing to the isothermal mode also requires some exotic phenomena to generate it [cf. Yoshimura (1983)]. Of course since the analysis there was limited to a special case, the argument is not conclusive. We should rather emphasize the fact that the isothermal perturbation can be generated from the adiabatic perturbation in principle, as stated in Chapter V.

Finally we make a brief comment on problems which were not treated or incompletely discussed in this article. The first problem is the limit of the linear perturbation theory and the estimation of non-linear effects. Though we made a tentative analysis on the criterion for the smallness of the perturbation in §III-2, it was yet far from complete. Further we could not discuss how to go beyond the linear perturbation theory retaining the idea of treating only the gauge-invariant quantities. The second problem is the generation of perturbations by quantum phenomena. Since the formalism developed in this article is based on classical general relativity, this problem is out of scope from the start. However, since it is very difficult to find a trigger in classical or quasi-classical phenomena for generating the seed perturbation for the present structure of the universe, as has been shown in this article, it seems to be crucially important to develop a formalism which enables us to treat quantum phenomena also [cf. Lukash and Novikov (1983)]. We hope we will be able to discuss these problems in a near future.

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Appendix A

Geometrical Quantities in a Robertson-Walker Spacetime

Metric:

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -dt^2 + a^2 d\sigma^2 = a^2(-d\eta^2 + d\sigma^2), \tag{A·1}$$

$$d\sigma^2 = \gamma_{ij}dx^i dx^j = \frac{dr^2}{1-Kr^2} + r^2 d\Omega^2_{n-1}. \tag{A·2}$$

Christoffel symbols:

$$\begin{array}{ll} \langle t\text{-coordinate}\rangle & \langle \eta\text{-coordinate}\rangle \end{array}$$

$$\Gamma^0{}_{00} = 0, \qquad \Gamma^0{}_{00} = a'/a, \tag{A·3a}$$

$$\Gamma^0{}_{0j} = 0, \qquad \Gamma^0{}_{0j} = 0, \tag{A·3b}$$

$$\Gamma^0{}_{ij} = a\dot{a}\gamma_{ij}, \qquad \Gamma^0{}_{ij} = (a'/a)\gamma_{ij}, \tag{A·3c}$$

$$\Gamma^j{}_{00} = 0, \qquad \Gamma^j{}_{00} = 0, \tag{A·3d}$$

$$\Gamma^i{}_{0j} = (\dot{a}/a)\delta^i{}_j, \qquad \Gamma^i{}_{0j} = (a'/a)\delta^i{}_j, \tag{A·3e}$$

$$\Gamma^i{}_{jk} = {}^s\Gamma^i{}_{jk}, \qquad \Gamma^i{}_{jk} = {}^s\Gamma^i{}_{jk}, \tag{A·3f}$$

where ${}^s\Gamma^i{}_{jk}$ denotes the Christoffel symbol for the metric γ_{ij} of the invariant n-space.

Curvature tensor:

$$\begin{array}{ll} \langle t\text{-coordinate}\rangle & \langle \eta\text{-coordinate}\rangle \end{array}$$

$$R^0{}_{i0j} = a\ddot{a}\gamma_{ij}, \qquad R^0{}_{i0j} = (a'/a)'\gamma_{ij}, \tag{A·4a}$$

$$R^i{}_{00j} = (\ddot{a}/a)\delta^i{}_j, \qquad R^i{}_{00j} = (a'/a)'\delta^i{}_j, \tag{A·4b}$$

$$\begin{aligned} R^i{}_{jkm} &= {}^sR^i{}_{jkm} + \dot{a}^2(\delta^i{}_k\gamma_{jm} - \delta^i{}_m\gamma_{jk}) \\ &= {}^sR^i{}_{jkm} + (a'/a)^2(\delta^i{}_k\gamma_{jm} - \delta^i{}_m\gamma_{jk}), \end{aligned} \tag{A·4c}$$

$$R^0{}_{00j} = R^0{}_{0ij} = R^0{}_{ijk} = R^i{}_{0jk} = R^i{}_{j0k} = 0, \tag{A·4d}$$

where ${}^sR^i{}_{jkm}$ is the Riemannian curvature tensor of the invariant n-space:

$${}^sR^i{}_{jkm} = K(\delta^i{}_k\gamma_{jm} - \delta^i{}_m\gamma_{jk}). \tag{A·5}$$

Ricci tensor:

$$R^0{}_0 = n\frac{\ddot{a}}{a} = n\frac{1}{a^2}\left(\frac{a'}{a}\right)', \tag{A·6a}$$

$$R^i{}_j = \left[\frac{\ddot{a}}{a} + (n-1)\left\{\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right\}\right]\delta^i{}_j$$

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$$=\frac{1}{a^2}\left[\frac{a''}{a}+(n-2)\left(\frac{a'}{a}\right)^2+(n-1)K\right]\delta^i{}_j\,, \tag{A·6b}$$

$$R^0{}_j=R^j{}_0=0\,, \tag{A·6c}$$

$$R=2n\frac{\ddot{a}}{a}+n(n-1)\left\{\left(\frac{\dot{a}}{a}\right)^2+\frac{K}{a^2}\right\}$$

$$=\frac{n}{a^2}\left[2\frac{a''}{a}+(n-3)\left(\frac{a'}{a}\right)^2+(n-1)K\right]. \tag{A·7}$$

Einstein tensor: $G_{\mu\nu}\equiv R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R$

$$G^0{}_0=-\frac{1}{2}n(n-1)\left\{\left(\frac{\dot{a}}{a}\right)^2+\frac{K}{a^2}\right\}$$

$$=-\frac{n(n-1)}{2a^2}\left\{\left(\frac{a'}{a}\right)^2+K\right\}, \tag{A·8a}$$

$$G^i{}_j=-(n-1)\left[\frac{\ddot{a}}{a}+\frac{n-2}{2}\left\{\left(\frac{\dot{a}}{a}\right)^2+\frac{K}{a^2}\right\}\right]\delta^i{}_j$$

$$=-\frac{n-1}{a^2}\left[\frac{a''}{a}+\frac{n-4}{2}\left(\frac{a'}{a}\right)^2+\frac{n-2}{2}K\right]\delta^i{}_j\,, \tag{A·8b}$$

$$G^0{}_j=G^j{}_0=0\,, \tag{A·8c}$$

$$G^i{}_i-nG^0{}_0=\frac{n(n-1)}{a^2}\left[\left(\frac{a'}{a}\right)^2-\left(\frac{a'}{a}\right)'+K\right]. \tag{A·8d}$$

Einstein equations:

$$G^\mu{}_\nu=\chi^2T^\mu{}_\nu\,;\quad T^\mu{}_\nu=(\rho+p)u^\mu u_\nu+p\delta^\mu{}_\nu\,, \tag{A·9}$$

$G^0{}_0=\chi^2T^0{}_0$:

$$\left(\frac{\dot{a}}{a}\right)^2+\frac{K}{a^2}=\frac{2\chi^2}{n(n-1)}\rho\,, \tag{A·9a}$$

$$\left(\frac{a'}{a}\right)^2+K=\frac{2a^2\chi^2}{n(n-1)}\rho\,. \tag{A·9a$'$}$$

$G^i{}_j=\chi^2T^i{}_j$:

$$(n-1)\frac{\ddot{a}}{a}+\frac{1}{2}(n-1)(n-2)\left\{\left(\frac{\dot{a}}{a}\right)^2+\frac{K}{a^2}\right\}=-p\,, \tag{A·9b}$$

$$(n-1)\frac{a''}{a}+\frac{1}{2}(n-1)(n-4)\left(\frac{a'}{a}\right)^2+\frac{1}{2}(n-1)(n-2)K=-a^2p\,. \tag{A·9b$'$}$$

Equation of motion: $G^{\mu\nu}{}_{;\nu}=0$

$$\dot{\rho}=-n\frac{\dot{a}}{a}(1+w)\rho\,, \tag{A·10}$$

$$\rho'=-n\frac{a'}{a}(1+w)\rho\,, \tag{A·10$'$}$$

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where

$$w \equiv p/\rho \,. \tag{A·11}$$

Note that Eq. (A·10) can be derived from Eqs. (A·9a) and (A·9b). Equations (A·9) yield various useful formulas for higher-order time-derivatives of the cosmic scale factor a:

$$\frac{\ddot{a}}{a} = -\frac{\chi^2}{n-1}\left(p + \frac{n-2}{n}\rho\right) = -\frac{1}{2}(nw+n-2)\left[\left(\frac{\dot{a}}{a}\right)^2 + \frac{K}{a^2}\right], \tag{A·12}$$

$$\frac{a''}{a} = -\frac{a^2\chi^2}{n-1}\left(p + \frac{n-4}{n}\rho\right) - K = -\frac{1}{2}(nw+n-4)\left(\frac{a'}{a}\right)^2 - \frac{1}{2}(nw+n-2)K, \tag{A·12$'$}$$

$$\left(\frac{\dot{a}}{a}\right)^{\cdot} = \frac{K}{a^2} - \frac{1+w}{n-1}\chi^2\rho = -\frac{n(1+w)}{2}\left(\frac{\dot{a}}{a}\right)^2 - \frac{nw+n-2}{2}\frac{K}{a^2}, \tag{A·13}$$

$$\left(\frac{a'}{a}\right)' = -\frac{nw+n-2}{n(n-1)}\chi^2 a^2\rho = -\frac{1}{2}(nw+n-2)\left\{\left(\frac{a'}{a}\right)^2 + K\right\}. \tag{A·13$'$}$$

The time-derivative of w is given by

$$\dot{w} = n(c_s{}^2 - w)(1+w)\frac{\dot{a}}{a}, \tag{A·14}$$

$$w' = n(c_s{}^2 - w)(1+w)\frac{a'}{a}, \tag{A·14$'$}$$

where

$$c_s{}^2 \equiv \dot{p}/\dot{\rho} = p'/\rho' \,. \tag{A·15}$$

Appendix B

Proof of the Decomposition Theorem

In this appendix we prove that a covariant linear differential equation of the second-order at most on the invariant n-space Σ can be decomposed into mutually decoupled equations, each of which contains only one type of components of the unknowns. In this appendix we omit the suffix s labelling the quantities associated with the invariant n-space Σ and its intrinsic metric γ_{ij}. Further we write the Laplace-Beltrami operator on Σ simply as Δ here since there is no fear of it being confused with the gauge-invariant amplitude for the density perturbation.

With the aid of the formulas

$$(\nabla_i\Delta - \Delta\nabla_i)f = -R_{ij}\nabla^j f\,, \tag{B·1a}$$

$$(\nabla_i\Delta - \Delta\nabla_i)v_j = 2R_i{}^k{}_j{}^m\nabla_k v_m - R_{ik}\nabla^k v_j + (\nabla_k R_i{}^k{}_j{}^m)v_m\,, \tag{B·1b}$$

$$\begin{aligned}(\nabla_i\Delta - \Delta\nabla_i)t_{jk} &= 2R_i{}^m{}_j{}^p\nabla_m t_{pk} + 2R_i{}^m{}_k{}^p\nabla_m t_{jp}\\ &\quad - R_{im}\nabla^m t_{jk} + (\nabla_m R_i{}^m{}_k{}^p)t_{jp} + (\nabla_m R_i{}^m{}_j{}^p)t_{pk}\,,\end{aligned} \tag{B·1c}$$

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which are obtained from

$$(\nabla_i\nabla_j-\nabla_j\nabla_i)v_k=R_{ijk}{}^m v_m\,, \tag{B·2}$$

we find that scalar, vector and tensor quantities obtained by operating differential operators of the second-order at most on scalar f, vector v^i and tensor $t_{ij}(=t_{ji})$ are written as follows:

Scalar:

$$f,\quad \nabla_i v^i=\Delta v\,,\quad \gamma_{ij}t^{ij}=t,\quad \Delta t\,,$$

$$\nabla_i\nabla_j\left(t^{ij}-\frac{1}{n}\gamma^{ij}t\right)=\frac{n-1}{n}(\Delta+nK)\Delta s\,. \tag{B·3}$$

Vector:

$$\nabla_i f,\quad v^i=v_*{}^i+\nabla^i v\,,$$

$$\Delta v^i=\Delta v_*{}^i+\nabla^i[\Delta v+(n-1)Kv],$$

$$\nabla_i t\,,\quad \nabla_i\Delta t\,,$$

$$\nabla_j\left(t^{ij}-\frac{1}{n}\gamma^{ij}t\right)=[\Delta+(n-1)K]t_*{}^i+\frac{n-1}{n}\nabla^i(\Delta+nK)s\,. \tag{B·4}$$

Tensor:

$$\gamma_{ij}f,\quad \left(\nabla_i\nabla_j-\frac{1}{n}\gamma_{ij}\Delta\right)f,\ \gamma_{ij}\Delta f\,,$$

$$\gamma_{ij}\nabla_k v^k=\gamma_{ij}\Delta v\,,$$

$$\nabla_i v_j+\nabla_j v_i=(\nabla_i v_{*j}+\nabla_j v_{*i})+2\left(\nabla_i\nabla_j-\frac{1}{n}\gamma_{ij}\Delta\right)v+\frac{2}{n}\gamma_{ij}\Delta v\,,$$

$$t^{ij}=t_*^{ij}+(\nabla^i t_*{}^j+\nabla^j t_*{}^i)+\left(\nabla^i\nabla^j-\frac{1}{n}\gamma^{ij}\Delta\right)s+\frac{1}{n}t\gamma^{ij}\,,$$

$$\gamma^{ij}\gamma_{km}t^{km}=\gamma^{ij}t,\quad \gamma^{ij}\Delta t\,,$$

$$\nabla^i\nabla^j t=\left(\nabla^i\nabla^j-\frac{1}{n}\gamma^{ij}\Delta\right)t+\frac{1}{n}\gamma^{ij}\Delta t\,,$$

$$\nabla^i\nabla_k t^{jk}+\nabla^j\nabla_k t^{ik}-\frac{2}{n}\nabla^i\nabla^j t$$

$$=\nabla^i[\{\Delta+(n-1)K\}t_*{}^j]+\nabla^j[\{\Delta+(n-1)K\}t_*{}^i]+\frac{2(n-1)}{n}\nabla^i\nabla^j(\Delta+nK)s\,,$$

$$\gamma_{ij}\nabla_k\nabla_m t^{km}=\frac{2(n-2)}{n}\Delta(\Delta+nK)s\gamma_{ij}+\frac{2}{n}\gamma_{ij}\Delta t\,,$$

$$\Delta t_{ij}=\Delta t_{*ij}+\nabla_i[\Delta t_{*j}-(n-1)t_{*j}]+\nabla_j[\Delta t_{*i}-(n-1)t_{*i}]$$

$$+\left(\nabla_i\nabla_j-\frac{1}{n}\gamma_{ij}\right)[\Delta+(2n-1)K]s+\frac{1}{n}\{(n-1)K\Delta s+\Delta t\}\gamma_{ij}\,. \tag{B·5}$$

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In these equations v and $v_*{}^i$ are the scalar and the vector components of v^i defined in Eq. (1·1), respectively, and $(s,\ t)$, $t_*{}^i$ and $t_*{}^{ij}$ are the scalar, the vector and the tensor components of t^{ij} defined in Eq. (1·3), respectively.

In Eq. (B·3) all the quantities are expressed in terms of the scalar components of the original quantities f, v^i and t^{ij}. Since $\nabla_i \Delta v^i = 0$ if $\nabla_i v^i = 0$ from Eqs. (B·1b) and (A·5), the divergenceless vector parts and the scalar parts of all the quantities in Eq. (B·4) consist only of the divergenceless vector components and only of the scalar components of the original variables, respectively. Similarly, since $\nabla_i \Delta t^{ij} = 0$ if $t^i{}_i = 0$ and $\nabla_i t^{ij} = 0$ from Eqs. (B·1c) and (A·5), the divergenceless traceless tensor parts, the divergenceless vector parts and the scalar parts of all the quantities in Eq. (B·5) consist only of the corresponding components of the original variables, respectively. This property also holds for any covariant linear differential equation of the second-order at most since it is written as a linear combination of the quantities given in Eqs. (B·3)~(B·5). Since v and $v_*{}^i(s,\ t,\ t_*{}^i$ and $t_*{}^{ij})$ all vanish if $v^i(t^{ij})$ vanishes from the inversion formulas in Eqs. (II-1·1) and (II-1·3), this property implies that the original equation is decomposed into equations containing only one type of components. Note that the constant curvature property of the background n-space plays an essential role in the proof.

Appendix C

Formulas for Harmonic Functions

Scalar harmonic functions:

$$(\Delta + k^2)\mathrm{Y} = 0\,. \tag{C·1}$$

$$\text{Vector:}\quad \mathrm{Y}_j \equiv -k^{-1}\mathrm{Y}_{|j}\,, \tag{C·2}$$

$$\text{Tensor:}\quad \mathrm{Y}_{ij} \equiv k^{-2}\mathrm{Y}_{|ij} + \frac{1}{n}\gamma_{ij}\mathrm{Y}\,. \tag{C·3}$$

Properties:

$$\mathrm{Y}_i{}^{|i} = k\mathrm{Y}\,, \tag{C·4}$$

$$\Delta \mathrm{Y}_j = -[k^2 - (n-1)K]\mathrm{Y}_j\,, \tag{C·5}$$

$$\mathrm{Y}_{i|j} = -k\left(\mathrm{Y}_{ij} - \frac{1}{n}\gamma_{ij}\mathrm{Y}\right), \tag{C·6}$$

$$\mathrm{Y}^i{}_i = 0\,, \tag{C·7}$$

$$\mathrm{Y}_{ij}{}^{|j} = \frac{n-1}{n}k^{-1}(k^2 - nK)\mathrm{Y}_i\,, \tag{C·8}$$

$$\mathrm{Y}_{im}{}^{|m}{}_{|j} = \frac{n-1}{n}(nK - k^2)\left(\mathrm{Y}_{ij} - \frac{1}{n}\gamma_{ij}\mathrm{Y}\right), \tag{C·9}$$

$$\Delta \mathrm{Y}_{ij} = -(k^2 - 2nK)\mathrm{Y}_{ij}\,, \tag{C·10}$$

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$$Y_{ij|m}-Y_{im|j}=\frac{k}{n}\left(1-\frac{nK}{k^2}\right)(\gamma_{im}Y_j-\gamma_{ij}Y_m). \tag{C·11}$$

Vector harmonic functions:

$$(\varDelta+k^2)Y^{(1)}{}_i=0\,; \tag{C·12}$$

$$Y^{(1)i}{}_{|i}=0\,. \tag{C·12a}$$

$$\text{Tensor:}\quad Y^{(1)}_{ij}\equiv-\frac{1}{2k}(Y^{(1)}_{i|j}+Y^{(1)}_{j|i}). \tag{C·13}$$

Properties:

$$Y^{(1)i}{}_i=0\,, \tag{C·14}$$

$$Y^{(1)}{}_{ij}{}^{|j}=\frac{1}{2k}\{k^2-(n-1)K\}Y_i{}^{(1)}\,, \tag{C·15}$$

$$Y^{(1)}{}_{im}{}^{|m}{}_{|j}+Y^{(1)}{}_{jm}{}^{|m}{}_{|i}=-\{k^2-(n-1)K\}Y^{(1)}_{ij}\,, \tag{C·16}$$

$$\varDelta Y^{(1)}{}_{ij}=-[k^2-(n+1)K]Y^{(1)}{}_{ij}\,. \tag{C·17}$$

Tensor harmonic functions:

$$(\varDelta+k^2)Y^{(2)}{}_{ij}=0\,; \tag{C·18}$$

$$Y^{(2)i}{}_i=0\,, \tag{C·18a}$$

$$Y^{(2)ij}{}_{|j}=0\,. \tag{C·18b}$$

Appendix D

Perturbation Formulas for Geometrical Quantities

(1) Scalar perturbations

Metric:

$$\tilde{g}_{00}=-a^2[1+2AY], \tag{D·1a}$$

$$\tilde{g}_{0j}=-a^2BY_j\,, \tag{D·1b}$$

$$\tilde{g}_{ij}=a^2[\gamma_{ij}+2H_LY\gamma_{ij}+2H_TY_{ij}], \tag{D·1c}$$

$$\tilde{g}^{00}=-a^{-2}[1-2AY], \tag{D·2a}$$

$$\tilde{g}^{0j}=-a^{-2}BY^j\,, \tag{D·2b}$$

$$\tilde{g}^{ij}=a^{-2}[\gamma^{ij}-2H_LY\gamma^{ij}-2H_TY^{ij}]. \tag{D·2c}$$

Christoffel symbols:

$$\delta\Gamma^0{}_{00}=A'Y\,, \tag{D·3a}$$

$$\delta\Gamma^0{}_{0j}=-\{kA+(a'/a)B\}Y_j\,, \tag{D·3b}$$

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$$\delta\Gamma^j{}_{00} = -\{kA + B' + (a'/a)B\}Y^j\,, \tag{D·3c}$$

$$\delta\Gamma^i{}_{0j} = H_L'\delta^i{}_j Y + H_T' Y^i{}_j\,, \tag{D·3d}$$

$$\begin{aligned}\delta\Gamma^0{}_{ij} = &\{-2(a'/a)A + (k/n)B + a^{-2}(a^2 H_L)'\}\gamma_{ij}Y \\ &+\{-kB + a^{-2}(a^2 H_T)'\}Y_{ij}\,,\end{aligned} \tag{D·3e}$$

$$\begin{aligned}\delta\Gamma^i{}_{jm} = &-kH_L(\delta^i{}_j Y_m + \delta^i{}_m Y_j - \gamma_{jm}Y^i) + (a'/a)B\gamma_{jm}Y^i \\ &+ H_T(Y^i{}_{j|m} + Y^i{}_{m|j} - Y_{jm}{}^{|i}).\end{aligned} \tag{D·3f}$$

Curvature tensor:

$$\delta R^0{}_{00j} = -(a'/a)'BY_j\,, \tag{D·4a}$$

$$\delta R^0{}_{0ij} = 0\,, \tag{D·4b}$$

$$\begin{aligned}\delta R^0{}_{i0j} = &\Big[-2\Big(\frac{a'}{a}\Big)'A - \frac{a'}{a}A' + \frac{k^2}{n}A + \frac{k}{n}\Big(B' + \frac{a'}{a}B\Big) \\ &+ H_L'' + \frac{a'}{a}H_L' + 2\Big(\frac{a'}{a}\Big)'H_L\Big]\gamma_{ij}Y \\ &+\Big[-k^2 A - k\Big(B' + \frac{a'}{a}B\Big) + H_T'' + \frac{a'}{a}H_T' + 2\Big(\frac{a'}{a}\Big)'H_T\Big]Y_{ij}\,,\end{aligned} \tag{D·4c}$$

$$\delta R^0{}_{ijm} = k\Big[-\frac{a'}{a}A + \frac{k}{n}B + H_L' + \frac{1}{n}\Big(1 - \frac{nK}{k^2}\Big)(H_T' - kB)\Big](\gamma_{ij}Y_m - \gamma_{im}Y_j), \tag{D·4d}$$

$$\begin{aligned}\delta R^i{}_{00j} = &\Big[\frac{k^2}{n}A - \frac{a'}{a}A' + \frac{k}{n}\Big(B' + \frac{a'}{a}B\Big) + H_L'' + \frac{a'}{a}H_L'\Big]\delta^i{}_j Y \\ &+\Big[-k\Big(B' + \frac{a'}{a}B\Big) - k^2 A + H_T'' + \frac{a'}{a}H_T'\Big]Y^i{}_j\,,\end{aligned} \tag{D·4e}$$

$$\delta R^i{}_{0jm} = \Big[kH_L' - \frac{a'}{a}\Big(kA + \frac{a'}{a}B\Big) + \frac{k}{n}\Big(1 - \frac{nK}{k^2}\Big)H_T'\Big](\delta^i{}_j Y_m - \delta^i{}_m Y_j), \tag{D·4f}$$

$$\begin{aligned}\delta R^i{}_{j0m} = &\Big[\frac{a'}{a}\Big(kA + \frac{a'}{a}B\Big) - kH_L' - \frac{k}{n}\Big(1 - \frac{nK}{k^2}\Big)H_L'\Big](\delta^i{}_m Y_j - \gamma_{jm}Y^i) \\ &+\Big(\frac{a'}{a}\Big)'B\gamma_{jm}Y^i\,,\end{aligned} \tag{D·4g}$$

$$\begin{aligned}\delta R^i{}_{jmp} = &\Big[-2\Big(\frac{a'}{a}\Big)^2 A + \frac{2k}{n}\frac{a'}{a}B + 2\frac{a'}{a}H_L' + 2\Big(\frac{a'}{a}\Big)^2 H_L + \frac{2}{n}k^2 H_L\Big](\delta^i{}_m\gamma_{jp} - \delta^i{}_p\gamma_{jm})Y \\ &+\Big[k^2 H_L - \frac{a'}{a}(H_T' - kB)\Big](\delta^i{}_p Y_{jm} - \delta^i{}_m Y_{jp} + Y^i{}_p\gamma_{jm} - Y^i{}_m\gamma_{jp}) \\ &+2\Big(\frac{a'}{a}\Big)^2 H_T(\delta^i{}_m Y_{jp} - \delta^i{}_p Y_{jm}) \\ &+H_T(Y^i{}_{j|pm} - Y^i{}_{j|mp} + Y^i{}_{p|jm} - Y^i{}_{m|jp} + Y_{jm}{}^{|i}{}_{|p} - Y_{jp}{}^{|i}{}_{|m}).\end{aligned} \tag{D·4h}$$

Ricci tensor:

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$$\delta R_{00}=-\Big[k^2A-n\frac{a'}{a}A'+k\Big(B'+\frac{a'}{a}B\Big)+nH_L''+n\frac{a'}{a}H_L'\Big]Y\,, \tag{D·5a}$$

$$\delta R_{0j}=\Big[-\Big\{\Big(\frac{a'}{a}\Big)'+(n-1)\Big(\frac{a'}{a}\Big)^2\Big\}B-(n-1)k\frac{a'}{a}A+(n-1)kH_L'$$
$$+\frac{n-1}{n}\frac{1}{k}(k^2-nK)H_T'\Big]Y_j\,, \tag{D·5b}$$

$$\delta R_{ij}=\Big[-2\Big\{\Big(\frac{a'}{a}\Big)'+(n-1)\Big(\frac{a'}{a}\Big)^2\Big\}A-\frac{a'}{a}A'+\frac{k^2}{n}A$$
$$+\frac{k}{n}\Big(B'+\frac{a'}{a}B\Big)+\frac{2(n-1)}{n}\frac{a'}{a}kB+H_L''+(2n-1)\frac{a'}{a}H_L'$$
$$+2\Big(\frac{a'}{a}\Big)'H_L+2(n-1)\Big(\frac{a'}{a}\Big)^2H_L+\frac{2(n-1)}{n}k^2H_L$$
$$+\frac{2(n-1)}{n^2}(k^2-nK)H_T\Big]\gamma_{ij}Y$$
$$+\Big[-k^2A-k\Big(B'+\frac{a'}{a}B\Big)+H_T''+\frac{a'}{a}H_T'+2\Big(\frac{a'}{a}\Big)'H_T$$
$$+(2-n)k^2H_L+2(n-1)\Big(\frac{a'}{a}\Big)^2H_T+(n-2)\frac{a'}{a}(H_T'-kB)$$
$$+\Big\{\frac{2-n}{n}k^2+2(n-1)K\Big\}H_T\Big]Y_{ij}\,. \tag{D·5c}$$

$$\delta R=a^{-2}\Big[-4n\frac{a''}{a}A-2n(n-3)\Big(\frac{a'}{a}\Big)^2A-2n\frac{a'}{a}A'+2k^2A$$
$$+2k\Big(B'+\frac{a'}{a}B\Big)+2(n-1)\frac{a'}{a}kB+2na^{-1}(aH_L')'$$
$$+2n(n-1)\frac{a'}{a}H_L'+2(n-1)(k^2-nK)\Big(H_L+\frac{H_T}{n}\Big)\Big]Y$$
$$=-2na^{-2}\Big[\Big\{2\Big(\frac{a'}{a}\Big)'+(n-1)\Big(\frac{a'}{a}\Big)^2\Big\}\mathcal{A}+\frac{a'}{a}\mathcal{A}'$$
$$-\frac{n-1}{n}k\frac{a'}{a}\mathcal{B}-\frac{k^2}{n}\Psi+(1-nc_s{}^2)\Big\{\Big(\frac{a'}{a}\Big)^2-\Big(\frac{a'}{a}\Big)'+K\Big\}\mathfrak{R}\Big]Y\,. \tag{D·6}$$

Einstein tensor:

$$\delta G^0{}_0=\frac{n-1}{a^2}\Big[n\Big(\frac{a'}{a}\Big)^2A-\frac{a'}{a}kB-n\frac{a'}{a}H_L'-(k^2-nK)\Big(H_L+\frac{H_T}{n}\Big)\Big]Y$$
$$=\frac{n-1}{a^2}\Big[n\Big(\frac{a'}{a}\Big)^2\mathcal{A}-\frac{a'}{a}k\mathcal{B}+n\Big\{\Big(\frac{a'}{a}\Big)^2-\Big(\frac{a'}{a}\Big)'+K\Big\}\mathfrak{R}\Big]Y\,, \tag{D·7a}$$

$$\delta G^0{}_j=\frac{n-1}{a^2}\Big[k\frac{a'}{a}A-KB-kH_L'-\frac{k^2-nK}{nk}H_T'\Big]Y_j$$

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$$=\frac{n-1}{a^2}\left[k\frac{a'}{a}\mathcal{A}-K\mathcal{B}+k\left(\frac{a'}{a}\right)^{-1}\left\{\left(\frac{a'}{a}\right)^2-\left(\frac{a'}{a}\right)'+K\right\}\mathcal{R}\right]\mathrm{Y}_j, \tag{D$\cdot$7b}$$

$$\delta G^i{}_0=\frac{n-1}{a^2}\left[\left\{\left(\frac{a'}{a}\right)'-\left(\frac{a'}{a}\right)^2\right\}B-k\frac{a'}{a}A+kH_L'+\frac{k^2-nK}{nk}H_T'\right]\mathrm{Y}^i$$

$$=\frac{n-1}{a^2}\left[-k\frac{a'}{a}\mathcal{A}+\left\{\left(\frac{a'}{a}\right)'-\left(\frac{a'}{a}\right)^2\right\}\mathcal{B}-\frac{1}{k}\left\{\left(\frac{a'}{a}\right)^2-\left(\frac{a'}{a}\right)'+K\right\}H_T'\right]\mathrm{Y}^i, \tag{D$\cdot$7c}$$

$$\delta G^i{}_j=\frac{n-1}{a^2}\left[\left\{2\frac{a''}{a}+(n-4)\left(\frac{a'}{a}\right)^2\right\}A+\frac{a'}{a}A'-\frac{k^2}{n}A\right.$$

$$-\frac{k}{n}\left(B'+\frac{a'}{a}B\right)-\frac{n-2}{n}k\frac{a'}{a}B-a^{-1}(aH_L')'$$

$$\left.-(n-2)\frac{a'}{a}H_L'-\frac{n-2}{n}(k^2-nK)\left(H_L+\frac{H_T}{n}\right)\right]\delta^i{}_j\mathrm{Y}$$

$$+\frac{1}{a^2}\left[-k^2A-k\left(B'+\frac{a'}{a}B\right)+a^{-1}(aH_T')'\right.$$

$$\left.+(n-2)\frac{a'}{a}(H_T'-kB)-(n-2)k^2\left(H_L+\frac{H_T}{n}\right)\right]\mathrm{Y}^i{}_j$$

$$=\frac{n-1}{a^2}\left[\left\{2\left(\frac{a'}{a}\right)'+(n-2)\left(\frac{a'}{a}\right)^2\right\}\mathcal{A}+\frac{a'}{a}\mathcal{A}'-\frac{n-2}{n}k\frac{a'}{a}\mathcal{B}\right.$$

$$\left.-\frac{k^2}{n}\Psi-nc_s{}^2\left\{\left(\frac{a'}{a}\right)^2-\left(\frac{a'}{a}\right)'+K\right\}\mathcal{R}\right]\delta^i{}_j\mathrm{Y}$$

$$-\frac{k^2}{a^2}\left[\mathcal{A}+\frac{1}{k}\frac{1}{a^{n-1}}(a^{n-1}\mathcal{B})'\right]\mathrm{Y}^i{}_j. \tag{D$\cdot$7d}$$

(2) Vector perturbations

For simplicity we omit here the suffix (1) used to distinguish quantities associated with a vector perturbation.

$$\tilde{g}_{00}=-a^2, \tag{D$\cdot$8a}$$

$$\tilde{g}_{0j}=-a^2B\mathrm{Y}_j, \tag{D$\cdot$8b}$$

$$\tilde{g}_{ij}=a^2[\gamma_{ij}+2H_T\mathrm{Y}_{ij}], \tag{D$\cdot$8c}$$

$$\tilde{g}^{00}=-a^{-2}, \tag{D$\cdot$9a}$$

$$\tilde{g}^{0j}=-a^{-2}B\mathrm{Y}^j, \tag{D$\cdot$9b}$$

$$\tilde{g}^{ij}=a^{-2}[\gamma^{ij}-2H_T\mathrm{Y}^{ij}]. \tag{D$\cdot$9c}$$

Christoffel symbols:

$$\delta\Gamma^0{}_{00}=0, \tag{D$\cdot$10a}$$

$$\delta\Gamma^0{}_{0j}=-(a'/a)B\mathrm{Y}_j, \tag{D$\cdot$10b}$$

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$$\delta\Gamma^i{}_{00} = -[B' + (a'/a)B]Y^i\,, \tag{D·10c}$$

$$\delta\Gamma^i{}_{0j} = H_T'Y^i{}_j + (1/2)B(Y_j{}^{|i} - Y^i{}_{|j}), \tag{D·10d}$$

$$\delta\Gamma^0{}_{ij} = [-kB + a^{-2}(a^2H_T)']Y_{ij}\,, \tag{D·10e}$$

$$\delta\Gamma^i{}_{jm} = (a'/a)B\gamma_{jm}Y^i + H_T(Y^i{}_{j|m} + Y^i{}_{m|j} - Y_{jm}{}^{|i}). \tag{D·10f}$$

Curvature tensor:

$$\delta R^0{}_{00j} = -(a'/a)'BY_j\,, \tag{D·11a}$$

$$\delta R^0{}_{0ij} = 0\,, \tag{D·11b}$$

$$\delta R^0{}_{i0j} = [-k\{B' + (a'/a)B\} + H_T'' + (a'/a)H_T' + 2(a'/a)'H_T]Y_{ij}\,, \tag{D·11c}$$

$$\delta R^0{}_{ijm} = -(H_T' - kB)(Y_{ij|m} - Y_{im|j}), \tag{D·11d}$$

$$\delta R^i{}_{00j} = [-k\{B' + (a'/a)B\} + H_T'' + (a'/a)H_T']Y^i{}_j\,, \tag{D·11e}$$

$$\delta R^i{}_{0jm} = -[K + (a'/a)^2]B(\delta^i{}_jY_m - \delta^i{}_mY_j) + (H_T' - kB)(Y^i{}_{m|j} - Y^i{}_{j|m}), \tag{D·11f}$$

$$\begin{aligned}\delta R^i{}_{j0m} = {}& H_T'(Y^i{}_{m|j} - Y_{jm}{}^{|i}) + (a'/a)^2B(\delta^i{}_mY_j - \gamma_{jm}Y^i)\\ &+ (a'/a)'B\gamma_{jm}Y^i - (1/2)B(Y_j{}^{|i} - Y^i{}_{|j})_{|m}\,,\end{aligned} \tag{D·11g}$$

$$\begin{aligned}\delta R^i{}_{jmp} = {}& 2(a'/a)^2H_T(\delta^i{}_mY_{jp} - \delta^i{}_pY_{jm})\\ &+ H_T(Y^i{}_{j|pm} - Y^i{}_{j|mp} + Y^i{}_{p|jm} - Y^i{}_{m|jp} + Y_{jm}{}^{|i}{}_{|p} - Y_{jp}{}^{|i}{}_{|m})\\ &+ (a'/a)(H_T' - kB)(\delta^i{}_mY_{jp} - \delta^i{}_pY_{jm} - \gamma_{jm}Y^i{}_p + \gamma_{jp}Y^i{}_m).\end{aligned} \tag{D·11h}$$

Ricci tensor:

$$\delta R_{00} = 0\,, \tag{D·12a}$$

$$\begin{aligned}\delta R_{0j} = -\Bigg[&\left\{\frac{(n-1)K + k^2}{2} + (n-1)\left(\frac{a'}{a}\right)^2 + \left(\frac{a'}{a}\right)'\right\}B\\ &+ \frac{(n-1)K - k^2}{2k}H_T'\Bigg]Y_j\,,\end{aligned} \tag{D·12b}$$

$$\begin{aligned}\delta R_{ij} = \Bigg[&-k\left(B' + \frac{a'}{a}B\right) + H_T'' + \frac{a'}{a}H_T' + 2\left(\frac{a'}{a}\right)'H_T\\ &+ 2(n-1)\left\{\left(\frac{a'}{a}\right)^2 + K\right\}H_T + (n-2)\frac{a'}{a}(H_T' - kB)\Bigg]Y_{ij}\,.\end{aligned} \tag{D·12c}$$

$$\delta R = 0\,. \tag{D·13}$$

Einstein tensor:

$$\delta G^0{}_0 = 0\,, \tag{D·14a}$$

$$\delta G^0{}_j = \frac{(n-1)K - k^2}{2a^2k}(H_T' - kB)Y_j\,, \tag{D·14b}$$

$$\delta G^i{}_0 = a^{-2}\left[-\left\{\frac{(n-1)K - k^2}{2} + (n-1)\left[-\left(\frac{a'}{a}\right)' + \left(\frac{a'}{a}\right)^2\right]\right\}B\right.$$

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$$-\frac{(n-1)K-k^2}{2k}H_T'\Big]Y^i\,, \tag{D·14c}$$

$$\delta G^i{}_j=a^{-2}\Big[-k\Big(B'+\frac{a'}{a}B\Big)+a^{-1}(aH_T')'+(n-2)\frac{a'}{a}(H_T'-kB)\Big]Y^i{}_j\,. \tag{D·14d}$$

(3) Tensor perturbations

We omit the suffix (2) here.

Metric:

$$\tilde{g}_{00}=-a^2\,, \tag{D·15a}$$

$$\tilde{g}_{0j}=0\,, \tag{D·15b}$$

$$\tilde{g}_{ij}=a^2[\gamma_{ij}+2H_T Y_{ij}], \tag{D·15c}$$

$$\tilde{g}^{00}=-a^{-2}\,, \tag{D·16a}$$

$$\tilde{g}^{0j}=0\,, \tag{D·16b}$$

$$\tilde{g}^{ij}=a^{-2}[\gamma^{ij}-2H_T Y^{ij}]. \tag{D·16c}$$

Christoffel symbols:

$$\delta\Gamma^0{}_{00}=\delta\Gamma^0{}_{0j}=\delta\Gamma^i{}_{00}=0\,, \tag{D·17a}$$

$$\delta\Gamma^i{}_{0j}=H_T'Y^i{}_j\,, \tag{D·17b}$$

$$\delta\Gamma^0{}_{ij}=a^{-2}(a^2H_T)'Y_{ij}\,, \tag{D·17c}$$

$$\delta\Gamma^i{}_{jm}=H_T(Y^i{}_{j|m}+Y^i{}_{m|j}-Y_{jm}{}^{|i}). \tag{D·17d}$$

Curvature tensor:

$$\delta R^0{}_{00j}=\delta R^0{}_{0ij}=0\,, \tag{D·18a}$$

$$\delta R^0{}_{i0j}=[H_T''+(a'/a)H_T'+2(a'/a)'H_T]Y_{ij}\,, \tag{D·18b}$$

$$\delta R^0{}_{ijm}=-H_T'(Y_{ij|m}-Y_{im|j}), \tag{D·18c}$$

$$\delta R^i{}_{00j}=[H_T''+(a'/a)H_T']Y^i{}_j\,, \tag{D·18d}$$

$$\delta R^i{}_{0jm}=H_T'(Y^i{}_{m|j}-Y^i{}_{j|m}), \tag{D·18e}$$

$$\delta R^i{}_{j0m}=H_T'(Y^i{}_{m|j}-Y_{jm}{}^{|i}), \tag{D·18f}$$

$$\begin{aligned}\delta R^i{}_{jmp}=&\,2(a'/a)^2H_T(\delta^i{}_mY_{jp}-\delta^i{}_pY_{jm})\\&+H_T(Y^i{}_{j|pm}-Y^i{}_{j|mp}+Y^i{}_{p|jm}-Y^i{}_{m|jp}+Y_{jm}{}^{|i}{}_{|p}-Y_{jp}{}^{|i}{}_{|m})\\&+(a'/a)H_T'(\delta^i{}_mY_{jp}-\delta^i{}_pY_{jm}-\gamma_{jm}Y^i{}_p+\gamma_{jp}Y^i{}_m).\end{aligned} \tag{D·18g}$$

Ricci tensor:

$$\delta R_{00}=\delta R_{0j}=0\,, \tag{D·19a}$$

$$\delta R_{ij}=[H_T''+(n-1)(a'/a)H_T'+2(a'/a)'H_T$$

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$$+2(n-1)(a'/a)^2 H_T+(k^2+2nK)H_T]\mathrm{Y}_{ij}\,, \tag{D·19b}$$

$$\delta R=0\,. \tag{D·20}$$

Einstein tensor:

$$\delta G^0{}_0=\delta G^0{}_j=\delta G^i{}_0=0\,, \tag{D·21a}$$

$$\delta G^i{}_j=a^{-2}[a^{-1}(aH_T{}')'+(n-2)(a'/a)H_T{}'+(k^2+2K)H_T]\mathrm{Y}^i{}_j\,. \tag{D·21b}$$

Appendix E

Derivation of Perturbation Equations for the Baryon-Photon System

In this appendix, we derive perturbation equations for the baryon-photon system in the case baryons interact with photons through Thomson scattering of photons by electrons but the internal energy of baryons can be neglected. The derivation is based on the relativistic kinetic theory. For general formulation of the theory, see for example a lecture note written by Stewart (1971).

In relativistic kinetic theory, an invariant space-like volume element on coordinate space is given by

$$\sigma_\alpha=-\frac{1}{3!}u_\alpha u^\mu \varepsilon_{\mu\nu\lambda\sigma}dx^\mu dx^\nu dx^\lambda dx^\sigma\,, \tag{E·1}$$

where u^μ is an arbitrary unit time-like vector field and $\varepsilon_{\mu\nu\lambda\sigma}$ is the completely antisymmetric tensor with $\varepsilon_{0123}=\sqrt{-g}$. While, an invariant volume element on momentum space for a particle species with rest mass m is

$$\pi_q=\frac{2}{4!}\theta(q^0)\delta(g_{\mu\nu}q^\mu q^\nu+m^2)\varepsilon_{\mu\nu\lambda\sigma}dx^\mu dx^\nu dx^\lambda dx^\sigma\,. \tag{E·2}$$

Then the distribution function $f(x^\alpha, q^\beta)$ is defined such that the number of particles within a unit momentum space volume π_q crossing a unit hypersurface element σ_α is given by

$$dn=f(x^\alpha, q^\beta)q^\mu\sigma_\mu\pi_q\,. \tag{E·3}$$

Because of the Liouville theorem, $q^\mu\sigma_\mu\pi_q$ is conserved along the trajectories of particles. Therefore, denoting the rate of change in dn due to collisions by

$$C[f]q^\mu\sigma_\mu\pi_q\,, \tag{E·4}$$

the relativistic Boltzmann equation takes the form

$$\mathcal{L}(f)=C[f], \tag{E·5}$$

where

$$\mathcal{L}\equiv\frac{d}{d\lambda}=q^\mu\frac{\partial}{\partial x^\mu}+\frac{dq^\mu}{d\lambda}\frac{\partial}{\partial q^\mu}\,, \tag{E·6}$$

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is the Liouville operator with λ being an affine parameter along the trajectory and

$$q^{\mu}=\frac{dx^{\mu}}{d\lambda}, \qquad \frac{dq^{\mu}}{d\lambda}=-\Gamma^{\mu}{}_{\alpha\beta}q^{\alpha}q^{\beta}. \tag{E·7}$$

The quantity $C[f]$ generally involves integrals over momentum space and is called the collision integral. For the time being, we proceed without specifying the explicit form of $C[f]$.

As usual, macroscopic quantities are calculated by integrating the corresponding microscopic quantities over momentum space with weight f. For example, from Eq. (E·3), we find that the particle flux density n^{μ} and the energy-momentum tensor $T^{\mu\nu}$ are given by

$$n^{\mu}=\int q^{\mu}f\pi_q, \tag{E·8a}$$

$$T^{\mu\nu}=\int q^{\mu}q^{\nu}f\pi_q. \tag{E·8b}$$

In order to evaluate these integrals, it is convenient to introduce a tetrad frame on which the momentum q^{μ} can be treated as on the flat space background. For our purpose of cosmological perturbations, a relevant choice of tetrad is

$$\tilde{e}^{\hat{0}}{}_{\mu}=-\tilde{N}_{\mu}, \tag{E·9a}$$

$$\tilde{e}^{\hat{i}}{}_{\mu}=a\left(\delta^{i}{}_{\mu}+\frac{1}{2}H^{i}{}_{\mu}+\frac{1}{2}\delta^{0}{}_{\mu}\beta^{i}\right), \tag{E·9b}$$

where $\tilde{N}^{\mu}$ is the unit time-like vector normal to the hypersurface $\eta=$const whose components are

$$\tilde{N}_{\mu}=(-\tilde{a},\ 0), \tag{E·10a}$$

$$\tilde{N}^{\mu}=\left(\frac{1}{\tilde{a}},\ -\frac{\beta^{i}}{\tilde{a}}\right), \tag{E·10b}$$

and $H_{\alpha\beta}$ represents the perturbation in the spatial parts of the metric,

$$H_{\alpha\beta}\equiv\frac{1}{a^2}\delta(\tilde{g}_{\alpha\beta}+\tilde{N}_{\alpha}\tilde{N}_{\beta}), \tag{E·11}$$

and raising and lowering indices of $H_{\alpha\beta}$ and β^{i} are to be done with the flat metric, i.e.,

$$H^{\alpha}{}_{\beta}=\eta^{\alpha\mu}H_{\mu\beta}, \quad \beta_i=\delta_{ij}\beta^{j}, \quad \text{etc.} \tag{E·12}$$

Note that $H_{i0}=\beta_i$ and $H_{00}=0$ at first order. Here and in what follows a tilded quantity represents the perturbed one and the background is assumed to be spatially flat ($K=0$) for simplicity. From Eqs. (E·10)~(E·12), the components of the inverse of the tetrad are

$$\tilde{e}_{\hat{0}}{}^{\mu}=\tilde{N}^{\mu}, \tag{E·13a}$$

$$\tilde{e}_{\hat{i}}{}^{\mu}=\frac{1}{a}\left(\delta_i{}^{\mu}-\frac{1}{2}H_i{}^{\mu}-\frac{1}{2}\delta_0{}^{\mu}\beta_i\right). \tag{E·13b}$$

Using such a tetrad frame, the momentum space volume element π_q may be expressed

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as

$$\pi_q = 2\theta(q^{\hat{0}})\delta(\eta_{\alpha\beta}q^{\hat{\alpha}}q^{\hat{\beta}}+m^2)d^{\hat{4}}q$$
$$=\frac{\theta(q^{\hat{0}})}{|q^{\hat{0}}|}\{\delta(q^{\hat{0}}-E_q)+\delta(q^{\hat{0}}+E_q)\}d^{\hat{4}}q$$
$$=\frac{d^{\hat{3}}q}{E_q}, \tag{E·14}$$

where $q^{\hat{\alpha}}=e^{\hat{\alpha}}{}_{\beta}q^{\beta}$, $E_q=\sqrt{q^2+m^2}$ and $q=\sqrt{\delta_{ij}q^{\hat{i}}q^{\hat{j}}}$. Further, it is often more convenient to consider $\{q, \gamma^{\hat{i}}\}$ as the independent variables than $\{q^{\hat{i}}\}$, where $\gamma^i=q^{\hat{i}}/q$ is the direction cosine of the 3-momentum $q^{\hat{i}}$. Then

$$\pi_q=\frac{q^2dqd\Omega_q}{E_q}; \qquad d\Omega_q=\frac{1}{3!}\varepsilon_{ijk}\gamma^id\gamma^jd\gamma^k. \tag{E·15}$$

In terms of the new independent variables $\{q, \gamma^i\}$, the left-hand side of the Boltzmann equation (E·5) takes the form

$$\mathcal{L}(\tilde{f})=q^{\mu}\frac{\partial\tilde{f}}{\partial x^{\mu}}+\frac{dq}{d\lambda}\frac{\partial\tilde{f}}{\partial q}+\frac{d\gamma^i}{d\lambda}\frac{\partial\tilde{f}}{\partial\gamma^i}. \tag{E·16}$$

Since the background is homogeneous and isotropic, both $\partial\tilde{f}/\partial x^i$ and $\partial\tilde{f}/\partial\gamma^i$ are quantities of first order. Therefore their respective coefficients q^{μ} and $d\gamma^i/d\lambda$ may be replaced by those of the lowest order, i.e.,

$$q^{\mu}=\frac{1}{a}q^{\hat{\mu}}, \qquad \frac{d\gamma^i}{d\lambda}=\frac{d}{d\lambda}\left(\frac{aq^i}{q}\right). \tag{E·17}$$

In particular, $d\gamma^i/d\lambda$ can be shown to vanish at the lowest order. Then, after a bit tedious manipulation of Eq. (E·16) we obtain

$$\frac{1}{q^0}\mathcal{L}(\tilde{f})=\tilde{f}'+\gamma^i\frac{\partial\tilde{f}}{\partial x^i}+\frac{a}{q}\frac{d(aq^0)}{d\lambda}\frac{\partial\tilde{f}}{\partial q}$$
$$=\tilde{f}'+\gamma^i\tilde{f}_{,i}-\left\{\frac{a'}{a}+\frac{q^{\hat{0}}}{q}\gamma^i\frac{\delta a_{,i}}{a}+\left(\frac{1}{2}H'_{ij}-\beta_{ij}\right)\gamma^i\gamma^j\right\}q\frac{\partial}{\partial q}\tilde{f}. \tag{E·18}$$

Having derived the perturbed Boltzmann equation, we now proceed to find the corresponding equations for macroscopic quantities. So far, we did not specify the species of particles. In the following, we concentrate our attention on the case of a photon fluid, in which $E_q=q$. The energy-momentum tensor of the fluid can be put into the form

$$\tilde{T}^{\mu\nu}=\tilde{\rho}\tilde{u}^{\mu}\tilde{u}^{\nu}+\tilde{p}\tilde{P}^{\mu\nu}+\tilde{\pi}^{\mu\nu}, \tag{E·19a}$$

where

$$\tilde{P}^{\mu\nu}\equiv\tilde{g}^{\mu\nu}+\tilde{u}^{\mu}\tilde{u}^{\nu} \tag{E·19b}$$

with $\tilde{u}^{\mu}$ being defined such that the energy flux in the rest frame of $\tilde{u}^{\mu}$ vanishes, i.e., $\tilde{P}^{\mu}{}_{\alpha}\tilde{T}^{\alpha\beta}\tilde{u}_{\beta}=0$, and $\tilde{\pi}^{\mu\nu}$ is the anisotropic stress which vanishes on the background and satisfies $\tilde{u}_{\mu}\tilde{\pi}^{\mu\nu}=\tilde{g}_{\mu\nu}\tilde{\pi}^{\mu\nu}=0$. Then from Eq. (E·8b), $\tilde{\rho}$, $\tilde{p}$ and $\tilde{\pi}^{\mu\nu}$ are given by

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$$\tilde{\rho}=\int(q^{\mu}\tilde{u}_{\mu})^2\tilde{f}qdqd\Omega_q\,, \tag{E·20a}$$

$$\tilde{p}=\frac{1}{3}\int(q^{\mu}q^{\nu}\tilde{P}_{\mu\nu})\tilde{f}qdqd\Omega_q\,, \tag{E·20b}$$

$$\tilde{\pi}^{\mu\nu}=\int q^{\alpha}q^{\beta}\Big(\tilde{P}_{\alpha}{}^{\mu}\tilde{P}_{\beta}{}^{\nu}-\frac{1}{3}\tilde{P}_{\alpha\beta}P^{\mu\nu}\Big)\tilde{f}qdqd\Omega_q\,, \tag{E·20c}$$

where $\tilde{u}_{\mu}q^{\mu}$ and $\tilde{P}^{\mu}{}_{\alpha}q^{\alpha}$ should be expressed in terms of $\{q,\gamma^i\}$; using Eqs. (E·9) and noting $E_q=q$, we obtain

$$q^{\mu}\tilde{u}_{\mu}=-q\{1-(\beta_i+v_i)\gamma^i\}, \tag{E·21a}$$

$$\tilde{P}^{i}{}_{\alpha}q^{\alpha}=\frac{q}{a}\Big\{\Big(\delta^{i}{}_{j}-\frac{1}{2}H^{i}{}_{j}\Big)\gamma^j-(\beta^i+v^i)\Big\}, \tag{E·21b}$$

$$\tilde{P}^{0}{}_{\alpha}q^{\alpha}=aq(\beta_i+v_i)\gamma^i\,, \tag{E·21c}$$

where $v^i=\tilde{u}^i/\tilde{u}^0$ is the fluid 3-velocity. Let $\tilde{f}=f+\delta f$ where f is the background distribution function. Then inserting Eqs. (E·21) into Eqs. (E·20) gives

$$\rho=\int fq^3dqd\Omega_q\,, \tag{E·22a}$$

$$\delta\rho=\int\delta fq^3dqd\Omega_q\,, \tag{E·22b}$$

$$p=\frac{1}{3}\int fq^3dqd\Omega_q=\frac{1}{3}\rho\,, \tag{E·22c}$$

$$\delta p=\frac{1}{3}\int\delta fq^3dqd\Omega_q=\frac{1}{3}\delta\rho\,, \tag{E·22d}$$

$$\Pi^{ij}=\int\delta f\Big(\gamma^i\gamma^j-\frac{1}{3}\delta^{ij}\Big)q^3dqd\Omega_q\,. \tag{E·22e}$$

In the last line, Π^{ij} is defined by $\Pi^{i}{}_{j}=\tilde{\pi}^{i}{}_{j}$ and their indices are to be raised or lowered by the flat 3-metric δ^{ij} or δ_{ij}.

Equations (E·22c) and (E·22d) show that there can be no entropy perturbation with respect to the photon fluid. In addition, these equations indicate that the integrated perturbed intensity defined by

$$\delta I(x^{\alpha},\Omega_q)\equiv4\pi\int\delta fq^3dq\,, \tag{E·23}$$

plays a central role in the present problem, which is a well-known fact. The equation for δI can be now easily derived. From Eqs. (E·5), (E·18) and (E·23), we obtain

$$\delta I'+\gamma^i\delta I_{,i}+4\frac{a'}{a}\delta I+4\rho\Big\{\Big(\frac{1}{2}H'_{ij}-\beta_{i,j}\Big)\gamma^i\gamma^j+(\ln a)_{,i}\gamma^i\Big\}=4\pi J\,, \tag{E·24}$$

where J is defined by

$$J\equiv\int\frac{1}{q^0}C[\tilde{f}]q^3dq\,. \tag{E·25}$$

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Then it is straightforward to write down the equations for the energy, the momentum and the stress of the fluid. They are

$$\delta\rho' + 4\frac{a'}{a}\delta\rho + F^i{}_{,i} + \frac{4}{3}\rho\left(\frac{1}{2}H' - \beta^i{}_{,i}\right) = \int J d\Omega_q\,, \tag{E·26a}$$

$$F^{i\prime} + 4\frac{a'}{a}F^i + F^{ij}{}_{,j} + \frac{4}{3}\rho\delta^{ij}(\ln\alpha)_{,j} = \int J\gamma^i d\Omega_q\,, \tag{E·26b}$$

$$F^{ij\prime} + 4\frac{a'}{a}F^{ij} + F^{ijk}{}_{,k} + \frac{4}{15}\rho\left(H^{ij\prime} + \frac{1}{2}\delta^{ij}H'\right)$$
$$- \frac{4}{15}\rho(\beta^i{}_{,k}\delta^{kj} + \beta^j{}_{,k}\delta^{ki} + \delta^{ij}\beta^k{}_{,k}) = \int J\gamma^i\gamma^j d\Omega_q\,, \tag{E·26c}$$

where

$$F^i \equiv \frac{1}{4\pi}\int \delta I\gamma^i d\Omega_q = \int \delta f\gamma^i q^3 dq d\Omega_q\,, \tag{E·27a}$$

$$F^{ij} \equiv \frac{1}{4\pi}\int \delta I\gamma^i\gamma^j d\Omega_q = \int \delta f\gamma^i\gamma^j q^3 dq d\Omega_q\,, \tag{E·27b}$$

$$F^{ijk} \equiv \frac{1}{4\pi}\int \delta I\gamma^i\gamma^j\gamma^k d\Omega_q = \int \delta f\gamma^i\gamma^j\gamma^k q^3 dq d\Omega_q\,. \tag{E·27c}$$

What we have to do now is to express F^i, F^{ij} and F^{ijk} in terms of $\tilde{\rho}$, $\tilde{u}^\mu$ and $\tilde{\pi}^{\mu\nu}$. First, let us consider F^i. From the definition of the 4-velocity $\tilde{u}^\mu$ we must have

$$-\tilde{P}^i{}_\mu\tilde{T}^{\mu\nu}\tilde{u}_\nu = -\int (q^\nu\tilde{u}_\nu)(\tilde{P}^i{}_\mu q^\mu)\tilde{f} q dq d\Omega_q$$
$$= \frac{1}{a}\int \{\delta f\gamma^i - f(\delta^i{}_j + \gamma^i\gamma_j)(\beta^j + v^j)\} q^3 dq d\Omega_q$$
$$= \frac{1}{a}\left\{F^i - \frac{4}{3}\rho(\beta^i + v^i)\right\} = 0\,.$$

Hence,

$$F^i = \frac{4}{3}\rho(\beta^i + v^i). \tag{E·28}$$

As for F^{ij}, one easily finds from Eqs. (E·22d) and (E·22e) that

$$F^{ij} = \Pi^{ij} + \delta p\delta^{ij}\,. \tag{E·29}$$

Finally, we consider the expression for F^{ijk}. However, this time we need a non-trivial assumption on the form of δf. This is because the Boltzmann equation, when expressed in terms of the moments of $\tilde{f}$ with respect to γ^i, gives an infinite sequence of equations for the moments of $\tilde{f}$. Therefore we have to truncate it at a certain order of the moments. In our case we should truncate it at the second order,

$$\delta f = \frac{1}{4\pi}(M + M_i\gamma^i + M_{ij}\gamma^i\gamma^j), \tag{E·30}$$

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where we may (actually "should") impose the traceless condition, $\delta^{ij}M_{ij}=0$, on M_{ij}. Then inserting Eq. (E·30) into Eqs. (E·27a) and (E·27c), we find

$$F^i=\frac{1}{3}\int M^i q^3 dq\,, \tag{E·31a}$$

$$F^{ijk}=\frac{1}{15}\int(\delta^{ij}M^k+\delta^{jk}M^i+\delta^{ki}M^j)q^3 dq\,. \tag{E·31b}$$

Hence,

$$F^{ijk}=\frac{1}{5}(\delta^{ij}F^k+\delta^{jk}F^i+\delta^{ki}F^j). \tag{E·32}$$

Using Eqs. (E·28), (E·29) and (E·32), we can rewrite Eqs. (E·26) in the familiar form

$$\delta\rho'+4\frac{a'}{a}\delta\rho+\frac{4}{3}\rho\left(v^i{}_{,i}+\frac{1}{2}H'\right)=\int J d\Omega_q\,, \tag{E·33a}$$

$$(v^i+\beta^i)'+\frac{3}{4\rho}(\delta p^{,i}+\pi^{ij}{}_{,j})+(\ln\alpha)^{,i}=\int J\gamma^i d\Omega_q\,, \tag{E·33b}$$

$$\Pi^{ij\prime}+4\frac{a'}{a}\Pi^{ij}+\frac{4}{15}\rho\left\{\left(v^{i,j}+v^{j,i}-\frac{2}{3}\delta^{ij}v^k{}_{,k}\right)+\left(H^{ij\prime}-\frac{1}{3}\delta^{ij}H'\right)\right\}$$
$$=\int J\gamma^i\gamma^j d\Omega_q\,, \tag{E·33c}$$

where indices of all the quantities appearing above should be raised or lowered by the flat 3-metric.

Now let us evaluate the collision integral $C[\tilde{f}]$. We consider the situation in which collisions are dominated by Thomson scattering of photons by electrons. We may assume the classical form of the collision integral under such a situation,

$$C[\tilde{f}]=\iiint\{F(p')\tilde{f}(q')(1+4\pi^3\tilde{f}(q))-F(p)\tilde{f}(q)(1+4\pi^3\tilde{f}(q'))\}$$
$$\times W(p+q\to p'+q')\pi_p\pi_{p'}\pi_{q'}\,. \tag{E·34}$$

Here $F(p)$ is the electron distribution function and $W(p+q\to p'+q')$ is the scattering probability density which is related to the differential cross section $d\sigma(u^\mu)$ measured in the rest frame of a 4-velocity u^μ as

$$W(p+q\to p'+q')\pi_{p'}\pi_{q'}=(-u_\mu q^\mu)(-u_\nu p^\nu)|v_p-v_q|d\sigma(u^\mu), \tag{E·35a}$$

where $|v_p-v_q|$ is the relative velocity of the incident particles as measured by the observer,

$$|v_p-v_q|=\frac{\|(u_\mu p^\mu)q^\nu-(u_\mu q^\mu)p^\nu\|}{(-u_\mu p^\mu)(-u_\mu q^\mu)}\,, \tag{E·35b}$$

and $\|\cdots\|$ denotes the norm of the vector. Since $W(p+q\to p'+q')$ is independent of the choice of u^μ, we may choose u^i to be parallel to the direction of the incident photons q^i in the rest frame of p^μ, i.e., u^μ to be a linear combination of p^μ and q^μ. In this case, $d\sigma(u^\mu)$ is known to be independent of u^μ and Eq. (E·35a) reduces to

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$$W(p+q\to p'+q')\pi_{p'}\pi_{q'}=(-p_\mu q^\mu)d\sigma$$
$$=(-p_\mu q^\mu)\sigma_s\delta^{(4)}(p+q-p'-q')\pi_{p'}\pi_{q'}, \tag{E·36}$$

where σ_s is called the scalar cross section. Then, integrating Eq. (E·34) with respect to p'^μ yields

$$C[\tilde{f}]=2\int\frac{d^{\hat{3}}p}{E_p}\int\frac{d^{\hat{3}}q'}{q'}\theta(q+p^{\hat{0}}-q')\delta((q+p-q')^2+m^2)$$
$$\times\sigma_s(-p_{\hat{\mu}}q^{\hat{\mu}})\{F(p')\tilde{f}(q')(1+4\pi^3\tilde{f}(q))-F(p)\tilde{f}(q)(1+4\pi^3\tilde{f}(q'))\}, \tag{E·37}$$

where all the momenta are those evaluated on the tetrad frame defined in Eqs. (E·13) and

$$p^{\hat{0}}=E_p=\sqrt{p^2+m^2}\ , \qquad p'^{\hat{\mu}}=q^{\hat{\mu}}+p^{\hat{\mu}}-q'^{\hat{\mu}}\ ,$$
$$(q+p-q')^2=\eta_{\alpha\beta}(q^{\hat{\alpha}}+p^{\hat{\alpha}}-q'^{\hat{\alpha}})(q^{\hat{\beta}}+p^{\hat{\beta}}-q'^{\hat{\beta}})$$

with m being the electron mass. The integral with respect to $q'^{\hat{i}}$ in Eq. (E·37) can be most simply evaluated by going into the rest frame of $p^{\hat{\mu}}$ in which

$$\bar{p}^{\hat{\mu}}=(m,\ 0,\ 0,\ 0), \tag{E·38a}$$

$$(q+p-q')^2+m^2=2m\left[\bar{q}'\left\{1+\frac{\bar{q}}{m}(1-\cos\bar{\theta})\right\}-\bar{q}\right], \tag{E·38b}$$

$$q+p^{\hat{0}}-q'=\bar{q}+m-\bar{q}'\ , \tag{E·38c}$$

$$-p_{\hat{\mu}}q^{\hat{\mu}}=m\bar{q}\ , \tag{E·38d}$$

where the bar denotes a quantity evaluated in the rest frame of $p^{\hat{\mu}}$ and $\bar{\theta}$ is the angle between $\bar{q}^{\hat{i}}$ and $\bar{q}'^{\hat{i}}$. Applying Eqs. (E·38) to Eq. (E·37), we obtain

$$C[\tilde{f}]=\int\frac{d^{\hat{3}}p}{E_p}\int d\Omega_{\bar{q}'}\bar{q}'^2\sigma_s\{F(p')\tilde{f}(q')(1+4\pi^3\tilde{f}(q))$$
$$-F(p)\tilde{f}(q)(1+4\pi^3\tilde{f}(q'))\}, \tag{E·39}$$

where

$$\bar{q}'=\frac{\bar{q}}{1+(\bar{q}/m)(1-\cos\bar{\theta})}\ .$$

While, σ_s for unpolarized electrons and photons is known to be given by

$$\sigma_s=\frac{d\sigma}{d\Omega_{\bar{q}'}}\frac{m\bar{q}}{\bar{q}'^2}\ ;\quad \frac{d\sigma}{d\Omega_{\bar{q}'}}=\frac{3}{16\pi}\left(\frac{\bar{q}'}{\bar{q}}\right)^2\left(\frac{\bar{q}'}{\bar{q}}+\frac{\bar{q}}{\bar{q}'}-\sin^2\bar{\theta}\right)\sigma_T\ , \tag{E·40}$$

where σ_T is the total Thomson cross section.

Now taking the non-relativistic limit $\bar{q}/m\to 0$ and noting that

$$\int p^\mu F(p)\pi_p=n_e u_e{}^\mu \tag{E·41}$$

for non-relativistic electrons, where n_e and $u_e{}^\mu$ are the number density and the 4-velocity of the electron fluid, we obtain

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$$C[\tilde{f}]=\frac{3}{8\pi}n_e\sigma_T\bar{q}\int d\Omega_{\bar{q}'}\frac{1+\cos^2\bar{\theta}}{2}(\tilde{f}(q')-\tilde{f}(q)), \tag{E·42}$$

where $\bar{q}'=\bar{q}$. Note that $\bar{q}$ in this expression is given by

$$\bar{q}=-q_\mu u_e{}^\mu=q\{1-(\beta_i+v_{ei})\gamma^i\}, \tag{E·43}$$

where $v_e{}^i$ is the electron 3-velocity. Then noting the equalities

$$q^3dq=\{1+4(\beta_i+v_{ei})\gamma^i\}\bar{q}^3d\bar{q}$$
$$=\{1+4(\beta_i+v_{ei})\gamma^i\}\bar{q}'^3d\bar{q}'$$

and

$$\cos^2\bar{\theta}=(\delta_{ij}\bar{\gamma}^i\bar{\gamma}'^j)^2,$$

and using the coordinate-invariant nature of the distribution function, the quantity J defined by Eq. (E·25) can be evaluated as

$$J=\frac{3}{8\pi}an_e\sigma_T\Big[\{1+4(\beta_i+v_{ei})\gamma^i\}\int\frac{1+\cos^2\bar{\theta}}{2}\tilde{f}(q')\bar{q}'^3d\bar{q}'d\Omega_{\bar{q}'}$$
$$-\int\frac{1+\cos^2\bar{\theta}}{2}d\Omega_{\bar{q}'}\int\tilde{f}(q)q^3dq\Big]$$
$$=\frac{1}{4\pi}an_e\sigma_T[\delta\rho-\delta I+4\rho(\beta_i+v_{ei})\gamma^i]. \tag{E·44}$$

Therefore we readily find

$$\int Jd\Omega_q=0\,, \tag{E·45a}$$

$$\int J\gamma^i d\Omega_q=\frac{4}{3}an_e\sigma_T\rho(v_e{}^i-v^i), \tag{E·45b}$$

$$\int J\Big(\gamma^i\gamma^j-\frac{1}{3}\delta^{ij}\Big)d\Omega_q=-an_e\sigma_T\Pi^{ij}\,, \tag{E·45c}$$

which are to be inserted into the right-hand sides of Eqs. (E·33).

Equations (E·33) can be then expanded in terms of relevant spatial harmonics. By extrapolating the present results to the space of non-zero background spatial curvature ($K\neq 0$), the scalar perturbation equations, in particular, are found to be given by

$$\delta_r{}'+\frac{4}{3}(kv_r+3H_L{}')=0\,, \tag{E·46a}$$

$$(v_r-B)'-\frac{1}{4}k\delta_r-kA+\frac{1}{6}k\Big(1-\frac{3K}{k^2}\Big)\Pi_r=an_e\sigma_T(v_m-v_r), \tag{E·46b}$$

$$\Pi_r{}'-\frac{8}{5}k\Big(v_r-\frac{1}{k}H_T{}'\Big)=-an_e\sigma_T\Pi_r\,, \tag{E·46c}$$

where the suffix r has been explicitly attached to every perturbation amplitude with respect to photons and the electron velocity has been replaced by the baryonic matter

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velocity v_m in the second equation, since electrons and baryons can be assumed to be tightly bound together for cosmological length-scales. By appealing to the conservation law of the total energy-momentum tensor, the associated equations of motion for the matter are given by

$$\delta_m'+kv_m+3H_L'=0\,, \qquad \text{(B·47a)}$$

$$(v_m-B)'+\frac{a'}{a}(1-3c_m^2)(v_m-B)-kA=\frac{4\rho_r}{3\rho_m}an_e\sigma_T(v_r-v_m), \qquad \text{(E·47b)}$$

where the effect of matter pressure has been taken into account in the form of sound velocity c_m even though the matter internal energy has been neglected. This is because the behavior of matter density perturbations is crucially affected by the matter pressure on scales within the matter sound horizon $\lambda \lesssim c_m t$ though the internal energy would not yet play a dynamically important role by itself. Equations (E·46) and (E·47), together with the Einstein equations for scalar perturbations, form the complete set of scalar perturbation equations for the baryon-photon system when the matter internal energy can be neglected.

Finally, for the sake of completeness, we list equations of motion for vector and tensor perturbations. For vector perturbations, they are

$$(v_r^{(1)}-B^{(1)})'+\frac{k^2-2K}{8k}\Pi_r^{(1)}=an_e\sigma_T(v_m^{(1)}-v_r^{(1)}), \qquad \text{(E·48a)}$$

$$\Pi_r^{(1)\prime}-\frac{8}{5}k\left(v_r^{(1)}-\frac{1}{k}H_T^{(1)\prime}\right)=-an_e\sigma_T\Pi_r^{(1)}, \qquad \text{(E·48b)}$$

$$(v_m^{(1)}-B^{(1)})'+\frac{a'}{a}(1-3c_m^2)(v_m^{(1)}-B^{(1)})=\frac{4\rho_r}{3\rho_m}an_e\sigma_T(v_r^{(1)}-v_m^{(1)}). \qquad \text{(E·48c)}$$

For tensor perturbations, we have only one equation:

$$\Pi_r^{(2)\prime}+\frac{8}{5}H_T^{(2)\prime}=-an_e\sigma_T\Pi_r^{(2)}\,. \qquad \text{(E·49)}$$

Appendix F

Perturbation Formulas for a Classical Scalar Field

The energy-momentum tensor of a single-component real scalar field is given by

$$\tilde{T}^\mu{}_\nu=\tilde{\nabla}^\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi}-\frac{1}{2}[\tilde{\nabla}^\lambda\tilde{\phi}\tilde{\nabla}_\lambda\tilde{\phi}+2\tilde{U}]\delta^\mu{}_\nu$$
$$+\xi[\tilde{G}^\mu{}_\nu\tilde{\phi}^2-\tilde{\nabla}^\mu\tilde{\nabla}_\nu(\tilde{\phi}^2)+\delta^\mu{}_\nu\tilde{\Box}(\tilde{\phi}^2)]. \qquad \text{(F·1)}$$

In order to obtain the formula for the perturbation of this energy-momentum tensor, we first write down the perturbation expressions for each term.

$\delta(\tilde{\nabla}^\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi})$:

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$$\begin{pmatrix}0 & \\ & 0\end{pmatrix}\text{-component}=2a^{-2}[A\phi'^2-\chi'\cdot\phi']\mathrm{Y}\,, \tag{F·2a}$$

$$\begin{pmatrix}0 & \\ & j\end{pmatrix}\text{-component}=ka^{-2}\phi'\cdot\chi\mathrm{Y}_j\,, \tag{F·2b}$$

$$\begin{pmatrix}i & \\ & 0\end{pmatrix}\text{-component}=-a^{-2}[B\phi'^2+k\phi'\cdot\chi]\mathrm{Y}^i\,, \tag{F·2c}$$

$$\begin{pmatrix}i & \\ & j\end{pmatrix}\text{-component}=0\,. \tag{F·2d}$$

Hence the perturbation of the energy-momentum tensor for the minimally coupled field ($\xi=0$),

$$\delta(\tilde{\nabla}^\mu\tilde{\phi}\tilde{\nabla}_\nu\tilde{\phi}-\frac{1}{2}[\tilde{\nabla}^\lambda\tilde{\phi}\tilde{\nabla}_\lambda\tilde{\phi}+2\tilde{U}]\delta^\mu{}_\nu)\,,$$

is expressed as

$$\begin{aligned}\begin{pmatrix}0 & \\ & 0\end{pmatrix}\text{-component}=a^{-2}\Big[&-\phi'\cdot X'-a^2U_\phi\cdot X+\Big(\mathcal{A}+\frac{1}{k}\mathcal{B}'\Big)\phi'^2\\&+\frac{1}{k}(\phi''+a^2U_\phi)\cdot\phi'\mathcal{B}\\&+\Big\{(\phi')^2-\Big(\frac{a'}{a}\Big)^{-1}\phi'\cdot\phi''-\Big(\frac{a'}{a}\Big)^{-1}a^2U_\phi\cdot\phi'\Big\}\mathfrak{R}\Big]\mathrm{Y}\,,\end{aligned} \tag{F·3a}$$

$$\begin{aligned}\begin{pmatrix}i & \\ & j\end{pmatrix}\text{-component}=a^{-2}\Big[&\phi'\cdot X'-a^2U_\phi\cdot X-\Big(\mathcal{A}+\frac{1}{k}\mathcal{B}'\Big)\phi'^2\\&+\frac{1}{k}(a^2U_\phi-\phi'')\cdot\phi'\mathcal{B}\\&-\Big\{(\phi')^2+\Big(\frac{a'}{a}\Big)^{-1}a^2U_\phi\cdot\phi'-\Big(\frac{a'}{a}\Big)^{-1}\phi'\cdot\phi''\Big\}\mathfrak{R}\Big]\delta^i{}_j\mathrm{Y}\,.\end{aligned} \tag{F·3b}$$

The $({}^0{}_j)$-component and the $({}^i{}_0)$-component are given by Eqs. (F·2b) and (F·2c), respectively.

The perturbations of the terms which appear additionally for a non-minimally coupled field are given as follows:

$\delta(\tilde{\nabla}^\mu\tilde{\nabla}_\nu\tilde{\phi}^2)$:

$$\begin{aligned}\begin{pmatrix}0 & \\ & 0\end{pmatrix}\text{-component}=a^{-2}\Big[&2(\phi^2)''A+\Big(-2\frac{a'}{a}A+A'\Big)(\phi^2)'\\&-2(\phi\cdot\chi)''+2\frac{a'}{a}(\phi\cdot\chi)'\Big]\mathrm{Y}\,,\end{aligned} \tag{F·4a}$$

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$$\begin{pmatrix}0 & \\ & j\end{pmatrix}\text{-component}=2a^{-2}k\left[(\phi\cdot\chi)'-\frac{a'}{a}\phi\cdot\chi-\frac{1}{2}(\phi^2)'A\right]Y_j\,, \tag{F·4b}$$

$$\begin{pmatrix}i & \\ & 0\end{pmatrix}\text{-component}=a^{-2}\left[\left\{-(\phi^2)''+2\frac{a'}{a}(\phi^2)'\right\}B+k(\phi^2)'A\right.$$
$$\left.-2k(\phi\cdot\chi)'+2\frac{a'}{a}k\phi\cdot\chi\right]Y^i\,, \tag{F·4c}$$

$$\begin{pmatrix}i & \\ & j\end{pmatrix}\text{-component}=-a^{-2}\left[\left\{-2\frac{a'}{a}A+\frac{k}{n}B+H_L'\right\}(\phi^2)'+2\left\{\frac{k^2}{n}\phi\cdot\chi+\frac{a'}{a}(\phi\cdot\chi)'\right\}\right]\delta^i{}_jY$$
$$+a^{-2}[(kB-H_T')(\phi^2)'+2k^2\phi\cdot\chi]Y^i{}_j\,. \tag{F·4d}$$

$\delta(\tilde{G}^\mu{}_\nu\tilde{\phi}^2)$:

$$\begin{pmatrix}0 & \\ & 0\end{pmatrix}\text{-component}=n(n-1)a^{-2}\left[\left\{\left(\frac{a'}{a}\right)^2\mathcal{A}-\frac{k}{n}\frac{a'}{a}\mathcal{B}+\left[\left(\frac{a'}{a}\right)^2-\left(\frac{a'}{a}\right)'+K\right]\mathfrak{R}\right\}\phi^2\right.$$
$$\left.-\left\{\left(\frac{a'}{a}\right)^2+K\right\}\phi\cdot\chi\right]Y\,, \tag{F·5a}$$

$$\begin{pmatrix}0 & \\ & j\end{pmatrix}\text{-component}=(n-1)a^{-2}\left[k\frac{a'}{a}\mathcal{A}-K\mathcal{B}\right.$$
$$\left.+k\left(\frac{a'}{a}\right)^{-1}\left\{\left(\frac{a'}{a}\right)^2-\left(\frac{a'}{a}\right)'+K\right\}\mathfrak{R}\right]\phi^2Y_j\,, \tag{F·5b}$$

$$\begin{pmatrix}i & \\ & 0\end{pmatrix}\text{-component}=(n-1)a^{-2}\left[-k\frac{a'}{a}\mathcal{A}+\left\{\left(\frac{a'}{a}\right)'-\left(\frac{a'}{a}\right)^2\right\}\mathcal{B}\right.$$
$$\left.-\frac{1}{k}\left\{\left(\frac{a'}{a}\right)^2-\left(\frac{a'}{a}\right)'+K\right\}H_T'\right]\phi^2Y^i\,, \tag{F·5c}$$

$$\begin{pmatrix}i & \\ & j\end{pmatrix}\text{-component}=(n-1)a^{-2}\left[\left\{\left[2\left(\frac{a'}{a}\right)'+(n-2)\left(\frac{a'}{a}\right)^2\right]\mathcal{A}+\frac{a'}{a}\mathcal{A}'\right.\right.$$
$$-\frac{n-2}{n}k\frac{a'}{a}\mathcal{B}-\frac{k^2}{n}\Psi-nc_s{}^2\left[\left(\frac{a'}{a}\right)^2-\left(\frac{a'}{a}\right)'+K\right]\mathfrak{R}\Big\}\phi^2$$
$$\left.-\left\{2\left(\frac{a'}{a}\right)'+(n-2)\left(\frac{a'}{a}\right)^2+(n-2)K\right\}\phi\cdot\chi\right]Y\delta^i{}_j$$
$$-k^2a^{-2}\left[\mathcal{A}+\frac{1}{k}\frac{1}{a^{n-1}}(a^{n-1}\mathcal{B})'\right]\phi^2Y^i{}_j\,. \tag{F·5d}$$

Hence the correction to the perturbation formulas (F·3) is given by ξ times

$\delta[\tilde{G}^\mu{}_\nu\tilde{\phi}^2-\tilde{\nabla}^\mu\tilde{\nabla}_\nu(\tilde{\phi}^2)+\delta^\mu{}_\nu\tilde{\Box}(\tilde{\phi}^2)]$:

$$\begin{pmatrix}0 & \\ & 0\end{pmatrix}\text{-component}=na^{-2}\left[\left\{(n-1)\left(\frac{a'}{a}\right)^2\phi^2+\frac{a'}{a}(\phi^2)'\right\}\mathcal{A}+\frac{a'}{a}(\phi^2)'\Psi\right.$$
$$+\left\{-\frac{n-1}{n}k\frac{a'}{a}\phi^2+\frac{1}{2k}\left[(n-3)\left(\frac{a'}{a}\right)^2+(n-1)K\right](\phi^2)'+\frac{1}{k}\frac{a'}{a}(\phi^2)''\right\}\mathcal{B}$$

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$$-2\frac{a'}{a}(\phi\cdot X)'-\left\{\left(\frac{a'}{a}\right)^2+K+\frac{2k^2}{n}\right\}\phi\cdot X$$

$$-\left\{-(n-1)\left[\left(\frac{a'}{a}\right)^2-\left(\frac{a'}{a}\right)'+K\right]\phi^2+(\phi^2)''\right.$$

$$\left.+\frac{(n-5)(a'/a)^2+2(a'/a)'+(n-1)K}{2(a'/a)}(\phi^2)'\right\}\mathcal{R}\Big]\mathrm{Y}\,, \qquad (\mathrm{F}\cdot 6\mathrm{a})$$

$$\binom{i}{\ \ 0}\text{-component}=a^{-2}\Big[\Big\{-(n-1)k\frac{a'}{a}\mathcal{A}+(n-1)\left[\left(\frac{a'}{a}\right)'-\left(\frac{a'}{a}\right)^2\right]\mathcal{B}$$

$$-\frac{n-1}{k}\left[\left(\frac{a'}{a}\right)^2-\left(\frac{a'}{a}\right)'+K\right]H_T'\Big\}\phi^2+2k(\phi\cdot X)'$$

$$-2k\frac{a'}{a}\phi\cdot X-k(\phi^2)'\Psi+\left\{(\phi^2)''-2\frac{a'}{a}(\phi^2)'\right\}\frac{1}{k}H_T'\Big]\mathrm{Y}^i\,, \qquad (\mathrm{F}\cdot 6\mathrm{b})$$

$\binom{i}{\ \ j}$-component;

$$a^{-2}\mathrm{Y}\delta^i{}_j\ \text{part}=(n-1)\Big[\Big\{2\left(\frac{a'}{a}\right)'+(n-2)\left(\frac{a'}{a}\right)^2\Big\}\mathcal{A}+\frac{a'}{a}\mathcal{A}'-\frac{n-1}{n}k\frac{a'}{a}\mathcal{B}$$

$$-\frac{k^2}{n}\Psi\Big]\phi^2-(n-1)\left[2\left(\frac{a'}{a}\right)'+(n-2)\left(\frac{a'}{a}\right)^2+(n-2)K\right]\phi\cdot X$$

$$-2(\phi\cdot X)''-2(n-2)\frac{a'}{a}(\phi\cdot X)'-2\frac{n-1}{n}k^2\phi\cdot X$$

$$+\frac{1}{k}\Big[(n-1)\Big\{2\left(\frac{a'}{a}\right)'+(n-2)\left(\frac{a'}{a}\right)^2+(n-2)K\Big\}\frac{1}{2}(\phi^2)'$$

$$+(\phi^2)'''+(n-2)\frac{a'}{a}(\phi^2)''\Big]\mathcal{B}+\frac{1}{k}(\phi^2)'\mathcal{B}''+\frac{1}{k}\left[2(\phi^2)''+(n-2)\frac{a'}{a}(\phi^2)'\right]\mathcal{B}'$$

$$+(\phi^2)'\mathcal{A}'+2\left[(\phi^2)''+(n-2)\frac{a'}{a}(\phi^2)'\right]\mathcal{A}$$

$$-\left(\frac{a'}{a}\right)^{-1}\Big[\Big\{(2n-3)\left(\frac{a'}{a}\right)'+\frac{1}{2}(n-2)(n-5)\left(\frac{a'}{a}\right)^2$$

$$+\frac{1}{2}(n-1)(n-2)\Big\}(\phi^2)'+n(n-1)c_s{}^2\Big\{\left(\frac{a'}{a}\right)^2-\left(\frac{a'}{a}\right)'+K\Big\}\frac{a'}{a}\phi^2$$

$$+(n-4)\left(\frac{a'}{a}\right)^2(\phi^2)'+(\phi^2)'''\Big]\mathcal{R}\,, \qquad (\mathrm{F}\cdot 6\mathrm{c})$$

$$a^{-2}k^2\mathrm{Y}^i{}_j\ \text{part}=-[\mathcal{A}+k^{-1}a^{1-n}(a^{n-1}\mathcal{B})']\phi^2-2\phi\cdot X\,. \qquad (\mathrm{F}\cdot 6\mathrm{d})$$

Finally we write down the perturbation formulas used to derive the perturbed field equation (VI-1·36):

$$\delta(\tilde{\Box}\tilde{\phi})=a^{-2}\Big[-X''-(n-1)\frac{a'}{a}X'-k^2X$$

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$$+2\left\{\phi''+(n-1)\frac{a'}{a}\phi'\right\}\mathcal{A}+\phi'\mathcal{A}'-n\phi'\mathcal{R}'$$

$$-\frac{1}{k}\left\{\phi'''+(n-1)\frac{a'}{a}\phi''\right\}\sigma_g-\frac{1}{k}\left\{2\phi''+(n-1)\frac{a'}{a}\phi'\right\}\sigma_g'-\frac{1}{k}\phi'\sigma_g''\Bigg]\mathrm{Y}\,, \qquad (\mathrm{F}\cdot 7)$$

$$\delta(\tilde{R}\tilde{\phi})=-2na^{-2}\left[\left\{2\left(\frac{a'}{a}\right)'+(n-1)\left(\frac{a'}{a}\right)^2\right\}\mathcal{A}+\frac{a'}{a}\mathcal{A}'-\frac{k^2}{n}\Psi\right.$$

$$\left.-\frac{n-1}{n}k\frac{a'}{a}\mathcal{B}+(1-nc_s{}^2)\frac{1}{k}\frac{a'}{a}\left\{\left(\frac{a'}{a}\right)^2-\left(\frac{a'}{a}\right)'+K\right\}\mathcal{B}\right]\phi\mathrm{Y}$$

$$+na^{-2}\left[2\left(\frac{a'}{a}\right)'+(n-1)\left(\frac{a'}{a}\right)^2+(n-1)K\right]X\mathrm{Y}+(R\phi)'\frac{1}{k}\sigma_g\mathrm{Y}\,. \qquad (\mathrm{F}\cdot 8)$$

Appendix G

List of Symbols

Background quantities

⟨Metric⟩

a (cosmic scale factor)	Eq. (II-0·1),	p. 8
γ_{ij} (metric of an invariant n-space)	Eq. (II-0·2),	p. 8
K (curvature of the invariant n-space)	Eq. (II-0·3),	p. 8

⟨Total matter⟩

ρ (proper energy density)	Eq. (II-0·5),	p. 9
p (pressure)	Eq. (II-0·5),	p. 9
h ($\equiv\rho+p$)	Eq. (II-0·7),	p. 9
w ($\equiv p/\rho$)	Eq. (II-3·32),	p. 18
c_s (sound velocity)	Eq. (II-3·34),	p. 18
u^μ (($n+1$)-velocity)	Eq. (II-0·5),	p. 9

⟨α-component⟩

ρ_α (proper energy density)	Eq. (II-5·4),	p. 28
p_α (pressure)	Eq. (II-5·4),	p. 28
h_α ($\equiv\rho_\alpha+p_\alpha$)	Eq. (II-5·8),	p. 28
w_α ($\equiv p_\alpha/\rho_\alpha$)	Eq. (II-5·25),	p. 30
c_α (sound velocity)	Eq. (II-5·26),	p. 30
$Q_{(\alpha)\mu}$ (energy source)	Eq. (II-5·1),	p. 27
Q_α (energy transfer rate)	Eq. (II-5·6),	p. 28
q_α	Eq. (II-5·9),	p. 28
$u_{(\alpha)}{}^\mu$ (($n+1$)-velocity)	Eq. (II-5·4),	p. 28

Gauge-invariant variables

⟨Metric⟩

Scalar perturbation

$\mathcal{A}$	Eq. (II-3·4),	p. 15

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