Cost Allocation for Steiner Trees

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ABSTRACT

A set of points, called consumers, and another point called central supplier, are located in a Euclidean plane. The cost of constructing a connection between two points is proportional to the distance between them. The minimum cost required for connecting all the consumers to the supplier is given by a minimal Steiner tree. An example is given in which for every allocation of the total cost of the tree to the consumers, a coalition of consumers exists, which is charged more than the cost required for connecting its members to the central supplier.

INTRODUCTION

The following question was raised by Claus and Kleitman [2]. A network G is given, whose set of nodes is $N \cup \{0\}$, where $N = \{1,2,\ldots,n\}$ corresponds to a set of consumers and 0 corresponds to a central supplier. The length $d(i,j)$ of an arc (i,j) of G denotes the cost of connecting i to j. The minimum cost required to connect all the consumers to the central supplier (using arcs of the network) is the length of a shortest spanning tree of G. The question is how to allocate the total cost of a shortest spanning tree T to the consumers. Several suggestions are given in [2]. Claus and Granot [3] suggest a game-theoretic approach to the problem. For every $S \subset N$ let T_S be a shortest tree of G whose set of nodes is $S \cup \{0\}$, and let $v(S)$ denote the total cost of T_S. Thus, a cooperative game $(N;v)$ in characteristic function form (see [7]) is associated with the problem, and solution concepts known in game theory can be employed. Granot and Huberman [5] prove that the core of this game is never empty. In fact, the following cost allocation belongs to the core. For every consumer i let $j(i)$ denote the node which follows i on the path

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in T which leads from i to 0. Let $x_i = d(i,j(i))$. Then the allocation $x = (x_1,\ldots,x_n)$ belongs to the core, i.e., $\sum_{i\in N} x_i = v(N)$ and $\sum_{i\in S} x_i \leq v(S)$ for all $S \subset N$.

It is interesting to point here that Granot and Huberman's proof can be simplified and their result can be strengthened in the following way. Suppose that a coalition S of consumers is allowed to use not only arcs that connect two members of S, or a member of S and 0, but also every other arc of the network G. Accordingly, let T_S denote a shortest tree of G which spans the set $S \cup \{0\}$ (i.e. T_S is a shortest Steiner tree of G w.r.t. $S \cup \{0\}$) and let u(S) denote the total cost of T_S. Obviously, $u(N) = v(N)$ and $u(S) \leq v(S)$ for every $S \subset N$. Thus, the core of (N;u) is contained in the core of (N;v). We claim that the cost allocation x belongs to the core of (N;u). This follows from the fact that $u(S) = \min\{v(R)\colon S \subset R \subset N\}$ and the fact that x belongs to the core of (N;v), but a direct proof is as follows. Suppose, per absurdum, that $S \subset N$ is a coalition such that $\sum_{i\in S} x_i > u(S)$. Consider the subgraph G_S of G such that (i,j) is an arc of G_S if and only if either (i,j) belongs to T_S or (i,j) belongs to T and $i \notin S$. It is easily verified that G_S spans $N \cup \{0\}$ and the total cost of G_S is less than that of T. This implies that T is not a shortest spanning tree of G, and hence a contradiction.

In this paper we deal with a more general setup of the problem. We assume that $N \cup \{0\}$ is just a *subset* of the set of nodes of G, i.e., the consumers are not limited to use only arcs linking two consumers or arcs linking a consumer to the supplier, but may use some additional arcs. It is well-known (see [1,p.143]) that this formulation enables us to deal with the case in which the consumers, located in a Euclidean plane, can use any path for connecting themselves to each other or to the central supplier. The minimum cost required for connecting all the consumers to the supplier is the length of a shortest Euclidean Steiner tree for the set $N \cup \{0\}$ (see [1]). The value u(S) could be defined as the length of a shortest Euclidean Steiner tree for the set $S \cup \{0\}$.

In view of Granot and Huberman's result, and the stronger version stated above, we find it interesting to report here that, in contrast to the minimum spanning tree game, the core of the Steiner tree game *can* be empty. Consider, for demonstration, an application to cable TV network, mentioned by Claus and Kleitman, where the users wish to be connected to the station in the cheapest way. Thus, they will construct a shortest Steiner tree. Unfortunately, as is shown in the example below, it may happen that they will not be able to split the cost in such a way that no coalition S of users is charged more than the minimum cost required for connecting all members of S to the station.

EXAMPLE

Consider a case of five consumers located symmetrically in the Euclidean plane around the central supplier. We shall prove (see "The Proof" below) that a shortest Steiner tree, which connects the five consumers to the supplier, is composed of two components, namely, a shortest Steiner tree linking two adjacent consumers to the supplier, and a shortest Steiner tree linking the other three consumers to the supplier. In other words, $u(\{1,2,3,4,5\}) = u(\{1,2\}) + u(\{3,4,5\})$. Since the core is a convex set, it follows by the symmetry (w.r.t. permutations of the consumers set) that if the core is not empty then the equal cost allocation must belong to the core. However, computation shows that if the distance between adjacent consumers is 1.0, then $u(\{1,2\}) = 1.5542$ and $u(\{3,4,5\}) = 2.3928$. It follows that in the equal cost allocation the coalition $\{1,2\}$ is charged more than $u(\{1,2\})$. This implies that the equal cost allocation is not in the core and hence the core of this game is empty. The shortest tree is shown in Figure 1.

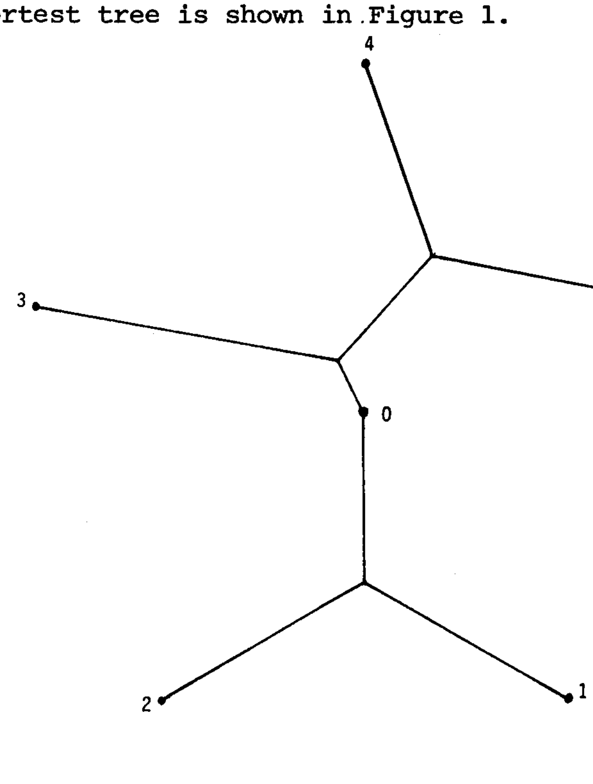

Fig. 1

THE PROOF

We shall prove that the tree shown in Figure 1 is the shortest Steiner tree that connects the vertices of the pentagon to its center.

Suppose that T is a shortest Steiner tree for the set {0,1,...,5}. It is well-known (see [4] for proofs of all properties of minimal Steiner trees mentioned in the present proof) that the degree of any node in T is not greater than 3. Moreover, since the angle between any two arcs incident at a node of T is not less than 120^{0}, it follows that the degree of every consumer i ($1 \leq i \leq 5$) in T is exactly 1. We shall prove that the degree of 0 in T is exactly 2.

First, suppose that the degree of 0 in T is 3. This implies that T (or an equivalent tree) is composed of three shortest Steiner trees with respect to the sets {0,1,2}, {0,3,4}, {0,5}. However, in that case the angle between the link from 0 to 5 and the link from 0 that belongs to the tree of {0,1,2} is 108^{0}, in contradiction to the property mentioned before.

Next, suppose that the degree of 0 in T is 1. In this case T has exactly ten nodes; this follows from the fact that the degree of every node of T which is not in {0,1,...,5} is 3, and if there are x such nodes then $(3x+6)/2 = (x+6) - 1$. It is easy to verify that T must be isomorphic to either of the trees shown in Figures 2 and 3. All the angles in these figures are of 120^{0}. We shall now reason why the shortest Steiner tree can be neither of the type of Figure 2 nor of the type of Figure 3. Notice that the center 0 in our example belongs to the convex hull of every set of four consumers. This implies, as can be easily seen, that neither of the points A,B,E,F,G,H,I,J,K,L can play the role of 0; for each one of these points there exists a set of four others in the same tree whose convex hull does not contain the point. It is only left to show that C cannot play the role of 0 (the case of D is symmetric). The straight line through s_1 and s_2 passes through the point which lies outside the pentagon of consumers and forms together with A and B an equilateral triangle (see [6]). Moreover, this line separates the center from the consumer corresponding to D. Similarly, the straight line through s_3 and s_4 passes through the point which lies outside the pentagon and forms together with E and F an equilateral triangle, and this line also separates the center from the consumer corresponding to D. However, since these two lines must be parallel, the structure as a whole is not feasible within the pentagon of consumers.

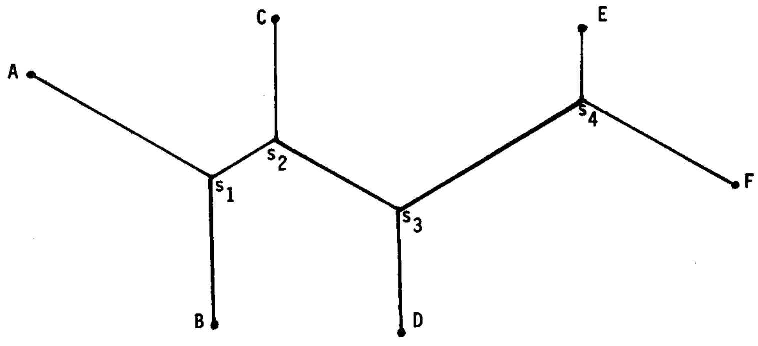

Fig. 2

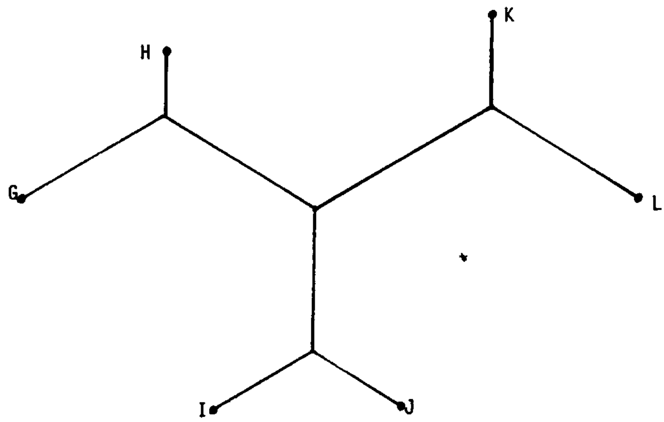

Fig. 3

Thus, the degree of 0 in T is 2. Obviously, a partition of {1,...,5} which yields a shortest tree is to the sets {1,2} and {3,4,5}. The shortest trees for the sets {0,1,2} and {0,3,4,5} can be easily found by the method of Melzak [6].

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NOTES ADDED IN PRESS

The idea of applying game theory to cost allocation for spanning trees, as well as the result of [5], appear in fact in C. G. Bird's "On Cost Allocation for a Spanning Tree: A Game Theoretic Approach," *Networks*, 6, 1976, pp. 335-350.

Another related paper, "Computational Complexity of the Game Theory Approach to Cost Allocation for a Tree," by the present author, is forthcoming in *Mathematics of Operations Research*.