Cotorsion pairs generated by modules of bounded projective dimension

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Abstract

We apply the theory of cotorsion pairs to study closure properties of classes of modules with finite projective dimension with respect to direct limit operations and to filtrations.

We also prove that if the ring is an order in an $\aleph_0$-noetherian ring Q of little finitistic dimension 0, then the cotorsion pair generated by the modules of projective dimension at most one is of finite type if and only if Q has big finitistic dimension 0. This applies, for example, to semiprime Goldie rings and to Cohen Macaulay noetherian commutative rings.

Our results allow us to give a positive answer to an open problem on the structure of divisible modules of projective dimension one over commutative domains posed in [23, Problem 6, p. 139]. We also give some insight on the structure of modules of finite weak dimension, giving a counterexample to [25, Open Problem 3, p. 187].

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[*] Supported by grant SAB2005-0139 of the Secretaría de Estado de Univesidades e Investigación del Ministerio de Educación y Ciencia de España. Partially supported by Università di Padova (Progetto di Ateneo CDPA048343 “Decomposition and tilting theory in modules, derived and cluster categories”).

[†] Partially supported by MEC-DGESIC (Spain) through Project MTM2005-00934, and by the Comissionat Per Universitats i Recerca de la Generalitat de Catalunya through Project 2005SGR00206.
While this paper was written, both authors were working within the Research Programme on Discrete and Continuous methods of Ring Theory at the CRM, Barcelona (Spain). They thank their host for its hospitality.
2000 Mathematics Subject Classification. Primary: 16D90; 16E30; Secondary: 16G99.

1 Introduction

In this paper we apply the theory of cotorsion pairs to study classes of modules with finite projective dimension. The first insight in this direction was made in [7] (see also [25, Chapter 7]). Our approach takes advantage, and it is based on, the recent developments in the area that had led, for example, to show that all tilting modules are of finite type [13], [14], [40], [15] and to solve the Baer splitting problem raised by Kaplansky in 1962 [2].

For a ring R, let $\mathcal{P}_n$ be the class of right R-modules of projective dimension at most n. Denote by mod-R the resolving class of right R-modules having a projective resolution consisting of finitely generated projective modules. We set mod-$R \cap \mathcal{P}_n := \mathcal{P}_n(\text{mod-}R)$.

A possible approach to understand the structure of the modules in $\mathcal{P}_n$ in terms of modules in $\mathcal{P}_n(\text{mod-}R)$, is to determine whether they belong to the direct limit closure of $\mathcal{P}_n(\text{mod-}R)$. However, as direct limits do not commute with the Ext functor, it is also convenient to turn the attention towards the smaller class of modules filtered by modules in $\mathcal{P}_n(\text{mod-}R)$ or, even better, towards direct summands of such modules. (See Fact 2.2.)

¿From the general theory of cotorsion pairs it follows that the modules in $\mathcal{P}_n$ are direct summands of $\mathcal{P}_n(\text{mod-}R)$-filtered modules if and only if the cotorsion pair $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is of finite type, that is, if and only if $\mathcal{P}_n^{\perp} = \mathcal{P}_n(\text{mod-}R)^{\perp}$. ($^{\perp}$ denotes the Ext-orthogonal, see § 2 for unexplained terms and notation).

We summarize these results, as well as the relation between filtrations and direct limits in Proposition 4.1. We give a new insight on this interaction in Theorem 4.6, where we show that if $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is of finite type, then any module in $\mathcal{P}_{n+1}$ is a direct limit of modules in $\mathcal{P}_{n+1}(\text{mod-}R)$. On the other hand, we also prove that the finite type of the cotorsion pair $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ for some $n \geq 1$ implies strong coherency/noetherianity conditions on the class $\mathcal{P}_n$, see Corollary 3.9.

The basic idea to show the finite type of the cotorsion pair $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$, is patterned in the method used to prove that tilting classes are of finite type. This means to follow a two-step procedure: First to show that $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is a cotorsion pair of countable type, and then conclude the finite type by proving that the Ext-orthogonal of the countably presented modules coincides with the Ext-orthogonal of the finitely presented ones.

After Raynaud a Gruson [37], it is well known that over an $\aleph_0$-noetherian ring any module of projective dimension at most n is filtered by countably generated (presented) modules of projective dimension at most n. Specializing to the case of projective dimension at most one, we observe in Proposition 5.5 that for right orders in $\aleph_0$-noetherian rings $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of countable type.

For rings with a two-sided (Ore) classical ring of quotients Q we look for descent type results. We consider the problem of getting information on $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ assuming that the right Q-modules of finite projective dimension are exactly the projective Q-modules.

We recall that, for a general ring R, the big finitistic dimension, F.dim R is the supremum of the projective dimension of right R-modules of finite projective dimension. The right little finitistic dimension, f.dim R is the supremum of the projective dimension of modules in mod-R with finite projective dimension. This is slightly different from other definitions of little finitistic dimension in which all finitely generated modules of finite projective dimension are considered. Our results indicate that the extra restriction in the definition is adequated.

In Theorem 6.7 we characterize rings with classical ring of quotients of little finitistic dimension zero. We show, for example, that if R has a two-sided classical ring of quotients Q then f.dim $Q = 0$ if and only if $\mathcal{P}_1(\text{mod-}R)^\perp$ coincides with the Ext-orthogonal of the set of modules $\{R/rR \mid \text{ for } r \text{ a nonzero divisor of } R\}$; this is to say that $\mathcal{P}_1(\text{mod-}R)^\perp = \mathcal{D}$ the class of divisible modules. Therefore, in this case, if $(\mathcal{P}_1, \mathcal{P}_1^\perp)$ is of finite type, then the modules in $\mathcal{P}_1$ are direct summands of modules filtered by cyclically presented modules of projective dimension at most one.

To work with a countably presented module $M \in \mathcal{P}_1$ we use the relative Mittag-Leffler conditions, that first appeared in [14] and were further developed in [4], as an effective way to characterize vanishing conditions of the functor Ext.

In Theorem 7.2 we patch together the results for countably presented modules with the ones giving the countable type proving that if the ring is an order in an $\aleph_0$-noetherian ring Q of little finitistic dimension 0, then the cotorsion pair generated by the modules of projective dimension at most one is of finite type if and only if Q has big finitistic dimension 0. As a consequence of our work we find, for example, that $(\mathcal{P}_1, \mathcal{D})$ is a cotorsion pair of finite type for orders in semisimple artinian rings (Corollary 8.1) so, in particular, for commutative domains (Corollary 8.2); our results answer in the affirmative an open problem posed by L. Fuchs and L. Salce [23, Problem 6, p. 139] on the structure of one dimensional divisible modules over domains. We also characterize the commutative noetherian rings for which $(\mathcal{P}_1, \mathcal{P}_1^\perp)$ is of finite type as the ones that are orders into artinian rings.

We remark that this kind of results had been only considered in the commutative domain setting. The cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^\perp)$ was known to be of finite type only in these two cases: the class of Prüfer domains and the class of Matlis domains. For the first class the key result [24, VI Theorem 6.5] is that a module of projective dimension at most one over a Prüfer domain is filtered by cyclic finitely presented modules (which are all of projective dimension at most one). For the second class, recall that a domain R is a Matlis domain provided that the quotient field Q of R has projective dimension one. If this is the case, then Matlis proved that the class of divisible module coincides with the class of epimorphic images of injective modules (see [34]). From this fact it easily follows that ${}^\perp\mathcal{D} = \mathcal{P}_1$ (see also [33]).

It is well known that any flat module is a direct limit of finitely generated free modules. If R is a commutative domain, then the class of modules of weak dimension at most one $\mathcal{F}_1$ coincides with the class $\varinjlim \mathcal{P}_1$ of modules that are direct limits of modules of projective dimension at most one [8]. In Theorem 6.7, we extend this result giving a general formula relating $\mathcal{F}_1$ and $\varinjlim \mathcal{P}_1$ (see also Corollary 6.8).

The paper is structured as follows: in Section 2 we introduce notations and some basic facts about cotorsion pairs. The notions concerning relative Mittag-Leffler modules are given in Section 3, where we also prove the results about these modules which will be needed in the sequel. We specialize to modules of bounded projective dimension in Section 4, and we examine the question of the countable type in Section 5.

In Sections 6 and 7, we assume that R has a classical ring of quotients with finitistic dimension 0 and we investigate the consequences on the class $\mathcal{P}_1$, proving Theorems 6.7 and

7.2 which are the main results of this part of the paper. We devote Section 8 to expose some applications of our work, and we finish in Section 9 with a discussion of examples and counterexamples that limit the scope for possible generalizations. In particular, we exhibit examples showing that $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ of finite type does not imply the finite type of $(\mathcal{P}_{n-1}, \mathcal{P}_{n-1}^{\perp})$ (Example 9.12 and Proposition 9.13). We also show in Example 9.2 (ii) that there are commutative (noetherian) domains with modules of weak dimension 2 that are not direct limits of modules of finite projective dimension at most 2. This gives a negative answer to [25, Open Problem 3, p. 187].

Acknowledgments. We kindly thank Santiago Zarzuela for providing us with Examples 9.2 (i) and Birge Huisgen-Zimmermann for suggesting the use of Example 9.12.

We are also grateful to the referee for the careful reading of the paper and her/his useful suggestions that lead to us, for example, in a straightened version of Theorem 6.7 and to the results concerning the class $\mathcal{F}_1$.

2 Preliminaries and notations

Let R be an associative ring with unit. For any infinite cardinal μ, mod_{μ}-R (R-mod_{μ}) and $\mathrm{mod}_{<\mu}$-R (R-$\mathrm{mod}_{<\mu}$) will be the classes of right (left) R-modules with a projective resolution consisting of $\leq \mu$-generated or $< \mu$-generated projective modules, respectively. We will simply write mod-R (R-mod) for $\mathrm{mod}_{<\aleph_0}$-R (R-$\mathrm{mod}_{<\aleph_0}$).

For any class $\mathcal{C}$ of right R-modules, $\mathcal{C}(\text{mod-}R)$ and $\mathcal{C}(\mathrm{mod}_{\aleph_0}\text{-}R)$ will denote the classes $\mathcal{C} \cap \text{mod-}R$ and $\mathcal{C} \cap \mathrm{mod}_{\aleph_0}\text{-}R$, respectively.

An ascending chain $(M_\alpha \mid \alpha < \mu)$ of submodules of a module M indexed by a cardinal μ is called *continuous* if $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$ for all limit ordinals $\alpha < \mu$. It is called a *filtration* of M if it is continuous, $M_0 = 0$ and $M = \bigcup_{\alpha<\mu} M_\alpha$.

Let $\mathcal{C}$ denote a class of modules. A module M is *$\mathcal{C}$-filtered* if it admits a filtration $(M_\alpha \mid \alpha < \mu)$ such that $M_{\alpha+1}/M_\alpha$ is isomorphic to some module in $\mathcal{C}$ for every $\alpha + 1 < \mu$. In this case, $(M_\alpha \mid \alpha < \mu)$ is a *$\mathcal{C}$-filtration* of M.

We denote by $\varinjlim \mathcal{C}$ the closure of $\mathcal{C}$ by direct limits.

For every class $\mathcal{C}$ of right R-modules set

$$\mathcal{C}^{\perp} = \{X \in \text{Mod-}R \mid \mathrm{Ext}^i_R(C, X) = 0 \text{ for all } C \in \mathcal{C} \text{ for all } i \geq 1\}$$

$$^{\perp}\mathcal{C} = \{X \in \text{Mod-}R \mid \mathrm{Ext}^i_R(X, C) = 0 \text{ for all } C \in \mathcal{C} \text{ for all } i \geq 1\}$$

$$\mathcal{C}^{\perp_1} = \{X \in \text{Mod-}R \mid \mathrm{Ext}^1_R(C, X) = 0 \text{ for all } C \in \mathcal{C}\}$$

$$^{\perp_1}\mathcal{C} = \{X \in \text{Mod-}R \mid \mathrm{Ext}^1_R(X, C) = 0 \text{ for all } C \in \mathcal{C}\}$$

A pair of classes of modules $(\mathcal{A}, \mathcal{B})$ is a *cotorsion pair* provided that $\mathcal{A} = {}^{\perp_1}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp_1}$. Note that for every class $\mathcal{C}$, ${}^{\perp}\mathcal{C}$ is a *resolving* class, that is, it is closed under extensions, kernels of epimorphisms and contains the projective modules. In particular, it is syzygy-closed. Dually, $\mathcal{C}^{\perp}$ is *coresolving*: it is closed under extensions, cokernels of monomorphisms and contains the injective modules. In particular, it is cosyzygy-closed. A pair $(\mathcal{A}, \mathcal{B})$ is called a *hereditary cotorsion pair* if $\mathcal{A} = {}^{\perp}\mathcal{B}$ and $\mathcal{B} = \mathcal{A}^{\perp}$. It is easy to see that $(\mathcal{A}, \mathcal{B})$ is a hereditary cotorsion pair if and only if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair such that $\mathcal{A}$ is resolving, if and only if $(\mathcal{A}, \mathcal{B})$ is a cotorsion pair such that $\mathcal{B}$ is coresolving.

A cotorsion pair $(\mathcal{A}, \mathcal{B})$ is *complete* provided that every right R-module M admits a *special $\mathcal{A}$-precover*, that is, there exists an exact sequence of the form $0 \to B \to A \to M \to 0$ with $B \in \mathcal{B}$ and $A \in \mathcal{A}$. For a class $\mathcal{C}$ of right modules, the pair $({}^{\perp}(\mathcal{C}^{\perp}), \mathcal{C}^{\perp})$ is a (hereditary) cotorsion pair; it is called the cotorsion pair *generated* by $\mathcal{C}$. Clearly, ${}^{\perp}(\mathcal{C}^{\perp}) = {}^{\perp_1}(\mathcal{C}^{\perp_1})$ provided that a first syzygy of M is contained in $\mathcal{C}$ whenever $M \in \mathcal{C}$.

Every cotorsion pair generated by a set of modules is complete, [19]. If all the modules in $\mathcal{C}$ have projective dimension $\leq n$, then ${}^{\perp}(\mathcal{C}^{\perp}) \subseteq \mathcal{P}_n$ as well.

In computing Ext-orthogonal classes of $\mathcal{C}$-filtered modules the following, known as Eklof's Lemma, is essential.

Fact 2.1 [18, XII.1.5] *Let R be a ring and let M, N be right R-modules. Assume that M has a filtration $(M_\alpha \mid \alpha < \mu)$ such that $Ext^1_R(M_{\alpha+1}/M_\alpha, N) = 0$ for all $\alpha + 1 < \mu$. Then $Ext^1_R(M, N) = 0$.*

We recall also the following useful description of the modules in the first component of a cotorsion pair

Fact 2.2 [42, Theorem 2.2] *Let $\mathcal{C}$ be a set of right R-modules. A right R-module M belongs to ${}^{\perp}(\mathcal{C}^{\perp})$ if and only if it is a direct summand of a $\mathcal{C}'$-filtered module where $\mathcal{C}' = \mathcal{C} \cup \{R\}$.*

A hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ in Mod-R is of *countable type* (*finite type*) provided that there is a class $\mathcal{S}$ of modules in $\mathrm{mod}_{\aleph_0}$-R (mod-R) such that $\mathcal{S}$ generates $(\mathcal{A}, \mathcal{B})$, that is, $\mathcal{S}^{\perp} = \mathcal{B}$ (hence $\mathcal{A} = {}^{\perp}(\mathcal{S}^{\perp})$).

We denote by $\mathcal{P}$ the class of right R-modules of finite projective dimension, and for every $n \geq 0$, we denote by $\mathcal{P}_n$ the class of right R-modules of projective dimension at most n. In case we need to stress the ring R we shall write $\mathcal{P}(R)$ and $\mathcal{P}_n(R)$, respectively.

In [1] it is shown that, for every $n \in \mathbb{N}$, $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is a hereditary cotorsion pair; moreover it is complete, since it is generated by a set of representatives of the modules in the class $\mathcal{P}_n(\mathrm{mod}_\mu\text{-}R)$ where $\mu = \max\{\mathrm{card}\, R, \aleph_0\}$.

For every $n \in \mathbb{N}$, consider also the cotorsion pair generated by the class $\mathcal{P}_n(\text{mod-}R)$; that is, the cotorsion pair $({}^{\perp}(\mathcal{P}_n(\text{mod-}R)^{\perp}), \mathcal{P}_n(\text{mod-}R)^{\perp})$. By definition, it is of finite type, and it is also hereditary because the class $\mathcal{P}_n(\text{mod-}R)$ is resolving. Clearly, the class ${}^{\perp}(\mathcal{P}_n(\text{mod-}R)^{\perp})$ is contained in $\mathcal{P}_n$.

We are interested in cotorsion pairs generated by subclasses of $\mathcal{P}_n$ and in n-tilting cotorsion pairs. Recall that for $n < \omega$, a module T is *n-tilting* provided

(T1) $T \in \mathcal{P}_n$,

(T2) $\mathrm{Ext}^i_R(T, T^{(I)}) = 0$ for each $i \geq 1$ and any set I, and

(T3) there exist $r \geq 0$ and a long exact sequence

$$0 \to R \to T_0 \to \cdots \to T_r \to 0$$

such that $T_i \in \mathrm{Add}(T)$ for each $0 \leq i \leq r$.

Here, $\mathrm{Add}(T)$ denotes the class of all direct summands of arbitrary direct sums of copies of T.

An n-tilting cotorsion pair is the hereditary cotorsion pair $({}^{\perp}(T^{\perp}), T^{\perp})$ generated by an n-tilting module T. The class $T^{\perp}$ is then called n-tilting class. If $\mathcal{S}$ is a subclass of $\mathcal{P}_n(\text{mod-}R)$ then the hereditary cotorsion pair generated by $\mathcal{S}$, that is $({}^{\perp}(\mathcal{S}^{\perp}), \mathcal{S}^{\perp})$, is an n-tilting cotorsion pair. By results in [13], [14] , [40] and [15] all n-tilting cotorsion pairs can be generated in this way, namely, they are of finite type. Consequently, the class $(\mathcal{P}_n(\text{mod-}R))^{\perp}$ is the smallest n-tilting class.

We will consider also Tor orthogonal classes. For every class $\mathcal{C}$ of right R modules we set

$$\mathcal{C}^{\intercal} = \{X \in R\text{-Mod} \mid \mathrm{Tor}_1^R(C, X) = 0 \text{ for all } C \in \mathcal{C}\}$$

Analogously, we define ${}^{\intercal}\mathcal{C}'$ and for $\mathcal{C}'$ a class of left R-modules.

Lemma 2.3 *Let R be a ring and let $\mathcal{C}_1$, $\mathcal{C}_2$ be classes of right R-modules.*

(i) If $\mathcal{C}_1^{\perp_1} \subseteq \mathcal{C}_2^{\perp_1}$, then $\mathcal{C}_1^{\intercal} \subseteq \mathcal{C}_2^{\intercal}$. Moreover, $\mathcal{C}_1^{\perp_1} = \mathcal{C}_2^{\perp_1}$ implies $\mathcal{C}_1^{\intercal} = \mathcal{C}_2^{\intercal}$.

(ii) Assume that $\mathcal{C}_i$ are subclasses of mod-R*, for $i = 1, 2$. Then $\mathcal{C}_1^{\intercal} \subseteq \mathcal{C}_2^{\intercal}$ if and only if $\mathcal{C}_1^{\perp_1} \subseteq \mathcal{C}_2^{\perp_1}$. Hence $\mathcal{C}_1^{\intercal} = \mathcal{C}_2^{\intercal}$ if and only if $\mathcal{C}_1^{\perp_1} = \mathcal{C}_2^{\perp_1}$.*

Proof. (i). Let B be a left module, and denote by B^* its character module. Let C be a right R-module. The well-known isomorphism

$$\mathrm{Ext}_R^1(C, B^*) \cong \mathrm{Tor}_1^R(C, B)^*$$

yields that $B \in C^{\intercal}$ if and only if $B^* \in C^{\perp_1}$.

Now $\mathcal{C}_1^{\perp_1} \subseteq \mathcal{C}_2^{\perp_1}$ implies, in particular, that all character modules contained in $\mathcal{C}_1^{\perp_1}$ are also in $\mathcal{C}_2^{\perp_1}$. By the remark above we deduce $\mathcal{C}_1^{\intercal} \subseteq \mathcal{C}_2^{\intercal}$.

(ii). Assume $\mathcal{C}_1^{\intercal} \subseteq \mathcal{C}_2^{\intercal}$.

Observe first that, for $i = 1, 2$, $\mathcal{C}_i^{\perp_1} = \mathcal{D}_i^{\perp_1}$ where $\mathcal{D}_i = {}^{\perp_1}(\mathcal{C}_i^{\perp_1}) \cap \text{mod-}R$. By (i), also $\mathcal{D}_1^{\intercal} \subseteq \mathcal{D}_2^{\intercal}$. This allows us to assume, without loss of generality, that, for $i = 1, 2$ the classes $\mathcal{C}_i$ are closed under isomorphism, extensions and that they contain R.

On the other hand, for $i = 1, 2$, $\mathcal{C}_i \subseteq \text{mod-}R$ implies that $\mathcal{C}_i^{\perp_1}$ is a definable class; hence, to prove $\mathcal{C}_1^{\perp_1} \subseteq \mathcal{C}_2^{\perp_1}$ it is enough to show that the pure injective modules in $\mathcal{C}_1^{\perp_1}$ are also in $\mathcal{C}_2^{\perp_1}$ (cf. [25, Lemma 3.1.10 and Example 3.1.11]).

By [8, Theorem 2.3] or by [25, Theorem 4.5.6], $\varinjlim \mathcal{C}_1 = {}^{\intercal}(\mathcal{C}_1^{\intercal})$ and we deduce that any module $C \in \mathcal{C}_2$ is a direct limit of modules in $\mathcal{C}_1$. As for any pure injective module E, $\mathrm{Ext}_R^1(-, E)$ commutes with direct limits, if E is a pure injective module in $\mathcal{C}_1^{\perp_1}$ then $\mathrm{Ext}_R^1(C, E) = 0$ for any $C \in \mathcal{C}_2$. Hence, $E \in \mathcal{C}_2^{\perp_1}$ as we wanted to prove.

The other implication is a consequence of (i). ■

3 Relative Mittag-Leffler conditions

The definition of Mittag-Leffler inverse systems goes back to Grothendieck [26, Proposition 13.1.1]. Raynaud and Gruson in [37] realized the strong connection between this concept and the notion of Mittag-Leffler module.

We recall here a weaker notion, that is the Mittag-Leffler condition restricted to particular classes.

Definition 3.1 *Let M be a right module over a ring R, and let $\mathcal{Q}$ be a class of left R-modules. We say that M is a* $\mathcal{Q}$-Mittag-Leffler module *if the canonical map*

$$\rho\colon M\bigotimes_R\prod_{i\in I}Q_i\to\prod_{i\in I}(M\bigotimes_R Q_i)$$

is injective for any family $\{Q_i\}_{i\in I}$ of modules in $\mathcal{Q}$.

Taking $\mathcal{Q}=R$-Mod we recover Raynaud and Gruson's notion of Mittag-Leffler modules. In case $\mathcal{Q}=\{Q\}$, we will simply say that the module is Q-Mittag-Leffler.

Relative Mittag-Leffler modules can be characterized in the following way.

Theorem 3.2 ([4, Theorem 5.1]) *Let $\mathcal{Q}$ be a class of left R-modules. For a right R-module M, the following statements are equivalent:*

(1) M is $\mathcal{Q}$-Mittag-Leffler.

(2) Every direct system of finitely presented right R-modules $(F_\alpha,u_{\beta\,\alpha})_{\beta\,\alpha\in I}$ with $M=\varinjlim(F_\alpha,u_{\beta\alpha})_{\beta,\alpha\in I}$ has the property that for any $\alpha\in I$ there exists $\beta\geq\alpha$ such that, for any $Q\in\mathcal{Q}$, $\operatorname{Ker}(u_{\beta\alpha}\otimes_R Q)=\operatorname{Ker}(u_\alpha\otimes_R Q)$, where $u_\alpha\colon F_\alpha\to M$ denotes the canonical map.

(3) There exists direct system of finitely presented right R-modules $(F_\alpha,u_{\beta\,\alpha})_{\beta\,\alpha\in I}$ with $M=\varinjlim(F_\alpha,u_{\beta\alpha})_{\beta,\alpha\in I}$ satisfying that for any $\alpha\in I$ there exists $\beta\geq\alpha$ such that, for any $Q\in\mathcal{Q}$, $\operatorname{Ker}(u_{\beta\alpha}\otimes_R Q)=\operatorname{Ker}(u_\alpha\otimes_R Q)$, where $u_\alpha\colon F_\alpha\to M$ denotes the canonical map.

The relation between relative Mittag-Leffler modules and cotorsion pairs of finite type is given by the following result.

Theorem 3.3 ([4, Theorem 9.5], [14, Theorem 5.1]) *Let R be an arbitrary ring. Let $(\mathcal{A},\mathcal{B})$ be a cotorsion pair of finite type. Let $\mathcal{S}=\mathcal{A}\bigcap\text{mod-}R$ and let $\mathcal{C}=\mathcal{A}^\intercal$. Then every module in $\mathcal{A}$ is $\mathcal{C}$-Mittag-Leffler. If M is a countably presented right R-module that is a direct limit of modules in $\mathcal{S}$, then M is in $\mathcal{A}$ if and only if it is $\mathcal{C}$-Mittag-Leffler.*

We illustrate now some closure properties of $\mathcal{Q}$-Mittag-Leffler modules that will be used later on. It will be useful to keep in mind the following auxiliary Lemma.

Lemma 3.4 *Let $\mu\colon A\to B$ be a morphism of right R-modules. Let A' and B' denote submodules of A and B, respectively, such that $\mu(A')\subseteq B'$. Let $\mu'\colon A'\to B'$ be the restriction of μ. Then the kernel of the induced map $f\colon B'/\mu'(A')\to B/\mu(A)$ is $\ker f=(\mu(A)\cap B')/\mu'(A')$.*

Proposition 3.5 *Let R be a ring, and let $M_R \in \mathcal{P}_1$. Assume that M is a $\mathcal{Q}$-Mittag-Leffler module where $\mathcal{Q}$ is a class of left R-modules contained in $M^{\intercal}$. Let $\mathcal{Y}$ be a class of left R-modules consisting of submodules of modules in $\mathcal{Q}$ such that $\mathcal{Y} \subseteq M^{\intercal}$. Then M is a $\mathcal{Y}$-Mittag-Leffler module.*

Proof. Using the Eilenberg trick, if needed, we can assume that M has a free presentation

$$(1) \quad 0 \to R^{(J)} \xrightarrow{\mu} R^{(I)} \to M \to 0,$$

where I and J are sets.

Since finitely presented modules are Mittag-Leffler, we can assume that either I or J is infinite. As we are stating a property on $M^{\intercal}$ and free modules are Mittag-Leffler, we can cancel the free direct summands of M. Hence, without loss of generality, we may assume that the image of μ has non zero projection on all the direct summands of $R^{(I)}$, and therefore that J is an infinite set.

For every finite subset F of J, let μ_F be the restriction of μ to R^F and let G_F be the smallest subset of I such that $\mu_F(R^F) \leq R^{G_F}$. Let C_F be the finitely presented right R-module $R^{G_F}/\mu_F(R^F)$; then $C_F \in \mathcal{P}_1(\text{mod-}R)$. Let $\mathcal{F}$ be the family of the finite subsets of J and consider the direct system $(C_F; f_{KF})_{F\subseteq K\in\mathcal{F}}$ where the structural morphisms $f_{KF}\colon C_F \to C_K$ are induced by the injections of R^{G_F} into R^{G_K}. Then, M_R is isomorphic to the direct limit of the direct system $(C_F; f_{KF})_{F\subseteq K\in\mathcal{F}}$. Let $f_F\colon C_F \to M \cong \varinjlim_F C_F$ be the canonical morphisms.

For every $F \leq K \in \mathcal{F}$ and every left R-module N, we have a commutative diagram:

$$\begin{tikzcd}
C_F \otimes_R N \arrow[r, "f_F\otimes_R 1_N"] \arrow[d, "f_{KF}\otimes_R 1_N"'] & M \otimes_R N \; . \\
C_K \otimes_R N \arrow[ur, "f_K\otimes_R 1_N"'] &
\end{tikzcd}$$

By the definitions of the finitely presented modules C_F and of the maps f_F and f_{KF}, Lemma 3.4 allows us to conclude

$$(a) \qquad \ker(f_F \otimes_R 1_N) = \frac{\mu \otimes_R 1_N \left(N^{(J)}\right) \cap N^{G_F}}{\mu_F \otimes_R 1_N \left(N^F\right)}$$

and

$$(b) \qquad \ker(f_{KF} \otimes_R 1_N) = \frac{\mu_K \otimes_R 1_N \left(N^K\right) \cap N^{G_F}}{\mu_F \otimes_R 1_N \left(N^F\right)}$$

where, for any set L, we identify $R^{(L)} \otimes_R N$ with $N^{(L)}$.

By Theorem 3.2(2), the assumption that M is a $\mathcal{Q}$-Mittag-Leffler module implies the following

(*) for every, $F \in \mathcal{F}$ there is a subset $l(F) \in \mathcal{F}$, $l(F) \supseteq F$ such that

$$\left[\mu \otimes_R 1_Q \left(Q^{(J)}\right)\right] \cap Q^{G_F} = \left[\mu_{l(F)} \otimes_R 1_Q \left(Q^{l(F)}\right)\right] \cap Q^{G_F},$$

for every $Q \in \mathcal{Q}$.

Let now ${}_RY \in \mathcal{Y}$ be a submodule of some module $Q \in \mathcal{Q}$. We claim that

$$\left[\mu \otimes_R 1_Y \left(Y^{(J)}\right)\right] \cap Y^{G_F} = \left[\mu_{l(F)} \otimes_R 1_Y \left(Y^{l(F)}\right)\right] \cap Y^{G_F}.$$

Observe that only the inclusion $\subseteq$ of the claim needs to be proved. Consider the commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & Y^{(J)} \arrow[r, "\mu\otimes_R 1_Y"] \arrow[d, "\sigma"'] & Y^{(I)} \arrow[r] \arrow[d, "\tau"] & M \otimes_R Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Q^{(J)} \arrow[r, "\mu\otimes_R 1_Q"'] & Q^{(I)} \arrow[r] & M \otimes_R Q \arrow[r] & 0
\end{tikzcd}$$

where the rows are exact by the hypothesis that $Q, Y \in M^{\intercal}$

Condition (*) and the commutativity of the above diagram yield:

$$\tau\left(\mu \otimes_R 1_Y \left(Y^{(J)}\right) \cap Y^{G_F}\right) = \tau\left(\mu \otimes_R 1_Y \left(Y^{(J)}\right)\right) \cap \tau(Y^{G_F}) \leq \mu \otimes_R 1_Q \left(Q^{(J)}\right) \cap \; Q^{G_F} =$$

$$= \mu_{l(F)} \otimes_R 1_Q \left(Q^{l(F)}\right) \cap Q^{G_F}$$

Let $\underline{y} \in Y^{(J)}$ be such that $\mu \otimes_R 1_Y(\underline{y}) \in Y^{G_F}$. By the above inclusion,

$$\tau\left(\mu \otimes_R 1_Y(\underline{y})\right) = \mu_{l(F)} \otimes_R 1_Q(\underline{z}),$$

for some $\underline{z} \in Q^{l(F)}$ with $\mu_{l(F)} \otimes_R 1_Q(\underline{z}) \in Q^{G_F}$. Since $\mu_{l(F)} \otimes_R 1_Q$ is the restriction of $\mu \otimes_R 1_Q$, $\mu_{l(F)} \otimes_R 1_Q(\underline{z}) = \mu \otimes_R 1_Q(\underline{z})$. Thus,

$$\tau\left(\mu \otimes_R 1_Y(\underline{y})\right) = \mu \otimes_R 1_Q\left(\sigma(\underline{y})\right) = \mu \otimes_R 1_Q(\underline{z}) \in Q^{G_F}.$$

By the injectivity of $\mu \otimes_R 1_Q$ we conclude that $\sigma(\underline{y}) = \underline{z}$, hence $\underline{y} \in Y^{l(F)}$. This proves the claim.

Then, taking into account (a) and (b), we conclude that M is a $\mathcal{Y}$-Mittag-Leffler module using Theorem 3.2(3). ■

A countably generated R-Mittag-Leffler module is countably presented [4, Corollary 5.3]; since an R-Mittag-Leffler module has an abundant supply of countably generated R-Mittag-Leffler submodules [4, Theorem 5.1 (4)], we can extend the coherence conclusion of the countably generated case to any $< \mu$-generated R-Mittag-Leffler module. We start with a couple of Lemmas.

Lemma 3.6 *Let M be a right R-module and let μ be an infinite cardinal. Then M is $< \mu$-presented if and only if there exists a direct system $(C_\alpha, u_{\beta\,\alpha} \colon C_\alpha \to C_\beta)_{\alpha \leq \beta \in \Lambda}$ of finitely presented modules such that $M = \varinjlim C_\alpha$ and the cardinality of Λ is strictly smaller than μ.*

Proof. If $\mu = \aleph_0$ the claim is obvious. Assume that M is $< \mu$-presented and $\mu > \aleph_0$. Let $\{x_i\}_{i\in I}$ be a generating set of M such that $|\, I| < \mu$. Consider the exact sequence

$$0 \to L \xrightarrow{g} R^{(I)} \xrightarrow{f} M \to 0$$

where, if $\{e_i\}_{i\in I}$ denotes the canonical basis of $R^{(I)}$, $f(e_i) = x_i$ for any $i \in I$. By hypothesis we can choose a generating set $\{y_j\}_{j\in J}$ of L such that $|\, J| < \mu$.

For any finite subset F of J there exists a finite subset $I(F)$ of I such that $g(\sum_{j\in F} y_j R) \subseteq R^{I(F)}$. Setting $C_F = R^{I(F)}/\sum_{j\in F} y_j R$, we obtain a direct system of finitely presented modules with limit M indexed by the set $\mathcal{F}$ of finite subsets of I. $\mathcal{F}$ has less than μ elements.

For the converse, let $(C_\alpha, u_{\beta\,\alpha})_{\alpha\le\beta\in\Lambda}$ be a direct system of finitely presented modules such that $M = \varinjlim C_\alpha$; assume $|\, \Lambda| < \mu$. The canonical presentation of the direct limit (see [43, Proposition 2.6.8])

$$\bigoplus_{\alpha\le\beta} C_{\beta\,\alpha} \xrightarrow{\Phi} \bigoplus_{\alpha\in\Lambda} C_\alpha \to M \to 0$$

where for every $\alpha \le \beta$, $C_{\beta\,\alpha} = C_\alpha$, gives a pure exact sequence

$$0 \to \mathrm{Im}\Phi \to \bigoplus_{\alpha\in\Lambda} C_\alpha \to M \to 0.$$

Since $\oplus_{\alpha\le\beta} C_{\beta\,\alpha}$ is $< \mu$-generated, so is $\mathrm{Im}\Phi$. Moreover, since $\oplus_{\alpha\in\Lambda} C_\alpha$ is $< \mu$-presented we conclude that M is $< \mu$-presented. ■

Lemma 3.7 *Let M be a right R-module, and let $\mu > \aleph_0$ be a cardinal such that M is $< \mu$-generated. Assume that any countably generated submodule of M can be embedded in a countably presented submodule of M, then M is $< \mu$-presented.*

Proof. If M is countably generated then, by hypothesis, M is countably presented. Therefore we may assume that $\mu > \aleph_1$. For the rest of the proof, fix $X = \{x_i\}_{i\in I}$ a set of generators of M of cardinality κ such that $\aleph_0 < \kappa < \mu$.

We shall construct an upward directed set $\mathcal{F}$ of cardinality κ and a directed family $\{N_F\}_{F\in\mathcal{F}}$ of countably presented submodules of M such that $\bigcup_{F\in\mathcal{F}} N_F = M$. Once this is done, as each N_F is a countable direct limit of finitely presented modules, M is the direct limit of a direct system of finitely presented modules indexed by a set of cardinality κ. By Lemma 3.6, it will follow that M is $< \mu$-presented.

We define $\mathcal{F}$ as the direct limit of a countable direct system, with injective maps, of the form

$$\mathcal{F}_1 \xrightarrow{\varepsilon_1} \mathcal{F}_2 \xrightarrow{\varepsilon_2} \dots \mathcal{F}_n \xrightarrow{\varepsilon_n} \mathcal{F}_{n+1} \dots$$

where $\mathcal{F}_n$ has cardinality κ for any $n \in \mathbb{N}$. We shall also construct the family $\{N_F\}_{F\in\mathcal{F}}$ inductively, by constructing $\{N_F\}_{F\in\mathcal{F}_n}$ satisfying that, for any $F \in \mathcal{F}_n$, $N_F = N_{\varepsilon_n(F)}$.

Let $\mathcal{F}_1$ be the set of finite subsets of X; for each $F \in \mathcal{F}_1$ we set N_F to be a countably presented submodule of M containing $\sum_{i\in F} x_i R$. Assume that $\mathcal{F}_n$ and $\{N_F\}_{F\in\mathcal{F}_n}$ are constructed satisfying the desired conditions. Set $\mathcal{F}_{n+1}$ to be the set of finite subsets of

$\mathcal{F}_n$, and define $\varepsilon_n \colon \mathcal{F}_n \to \mathcal{F}_{n+1}$ as $\varepsilon_n(F) = \{F\}$, for any $F \in \mathcal{F}_n$. For any $F \in \mathcal{F}_n$, define $N_{\varepsilon_n(F)} = N_F$, and for $F \in \mathcal{F}_{n+1} \setminus \varepsilon_n(\mathcal{F}_n)$ define N_F to be a countably presented submodule of M containing $\sum_{F' \in F} N_{F'}$. This completes the inductive step of the construction. ■

Proposition 3.8 *Let μ be an infinite cardinal, and let M be a $< \mu$-generated R-Mittag-Leffler right R-module. Then M is $< \mu$-presented.*

Proof. Assume first that $\mu = \aleph_0$, so that M is a finitely generated module. To prove that M is finitely presented we only need to show that the natural map

$$\rho_J \colon M \otimes R^J \to (M \otimes R)^J$$

is bijective for any set J (cf.[20, Theorem 3.2.22]). Since M is finitely generated, for any set J, ρ_J is onto [20, Lemma 3.2.21] and by our assumption ρ_J is also injective, hence bijective.

Assume now $\mu > \aleph_0$. By [4, Theorem 5.1 (4)], any countably generated submodule of M is contained in a countably presented R-Mittag-Leffler module. We conclude by Lemma 3.7. ■

Corollary 3.9 *Let $(\mathcal{A}, \mathcal{B})$ be a hereditary cotorsion pair of finite type, and let μ be an infinite cardinal. If M is a right R-module in $\mathcal{A}$ that is $< \mu$-generated then $M \in \mathcal{A}(\mathrm{mod}_{<\mu}\text{-}R)$*

Proof. First observe that, since the cotorsion pair is hereditary, the class $\mathcal{A}$ is resolving; so to prove the statement it is enough to show that if $M \in \mathcal{A}$ is $< \mu$-generated then it is $< \mu$-presented.

Since $M \in \mathcal{A}$, it is $\mathcal{A}^{\intercal}$-Mittag-Leffler by Theorem 3.3. Hence M is R-Mittag-Leffler and thus the conclusion follows by Proposition 3.8. ■

By [4, Proposition 9.2] the conclusion of Corollary 3.9 holds, more generally, for hereditary cotorsion pairs $(\mathcal{A}, \mathcal{B})$ of countable type and such that the class $\mathcal{B}$ is closed by direct sums.

4 The cotorsion pair $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$

We start characterizing when this cotorsion pair is of finite type.

Proposition 4.1 *Let R be a ring. The following conditions are equivalent:*

(i) The class $\mathcal{P}_n^{\perp}$ is closed under direct sums.

(ii) $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is an n-tilting cotorsion pair.

(iii) The cotorsion pair $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is of finite type.

(iv) $\mathcal{P}_n^{\perp} = \mathcal{P}_n(\mathrm{mod}\text{-}R)^{\perp}$.

(v) Every module in $\mathcal{P}_n$ is a direct summand of a $\mathcal{P}_n(\text{mod-}R)$-filtered module.

When the above equivalent conditions hold, then $\mathcal{P}_n \subseteq \varinjlim \mathcal{P}_n(\text{mod-}R)$.

Proof. (i) $\Rightarrow$ (ii). A hereditary cotorsion pair $(\mathcal{A}, \mathcal{B})$ is an n-tilting cotorsion pair if and only if it is complete, $\mathcal{A} \subseteq \mathcal{P}_n$ and $\mathcal{B}$ is closed under direct sums (see [3], [32] or [25]). Since $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is a complete cotorsion pair, condition (i) implies (ii).

(ii) $\Rightarrow$ (iii). Any n-tilting cotorsion pair $(\mathcal{A}, \mathcal{B})$ is of finite type, by [14] and [15].

(iii) $\Leftrightarrow$ (iv). By definition, a cotorsion pair $(\mathcal{A}, \mathcal{B})$ is of finite type if and only if it is generated by (representatives of) the modules in $\mathcal{A}(\text{mod-}R)$.

(iv)$\Leftrightarrow$ (v). Is a consequence of Fact 2.2.

(iii) $\Rightarrow$ (i). This follows by the fact that for every $M \in \text{mod-}R$, the functors $\operatorname{Ext}^i_R(M, -)$ commutes with direct sums.

If the conditions hold, then the rest of the claim follows from [8, proof of Theorem 2.3]. ■

Trivially, $(\mathcal{P}_0, \mathcal{P}_0^{\perp})$ is of finite type. Note that, in this case, condition (v) in Proposition 4.1 can be stated by saying that any projective right module is a direct summand of an R-filtered (hence free) module.

It is well known that $\mathcal{P}_1 \subseteq \varinjlim \mathcal{P}_1(\text{mod-}R)$. This can be seen as a consequence of the fact that $(\mathcal{P}_0, \mathcal{P}_0^{\perp})$ is of finite type. The rest of this section will be devoted to extending this result to arbitrary projective dimension. That is, $(\mathcal{P}_{n-1}, \mathcal{P}_{n-1}^{\perp})$ of finite type implies $\mathcal{P}_n \subseteq \varinjlim \mathcal{P}_n(\text{mod-}R)$. Our arguments follow the ones in [15].

First we state a Lemma.

Lemma 4.2 *Let R be a ring. Let*

$$0 \to H \to G \to C \to 0$$

be an exact sequence of right R-modules. Let μ be an infinite cardinal. Then,

(i) if there exists $n \geq 0$ such that H and C are in $\mathcal{P}_n(\text{mod}_{<\mu}\text{-}R)$ then also $G \in \mathcal{P}_n(\text{mod}_{<\mu}\text{-}R)$.

(ii) If H and G in $\mathcal{P}_{n-1}(\text{mod}_{<\mu}\text{-}R)$, for some $n \geq 1$, then $C \in \mathcal{P}_n(\text{mod}_{<\mu}\text{-}R)$.

Proof. Statement (i) follows by inductively applying the Horseshoe Lemma.

To prove (ii) we can assume that $n > 1$. Let $0 \to G_1 \to P_0 \to G \to 0$ be an exact sequence with P_0 a $< \mu$-generated projective module and $G_1 \in \mathcal{P}_{n-2}(\text{mod}_{<\mu}\text{-}R)$. By a pull-back argument we obtain the following commutative diagram:

$$\begin{tikzcd}
 & 0 & 0 & & \\
0 \arrow[r] & H \arrow[r] \arrow[u] & G \arrow[r] \arrow[u] & C \arrow[r] & 0 \\
0 \arrow[r] & X \arrow[r] \arrow[u] & P_0 \arrow[r] \arrow[u] & C \arrow[r] \arrow[u, equal] & 0. \\
 & G_1 \arrow[r, equal] \arrow[u] & G_1 \arrow[u] & & \\
 & 0 \arrow[u] & 0 \arrow[u] & &
\end{tikzcd}$$

Applying (i) to the exact sequence $0 \to G_1 \to X \to H \to 0$ we deduce that $X \in \mathcal{P}_{n-1}(\text{mod}_{<\mu}\text{-}R)$. Hence $C \in \mathcal{P}_n(\text{mod}_{<\mu}\text{-}R)$. ■

Following the ideas in [15], we look at conditions on the syzygy module of $M \in \mathcal{P}_n$. To this aim, we state a result for $\mathcal{C}$-filtered modules, where $\mathcal{C}$ is a class of $< \mu$-presented modules for some infinite cardinal μ. The proof of this result for the case of $\mu \geq \aleph_1$ appears in [18, XII.1.14] and in [15, Proposition 3.1] for the case $\mu = \aleph_0$.

These proofs are variations of the original ideas by Eklof, Fuchs and Shelah in [17, Theorem 10]. An alternative proof appears in [41, Theorem 6] inspired by an idea by Hill [27] which was further developed by Fuchs and Lee [22].

Proposition 4.3 ([18, XII 1.14], [15, Prop. 3.1], [41, Theorem 6]) *Let μ be an infinite cardinal. Let M be a $\mathcal{C}$-filtered module where $\mathcal{C}$ is a family of $< \mu$-presented modules. Then there exists a subset $\mathcal{S}$ of $\mathcal{C}$-filtered submodules of M satisfying the following properties:*

(1) $0 \in \mathcal{S}$.

(2) $\mathcal{S}$ is closed under unions of arbitrary chains.

(3) For every $N \in \mathcal{S}$, N and M/N are $\mathcal{C}$-filtered.

(4) For every subset $X \subseteq M$ of cardinality $< \mu$, there is a $< \mu$-presented module $N \in \mathcal{S}$ such that $X \subseteq N$.

An immediate consequence of conditions (2) and (4) in Proposition 4.3 is the following.

Corollary 4.4 *Let μ be an infinite cardinal. Let M be a μ-generated $\mathcal{C}$-filtered module where $\mathcal{C}$ is a family of $< \mu$-presented modules. Then there is a filtration $(M_\alpha \mid \alpha \in \mu)$ of M consisting of $< \mu$-presented submodules of M such that M_α and M/M_α are $\mathcal{C}$-filtered for every $\alpha \in \mu$.*

The next result is a straight generalization of [15, Lemma 3.3]

Lemma 4.5 *Let μ be an infinite cardinal, and let $\mathcal{C}$ be a family of $< \mu$-presented right modules containing the regular module R. Let M be a μ-presented right module, and let*

$$0 \to K \to F \to M \to 0$$

be a free presentation of M with F and K μ-generated. Assume that K is a direct summand of a $\mathcal{C}$-filtered module. Then, there exists an exact sequence:

$$0 \to H \to G \to M \to 0$$

where H and G are μ-generated $\mathcal{C}$-filtered modules.

Proof. Let K be a summand of a $\mathcal{C}$-filtered module P. Since K is μ-generated, Proposition 4.3 implies that K is contained in a μ-generated $\mathcal{C}$-filtered submodule of P; thus we may assume that P is μ-generated. By Eilenberg's trick, $K \oplus P^{(\omega)} \cong P^{(\omega)}$. Consider the exact sequence

$$0 \to K \oplus P^{(\omega)} \to F \oplus P^{(\omega)} \to M \to 0$$

and let $H = K \oplus P^{(\omega)} \cong P^{(\omega)}$, $G = F \oplus P^{(\omega)}$. Then G and H are μ-generated $\mathcal{C}$-filtered modules. ■

Now we are ready to prove the announced result.

Theorem 4.6 *Let R be a ring, and let $n \geq 1$.*

(i) If the cotorsion pair generated by $\mathcal{P}_{n-1}(\mathrm{mod}_{\aleph_0}\text{-}R)$ is of finite type, then every module in $\mathcal{P}_n(\mathrm{mod}_{\aleph_0}\text{-}R)$ is a direct limit of modules in $\mathcal{P}_n(\mathrm{mod}\text{-}R)$.

(ii) If the cotorsion pair $(\mathcal{P}_{n-1}, \mathcal{P}_{n-1}^{\perp})$ is of finite type, then every module in $\mathcal{P}_n$ is a direct limit of modules in $\mathcal{P}_n(\mathrm{mod}\text{-}R)$.

Proof. Statements (i) and (ii) are clear for $n = 1$. Hence we may assume that $n > 1$.

(i) Let $M \in \mathcal{P}_n(\mathrm{mod}_{\aleph_0}\text{-}R)$. Then there is an exact sequence

$$0 \to K \to F_0 \to M \to 0$$

where F_0 is an $\aleph_0$-generated free module and $K \in \mathcal{P}_{n-1}(\mathrm{mod}_{\aleph_0}\text{-}R)$. By assumption K is a direct summand of a $\mathcal{P}_{n-1}(\mathrm{mod}\text{-}R)$-filtered module.

By Lemma 4.5 applied to the family $\mathcal{P}_{n-1}(\mathrm{mod}\text{-}R)$ for the case $\mu = \aleph_0$, there exists an exact sequence

$$0 \to H \to G \to M \to 0$$

where H and G are countably generated $\mathcal{P}_{n-1}(\mathrm{mod}\text{-}R)$-filtered modules. By Corollary 4.4, H and G admit filtrations $(H_i \mid i \in \mathbb{N})$ and $(G_j \mid j \in \mathbb{N})$, respectively, consisting of finitely presented $\mathcal{P}_{n-1}(\mathrm{mod}\text{-}R)$-filtered submodules. Without loss of generality we can assume that H is a submodule of G. Given $i < \omega$, there is an $j(i)$ such that $H_i \subseteq G_{j(i)}$; and we can choose the sequence $(j(i))_{i<\omega}$ to be strictly increasing. Consider the exact sequence

$$0 \to H_i \to G_{j(i)} \to C_i \to 0$$

For every $i \in \mathbb{N}$, the modules H_i and $G_{j(i)}$ are finitely presented and they belong to ${}^{\perp}(\mathcal{P}_{n-1}(\text{mod-}R)^{\perp})$, by Fact 2.2. By Corollary 3.9 they belong to $\mathcal{P}_{n-1}(\text{mod-}R)$. Thus, by Lemma 4.2, $C_i \in \mathcal{P}_n(\text{mod-}R)$. Moreover, $M \cong \varinjlim C_i$ by construction, hence (i) follows.

(ii) By way of contradiction, assume that the result is not true and let μ be the least cardinal for which there exists an R-module $M \in \mathcal{P}_n(\text{mod}_{\mu}\text{-}R)$ which is not a direct limit of modules in $\mathcal{P}_n(\text{mod-}R)$. By (i), $\mu > \aleph_0$.

There exists an exact sequence

$$0 \to K \to F_0 \to M \to 0$$

where F_0 is a μ-generated free module and $K \in \mathcal{P}_{n-1}(\text{mod}_{\mu}\text{-}R)$. By assumption K is a direct summand of a $\mathcal{P}_{n-1}(\text{mod-}R)$-filtered module.

By Lemma 4.5 applied to the family $\mathcal{P}_{n-1}(\text{mod-}R)$, there exists an exact sequence

$$0 \to H \to G \to M \to 0$$

where H and G are μ-generated $\mathcal{P}_{n-1}(\text{mod-}R)$-filtered modules. Without loss of generality we can assume that H is a submodule of G. By Corollary 4.4, H and G admit filtrations $(H_\alpha \mid \alpha < \mu)$ and $(G_\alpha \mid \alpha < \mu)$, respectively, consisting of $< \mu$-presented $\mathcal{P}_{n-1}(\text{mod-}R)$-filtered submodules. Using the system $\mathcal{S}$ from Proposition 4.3, it is not hard to see that we can modify the filtration $(G_\alpha \mid \alpha < \mu)$ so that $H_\alpha \subseteq G_\alpha$ for all $\alpha < \mu$. Consider the exact sequence

$$0 \to H_\alpha \to G_\alpha \to C_\alpha \to 0$$

Now, for every $\alpha < \mu$, the modules H_α and G_α are $< \mu$-presented and in ${}^{\perp}(\mathcal{P}_{n-1}(\text{mod-}R)^{\perp})$, by Fact 2.2. By Corollary 3.9, they belong to $\mathcal{P}_{n-1}(\text{mod}_{<\mu}\text{-}R)$. Thus, by Lemma 4.2, $C_\alpha \in \mathcal{P}_n(\text{mod}_{<\mu}\text{-}R)$. By the minimality of μ, C_α is a direct limit of objects in $\mathcal{P}_n(\text{mod-}R)$. Now, $M \cong \varinjlim_{\alpha \in \mu} C_\alpha$, by construction, hence M is a direct limit of objects in $\mathcal{P}_n(\text{mod-}R)$, too. A contradiction. ■

Remark 4.7 As $(\mathcal{P}_0, \mathcal{P}_0^{\perp})$ is always of finite type, it is easy to find examples showing that, in general, the finite type of $(\mathcal{P}_{n-1}, \mathcal{P}_{n-1}^{\perp})$ does not imply the finite type of $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$. More involved examples will be given in Examples 9.2.

Moreover, the finite type has not a descent property. In fact, we will show in Proposition 9.13, that there exist artin algebras with the property that $(\mathcal{P}_2, \mathcal{P}_2^{\perp})$ is of finite type, while $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is not.

5 Countable Type

We recall the following results on the countable type of the cotorsion pair $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$.

Fact 5.1 If R is a commutative domain then, by [24, VI 6], every module in $\mathcal{P}_1$ admits a $\mathcal{P}_1(\text{mod}_{\aleph_0}\text{-}R)$-filtration. Hence the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of countable type.

If R is a right $\aleph_0$-noetherian ring (that is all the right ideals of R are at most $\aleph_0$-generated), then Raynaud and Gruson in [37, Corollary 3.2.5] proved that the cotorsion pair $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is of countable type. This result appears also in [1] and [29, Proposition 2.1].

In the one dimensional case, these two cases can be seen in a common setting.

Definition 5.2 *Let R be a ring, and let Σ denote the multiplicative set of the non zero divisors of R. A right R-module D is said to be divisible if* $\mathrm{Ext}^1_R(R/rR, D) = 0$*, for every* $r \in \Sigma$*. A left R-module Y is said to be torsion free if* $\mathrm{Tor}^R_1(R/rR, Y) = 0$*, for every* $r \in \Sigma$*. Divisible left R-modules and torsion free right R-modules are defined in an analogous way.*

We denote by $\mathcal{D}$ the class of all divisible right R-module and by $\mathcal{TF}$ the class of all torsion free left R-modules.

Thus, a right (left) R-module D is divisible if and only if right (left) multiplication by an element of Σ is a surjective map and a left (right) R-module Y is torsion free if and only if left (right) multiplication by an element of Σ is an injective map. Moreover, if $\mathcal{C} = \{R/rR \mid r \in \Sigma\} \cup \{R\}$, then $\mathcal{D} = \mathcal{C}^{\perp}$ and $\mathcal{TF} = \mathcal{C}^{\intercal}$.

Examples of torsion free R-modules are the submodules of free R-modules.

If S is a multiplicative subset of Σ that satisfies the left Ore condition, then $S^{-1}R/R$ is a direct limit of R/sR, for $s \in S$. Dually, if S is a multiplicative subset of Σ that satisfies the right Ore condition, then RS^{-1}/R is a direct limit of R/Rs for $s \in S$. Hence we have the following well known fact.

Lemma 5.3 *Let R be a ring, and let S be a multiplicative subset of* Σ.

- *(i) If S satisfies the left Ore condition, then* $\mathrm{Tor}^R_1(S^{-1}R/R, K) = 0$*, for any torsion free left R-module K. In particular, K is embedded in $S^{-1}R \otimes_R K$ via the assignment $y \mapsto 1 \otimes_R y$, for any $y \in K$.*
- *(ii) If S satisfies the right Ore condition, then* $\mathrm{Tor}^R_1(K, RS^{-1}/R) = 0$*, for any torsion free right R-module K. In particular, K is embedded in $K \otimes_R RS^{-1}$ via the assignment $y \mapsto y \otimes_R 1$, for any $y \in K$.*

Lemma 5.4 *Let R be a ring, and let S be a multiplicative subset of Σ that satisfies the right Ore condition. Set $Q = RS^{-1}$. If F is a free right R-module and $K \leq F$ is such that $K \otimes_R Q$ is countably generated right Q-module, then K is contained in a countably generated direct summand of F.*

PROOF. Let $(e_i; i \in I)$ be a basis of F. For every $i \in I$ denote by $\pi_i \colon F \to e_iR$ the canonical projection. For every subset X of F, define the support of X as

$$\mathrm{supp}(X) = \{i \in I \mid \pi_i(x) \neq 0, \text{ for some } x \in X\}.$$

Choose a set of Q-generators of $K \otimes_R Q$ of the form $\{y_n \otimes_R 1 \mid n \in \mathbb{N}\}$, where $y_n \in K$ for every $n \in \mathbb{N}$. We claim that $\mathrm{supp}(K) = \mathrm{supp}(\sum_{n\in\mathbb{N}} y_nR)$, hence countable. It is clear that $\mathrm{supp}(\sum_{n\in\mathbb{N}} y_nR) \subseteq \mathrm{supp}(K)$. For the converse, let $y \in K$. There exist $r_1, \dots, r_\ell \in R$ and $s \in S$ such that $y \otimes_R 1 = \sum_{i=1}^{\ell} y_ir_i \otimes_R s^{-1}$. As K is torsion free, we deduce from Lemma 5.3 that $ys = \sum_{i=1}^{\ell} y_ir_i$. Since s is not a zero divisor

$$\mathrm{supp}(y) = \mathrm{supp}(ys) \subseteq \mathrm{supp}(\sum_{n\in\mathbb{N}} y_nR).$$

This finishes the proof of our claim. Now $K \subseteq \bigoplus_{i \in \mathrm{supp}(K)} e_i R$ which is a countably generated direct summand of F. ■

Proposition 5.5 *Let R be a ring, and let S be a multiplicative subset of Σ that satisfies the right Ore condition. If $Q = RS^{-1}$ is right $\aleph_0$-noetherian then the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of countable type.*

Proof. The result follows by Lemma 5.4 using a back and forth argument in the projective resolution of a module, taking into account that $\mathrm{Tor}_1^R(M, Q) = 0$, for every right R-module M. ■

Remark 5.6 By [41, Corollary 11] the cotorsion pair $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is of countable type if and only if every module in $\mathcal{P}_n$ is $\mathcal{P}_n(\mathrm{mod}_{\aleph_0}\text{-}R)$-filtered.

We show now by an example that the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is not, in general, of countable type and also that Proposition 5.5 cannot be extended to arbitrary finite projective dimension.

Example 5.7 Observe first that if $\mathbf{m}$ is a maximal right ideal of a ring R then the simple right module $R/\mathbf{m}$ is $\mathrm{mod}_{\aleph_0}$-R-filtered if and only if $\mathbf{m} \in \mathrm{mod}_{\aleph_0}\text{-}R$.

1). Let R be the K-free algebra generated over the field K by an uncountable set X. Then the twosided ideal generated by X is an uncountably generated maximal right (or left) ideal $\mathbf{m}$ of R. Since, R is a hereditary ring, we infer that the simple module $R/\mathbf{m}$ has projective dimension 1. In view of Remark 5.6, $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ cannot be of countable type since $R/\mathbf{m}$ is not $P_1(\mathrm{mod}_{\aleph_0}\text{-}R)$-filtered.

2). Let R be a commutative valuation domain such that its maximal ideal $\mathbf{m}$ is $\aleph_n$-generated. By a result of Osofsky [24, Theorem 3.2], the projective dimension of $\mathbf{m}$ is $n+1$, so that the projective dimension of $R/\mathbf{m}$ is $n+2$. If $n > 0$ then $R/\mathbf{m}$ is not $P_{n+2}(\mathrm{mod}_{\aleph_0}\text{-}R)$-filtered, hence $(\mathcal{P}_{n+2}, \mathcal{P}_{n+2}^{\perp})$ is not of countable type.

6 Finitistic dimensions of classical rings of quotients

We recall the notions of little and big finitistic dimension of a ring R. For later convenience, we introduce also an intermediate notion.

Definition 6.1 *The (right)* little finitistic dimension*, f.dim R, is the supremum of the projective dimension of the right R-modules in $\mathcal{P}(\mathrm{mod}\text{-}R)$.*

The (right) big finitistic dimension*, F.dim R, is the supremum of the projective dimension of the right R-modules in $\mathcal{P}$.*

We denote by $f_{\aleph_0}$.dim R the supremum of the projective dimension of the right R-modules in $\mathcal{P}(\mathrm{mod}_{\aleph_0}\text{-}R)$,

Clearly, f.dim $R \leq f_{\aleph_0}$.dim $R \leq$ F.dim R.

These dimensions were already considered by Bass in his fundamental paper [11]. We note however that our little finitistic dimension is considered only over modules in mod-R, instead of over all finitely generated modules.

As in the previous section, we denote by Σ the multiplicative set of the non zero divisors of the ring R. We say R has *classical ring of quotients* if it satisfies the left and right Ore condition with respect to Σ. From now on, for a ring with classical ring of quotients, $Q = R\Sigma^{-1} = \Sigma^{-1}R$, and we identify R with its image in Q.

In what follows, $\mathcal{P}_n(R)$ and $\mathcal{P}_n(Q)$ will denote the classes of right modules of projective dimension at most n over R and Q, respectively.

Lemma 6.2 *Let R be a ring with classical ring of quotients Q. Then, a right Q-module V belongs to $\mathcal{P}_1(Q)$ if and only if there is $M_R \in \mathcal{P}_1(R)$ such that $V = M \otimes_R Q$.*

Proof. The sufficiency is clear. For the only if part, let $V \in \mathcal{P}_1(Q)$. Without loss of generality we can assume that there is a short exact sequence

$$0 \to Q^{(\alpha)} \xrightarrow{\mu} Q^{(\beta)} \to V \to 0,$$

for some cardinals α, β.

Let $(d_i \colon i \in \alpha)$ be the canonical basis of the right Q-free module $Q^{(\alpha)}$. The injection μ is represented by a column finite matrix A' with entries in $Q = R\Sigma^{-1}$ acting as left multiplication on the basis elements d_i. For every $i \in \alpha$, let $r_i \in \Sigma$ be a common right denominator of the elements of the i^{th}-column of A'. Changing the basis $(d_i \mid i \in \alpha)$ with the basis $(r_i d_i \colon i \in \alpha)$, we can assume that the monomorphism μ is represented by a column finite matrix A with entries in R. As R is inside Q, we get the short exact sequence

$$0 \to R^{(\alpha)} \xrightarrow{\nu} R^{(\beta)} \to M \to 0,$$

where the map ν is represented by the matrix A. Then it is clear that $M \otimes_R Q \cong V$. ■

A characterization of the rings with classical ring of quotients Q of big finitistic dimension zero is now immediate.

Proposition 6.3 *Let R be a ring with classical ring of quotients Q. Then, the following statements are equivalent:*

(i) For every right R-module $M \in \mathcal{P}_1(R)$, $M \otimes_R Q \in \mathcal{P}_0(Q)$;

(ii) F.dim $Q = 0$.

Proof. (i)⇒ (ii). Assume by way of contradiction that F.dim $Q > 0$. Let n be the least natural number such that there is a non projective right module $V \in \mathcal{P}_n(Q)$. Consider a free presentation $0 \to V_1 \to Q^{(\alpha)} \to V \to 0$ of V. Then $V_1 \in \mathcal{P}_{n-1}(Q)$, hence V_1 is projective. So V has projective dimension one. By Lemma 6.2 and condition (i) we get a contradiction.

(ii) ⇒ (i). Obvious because Q is flat as a left R-module. ■

Now we will characterize rings with classical ring of quotients of little finitistic dimension zero. We note the following easy but useful lemma.

Lemma 6.4 *Let R be a ring and let $C \in \mathcal{P}_1(\text{mod-}R)$. There is a finitely generated projective module P and a short exact sequence*

$$0 \to R^m \to R^n \to C \oplus P \to 0.$$

Proof. If C is projective, the claim is obvious with $m = 0$. Let p.d.$C = 1$. By assumption, there exists a short exact sequence $0 \to P \to R^k \to C \to 0$ for some $k > 0$ and some finitely generated projective module P. Let P' be a projective module such that $P \oplus P' \cong R^m$ for some $m > 0$. Then $R^k \oplus P' \oplus P \cong R^{k+m}$ and thus the short exact sequence

$$0 \to P \oplus P' \to R^k \oplus P' \oplus P \to C \oplus P \to 0$$

satisfies the requirements. ∎

Let $\mathcal{C} = \{R/rR \mid r \in \Sigma\} \cup R$. Then $\mathcal{D} = \mathcal{C}^{\perp}$ and $\mathcal{TF} = \mathcal{C}^{\intercal}$, where $\mathcal{D}$ is the class of divisible right R-modules and $\mathcal{TF}$ is the class of torsion free left R-modules. Clearly, $\mathcal{C} \subseteq \mathcal{P}_1(\text{mod-}R)$.

Proposition 6.5 *Let R be a ring with classical ring of quotients Q. Assume that f.dim Q $=0$. Then the following hold.*

(i) The class $\mathcal{D}$ of divisible right modules is a 1-tilting class and it coincides with $\mathcal{P}_1(\text{mod-}R)^{\perp}$.

(ii) The class $\mathcal{TF}$ of torsion free left modules coincides with $\mathcal{P}_1(\text{mod-}R)^{\intercal}$.

Proof. Let $C_R \in \mathcal{P}_1(\text{mod-}R)$. By adding a finitely generated projective module and using Lemma 6.4, we may assume, w.l.o.g. that C_R fits in a short exact sequence of the form

$$0 \to R^m \xrightarrow{\mu} R^n \to C \to 0. \tag{1}$$

where the injection μ can be represented by a $n \times m$ matrix A with entries in R and acting on the elements of R^m viewed as columns vectors. Tensoring the exact sequence (1) by the flat left R-module Q we get the short exact sequence

$$0 \to Q^m \xrightarrow{A \otimes 1_Q} Q^n \to C \otimes_R Q \to 0$$

of right Q-modules. Using the assumption f.dim $Q = 0$, we conclude that $C \otimes_R Q$ is a projective right Q-module. Thus there is a splitting map $Q^n \to Q^m$ represented by an $m \times n$ matrix B' with entries in $Q = \Sigma^{-1}R$ such that $B'A = I_m$, where I_m is the $m \times m$ identity matrix. Let $r \in \Sigma$ be the product of the left denominators of the entries in B', then the matrix $B = rB'$ has entries in R, and $BA = rI_m$. Thus we have the following commutative diagram:

$$(*)\qquad\begin{tikzcd} R^m \arrow[r, "A"] \arrow[dr, "\overline{r}"'] & R^n \arrow[d, "B"] \\ & R^m \end{tikzcd}$$

where $\overline{r}$ denotes the map given by left multiplication by r.

(i) If we show that $\mathcal{D} = \mathcal{P}_1(\text{mod-}R)^{\perp}$, then we will have that $\mathcal{D}$ is a 1-tilting class, since the cotorsion pair $({}^{\perp}(\mathcal{P}_1(\text{mod-}R)^{\perp}), \mathcal{P}_1(\text{mod-}R)^{\perp})$ is a 1-tilting cotorsion pair. By definition, $\mathcal{D} \supseteq \mathcal{P}_1(\text{mod-}R)^{\perp}$. We need to show that $\mathrm{Ext}^1_R(C, D) = 0$, for every $C \in \mathcal{P}_1(\text{mod-}R)$ and for every $D_R \in \mathcal{D}$. Applying the functor $\mathrm{Hom}_R(-, D)$ to the sequence (1), we obtain the exact sequence

$$0 \to \mathrm{Hom}_R(C, D) \to D^n \overset{\mathrm{Hom}_R(A,D)}{\longrightarrow} D^m \to \mathrm{Ext}^1_R(C, D) \to 0 \tag{2}$$

where the map $\mathrm{Hom}_R(A, D)$ is represented by the matrix A acting by right multiplication on elements of D^n_R viewed as row vectors. Applying the functor $\mathrm{Hom}_R(-, D)$ to the commutative diagram (*), we obtain the commutative diagram:

$$\begin{tikzcd}
D^m \arrow[r, "\mathrm{Hom}(B,D)"] \arrow[dr, "\overline{r}"'] & D^n \arrow[d, "\mathrm{Hom}(A,D)"] \\
 & D^m
\end{tikzcd}$$

Since the right multiplication by r is surjective on D, we conclude that the group homomorphism $\mathrm{Hom}_R(A, D)$ is surjective. Hence, from sequence (2) we infer that $\mathrm{Ext}^1_R(C, D) = 0$.

(ii) By Lemma 2.3(i), condition (ii) is a consequence of (i). We find it interesting to give an independent proof.

By definition, $\mathcal{TF} \supseteq \mathcal{P}_1(\text{mod-}R)^{\intercal}$. Let $Y \in \mathcal{TF}$. Applying the functor $- \otimes_R Y$ to sequence (1), we obtain the exact sequence

$$0 \to \mathrm{Tor}^R_1(C, Y) \to Y^m \overset{A \otimes_R 1_Y}{\longrightarrow} Y^n \to C \otimes_R Y \to 0 \tag{3}$$

where the map $A \otimes_R 1_Y$ is represented by the matrix A acting as left multiplication on elements of ${}_RY^m$ viewed as columns vectors. Applying the functor $- \otimes_R Y$ to the commutative diagram (*), we obtain the commutative diagram:

$$\begin{tikzcd}
Y^m \arrow[r, "A \otimes_R 1_Y"] \arrow[dr, "\overline{r}"'] & Y^n \arrow[d, "B \otimes_R 1_Y"] \\
 & Y^m
\end{tikzcd}$$

Since the left multiplication by r is injective on Y, we conclude that the group homomorphism $A \otimes_R 1_Y$ is injective. Hence, from sequence (3) we infer that $\mathrm{Tor}^R_1(C, Y) = 0$. Hence $Y \in \mathcal{P}_1(\text{mod-}R)^{\intercal}$ as we wanted to show.

■

Remark 6.6 In the situation of Proposition 6.5, the class $\mathcal{TF}$ is the 1-cotilting torsion free class $\mathcal{C}^{\intercal}$ or, equivalently, $\mathcal{TF} = {}^{\perp}\mathcal{C}^*$, where $\mathcal{C}^*$ denotes the class of the character modules of the modules in $\mathcal{C}$. See [6] or [25] for the needed results and for the unexplained terminology.

We give now a characterization of rings with classical ring of quotients Q of little finitistic dimension 0. In this result, $\mathcal{F}_1$ denotes the class of right R-modules of weak dimension at most one.

Theorem 6.7 *Let R be a ring with classical ring of quotients Q. Then, the following statements are equivalent:*

(i) For every right R-module $C \in \mathcal{P}_1(\text{mod-}R)$, $C \otimes_R Q \in \mathcal{P}_0(\text{mod-}Q)$;

(ii) For every right R-module $C \in \mathcal{P}_1$, $C \otimes_R Q$ is flat;

(iii) f.dim $Q = 0$;

(iv) the class $\mathcal{D}$ of divisible right modules is a 1-tilting class and it coincides with $\mathcal{P}_1(\text{mod-}R)^{\perp}$;

(v) the class $\mathcal{TF}$ of torsion free left modules coincides with $\mathcal{P}_1(\text{mod-}R)^{\intercal}$;

(vi) $\varinjlim \mathcal{P}_1 = \varinjlim \mathcal{P}_1(\text{mod-}R) = \mathcal{F}_1 \bigcap {}^{\intercal}Q\text{-Mod}$;

(vii) $\mathcal{P}_1(\text{mod-}R)^{\intercal} \supseteq Q\text{-Mod}$.

Proof. (i) $\Leftrightarrow$ (ii). Follows from the fact that any module in $\mathcal{P}_1$ is a direct limit of modules in $\mathcal{P}_1(\text{mod-}R)$, combined with the fact that any finitely presented flat module is projective.

(i)$\Rightarrow$ (iii). Follows from Lemma 6.2 (cf. Proposition 6.3).

(iii)$\Rightarrow$ (iv). By Proposition 6.5 (ii).

(iv)$\Rightarrow$ (v). Follows from Lemma 2.3 (i).

(v)$\Rightarrow$ (vi). For any ring R, $\varinjlim \mathcal{P}_1 = \varinjlim \mathcal{P}_1(\text{mod-}R)$, since every module in $\mathcal{P}_1$ is a direct limit of modules in $\mathcal{P}_1(\text{mod-}R)$. Moreover, $\mathcal{F}_1 \supseteq \varinjlim \mathcal{P}_1$, since $\mathcal{P}_1 \subseteq \mathcal{F}_1$ and the Tor functor commutes with direct limits. By [8, Theorem 2.3] or by [25, Theorem 4.5.6], $\varinjlim \mathcal{P}_1 = {}^{\intercal}(\mathcal{P}_1(\text{mod-}R)^{\intercal})$ and, by (v), $\mathcal{P}_1(\text{mod-}R)^{\intercal} = \mathcal{TF} \supseteq Q\text{-Mod}$. Therefore, $\varinjlim \mathcal{P}_1 \subseteq \mathcal{F}_1 \bigcap {}^{\intercal}Q\text{-Mod}$. It also follows that to prove the other inclusion we only have to see that $\mathrm{Tor}_1^R(M, Y) = 0$, for $M \in \mathcal{F}_1 \bigcap {}^{\intercal}Q\text{-Mod}$ and every torsion free left R-module Y .

Let $M \in \mathcal{F}_1 \bigcap {}^{\intercal}Q\text{-Mod}$ and $Y \in \mathcal{TF}$. By Lemma 5.3, there is an exact sequence

$$0 \to Y \to Q \otimes_R Y \to (Q \otimes_R Y)/Y \to 0.$$

which, by the choice of M, yields the exact sequence

$$0 = \mathrm{Tor}_2^R(M, (Q \otimes_R Y)/Y) \to \mathrm{Tor}_1^R(M, Y) \to \mathrm{Tor}_1^R(M, Q \otimes_R Y) = 0.$$

Hence, $\mathrm{Tor}_1^R(M, Y) = 0$ as wanted.

(vi) $\Rightarrow$ (vii). By (vi), $\mathcal{P}_1(\text{mod-}R) \subseteq {}^{\intercal}Q\text{-Mod}$. Hence $Q\text{-Mod} \subseteq ({}^{\intercal}Q\text{-Mod})^{\intercal} \subseteq \mathcal{P}_1(\text{mod-}R)^{\intercal}$.

(vii)$\Rightarrow$ (i). Let C be a right R-module in $\mathcal{P}_1(\text{mod-}R)$ and let N be a left Q-module. By hypothesis $\mathrm{Tor}_1^R(C, N) = 0$. As the ring homomorphism $R \to Q$ is an epimorphism, $0 = \mathrm{Tor}_1^R(C, N) \cong \mathrm{Tor}_1^Q(C \otimes_R Q, N)$. So $C \otimes_R Q$ is a flat right Q-module, hence projective, since it is finitely presented. ■

As a consequence of Theorem 6.7 we obtain the following Corollary.

Corollary 6.8 *Let R be a ring with classical ring of quotients Q. Then $\mathcal{F}_1 = \varinjlim \mathcal{P}_1$ and f.dim $Q = 0$ if and only if any right Q-module of finite weak dimension is a flat Q-module.*

PROOF. Since Q is flat as a right and as a left R-module, we have that for any left Q-module N and any right R-module M, $\operatorname{Tor}_1^R(M,N) \cong \operatorname{Tor}_1^Q(M \otimes_R Q, N)$ and symmetrically, $\operatorname{Tor}_1^R(M,N) \cong \operatorname{Tor}_1^Q(M, Q \otimes_R N)$, for any right Q-module M and any left R-module N.

Assume any right Q-module of finite weak dimension is flat, then f.dim $Q = 0$ because finitely presented flat modules are projective. If $M \in \mathcal{F}_1$ and N is a left Q-module $\operatorname{Tor}_1^R(M,N) \cong \operatorname{Tor}_1^Q(M \otimes_R Q, N) = 0$. Hence, by Theorem 6.7 (vi), $M \in \varinjlim \mathcal{P}_1$.

Assume now that $\mathcal{F}_1 = \varinjlim \mathcal{P}_1$ and f.dim $Q = 0$. By Theorem 6.7, $\mathcal{F}_1 = \mathcal{F}_1 \bigcap {}^{\intercal}Q\text{-Mod}$. Let M be a right Q-module of weak dimension one. As $M_R \in \mathcal{F}_1$, $M \in {}^{\intercal}Q\text{-Mod} \bigcap \text{Mod-}Q$ is a flat module. ■

There are (commutative noetherian) rings R such that f.dim $R = 0$ but with modules of weak dimension bigger than zero, cf. Example 8.5.

Commutative domains and, more generally, orders in von Neumann regular rings are examples of rings such that $\mathcal{F}_1 = \varinjlim \mathcal{P}_1$. The case of commutative domains was already proved in [8, Theorem 3.5].

We consider now a situation which is intermediate between the ones considered in Theorem 6.7 and Proposition 6.3.

Proposition 6.9 *Let R be a ring with classical ring of quotients Q. Then, the following statements are equivalent:*

(i) For every right R-module $M \in \mathcal{P}_1(\text{mod}_{\aleph_0}\text{-}R)$, $M \otimes_R Q \in \mathcal{P}_0(\text{mod}_{\aleph_0}\text{-}Q)$;

(ii) $\text{f}_{\aleph_0}$.dim $Q = 0$;

(iii) f.dim $Q = 0$ and $M \otimes_R Q$ is a pure projective module, for every right R-module $M \in \mathcal{P}_1(\text{mod}_{\aleph_0}\text{-}R)$;

(iv) f.dim $Q = 0$ and $M \otimes_R Q$ is a Mittag-Leffler right Q-module, for every right R-module $M \in \mathcal{P}_1(\text{mod}_{\aleph_0}\text{-}R)$

PROOF. The equivalence (i) ⇔ (ii) follows by the definition of $\text{f}_{\aleph_0}$.dim $Q = 0$ and by Lemma 6.2.

(ii) ⇒ (iii). Condition (ii) clearly implies f.dim $Q = 0$. Moreover, for every right R-module $M \in \mathcal{P}_1(\text{mod}_{\aleph_0}\text{-}R)$, $M \otimes_R Q$ is pure projective right Q-module, since by hypothesis it is projective.

(iii)⇒ (i). Let $M_R \in \mathcal{P}_1(\text{mod}_{\aleph_0}\text{-}R)$. Then, as M_R is countably presented and of projective dimension at most one, it is a direct limit of a countable direct system of the form $(C_n; f_n\colon C_n \to C_{n+1})_{n \in \mathbb{N}}$, where the right R-modules $C_n \in \mathcal{P}_1(\text{mod-}R)$ ([14, Sec.2]). Hence M fits in a pure exact sequence of the form

$$0 \to \oplus_{n \in \mathbb{N}} C_n \xrightarrow{\phi} \oplus_{n \in \mathbb{N}} C_n \to M \to 0$$

where, for every $n \in \mathbb{N}$, $\phi\varepsilon_n = \varepsilon_n - \varepsilon_{n+1}f_n$ and $\varepsilon_n \colon C_n \to \oplus_{n\in\mathbb{N}}C_n$ denotes the canonical map. Tensoring by Q we get the pure exact sequence of right Q-modules

$$0 \to \oplus_{n\in\mathbb{N}}(C_n \otimes_R Q) \overset{\phi\otimes_R Q}{\to} \oplus_{n\in\mathbb{N}}(C_n \otimes_R Q) \to M \otimes_R Q \to 0,$$

which is splitting by the hypothesis that $M \otimes_R Q$ is pure projective. Thus $M \otimes_R Q$ is a direct summand of $\oplus_{n\in\mathbb{N}}(C_n \otimes_R Q)$ and for every $n \in \mathbb{N}$, $C_n \otimes_R Q$ is projective right Q-module, since f.dim $Q = 0$. Thus $M \otimes_R Q$ is projective, too.

(iii) ⇔ (iv). The equivalence follows by the well known fact that countably generated (hence countably presented) Mittag-Leffler right modules are pure projective [37, Corollaire 2.2.2]. ■

7 Orders in rings with finitistic dimension zero

We start by giving a characterization for the equality of the two classes $\mathcal{P}_1(\text{mod}_{\aleph_0}\text{-}R)^{\perp}$ and $\mathcal{P}_1(\text{mod-}R)^{\perp}$.

Proposition 7.1 *Let R be a ring with classical ring of quotients Q such that f.dim $Q = 0$. Then the following statements are equivalent.*

(i) $f_{\aleph_0}$.dim $Q = 0$

(ii) Every right R-module $M \in \mathcal{P}_1(\text{mod}_{\aleph_0}\text{-}R)$ is a direct summand of a $\mathcal{P}_1(\text{mod-}R)$-filtered module;

(iii) the cotorsion pair generated by $\mathcal{P}_1(\text{mod}_{\aleph_0}\text{-}R)$ is of finite type.

Proof. Conditions (ii) and (iii) are equivalent by Fact 2.2.

(i) ⇒ *(iii)*. Let $(\mathcal{A}, \mathcal{B})$ be the cotorsion pair of finite type generated by $\mathcal{P}_1(\text{mod-}R)$. We must show that every right R-module M in $\mathcal{P}_1(\text{mod}_{\aleph_0}\text{-}R)$ is in $\mathcal{A}$. As any module in $\mathcal{P}_1$ is a direct limit of modules in $\mathcal{P}_1(\text{mod-}R)$, by Theorem 3.3 we only need to show that a right R-module M in $\mathcal{P}_1(\text{mod}_{\aleph_0}\text{-}R)$ is Mittag-Leffler with respect to the class $\mathcal{P}_1(\text{mod-}R)^{\intercal}$. By Theorem 6.7, $\mathcal{P}_1(\text{mod-}R)^{\intercal}$ coincides with the class $\mathcal{TF}$ of torsion free left R-modules.

We show now that, under our hypothesis, every right R-module M in $\mathcal{P}_1(\text{mod}_{\aleph_0}\text{-}R)$ is $\mathcal{TF}$-Mittag-Leffler.

The assumption $\text{f}_{\aleph_0}$.dim $Q = 0$ implies that $M \otimes_R Q$ is a projective right Q-module, hence a Mittag-Leffler right Q-module.

We claim that M is $\mathcal{Q}$-Mittag-Leffler, where $\mathcal{Q} = Q$-Mod.

In fact, for every right R-module N and any left Q-module V, $N \otimes_R V \cong N \otimes_R (Q \otimes_Q V)$. Hence if $(V_i; i \in I)$ is a family of left Q-modules, the above remark and the fact that $M \otimes_R Q$ is a projective right Q-module imply that the map

$$\rho \colon M \bigotimes_R \prod_{i\in I} V_i \to \prod_{i\in I}(M \bigotimes_R V_i)$$

is injective.

Let now ${}_RY \in \mathcal{TF}$ and consider the exact sequence

$$(1) \quad 0 \to R \to Q \to Q/R \to 0.$$

By Lemma 5.3, $\mathrm{Tor}_1^R(Q/R, Y) = 0$. Thus, tensoring by Y the exact sequence (1) we obtain the embedding

$$(2) \quad 0 \to R \otimes_R Y \to Q \otimes_R Y.$$

Since $Q \otimes_R Y$ is a left Q-module, Proposition 3.5 implies that M is $\mathcal{TF}$-Mittag-Leffler.

(iii) ⇒ *(i)*. By Theorem 3.3, if the cotorsion pair generated by $\mathcal{P}_1(\text{mod}_{\aleph_0}\text{-}R)$ is of finite type then every module $M \in \mathcal{P}_1(\text{mod}_{\aleph_0}\text{-}R)$ is Mittag-Leffler with respect to the class $(\mathcal{P}_1(\text{mod-}R))^\intercal$. As f.dim $Q = 0$, Theorem 6.7 implies that $\mathcal{TF} = (\mathcal{P}_1(\text{mod-}R))^\intercal$ and, hence, Q-Mod is contained in $\mathcal{TF}$. Thus the right Q-module $M \otimes_R Q$ is Mittag-Leffler. The conclusion follows by Proposition 6.9. ■

We now patch together our results in the setting of orders into $\aleph_0$-noetherian rings.

In the next theorem ∂ denotes the Fuchs' divisible module defined in [24, VII.1] for the commutative case and in [5, §5] for the noncommutative setting. The module ∂ is a 1-tilting module generating the cotorsion pair $({}^\perp\mathcal{D}, \mathcal{D})$ (cf. [21] for the commutative case and [5, Proposition 5.5] for the general case).

Theorem 7.2 *Let R be a ring with an $\aleph_0$-noetherian classical ring of quotients Q. Assume that f.dim $Q = 0$. Then the following statements are equivalent*

(i) $\mathrm{f}_{\aleph_0}$*.dim* $Q = 0$

(ii) *F.dim* $Q = 0$

(iii) $(\mathcal{P}_1, \mathcal{P}_1^\perp)$ *is of finite type;*

(iv) *Every module of projective dimension at most one is a direct summand of a* $\mathcal{P}_1(\text{mod-}R)$*-filtered module.*

(v) *Every module of projective dimension at most one is a direct summand of a* $\mathcal{C}$*-filtered module, where* $\mathcal{C} = \{R/rR \mid r \in \Sigma\} \cup \{R\}$.

When the above equivalent statements hold then $(\mathcal{P}_1, \mathcal{P}_1^\perp) = (\mathcal{P}_1, \mathcal{D})$ *where* $\mathcal{D}$ *is the class of divisible modules; so that every divisible module of projective dimension at most one is a direct summand of a direct sum of copies of* ∂*. Moreover, every module of projective dimension at most two is a direct limit of modules in* $\mathcal{P}_2(\text{mod-}R)$*.*

Proof. (i)⇔ (ii). Follows from Fact 5.1 and Eklof's Lemma (Fact 2.1).

(i)⇔ (iii). If $\mathrm{f}_{\aleph_0}$.dim $Q = 0$ then, by Proposition 5.5 and Proposition 7.1, it follows that $(\mathcal{P}_1, \mathcal{P}_1^\perp)$ is of finite type. The converse follows from Proposition 7.1.

Statements (iii), (iv) and (v) are equivalent by Fact 2.2 and Proposition 6.5.

When the statements hold then $(\mathcal{P}_1, \mathcal{P}_1^\perp) = (\mathcal{P}_1, \mathcal{D})$ by Proposition 6.5. In this situation, ∂ is a 1-tilting module generating the cotorsion pair $(\mathcal{P}_1, \mathcal{D})$ [5, Proposition 5.5]. Therefore, by well known results on tilting cotorsion pairs, $\mathcal{P}_1 \bigcap \mathcal{D}$ is the class Add ∂ consisting of direct summands of direct sums of copies of ∂.

The statement on the modules of projective dimension two is a consequence of Theorem 4.6. ∎

8 Orders in semisimple artinian rings and noetherian rings

A semisimple artinian ring has global dimension 0 and it is artinian, therefore Theorem 7.2 applies immediately to orders into semisimple artinian rings, that is, to semiprime Goldie rings.

Corollary 8.1 *Let R be a semiprime Goldie ring then the conclusions of Theorem 7.2 hold for R. In particular, $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type.*

From the previous Corollary, we single out the case of commutative domain, as it completes the results obtained in [33] by S. B. Lee, and it gives a positive answer to [23, Problem 6, p. 139]

Corollary 8.2 *Let R be a commutative domain then the conclusions of Theorem 7.2 hold for R. In particular,*

(i) $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type;

(ii) every divisible module of projective dimension at most one is a direct summand of a direct sum of copies of ∂.

Our next goal is to characterize the commutative noetherian rings such that the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type as the ones that are orders into artinian rings. Therefore, in the commutative noetherian case, Theorem 7.2 gives the best possible result. We remark however that in Remark 9.7 we will see that the condition f.dim $Q = 0$ is not a necessary condition for the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ to be of finite type.

Lemma 8.3 *Let R be a noetherian commutative ring with classical ring of quotients Q. Then, f.dim $Q = 0$.*

Proof. It is well known that the set of zero divisors of a commutative noetherian ring R coincides with the union of the prime ideals of R associated to R. Let $\{P_1, P_2 \dots, P_n\}$ be the set of the prime ideals associated to R. For every $1 \leq i \leq n$, let P_iQ denote the extension of P_i in Q. Then $\{P_1Q, P_2Q \dots, P_nQ\}$ is the set of prime ideals of Q, and by [35, Theorem 6.2], it is the set of associated primes of Q. Let P_iQ be a maximal ideal in Q and consider the localization Q_{P_iQ} of Q at P_iQ. Again by [35, Theorem 6.2], the maximal ideal of Q_{P_iQ} is an associated prime of Q_{P_iQ}, hence it consists of zero divisors. This means that the regular sequences in Q_{P_iQ} are empty. Hence by the Auslander Buchsbaum Formula, [9] or [43, Theorem 4.4.15], f.dim $Q_{P_iQ} = 0$. Since this holds for all maximal ideals of Q, we conclude that any finitely generated (presented) module of finite projective dimension is flat and, hence, projective. Therefore, f.dim $Q = 0$. ∎

As a consequence of the above Lemma, Theorem 6.7 and Corollary 6.8 we get.

Corollary 8.4 *Let R be a commutative noetherian ring, and let Q be its classical ring of quotients. Then R satisfies all the equivalent statements of Theorem 6.7. In particular, the class $\mathcal{D}$ of divisible modules coincides with $\mathcal{P}_1(\text{mod-}R)^{\perp}$ and the class $\mathcal{TF}$ of torsion free modules coincides with $\mathcal{P}_1(\text{mod-}R)^{\intercal}$.*

Moreover $\mathcal{F}_1 = \varinjlim \mathcal{P}_1$ if and only if the Q-modules of finite weak Q-dimension are flat.

Example 8.5 Let R be a commutative noetherian ring of Krull dimension m. Let n the supremum of the weak dimension of the R modules with finite weak dimension, by [12, Corollary 5.3], $n \leq m \leq n+1$. Therefore if $n = 0$, the Krull dimension of R can be at most 1. This implies that if R is a commutative noetherian ring that it is its own classical ring of quotients and has Krull dimension ≥ 2 then R does not satisfy the conclusions of the above Corollary.

For example, let k be a field. Let R be the localization of the ring $k[x, y, z]/(z^2, zx, zy)$ at the maximal ideal $(\overline{x}, \overline{y}, \overline{z})$. Hence, R is a local noetherian ring of Krull dimension 2 such that any nonzero divisor is invertible.

The localization of R at the multiplicative closed system $\{1, \overline{x}, \overline{x}^2, \dots\}$ is isomorphic to $k(x)[y]$ and it is a flat R-module. Hence, $k(x) \cong k(x)[y]/yk(x)[y]$ is an R-module of weak dimension 1 which is not flat and $k(x) \in \mathcal{F}_1 \setminus \varinjlim \mathcal{P}_1$.

Theorem 8.6 *Let R be a commutative noetherian ring with classical ring of quotients Q. Then the following are equivalent.*

(i) The cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type.

(ii) F.dim $Q = 0$.

(iii) Q is artinian.

(iv) the set of prime ideals associated to R coincides with the set of minimal prime ideals of R.

Proof. Over any $\aleph_0$-noetherian ring the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of countable type. Thus for such rings, $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type if and only if the cotorsion pair generated by $\mathcal{P}_1(\text{mod}_{\aleph_0}\text{-}R)$ is of finite type, by Fact 2.1.

(i) ⇔ (ii). By Lemma 8.3, f.dim $Q = 0$. The above remark and Theorem 7.2 give the equivalence.

(ii) ⇔ (iii). A combination of a result by Bass [11] and one by Raynaud and Gruson [37] shows that, for a commutative noetherian ring, the big finitistic dimension equals the Krull dimension. Moreover, a commutative noetherian ring is artinian if and only if its Krull dimension is zero.

(iii) $\Leftrightarrow$ (iv). As noted in the proof of Lemma 8.3, the prime ideals of Q are exactly the extension at Q of the associated prime ideals of R. Hence the claim is immediate. $\blacksquare$

Noetherian Cohen-Macaulay rings have an artinian ring of quotients so they satisfy the above theorem.

Kaplansky's characterization of commutative rings with big finitistic dimension zero (see [11, page 1]) combined with Theorem 7.2 allows us to prove,

Corollary 8.7 *Let R be a commutative ring and Q its classical ring of quotients. Assume that Q is a perfect ring and $\aleph_0$-noetherian. Then, $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type.*

9 Examples

In this section we exhibit examples and counterexamples for the finite type of the cotorsion pairs $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$. Our first type of examples is based on the following observation.

Lemma 9.1 Let R be a ring such that f.dim $R = m <$ F.dim R. Then $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is not of finite type, for all $n > m$.

PROOF. By Auslander's Lemma, any direct summand of a $\mathcal{P}_n$(mod-R)-filtered module has projective dimension at most n. But, by assumption, for any $n > m$, $\mathcal{P}_n(\text{mod-}R) = \mathcal{P}_m(\text{mod-}R)$ and in $\mathcal{P}_n$ there exist modules of projective dimension greater than m. Therefore, for all $n > m$, $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is not of finite type. $\blacksquare$

In trying to generalize the results in Section 8 to the cotorsion pair $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$, for $n > 1$, the first thing to keep in mind are the next counterexamples showing that, even over commutative domains these cotorsion pairs are not of finite type, in general.

Let R be a ring, and let $n \geq 0$. We denote by $\mathcal{F}_n$ the class of (right) modules of weak dimension at most n. In the second example we show that, for commutative domains, the formula $\mathcal{F}_1 = \varinjlim \mathcal{P}_1$ from [8] (see also Corollary 6.8) is no longer true for $n = 2$. This gives a negative answer to [25, Open Problem 3, p. 187].

Examples 9.2 (i) There is a commutative local noetherian domain such that the cotorsion pair $(\mathcal{P}_2, \mathcal{P}_2^{\perp})$ is not of finite type.

(ii) There is a commutative local noetherian domain such that $\varinjlim \mathcal{P}_2$ coincides with $\varinjlim \mathcal{P}_1 = \mathcal{F}_1$ and $\mathcal{F}_1 \subsetneq \mathcal{F}_2$.

(iii) If R is a non Dedekind Prüfer domain, then $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is not of finite type, for all $n > 1$.

PROOF. Any commutative noetherian local ring of Krull dimension $n \geq 2$ and depth 1 is of the type claimed in (i). An explicit example is the non Cohen-Macaulay ring in [16, Ex.2.1.18, pag64]. Let $R = K[[X^4, X^3Y, XY^3, Y^4]] \subset K[[X, Y]]$, where K is a field and X, Y are indeterminates. R is a local noetherian domain of Krull dimension 2 and X^4, Y^4 is a system of parameters, but it is not a regular sequence. In fact, $Y^4(X^3Y)^2 = X^6Y^6 =$

$X^4(XY^3)^2$, but $(X^3Y)^2 \notin (X^4)$, so depth $R = 1$. Hence, by Auslander-Buchsbaum equality [9], f.dim $R = 1$ and by [37] F.dim $R = 2$. Now the conclusion follows from Lemma 9.1.

Note that a commutative noetherian local ring R of depth 1, satisfies that $\mathcal{P}_2(\text{mod-}R) = \mathcal{P}_1(\text{mod-}R)$, hence $\varinjlim \mathcal{P}_2 = \varinjlim \mathcal{P}_1$, since

$$\varinjlim \mathcal{P}_2 \subseteq \varinjlim \mathcal{P}_1 = \varinjlim \mathcal{P}_1(\text{mod-}R) = \varinjlim \mathcal{P}_2(\text{mod-}R) \subseteq \varinjlim \mathcal{P}_2.$$

If, moreover R is a domain, then by Corollary 8.4, $\varinjlim \mathcal{P}_2 = \varinjlim \mathcal{P}_1 = \mathcal{F}_1$.

By [12, Corollary 5.3], a commutative noetherian local ring of Krull dimension $n \geq 3$ has modules of weak dimension $n - 1$. This implies that a commutative noetherian local domain of Krull dimension $n \geq 3$ and depth 1 is an example of the type required in (ii).

A variation of the example given in (i) shows the existence of such rings. Let $S = K[[X^4, Y^4, Z^4, X^3Y, X^3Z, XY^3, XZ^3, Y^2Z^2]] \subset K[[X, Y, Z]]$, where K is a field and X, Y, Z are indeterminates. S is a local noetherian domain of Krull dimension 3 and X^4, Y^4, Z^4 is a system of parameters. Now X^4 is a regular sequence, and we claim that $T = S/(X^4)$ has depth 0. This will imply that X^4 is a maximal regular sequence of S and, by Rees Theorem ([16, Theorem 1.2.5]), the depth of S is exactly one.

Arguing as in (i), we have that $(X^3Y)^2 + (X^4)$ is a nonzero element in the annihilator of $Y^4 + (X^4)$. Moreover,

$$Z^4[(X^3Y)^2Z^4] = X^4(XZ^3)^2(Y^2Z^2)$$

while $(X^3Y)^2Z^4 = X^4(X^2Y^2Z^4)$ and $X^2Y^2Z^4 \notin S$. Hence, $(X^3Y)^2Z^4 + (X^4)$ is a nonzero element in the annihilator of the ideal $I = (Y^4, Z^4, X^4)/(X^4)$. Since the maximal ideal of T is nilpotent modulo I, we deduce that it does not contain a regular element.

To prove (iii) recall that finitely presented modules over a Prüfer domain R have projective dimension at most one, hence $\mathcal{P}_n(\text{mod-}R) = \mathcal{P}_1(\text{mod-}R)$, for every $n \geq 1$. Now our statement will follow from Lemma 9.1, once we have proved that in a non Dedekind Prüfer domain $\mathcal{P}_1 \subsetneq \mathcal{P}_2$.

To this aim note that a non Dedekind Prüfer domain is a non noetherian ring, hence it has a countably generated ideal I that is not finitely generated. Being R semihereditary, I is flat, and, since R is a domain, it is countably presented. As I is flat and countably presented it has projective dimension at most 1. Since in a domain the projective ideals are finitely generated, we deduce that I has projective dimension exactly 1. Therefore $R/I \in \mathcal{P}_2 \setminus \mathcal{P}_1$. ■

On the positive side, we consider the case of an Iwanaga-Gorenstein ring, that is a left and right noetherian ring R such that the right module R_R has finite injective dimension and the left module ${}_RR$ has also finite injective dimension. In this case, both dimensions coincide. The ring R is said to be an n-Iwanaga-Gorenstein ring if these dimensions are both n.

Example 9.3 If R is an n-Iwanaga-Gorenstein ring, then $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is of finite type.

Proof. It was shown in [6, Theorem 3.2] that if R is an n-Iwanaga- Gorenstein ring, then f.dim $R =$ F.dim $R = n$ and that the cotorsion pair generated by $\mathcal{P}(\text{mod-}R)$ is the n-tilting cotorsion pair corresponding to the n-tilting module $T = \bigoplus_{0\leq i\leq n} I_i$ where $0 \to R \to I_0 \to I_1 \to \cdots \to I_n \to 0$ is a minimal injective coresolution of R. Moreover, in [6] it is shown that $\mathcal{P} = {}^{\perp}(T^{\perp})$. Hence, $(\mathcal{P}, \mathcal{P}^{\perp}) = (\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is of finite type. ■

Example 9.4 If R is a commutative Gorenstein ring then it is Cohen-Macaulay. Hence, by Theorem 8.6 the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is always of finite type and it is generated by $\{R/rR \mid\ r \text{ regular element of } R\}$.

If R is n-Gorenstein, we do not know whether $(\mathcal{P}_m, \mathcal{P}_m^{\perp})$ is of finite type for $1 < m < n$, cf. Proposition 9.13.

Example 9.5 (i) If f.dim $R = 0$, then $\mathcal{P}_n(\text{mod-}R) = \mathcal{P}_0(\text{mod-}R)$, for every n. Hence, $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is of finite type if and only if F.dim $R = 0$.

(ii) If R is a right noetherian ring, right self-injective, then all projective right modules are injective. Hence F.dim $R = 0$ and so for every $n \in \mathbb{N}$, $(\mathcal{P}_n, \mathcal{P}_n^{\perp}) = (\mathcal{P}_0, \mathcal{P}_0^{\perp})$ is of finite type.

Next we consider the case of an artin algebra, that is a finitely generated algebra over a commutative artin ring.

Recall that a subclass $\mathcal{X}$ of $\mathcal{P}(\text{mod-}R)$ is said to be *contravariantly finite* if every $M \in$ mod-R admits an $\mathcal{X}$-precover (cover), that is there exist $X \in \mathcal{X}$ and a morphism $f\colon X \to M$ such that $\text{Hom}_R(X', X) \to \text{Hom}_R(X', M)$ is surjective for every $X' \in \mathcal{X}$.

Auslander and Reiten [10] proved a fundamental result, namely that if $\mathcal{P}(\text{mod-}R)$ is contravariantly finite, then the little finitistic dimension of R is finite.

Huisgen-Zimmermann and Smalø in [29] strengthened Auslander-Reiten's result by proving that, if $\mathcal{P}(\text{mod-}R)$ is contravariantly finite, then the big finitistic dimension of R coincides with its little finitistic dimension.

In [7, Theorem 4.3] Angeleri Hügel and Trlifaj showed that, for any right noetherian ring R, f.dim $R \leq n$ if and only if the cotorsion pair generated by $\mathcal{P}(\text{mod-}R)$ is an n-tilting cotorsion pair. Moreover, they prove that for an artin algebra R, $\mathcal{P}(\text{mod-}R)$ is contravariantly finite in mod-R if and only if the tilting module corresponding to the cotorsion pair generated by $\mathcal{P}(\text{mod-}R)$ can be taken to be finitely generated. Thus, as a consequence of all these results we have:

Example 9.6 Let R be an artin algebra. Assume that $\mathcal{P}(\text{mod-}R)$ is contravariantly finite in mod-R. Let f.dim $R = n(=$ F.dim $R)$. Then, $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is of finite type.

PROOF. By the preceding remarks and [7, Corollary 3.6]. ■

Remark 9.7 In contrast with our previous discussion on rings with classical ring of quotients with finitistic dimension 0, we note that an artin algebra coincides with its classical ring of quotients. So Example 9.6 shows that there exists a ring with classical ring of quotients of little finitistic dimension greater than zero such that the cotorsion pair $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is of finite type.

Since over right perfect rings, direct limits of module of finite projective dimension n are still of finite projective dimension n, we have the following general observation.

Proposition 9.8 *Let R be a right perfect ring. Assume that f.dim $R = n$ and F.dim $R > n$, for some $n \geq 1$. Then the cotorsion pair $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is not of finite type.*

Proof. By hypothesis, there exists a right module M of projective dimension exactly $n+1$. Assume, by way of contradiction that $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is of finite type. By Theorem 4.6, M_R is a direct limit of objects in $\mathcal{P}_{n+1}(\text{mod-}R)$ which coincides with $\mathcal{P}_n(\text{mod-}R)$, by assumption. Since R is right perfect, p.d.$M \leq n$ (see [11, Theorem P]), a contradiction. ■

In [39] Smalø constructs a family of examples of finite dimensional algebras R_n, such that f.dim $R_n = 1$ an F.dim $R_n = n$ for every $n \in \mathbb{N}$. So that, for $n > 1$, R_n satisfies the hypothesis of Proposition 9.8.

Example 9.9 In [31], Igusa, Smalø and Todorov construct an example of a finite dimensional monomial algebra such that f.dim $R = 1 =$ F.dim R. However, as proved in [8, §5], $(\mathcal{P}_1, \mathcal{P}_1^{\perp})$ is not of finite type.

We devote the rest of the section to give an example showing that the finite type property of $(\mathcal{P}_n, \mathcal{P}_n^{\perp})$ is not inherited, in general, by $(\mathcal{P}_{n-1}, \mathcal{P}_{n-1}^{\perp})$. We recall that this was mentioned in the second statement of Remark 4.7.

As the example will be a quotient of a path algebra, we find it more convenient to think the modules as representations of the associated quiver. So from now on our statements will involve left modules.

First we show that, over artin algebras, the functor Ext commutes inverse limits of finitely generated modules.

Lemma 9.10 *Let R be an artin algebra. Let $(M_\alpha, f_{\alpha\beta}\colon M_\beta \to M_\alpha)_{\beta>\alpha\in\Lambda}$ be an inverse system of finitely generated left R-modules. Then, for any left R-module A and for any $k \geq 0$, $\operatorname{Ext}_R^k(A, \varprojlim M_\alpha) \cong \varprojlim \operatorname{Ext}_R^k(A, M_\alpha)$.*

Proof. We can assume $k \geq 1$. The ring R has a duality that we denote by D, and any finitely generated module M satisfies that $M \cong D(D(M))$. Thus

$$\operatorname{Ext}_R^k(A, \varprojlim M_\alpha) \cong \operatorname{Ext}_R^k(A, D(\varinjlim DM_\alpha)) \cong D\operatorname{Tor}_k^R(A, \varinjlim DM_\alpha)$$

As Tor commutes with direct limits

$$\operatorname{Ext}_R^k(A, \varprojlim M_\alpha) \cong \varprojlim D\operatorname{Tor}_k^R(A, DM_\alpha) \cong \varprojlim \operatorname{Ext}_R^k(A, M_\alpha).$$

■

Corollary 9.11 *Let R be an artin algebra. Let $(M_\alpha, f_{\alpha\beta}\colon M_\beta \to M_\alpha)_{\beta>\alpha\in\Lambda}$ be an inverse system of finitely generated left R-modules. If, for each $\alpha \in \Lambda$, $M_\alpha \in \mathcal{P}_1(R\text{-mod})^{\perp}$ then $\varprojlim M_\alpha \in \mathcal{P}_1^{\perp}$.*

Proof. By Lemma 9.10, $\varprojlim M_\alpha \in \mathcal{P}_1(R\text{-mod})^\perp$. Let $A \in \mathcal{P}_1$, then $A = \varinjlim A_i$ for some A_i finitely generated modules of projective dimension at most one.

Since $\varprojlim M_n \cong D(\varinjlim D(M_n))$. This allows us to conclude that, being a dual modules, $\varprojlim M_n$ is pure injective, hence

$$\operatorname{Ext}^1_R(\varinjlim A_i, \varprojlim M_\alpha) \cong \varprojlim \operatorname{Ext}^1_R(A_i, \varprojlim M_\alpha) = 0.$$

■

Example 9.12 [Communicated by B. Huisgen-Zimmermann]
Consider the quiver Q given by

$$\begin{tikzcd} 3 \arrow[r, "\delta"] & 1 \arrow[r, bend left, "\alpha"] \arrow[r, bend right, "\beta"] & 2 \arrow[r, "\epsilon"] \arrow[l, bend left=80, "\gamma"] & 4 \end{tikzcd}$$

Let K be a field and consider the path algebra $R = KQ/I$ where the ideal I is generated by: $\epsilon\beta$, $\gamma\beta$, $\beta\delta$, $\epsilon\alpha\delta$; all paths leaving the vertex 1 that have length at least 3; all paths leaving the vertex 2 that have length at least 2. Then, the following hold:

1. By [28] and [30], $\mathcal{P}(R\text{-mod})$ is contravariantly finite and f.dim R =F. dim $R = 2$, so $(\mathcal{P}_2, \mathcal{P}_2^\perp)$ is of finite type, by Example 9.6.

2. By [30], $\mathcal{P}_1(R\text{-mod})$ fails to be contravariantly finite.

Proposition 9.13 *Let R be the finite dimensional algebra defined in Example 9.12. Then* $(\mathcal{P}_2(R\text{-Mod}), \mathcal{P}_2(R\text{-Mod})^\perp)$ *is of finite type, but* $(\mathcal{P}_1(R\text{-Mod}), \mathcal{P}_1(R\text{-Mod})^\perp)$ *fails to be of finite type.*

Proof. For the first claim, see Example 9.12 1. For the second, let $i \in \{1, 2, 3, 4\}$ and denote by $P_i = Re_i$ the indecomposable projective left modules of R and denote by $I_i = E(S_i)$ the indecomposable injective left modules.

Let $J = P_3 \oplus R\epsilon\alpha \oplus R\gamma\alpha \oplus R\epsilon \oplus R\gamma$. Note that J is a two-sided ideal of R and that R/J is isomorphic to the *Kronecker algebra* that we shall denote by Λ. The left Λ modules are left R modules via the projection $R \to R/J = \Lambda$.

Consider the simple regular modules over Λ:

$$V_\lambda = \begin{tikzcd} K \arrow[r, shift left, "\lambda"] \arrow[r, shift right, "1"'] & K \end{tikzcd} \quad \text{for every } \lambda \in K; \qquad V_\infty = \begin{tikzcd} K \arrow[r, shift left, "1"] \arrow[r, shift right, "0"'] & K \end{tikzcd}.$$

Then,
(i) For every $\lambda \in K$, V_λ is a finitely generated R-module of projective dimension 1.
(ii) $V_\infty \in \mathcal{P}_1(R\text{-mod})^\perp$.

In fact, as an R-module, $V_\lambda \cong P_1/R(\alpha - \lambda\beta)$ and $R(\alpha - \lambda\beta) \cong P_2$. Therefore (i) holds. To verify (ii), note that V_∞ is a quotient of I_4 and recall that $\mathcal{P}_1(R\text{-mod})^\perp$ contains the injective modules and is closed under epimorphic images.

For any $\lambda \in K$, denote by T_λ the corresponding Prüfer Λ-module and by t_λ the corresponding tube in Λ-mod. As T_λ and the modules in t_λ are filtered by V_λ, condition (i) above tells us that they are modules in $\mathcal{A} = {}^\perp(\mathcal{P}_1(R\text{-mod})^\perp)$.

As $\mathcal{B} = \mathcal{P}_1(R\text{-mod})^\perp$ is a tilting class, it is closed by direct limits and extensions. Hence, by condition (ii) above, all the modules in t_∞ and the Prüfer module T_∞ are in $\mathcal{B}$. By Corollary 9.11, we can also conclude that the adic module Z_∞ is in $\mathcal{B}$. Therefore, for any set I, $Z_\infty^{(I)} \in \mathcal{B}$.

Now we are ready to proceed as in [8] to conclude that $(\mathcal{P}_1(R\text{-Mod}), \mathcal{P}_1(R\text{-Mod})^\perp)$ is not of finite type.

By [38, Proposition 3], if T_λ is any of the Prüfer modules of the Kronecker algebra, then the generic module Q is a direct summand of $T_\lambda^{\mathbb{N}}$. Since for finite dimensional algebras, $\mathcal{P}_1$ is closed under products, taking $\lambda \in K$ we deduce that the generic module Q has projective dimension 1 viewed as an R-module. Since Z_∞ is the dual of a Prüfer module it is pure injective, however it is not Σ-pure injective. By results due to Okoh [36, Proposition 1 and Remark], $\mathrm{Ext}^1_\Lambda(Q, Z_\infty^{(\mathbb{N})}) = 0$ would imply $Z_\infty^{(\mathbb{N})}$ pure injective. We conclude that $\mathrm{Ext}^1_R(Q, Z_\infty^{(\mathbb{N})}) \neq 0$ and therefore, by Proposition 4.1, $(\mathcal{P}_1, \mathcal{P}_1^\perp)$ is not of finite type, since $\mathcal{P}_1^\perp \neq \mathcal{B}$. ∎

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