Couette flow solution for regularized 13 moment equations

Henning Struchtrup[*], Toby Thatcher[*] and Manuel Torrilhon[†]

[*]*Dept. of Mechanical Engineering, University of Victoria PO Box 3055 STN CSC, Victoria BC V8W 3P6, Canada, e-mail: struchtr@uvic.ca*
[†]*Applied and Computational Mathematics, Princeton University, 213 Fine Hall, Princeton, NJ, 08544 USA*

Abstract. The order of magnitude method offers an alternative to the methods of Chapman-Enskog and Grad to derive macroscopic transport equations for rarefied gas flows. This method yields the regularized 13 moment equations (R13) which are presented and discussed. Approximate solutions of the R13 equations are considered for steady state Couette flow. The order of magnitude method is used to derive the non-linear bulk equations, which are of second order in the Knudsen number. Knudsen layers result from the linearized R13 equations, which are of third order. Superpositions of bulk solutions and Knudsen layers show excellent agreement with DSMC calculations.

INTRODUCTION

Processes in rarefied gases are well described by the Boltzmann equation [1, 2], a non-linear integro-differential equation that describes the evolution of the particle distribution function f in phase space, i.e. on the microscopic level.

The most important scaling parameter to characterize processes in rarefied gases is the Knudsen number Kn, defined as the ratio between the mean free path of a particle and a relevant reference length scale (e.g. channel width, wavelength, etc.). If the Knudsen number is small, the Boltzmann equation can be reduced to simpler models, which allow faster solutions. Indeed, if Kn < 0.01 (say), the equations of ordinary hydrodynamics—the laws of Navier-Stokes and Fourier (NSF)—can be derived from the Boltzmann equation, e.g. by the Chapman-Enskog method [1, 2]. The NSF equations are macroscopic equations for mass density ρ, velocity v_i and temperature T, and thus pose a mathematically less complex problem than the Boltzmann equation.

Macroscopic equations for rarefied gas flows at Knudsen numbers above 0.01 are highly desirable, since they promise to replace the Boltzmann equation with simpler equations that still capture the relevant physics. The Chapman-Enskog expansion is the classical method to achieve this goal, but the resulting Burnett and super-Burnett equations are unstable [3]. Thus, one either has to find ways to stabilize the Burnett and super-Burnett equations [4]–[6], or to find alternatives to the Chapman-Enskog expansion.

A classical alternative is Grad's moment method [7] which extends the set of variables beyond the hydrodynamic ones by adding deviatoric stress tensor σ_{ij}, heat flux q_i and possibly higher moments of the phase density. The resulting equations are stable but lead to spurious discontinuities in shocks, and for given value of the Knudsen number it is not clear what set of moments one would have to consider [2].

Combinations of the Grad and Chapman-Enskog methods were presented by Reinecke and Kremer who rederived the Burnett equations [8], and by Struchtrup and Torrilhon who performed a Chapman-Enskog expansion around a non-equilibrium phase density of Grad type [9, 10, 2]. The latter approach led to the "regularized 13 moment equations" (R13 equations) which form a stable set of equations for the 13 variables ρ, v_i, T, σ_{ij}, q_i at super-Burnett order.

The weak point in the Grad method is that no statement is made to connect Knudsen number and relevant moments. As a consequence the original derivation of the R13 equations in Refs. [9, 10] required the assumption of different time scales for the basic 13 moments, and higher moments. While this assumption leads to a set of equations with desired behavior, it is difficult to justify since the characteristic times of all moments are, in fact, of the same order.

An alternative approach to the problem was presented by Struchtrup in Refs. [11, 12, 2], partly based on earlier work by Müller et al. [13]. The *order of magnitude method* considers not the Boltzmann equation itself, but its infinite system of moment equations. The method of finding the proper equations with *order of accuracy* λ_0 in the Knudsen number consist of the following three steps:

1. Determination of the *order of magnitude* λ of the moments.
2. Construction of a moment set with minimum number of moments at any order λ.

3. Deletion of all terms in all equations that would lead only to contributions of orders $\lambda > \lambda_0$ in the conservation laws for energy and momentum.

Step 1 is based on a Chapman-Enskog expansion where a moment ϕ is expanded according to

$$\phi = \phi_0 + Kn\phi_1 + Kn^2\phi_2 + Kn^3\phi_3 + \cdots,$$

and the leading order of ϕ is determined by inserting this ansatz into the complete set of moment equations. A moment is said to be of leading order λ if $\phi_\beta = 0$ for all $\beta < \lambda$. This first step agrees with the ideas of Ref. [13]. Alternatively, the order of magnitude of the moments can be found from the principle that *a single term in an equation cannot be larger in size by one or several orders of magnitude than all other terms* [15].

In Step 2 new variables are introduced by linear combination of the moments originally chosen. The new variables are constructed such that the number of moments at a given order λ is minimal. This step does not only simplify the later discussion, but gives an unambiguous set of moments at order λ. This ensures that the final result will be independent of the initial choice of moments.

Step 3 follows from the definition of the order of accuracy λ_0: A set of equations is said to be accurate of order λ_0, when stress σ_{ij} and heat flux q_i are known within the order $O\left(Kn^{\lambda_0}\right)$. The evaluation of this condition is based on the fact that all moment equations are coupled. This implies that each term in any of the moment equations has some influence on all other equations, in particular on the conservation laws. A theory of order λ_0 will consider only those terms in all equations whose leading order of *influence* in the conservation laws is $\lambda \leq \lambda_0$. Luckily, in order to evaluate this condition, it suffices to start with the conservation laws, and step by step, order by order, add the relevant terms that are required

The order of magnitude method was applied to the special cases of Maxwell molecules and the BGK model in Refs. [11, 2], and it was shown that it yields the Euler equations at zeroth order, the NSF equations at first order, and Grad's 13 moment equations (with omission of the non-linear term $\frac{\sigma_{ij}}{\rho}\frac{\partial\sigma_{jk}}{\partial x_k}$) at second order. The regularized 13 moment equations (R13) are obtained as the third order approximation. They consist of the conservation laws for mass, momentum and energy (with $\theta = RT$, and the ideal gas law $p = \rho\theta$; R is the gas constant),

$$\frac{\partial\rho}{\partial t} + \frac{\partial\rho v_k}{\partial x_k} = 0,$$

$$\rho\frac{\partial v_i}{\partial t} + \rho v_k\frac{\partial v_i}{\partial x_k} + \frac{\partial p}{\partial x_i} + \frac{\partial\sigma_{ik}}{\partial x_k} = 0, \tag{1}$$

$$\frac{3}{2}\rho\frac{\partial\theta}{\partial t} + \frac{3}{2}\rho v_k\frac{\partial\theta}{\partial x_k} + \frac{\partial q_k}{\partial x_k} = -(p\delta_{ij} + \sigma_{ij})\frac{\partial v_i}{\partial x_j},$$

and the balance laws for deviatoric stress σ_{ij} (with $\sigma_{ij} = \sigma_{ji}$ and $\sigma_{kk} = 0$) and heat flux q_k, which read

$$\frac{\partial\sigma_{ij}}{\partial t} + v_k\frac{\partial\sigma_{ij}}{\partial x_k} + \sigma_{ij}\frac{\partial v_k}{\partial x_k} + \frac{4}{5}\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + 2p\frac{\partial v_{\langle i}}{\partial x_{j\rangle}} + 2\sigma_{k\langle i}\frac{\partial v_{j\rangle}}{\partial x_k} + \frac{\partial m_{ijk}}{\partial x_k} = -\frac{p}{\mu}\sigma_{ij}, \tag{2}$$

$$\frac{\partial q_i}{\partial t} + v_k\frac{\partial q_i}{\partial x_k} + \frac{5}{2}p\frac{\partial\theta}{\partial x_i} + \frac{5}{2}\sigma_{ik}\frac{\partial\theta}{\partial x_k} + \theta\frac{\partial\sigma_{ik}}{\partial x_k} - \theta\sigma_{ik}\frac{\partial\ln\rho}{\partial x_k} + \frac{7}{5}q_k\frac{\partial v_i}{\partial x_k} + \frac{2}{5}q_k\frac{\partial v_k}{\partial x_i}$$
$$+ \frac{7}{5}q_i\frac{\partial v_k}{\partial x_k} + \frac{1}{2}\frac{\partial R_{ik}}{\partial x_k} + \frac{1}{6}\frac{\partial\Delta}{\partial x_i} + m_{ijk}\frac{\partial v_j}{\partial x_k} - \frac{\sigma_{ij}}{\rho}\frac{\partial\sigma_{jk}}{\partial x_k} = -\frac{2}{3}\frac{p}{\mu}q_i. \tag{3}$$

The equations are closed are by the expressions

$$\Delta = -\frac{\sigma_{ij}\sigma_{ij}}{\rho} - 12\frac{\mu}{p}\left[\theta\frac{\partial q_k}{\partial x_k} + \theta\sigma_{kl}\frac{\partial v_k}{\partial x_l} + \frac{5}{2}q_k\frac{\partial\theta}{\partial x_k} - q_k\theta\frac{\partial\ln\rho}{\partial x_k}\right],$$

$$R_{ij} = -\frac{4}{7}\frac{1}{\rho}\sigma_{k\langle i}\sigma_{j\rangle k} - \frac{24}{5}\frac{\mu}{p}\left[\theta\frac{\partial q_{\langle i}}{\partial x_{j\rangle}} + q_{\langle i}\frac{\partial\theta}{\partial x_{j\rangle}} - \theta q_{\langle i}\frac{\partial\ln\rho}{\partial x_{j\rangle}} + \frac{10}{7}\theta\sigma_{k\langle i}\frac{\partial v_{\langle j\rangle}}{\partial x_{k\rangle}}\right],$$

$$m_{ijk} = -2\frac{\mu}{p}\left[\theta\frac{\partial\sigma_{\langle ij}}{\partial x_{k\rangle}} - \sigma_{\langle ij}\theta\frac{\partial\ln\rho}{\partial x_{k\rangle}} + \frac{4}{5}q_{\langle i}\frac{\partial v_j}{\partial x_{k\rangle}}\right]. \tag{4}$$

In the above equations the indices between angular brackets refer to the symmetric and trace-free parts of tensors. Grad's original 13 moment equations are recovered by setting $\Delta = R_{ij} = m_{ijk} = 0$. μ denotes the viscosity with $\mu = \mu_0 (\theta/\theta_0)^\omega$; for Maxwell moelcules $\omega = 1$.

For general, i.e. non-Maxwellian, molecule types the order of magnitude method was performed to second order in Refs. [12, 2]; the derivation of the third order equations would be far more involved than for Maxwell molecules. Again the equations at zeroth and first order are the Euler and NSF equations with exact viscosity, heat conductivity and Prandtl number. The second order equations are a generalization of Grad's 13 moment equations.

The order of magnitude method reproduces the established results of the CE expansion at zeroth (Euler) and first (NSF) order. Moreover it provides a new link between the Knudsen number and Grad's 13 moment equations which turn out to be of second order in the Knudsen number, together with a generalization of these for non-Maxwellian molecules. Finally, the method provides a rational derivation of the R13 equations.

As was shown in Refs. [9, 10], the R13 equations have the following properties: They contain the Burnett and super-Burnett equations when expanded in a series in the Knudsen number. However, other than the Burnett and super-Burnett equations, the R13 equations are linearly stable for all wavelengths and frequencies. Dispersion relation and damping for the R13 equations agree better with experimental data than those for the Navier-Stokes-Fourier equations, or the original 13 moments system. They also allow the description of Knudsen boundary layers .

BULK EQUATIONS FOR COUETTE GEOMETRY

The scaling processes of Chapman-Enskog expansion and of the order of magnitude method are applied to the complete set of equations without further consideration of geometry, boundary conditions etc. Thus the basic scaling argument gives the maximum size (or minimum order of magnitude) of the moments involved. However, due to the particular geometry of a problem the vector and tensor components of σ_{ij} or q_i can be of different order of magnitude. This allows to further reduce the equations.

In this section the 13 moment equations will be further reduced for steady-state Couette flow: Two infinite parallel plates at constant distance L move with the constant velocities v_W^0, v_W^L relative to each other in their respective planes, and are kept at constant temperatures θ_W^0, θ_W^L. The coordinate frame is chosen such that the planes move into the direction $x = x_1$, and $y = x_2$ is the direction perpendicular to the plates.

We are interested only in the second order equations, and thus it is sufficient to base the argument on Grad's 13 moment equations; a similar treatment of the R13 equations would lead to the same result.

Due to the symmetry of the problem, all variables will depend only on the coordinate y. Since the walls are impermeable, the velocity of the gas must point into the x-direction, that is

$$v_i = \{v(y), 0, 0\}_i \quad \text{and thus} \quad \frac{\partial v_k}{\partial x_k} = 0, \ v_k \frac{\partial}{\partial x_k} = 0.$$

Furthermore, since the set-up is independent of the third space coordinate, $z = x_3$, neither stress nor heat flux should be associated with that direction, so that $\sigma_{13} = \sigma_{23} = q_3 = 0$, and

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11}(y) & \sigma_{12}(y) & 0 \\ \sigma_{12}(y) & \sigma_{22}(y) & 0 \\ 0 & 0 & -\sigma_{11}(y) - \sigma_{22}(y) \end{bmatrix}_{ij}, \quad q_i = \{q_1(y), q_2(y), 0\}_i. \tag{5}$$

The conservation laws (1) reduce to

$$0 = 0, \ \frac{d\sigma_{12}}{dy} = 0, \ \frac{d\rho\theta}{dy} + \frac{d\sigma_{22}}{dy} = 0, \ \frac{d\sigma_{32}}{dy} = 0, \ \frac{dq_2}{dy} = -\sigma_{12}\frac{dv}{dy}.$$

The first and fourth equation are trivial, and two of the remaining three can be integrated so that

$$\sigma_{12} = \sigma_{12}^0 = \text{const.}, \ \rho\theta + \sigma_{22} = P_0 = \text{const.}, \ \frac{dq_2}{dy} = -\sigma_{12}\frac{dv}{dy}. \tag{6}$$

The equations (2, 3) to second order (i.e. with $\Delta = R_{ij} = m_{ij} = 0$) for the relevant components of stress and heat flux can then be cast to read

$$\frac{2}{5}\frac{dq_1}{dy} + P_0\frac{dv}{dy} = -\frac{P_0 - \sigma_{22}}{\mu}\sigma_{12} \quad , \quad -\frac{6}{5}\sigma_{12}\frac{dv}{dy} = -\frac{P_0 - \sigma_{22}}{\mu}\sigma_{22} ,$$

$$\frac{7}{5}q_2\frac{dv}{dy} + \frac{7}{2}\sigma_{12}\frac{d\theta}{dy} = -\frac{2}{3}\frac{P_0 - \sigma_{22}}{\mu}q_1 \quad , \quad \frac{5}{2}P_0\frac{d\theta}{dy} + \sigma_{22}\frac{d\theta}{dy} + \theta\frac{d\sigma_{22}}{dy} + \frac{2}{5}q_1\frac{dv}{dy} = -\frac{2}{3}\frac{P_0 - \sigma_{22}}{\mu}q_2 .$$

In Couette flow the gradients of velocity and temperature are prescribed, while stress and heat flux result from the response of the gas to the given boundary conditions for temperature and velocity. To find the order of magnitude of the components of stress, σ_{ij}, and heat flux, q_i, for given values of the gradients and given Knudsen number, the scales of the quantities are made explicit by replacing

$$\mu \to \varepsilon\mu \ , \ \sigma_{12} \to \varepsilon_{12}\sigma_{12} \ , \ \sigma_{22} \to \varepsilon_{22}\sigma_{22} \ , \ q_1 \to \varepsilon_1 q_1 \ , \ q_2 \to \varepsilon_2 q_2 .$$

Here, ε is the Knudsen number, and ε_{12}, ε_{22}, ε_1, ε_2 are unknown coefficients that are to be determined from the equations. From the original derivation, or from the Chapman-Enskog expansion, we know that these scaling factors are at best of the order of the Knudsen number, but they might be smaller. Thus, they are of order $O(\varepsilon^a)$ with $a \geq 1$. The hydrodynamic variables and their gradients are not scaled.

The rescaled equations for σ_{12} and q_2 read

$$\varepsilon_1\frac{2}{5}\frac{dq_1}{dy} + P_0\frac{dv}{dy} = -\frac{\varepsilon_{12}}{\varepsilon}\frac{P_0 - \varepsilon_{22}\sigma_{22}}{\mu}\sigma_{12} \quad , \quad \varepsilon_{22}\left(\sigma_{22}\frac{d\theta}{dy} + \theta\frac{d\sigma_{22}}{dy}\right) + \varepsilon_1\frac{2}{5}q_1\frac{dv}{dy} + \frac{5}{2}P_0\frac{d\theta}{dy} = -\frac{2}{3}\frac{\varepsilon_2}{\varepsilon}\frac{P_0 - \varepsilon_{22}\sigma_{22}}{\mu}q_2.$$

Each equation has only one explicit term of order $O(\varepsilon^0)$. Since all unkown scaling parameters are at least of order $O(\varepsilon)$, by the principle that *a single term in an equation cannot be larger in size by one or several orders of magnitude than all other terms* [15], this term must be matched in order of magnitude by the term on the right, so that $\varepsilon_{12} = \varepsilon_2 = \varepsilon$.

With the previous results, the scaled equations for σ_{22} and q_1 read

$$-\varepsilon\frac{6}{5}\sigma_{12}\frac{dv}{dy} = -\frac{\varepsilon_{22}}{\varepsilon}\frac{P_0 - \varepsilon_{22}\sigma_{22}}{\mu}\sigma_{22} \quad , \quad \varepsilon\frac{7}{5}q_2\frac{dv}{dy} + \varepsilon\frac{7}{2}\sigma_{12}\frac{d\theta}{dy} = -\frac{2}{3}\frac{\varepsilon_1}{\varepsilon}\frac{P_0 - \varepsilon_{22}\sigma_{22}}{\mu}q_1 .$$

All terms on the left hand sides are at least of order $O(\varepsilon)$, hence it follows $\varepsilon_{22} = \varepsilon_1 = \varepsilon^2$.

The above arguments revealed the proper order of magnitude of all terms in the equations in powers of the Knudsen number ε. The properly scaled equations read

$$\varepsilon^2\frac{dq_1}{dy} + P_0\frac{dv}{dy} = -\frac{P_0 - \varepsilon^2\sigma_{22}}{\mu}\sigma_{12} \quad , \quad -\varepsilon\frac{6}{5}\sigma_{12}\frac{dv}{dy} = -\varepsilon\frac{P_0 - \varepsilon^2\sigma_{22}}{\mu}\sigma_{22} ,$$

$$\varepsilon\left(\frac{7}{5}q_2\frac{dv}{dy} + \frac{7}{2}\sigma_{12}\frac{d\theta}{dy}\right) = -\frac{2}{3}\varepsilon\frac{P_0 - \varepsilon^2\sigma_{22}}{\mu}q_1 \quad , \quad \varepsilon^2\left(\sigma_{22}\frac{d\theta}{dy} + \theta\frac{d\sigma_{22}}{dy} + \frac{2}{5}q_1\frac{dv}{dy}\right) + \frac{5}{2}P_0\frac{d\theta}{dy} = -\frac{2}{3}\frac{P_0 - \varepsilon^2\sigma_{22}}{\mu}q_2 .$$

The first order equations are obtained by removing all terms that carry a factor ε^a ($a \geq 1$). As one would expect, this gives the NSF equations for Couette flow, viz.

$$\sigma_{12} = -\mu\frac{\partial v}{\partial y} \ , \ q_2 = -\frac{15}{4}\mu\frac{d\theta}{dy} \ , \ \sigma_{22} = 0 \ , \ q_1 = 0 .$$

The second order equations are obtained by adding terms that carry the factor ε^1 and ignoring those with higher powers in ε. This gives again the NSF laws for σ_{12} and q_2 and the leading contributions for σ_{22} and q_1 [16, 2],

$$\sigma_{12} = -\mu\frac{\partial v}{\partial y} \ , \ \sigma_{22} = -\frac{6}{5}\frac{\sigma_{12}\sigma_{12}}{P_0} \ , \ q_1 = \frac{7}{2}\frac{\sigma_{12}q_2}{P_0} \ , \ q_2 = -\frac{5}{2}\frac{\mu}{\text{Pr}}\frac{d\theta}{dy} . \tag{7}$$

The equations (7) will be termed as "bulk equations". Indeed, Grad's 13 moment equations cannot describe Knudsen layers [2], and thus the equations are valid only outside the Knudsen layer.

Normal stress, σ_{22}, and heat flux parallel to the wall, q_1, both vanish in the NSF theory, and thus their non-zero values as given by (7) describe pure rarefaction effects. In particular it must be noted that there is no temperature gradient in the x-direction: q_1 is a heat flux that is not driven by a temperature gradient.

The solution of the bulk equations for Couette flow (1,7) requires jump and slip boundary conditions of second order that can be computed from Grad's 13 moment distribution [2],

$$v^{\alpha} - v_W^{\alpha} = \frac{-\frac{2-\chi}{\chi}\alpha_1\sqrt{\frac{\pi}{2}}\sqrt{\theta}\sigma_{12}n^{\alpha} - \frac{1}{5}\alpha_2 q_1}{\rho\theta + \frac{1}{2}\sigma_{22}} \quad , \quad \theta^{\alpha} - \theta_W^{\alpha} = -\frac{\frac{2-\chi}{2\chi}\beta_1\sqrt{\frac{\pi}{2}}\sqrt{\theta}q_2 n^{\alpha} + \frac{1}{4}\theta\sigma_{22}}{\rho\theta + \frac{1}{2}\sigma_{22}} + \frac{V^2}{4} \quad , \qquad (8)$$

with correction factors $\alpha_1, \alpha_2, \beta_1$ close to unity [1, 2]. The constant P_0 follows from the prescribed mass between the plates. Note that the solution of the full 13 moment equations would require an additional boundary condition for σ_{22}.

SUPERPOSITION OF BULK SOLUTION AND KNUDSEN LAYERS

For constructing the solution it is assumed that the non-equilibrium quantities can be split into the bulk (B) and Knudsen layer (L) contributions as $\phi = \phi_B + \phi_L$, where the Knudsen layer contributions vanish in some distance from the wall. The bulk contributions will be computed from the bulk equations of the previous section.

The Knudsen layer contribution is computed from the linearized R13 equations in Couette geometry. Together with the linearized conservation laws, the relevant equations can be reduced to

$$p_0\frac{dv}{dy} + \frac{2}{5}\frac{dq_1}{dy} = -\frac{p_0}{\mu_0}\sigma_{12} = \text{const.} \quad , \quad q_1 = \frac{9}{5}\left(\frac{\mu_0\sqrt{\theta_0}}{p_0}\right)^2\frac{d^2q_1}{dy^2} \quad ,$$

$$\frac{5}{2}\frac{d\theta}{dy} + \frac{\theta_0}{p_0}\frac{d\sigma_{22}}{dy} = -\frac{2}{3}\frac{q_2}{\mu_0} = \text{const.} \quad , \quad \sigma_{22} = \frac{6}{5}\left(\frac{\mu_0\sqrt{\theta_0}}{p_0}\right)^2\frac{d^2\sigma_{22}}{dy^2} \quad ,$$

so that velocity and temperature are obtained as

$$v = v_0 - \frac{\sigma_{12}}{\mu_0}\left(y - \frac{L}{2}\right) - \frac{2}{5}\frac{q_1}{p_0} \quad \text{with} \quad q_1 = A\sinh\left[\sqrt{\frac{5}{9}}\frac{\frac{y}{L} - \frac{1}{2}}{\text{Kn}}\right] + B\cosh\left[\sqrt{\frac{5}{9}}\frac{\frac{y}{L} - \frac{1}{2}}{\text{Kn}}\right] \quad ,$$

$$\theta = \theta_0 - \frac{4}{15}\frac{q_2}{\mu_0}\left(y - \frac{L}{2}\right) - \frac{2}{5}\frac{\theta_0\sigma_{22}}{p_0} \quad \text{with} \quad \sigma_{22} = C\sinh\left[\sqrt{\frac{5}{6}}\frac{\frac{y}{L} - \frac{1}{2}}{\text{Kn}}\right] + D\cosh\left[\sqrt{\frac{5}{6}}\frac{\frac{y}{L} - \frac{1}{2}}{\text{Kn}}\right] \quad .$$

v_0, σ_{12}, A, B and θ_0, q_1, C, D are constants of integration and $\text{Kn} = \frac{\mu_0\sqrt{\theta_0}}{p_0 L}$ is the Knudsen number. We can identify $\left(-\frac{2}{5}\frac{q_1}{p_0}\right)$ and $\left(-\frac{2}{5}\frac{\theta_0\sigma_{22}}{p_0}\right)$ as the Knudsen boundary layers for velocity and temperature according to the R13 equations.

Grad's 13 moment equations do not give linear Knudsen layers. The Burnett equations yield Knudsen layers for density ρ and σ_{22}, but not for v, θ and q_1. The super-Burnett equations give unphysical periodic solutions.

The superposition of bulk solution and Knudsen layers for the R13 equations gives

$$v = v_{|B} - \frac{2}{5}\frac{q_{1|L}}{P_0} \quad , \quad \theta = \theta_{|B} - \frac{2}{5}\frac{\theta_0\sigma_{22|L}}{P_0} \quad , \quad \sigma_{12} = \sigma_{12|B} \quad , \quad \sigma_{22} = \sigma_{22|B} + \sigma_{22|L} \quad ,$$

$$p = P_0 - \sigma_{22|L} \quad , \quad \rho = \frac{p}{\theta} \quad , \quad q_1 = q_{1|B} + q_{1|L} \quad , \quad q_2 = q_{2|B} \quad .$$

The constants of integration A, B, C, D, should be computed from additional boundary conditions for normal stress and parallel heat flux. Since at present no set of reliable boundary conditions is available, the constants were obtained by fitting to DSMC simulations for Maxwell molecules (the symmetry of the results shows that $B = C = 0$). Figures 1 and 2 compare results of DSMC calculations, NSF equations with first order jump and slip boundary conditions, and the R13 equations (superpositions) for $v_W^0 = 0$, $v_W^L = 200\frac{m}{s}$, $\theta_W^0 = \theta_W^L = 273K$ and Knudsen numbers Kn = 0.05, 0.1. The numerical method for the solution of the R13 equation is outlined in Ref. [16]. Recall that $\sigma_{22} = P_0 - p$.

The superpositions match the DSMC simulations quite well. Due to the quadratic term $\frac{V^2}{4}$ in the jump condition $(8)_2$, the temperature maximum is reproduced very well, while some differences can be observed at the boundaries. NSF, on the other hand, can neither describe Knudsen boundary layers nor the rarefaction effects described by σ_{22} and q_1. For Knudsen numbers ≥ 0.5 non-linear higher order terms become relevant which are not included in the second order bulk equations (results not shown).

In conclusion we state that the results presented in this paper show that the R13 equations can give reliable results for boundary value problems, including the description of Knudsen boundary layers.
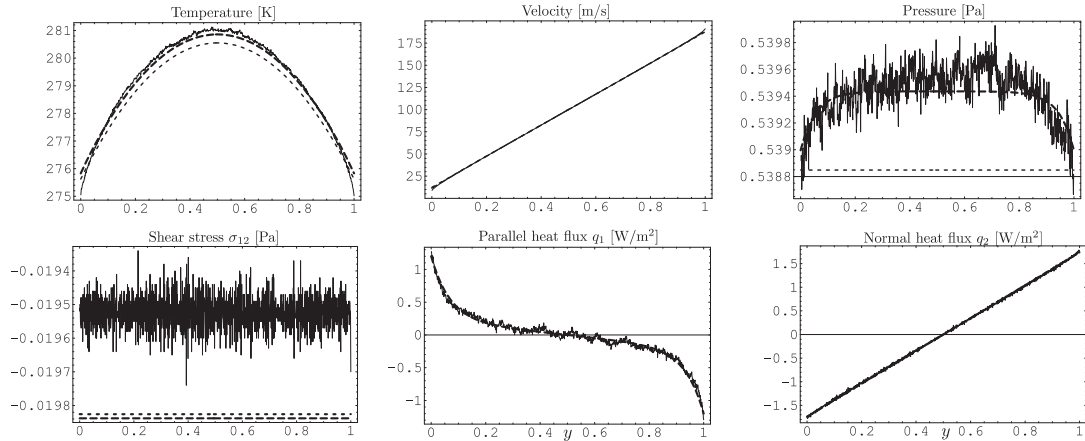
FIGURE 1. Couette flow at Kn $= 0.05$, with $v_W^L = 200\frac{\text{m}}{\text{s}}$. Continuous line: DSMC, finely dashed line (- - - -): NSF, dashed line ($----$): superposition of bulk solution and linear Knudsen layer. Recall that NSF implies $q_1 = \sigma_{22} = 0$ (curves not shown).
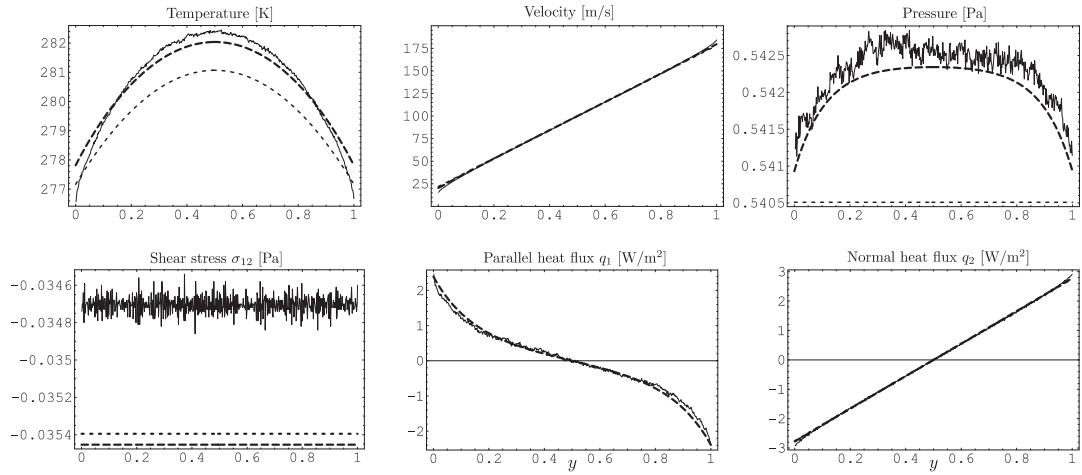

FIGURE 2. Couette flow at Kn $= 0.1$, with $v_W^L = 200\frac{\text{m}}{\text{s}}$. Continuous line: DSMC, finely dashed line (- - - -): NSF, dashed line ($----$): superposition of bulk solution and linear Knudsen layer. Recall that NSF implies $q_1 = \sigma_{22} = 0$ (curves not shown).

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