ADV. THEOR. MATH. PHYS.
Volume 23, Number 1, 75–166, 2019

Coulomb branches of $3d$ $\mathcal{N} = 4$ quiver gauge theories and slices in the affine Grassmannian

Alexander Braverman, Michael Finkelberg, and Hiraku Nakajima

Two appendices by Alexander Braverman Michael Finkelberg, Joel Kamnitzer, Ryosuke Kodera, Hiraku Nakajima, Ben Webster, and Alex Weekes

This is a companion paper of [Part II]. We study Coulomb branches of unframed and framed quiver gauge theories of type ADE. In the unframed case they are isomorphic to the moduli space of based rational maps from $\mathbb{P}^1$ to the flag variety. In the framed case they are slices in the affine Grassmannian and their generalization. In the appendix, written jointly with Joel Kamnitzer, Ryosuke Kodera, Ben Webster, and Alex Weekes, we identify the quantized Coulomb branch with the truncated shifted Yangian.

1. Introduction

In [Nak16] the third named author proposed an approach to define the Coulomb branch $\mathcal{M}_C$ of a $3d\,\mathcal{N}=4$ SUSY gauge theory in a mathematically rigorous way. The subsequent paper [Part II] by the present authors gives a mathematically rigorous definition of $\mathcal{M}_C$ as an affine algebraic variety. The purpose of this companion paper is to determine $\mathcal{M}_C$ for a framed/unframed quiver gauge theory of type ADE.

Let us first consider the unframed case. We are given a quiver $Q = (Q_0, Q_1)$ of type ADE and a Q_0-graded vector space $V = \bigoplus V_i$. Here Q_0 (resp. Q_1) is the set of vertices (resp. oriented arrows) of Q. We consider the corresponding quiver gauge theory: the gauge group is $\mathrm{GL}(V) = \prod \mathrm{GL}(V_i)$, its representation is $\mathbf{N} = \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{\mathrm{o}(h)}, V_{\mathrm{i}(h)})$, where $\mathrm{o}(h)$ (resp. $\mathrm{i}(h)$) is the outgoing (resp. incoming) vertex of an arrow $h \in Q_1$. Our first main result (Theorem 3.1) is $\mathcal{M}_C \cong \mathring{Z}^\alpha$, where $\mathring{Z}^\alpha$ is the moduli space of based rational maps from $\mathbb{P}^1$ to the flag variety $\mathcal{B} = G/B$ of degree α, where the group G is determined by the ADE type of Q, and α is given by the dimension vector $\dim V = (\dim V_i)_{i \in Q_0}$.

In physics literature, it is argued that $\mathcal{M}_C$ is the moduli space of monopoles on $\mathbb{R}^3$. See [HW97] for type A, [Ton99] in general. Here the gauge group $G_c = G_{ADE,c}$ is the maximal compact subgroup of G, hence is determined by the type of the quiver as above. It is expected that two moduli spaces are in fact isomorphic as complex manifolds. See Remark 1.1 below.

We also generalize this result to the case of an affine quiver (Theorem 3.22). We show that $\mathcal{M}_C$ is a partial compactification of the moduli space of parabolic framed G-bundles over $\mathbb{P}^1 \times \mathbb{P}^1$. In fact, we prove this under an assumption that the latter space is normal, which is known only for type A. This space is expected to be isomorphic to the Uhlenbeck partial compactification of the moduli space of G_c-calorons ($= G_c$-instantons on $\mathbb{R}^3 \times S^1$).

In order to show $\mathcal{M}_C \cong \mathring{Z}^\alpha$, we use the criterion in [Part II, Theorem 5.26]: Suppose that we have $\varPi\colon \mathcal{M} \to \mathfrak{t}(V)/\mathbb{W}$ such that $\mathcal{M}$ is a Cohen-Macaulay affine variety and $\varPi$ is flat. Here $\mathfrak{t}(V)$ is the Lie algebra of a maximal torus $T(V)$ of $\mathrm{GL}(V)$, and $\mathbb{W}$ is the Weyl group of $\mathrm{GL}(V)$. If we have a birational isomorphism $\Xi^\circ\colon \mathcal{M} \overset{\approx}{\dashrightarrow} \mathcal{M}_C$ over $\mathfrak{t}(V)/\mathbb{W}$, which is biregular up to codimension 2, it extends to the whole spaces. In order to apply this for $\mathcal{M} = \mathring{Z}^\alpha$, we shall check those required properties. To construct Ξ°, let us recall that $\mathcal{M}_C$ is birational to $T^*T(V)^\vee/\mathbb{W}$ over the complement of union of hyperplanes (generalized root hyperplanes) in $\mathfrak{t}(V)/\mathbb{W}$ (see [Part II, Corollary 5.21]). The same is true by the 'factorization' property of $\mathring{Z}^\alpha$. This defines Ξ° up to codimension 1, hence it is enough to check that Ξ° extends at a generic point in each hyperplane. This last check will be done again by using the factorization. The factorization property is well-known in the context of zastava space Z^α, which is a natural partial compactification of $\mathring{Z}^\alpha$. (See [BDF16] and references therein.)

Remark 1.1. Consider $\mathring{Z}^\alpha$ the moduli space of based rational maps $\mathbb{P}^1 \to G/B$ of degree α, and the moduli space of G_c-monopoles with monopole charge α. It is known that there is a bijection between two moduli spaces (given in [Hit83, Don84] for A_1, [HM89, Hur89] for classical groups and [Jar98] for general groups). It is quite likely that this is an isomorphism of complex manifolds, but not clear to authors whether the proofs give this stronger statement. For A_1, one can check it by using [Nak93], as it is easy to check that the bijection between the moduli space of solutions of Nahm's equation and the moduli space of based maps is an isomorphism.

For affine type A, a bijection is given by [Tak16] based on earlier works [Nye01, NS00, CH08, CH10]. The same question above arises here also.

Let us turn to the framed case. We take an additional Q_0-graded vector space $W = \bigoplus W_i$ and add $\bigoplus \operatorname{Hom}(W_i, V_i)$ to $\mathbf{N}$. The answer has been known in the physics literature [HW97, dBHOO97, dBHO$^+$97, CK98] (at least for type A): $\mathcal{M}_C$ is a moduli space of singular G_c-monopoles on $\mathbb{R}^3$. Two coweights $\lambda, \mu \colon S^1 \to G_c$ attached at 0, ∞ of $\mathbb{R}^3$ are given by dimension vectors for framed and ordinary vertices respectively. We will not use the moduli space of singular monopoles, hence we will not explain how λ and μ arise in this paper. (See the summary in [BDG17, App. A].) But let us emphasize that we need to take the Uhlenbeck partial compactification.[1] This point will be important as explained below.

On the other hand, it was conjectured that $\mathcal{M}_C$ is a framed moduli space of S^1-equivariant instantons on $\mathbb{R}^4$ in [Nak16, §3.2] when μ is dominant. There is a subtle difference between S^1-equivariant instantons and singular monopoles. (The third named author learned it after reading [BDG17]. See [Nak15, §5(iii)].) The former makes sense only when μ is dominant, but is expected to be isomorphic to the latter as complex manifolds.

[1]Uhlenbeck partial compactification is necessary, due to bubbling at 0. This is naturally understood by considering singular monopoles as S^1-equivariant instantons on the Taub-NUT space. See Remark 1.2 below. This bubbling is called *monopole bubbling* in physics literature. e.g.,[KW07, BDG17].

In order to identify $\mathcal{M}_C$, we use the criterion as above. In particular, we need a candidate $\mathcal{M}$ as an affine algebraic variety, or at least a complex analytic space. For this purpose, the moduli space of singular monopoles has a defect, as a complex structure on its Uhlenbeck partial compactification is not constructed in the literature except of type A. (See Remark 1.2 below for type A.)

By [BF10] the Uhlenbeck partial compactification of the framed moduli space of S^1-equivariant instantons on $\mathbb{R}^4$ is isomorphic to a slice $\overline{\mathcal{W}}^\lambda_\mu$ of a $G[[z]]$-orbit in the affine Grassmannian Gr_G in the closure of another orbit. This is a reasonable alternative, as it has a close connection to the zastava space Z^α, and lots of things are known in the literature.

The first half of this paper is devoted to the construction of a generalization of the slice $\overline{\mathcal{W}}^\lambda_\mu$, which makes sense even when μ is *not* dominant. We call it a *generalized slice*, and denote it by the same notation. There are several requirements for the generalized slice. It must be possible for us to check properties required in the criterion above. The most important one is the factorization. Also we should have a dominant birational morphism $\mathring{Z}^{\alpha^*} \to \overline{\mathcal{W}}^\lambda_\mu$, as a property of the Coulomb branch (see [Part II, Remark 5.14] and Remark 3.11 below). These properties naturally led the authors to our definition of generalized slices. The heart of the first half is Proposition 2.10 showing $\overline{\mathcal{W}}^\lambda_\mu$ is a certain affine blowup of the zastava space Z^{α^*} up to codimension 2.

We introduce $\overline{\mathcal{W}}^\lambda_\mu$ as a moduli space of G-bundles over $\mathbb{P}^1$ with trivialization outside 0 and B-structure. This definition originally appeared in [FM99]. We also observe that it has an embedding into $G(z)$ so that its image coincides with the space of scattering matrices of singular monopoles appearing in [BDG17].[2]

We conjecture that Coulomb branches of framed affine quiver gauge theories are Uhlenbeck partial compactifications of moduli spaces of instantons on the Taub-NUT space invariant under a cyclic group action. This is *not* precise yet as 1) we do not endow them with the structure of affine algebraic varieties, and 2) we do not specify the cyclic group action. Also we should recover the moduli spaces of singular monopoles by replacing the cyclic group with S^1. Therefore they must be isomorphic to the generalized slice $\overline{\mathcal{W}}^\lambda_\mu$, but we do not give a proof of this simple version of the conjecture.

Remark 1.2. Singular monopoles are S^1-equivariant instantons on the Taub-NUT space by [Kro85]. See also [BC11]. A moduli space of instantons

[2]This was explained to the authors by D. Gaiotto and J. Kamnitzer during the preparation of the manuscript.

on the Taub-NUT space is described as Cherkis bow variety [Che09, Che10], though mathematically rigorous proof of the completeness is still lacking as far as the authors know. Its S^1-fixed locus is also Cherkis bow variety of a different type. A bow variety is technically more tractable than the moduli space of singular monopoles. In [NT17], Takayama and the third named author will identify $\mathcal{M}_C$ for a framed quiver gauge theory of type A with Cherkis bow variety. This method is applicable for affine type A case, which conjecturally corresponds to a moduli space of $\mathbb{Z}/k\mathbb{Z}$-equivariant instantons on the Taub-NUT space.

The paper is organized as follows. In §2 we introduce a generalized slice as explained above. In §3 we identify the Coulomb branch of a quiver gauge theory of type ADE with generalized slices. We also treat the case of affine type, but without framing. Then we identify the Coulomb branch of a framed Jordan quiver gauge theory with a symmetric power of the surface $xy = w^l$ ($l \geq 0$). In §4 we study the folding of a quiver gauge theory. We show that the character of the coordinate ring of the fixed point loci of the Coulomb branch is given by the twisted monopole formula in [CFHM14]. In the appendices §§A,B written jointly with Joel Kamnitzer, Ryosuke Kodera, Ben Webster, and Alex Weekes, we study the embedding of the quantized Coulomb branch into the ring of difference operators. We find various difference operators known in the literature, such as Macdonald operators and those in representations of Yangian ([GKLO05, KWWY14]). In particular, we show that the quantized Coulomb branch of a framed quiver gauge theory of type ADE is isomorphic to the truncated shifted Yangian introduced in [KWWY14] under the dominance condition in §B.

Notation

We basically follow the notation in [Part II]. However a group G is used for a flag variety $\mathcal{B} = G/B$, while the group for the gauge theory is almost always a product of general linear groups and denoted by $\mathrm{GL}(V)$. Exceptions are Proposition 3.23 and §A.4, where the gauge theory for the adjoint representation of a reductive group is considered. We use W for a vector space for a quiver, while the Weyl group of G is denoted by $\mathbb{W}$.

Errata to [Part II]

1) Following references to `\subsubsection` in 3(vii) are broken, though their hyperlinks in the pdf file correctly work.

- p.1093, replace §3(vii)3(vii) by §3(vii)(d).
- p.1100, replace 3(vii) by (a).
- p.1102, replace 3(vii) by (c).
- p.1104, replace 3(vii) by (d) (twice).
- p.1106, replace §3(vii)3(vii) by §3(vii)(d) (twice).
- p.1107, replace §3(vii)3(vii) by §3(vii)(d).
- p.1116, replace §3(vii)3(vii) by §3(vii)(d).
- p.1117, replace §3(vii)3(vii) by §3(vii)(b).
- p.1119, replace §3(vii)3(vii) by §3(vii)(d).
- p.1136, replace §3(vii)3(vii),3(vii) by §3(vii)(a),(c).

2) In Proposition 6.2, the formula $f[\mathcal{R}_\lambda] * g[\mathcal{R}_\mu] = a_{\lambda,\mu} fg[\mathcal{R}_{\lambda+\mu}]$ is correct for $\operatorname{gr}\mathcal{A}$, but *not* true for $\operatorname{gr}\mathcal{A}_\hbar$. Replace g in the right hand side by $g(\bullet + \hbar\lambda)$. This shift appears when we exchange the order of g and r^λ in the proof.

Acknowledgments

We thank R. Bezrukavnikov, M. Bullimore, S. Cherkis, T. Dimofte, P. Etingof, B. Feigin, D. Gaiotto, D. Gaitsgory, V. Ginzburg, A. Hanany, A. Kuznetsov, A. Oblomkov, V. Pestun, L. Rybnikov, Y. Takayama, M. Temkin and A. Tsymbaliuk for the useful discussions. We also thank J. Kamnitzer, R. Kodera, B. Webster, and A. Weekes for the collaboration.

A.B. was partially supported by the NSF grant DMS-1501047. M.F. was partially funded within the framework of the HSE University Basic Research Program and the Russian Academic Excellence Project '5-100'. The research of H.N. is supported by JSPS Kakenhi Grant Numbers 23224002, 23340005, 24224001, 25220701.

2. Zastava and slices

2(i). Zastava

Let G be an adjoint simple simply laced complex algebraic group. We fix a Borel and a Cartan subgroup $G \supset B \supset T$. Let Λ be the coweight lattice, and let $\Lambda_+ \subset \Lambda$ be the submonoid spanned by the simple coroots α_i, $i \in Q_0$. Here the index set of simple coroots is identified with the set of vertices of the Dynkin diagram, i.e. Q_0. The involution $\alpha \mapsto -w_0\alpha$ of Λ restricts to an involution of Λ_+ and induces an involution $\alpha_i \mapsto \alpha_{i^*}$ of the set of the simple coroots. We will sometimes write $\alpha^* := -w_0\alpha$ for short. For $\alpha = \sum_{i\in Q_0} a_i\alpha_i$

let $Z^\alpha \supset \mathring{Z}^\alpha$ be the corresponding zastava space (moduli space of based quasimaps ϕ from $\mathbb{P}^1$ to the flag variety $\mathcal{B} = G/B$ such that ϕ has no defect at $\infty \in \mathbb{P}^1$ and $\phi(\infty) = B_- \in \mathcal{B}$, the opposite Borel subgroup to B sharing the same Cartan torus T) and its open moduli subspace of based maps. Recall the factorization map $\pi_\alpha\colon Z^\alpha \to \mathbb{A}^\alpha$ and its section $s_\alpha\colon \mathbb{A}^\alpha \hookrightarrow Z^\alpha$, see e.g. [BDF16]: the restriction of π_α to $\mathring{Z}^\alpha \subset Z^\alpha$ takes a based map $\phi\colon \mathbb{P}^1 \to \mathcal{B}$ to the pullback $\phi^*\mathfrak{S}$ of the Q_0-colored Schubert divisor (the boundary of the open B-orbit in $\mathcal{B}$). Recall that $\mathbb{A}^\alpha = \mathbb{A}^{|\alpha|}/S_\alpha$ where $\mathbb{A}^{|\alpha|} = \prod_{i\in Q_0} \mathbb{A}^{a_i}$, and S_α is the product of the symmetric groups $\prod_{i\in Q_0} S_{a_i}$. We define $\underline{Z}^\alpha := Z^\alpha \times_{\mathbb{A}^\alpha} \mathbb{A}^{|\alpha|}$, $\underline{\mathring{Z}}^\alpha := \mathring{Z}^\alpha \times_{\mathbb{A}^\alpha} \mathbb{A}^{|\alpha|}$. Clearly, S_α acts on both $\underline{Z}^\alpha$ and $\underline{\mathring{Z}}^\alpha$, and we have $Z^\alpha = \underline{Z}^\alpha/S_\alpha$, $\mathring{Z}^\alpha = \underline{\mathring{Z}}^\alpha/S_\alpha$.

We denote by $w_{i,r},\ i \in Q_0,\ 1 \le r \le a_i$ the natural coordinates on $\mathbb{A}^{|\alpha|}$. We define an open subset $\mathring{\mathbb{A}}^{|\alpha|} \subset \mathbb{A}^{|\alpha|}$ as the complement to all the diagonals $w_{i,r} = w_{j,s}$, and also $\mathring{\mathbb{A}}^\alpha := \mathring{\mathbb{A}}^{|\alpha|}/S_\alpha \subset \mathbb{A}^\alpha$. We also define a bigger open subset $\mathring{\mathbb{A}}^{|\alpha|} \subset \overset{\bullet}{\mathbb{A}}{}^{|\alpha|} \subset \mathbb{A}^{|\alpha|}$ as the complement to all the pairwise intersections of diagonals. We set $\overset{\bullet}{\mathbb{A}}{}^\alpha := \overset{\bullet}{\mathbb{A}}{}^{|\alpha|}/S_\alpha \subset \mathbb{A}^\alpha$.

Recall that $\pi_\alpha\colon Z^\alpha \to \mathbb{A}^\alpha$ is flat (since $\mathbb{A}^\alpha$ is smooth, Z^α has rational singularities and hence it is Cohen-Macaulay [BF17, Proposition 5.2], and all the fibers of π_α have the same dimension $|\alpha|$ [BFGM02, Propositions 2.6, 6.4, and the line right after 6.4]). Recall the regular functions $(w_{i,r}, y_{i,r})_{i\in Q_0,\ 1\le r\le a_i}$ on $\underline{Z}^\alpha$, see [BDF16, 2.2]. Note that $\pi_\alpha(w_{i,r}, y_{i,r}) = (w_{i,r})$. We have $\pi_\alpha^{-1}(\mathring{\mathbb{A}}^{|\alpha|}) \cong \mathring{\mathbb{A}}^{|\alpha|} \times \mathbb{A}^{|\alpha|}$ with coordinates $(y_{i,r})$ on the second factor. Recall that the boundary $\partial Z^\alpha = Z^\alpha \setminus \mathring{Z}^\alpha$ is the zero divisor of a regular function $F_\alpha \in \mathbb{C}[Z^\alpha]$ defined uniquely up to a multiplicative scalar; in terms of $(w_{i,r}, y_{i,r})$ coordinates we have

$$F_\alpha = \prod_{i,r} y_{i,r} \prod_{\substack{h:Q_1\sqcup\overline{Q_1}\\ \mathrm{o}(h)=i}}^{1\le s\le a_{\mathrm{i}(h)}} (w_{i,r} - w_{\mathrm{i}(h),s})^{-1/2},$$

(the inner product over all arrows $h \in Q_1$ or in the opposite orientation $\overline{Q_1}$ connected to i), see [BDF16, Theorem 1.6.(2)]. It follows that $\pi_\alpha^{-1}(\mathring{\mathbb{A}}^{|\alpha|}) \cap \underline{\mathring{Z}}^\alpha \cong \mathring{\mathbb{A}}^{|\alpha|} \times \mathbb{G}_m^{|\alpha|}$, and $\pi_\alpha^{-1}(\mathring{\mathbb{A}}^\alpha) \cap \mathring{Z}^\alpha \cong (\mathring{\mathbb{A}}^{|\alpha|} \times \mathbb{G}_m^{|\alpha|})/S_\alpha$ (with respect to the diagonal action).

In case $\alpha = \beta+\gamma$, $\beta = \sum_{i\in Q_0} b_i\alpha_i$, $\gamma = \sum_{i\in Q_0} c_i\alpha_i$, according to [BDF16, Theorem 1.6.(3)], the factorization isomorphism

$$\mathfrak{f}_{\beta,\gamma}\colon \underline{\mathring{Z}}^\alpha|_{(\mathbb{A}^{|\beta|}\times\mathbb{A}^{|\gamma|})_{\mathrm{disj}}} \xrightarrow{\sim} (\underline{\mathring{Z}}^\beta \times \underline{\mathring{Z}}^\gamma)|_{(\mathbb{A}^{|\beta|}\times\mathbb{A}^{|\gamma|})_{\mathrm{disj}}}$$

takes $(w_{i,r}, y_{i,r})_{i\in Q_0}^{1\le r\le a_i}$ to

$$\Bigg(\Bigg(w_{i,r}, y_{i,r}\prod_{b_i+1\le s\le a_i}(w_{i,r}-w_{i,s})\Bigg)_{i\in Q_0}^{1\le r\le b_i}, \qquad \Bigg(w_{i,r}, y_{i,r}\prod_{1\le s\le b_i}(w_{i,r}-w_{i,s})\Bigg)_{i\in Q_0}^{b_i+1\le r\le a_i}\Bigg). \tag{2.1}$$

Here $(\mathbb{A}^{|\beta|}\times\mathbb{A}^{|\gamma|})_{\mathrm{disj}}$ is the open subset of $\mathbb{A}^{|\beta|}\times\mathbb{A}^{|\gamma|}$ formed by all the configurations where none of the first $|\beta|$ points meets any of the last $|\gamma|$ points.

Remark 2.2. For a future use we recall the examples of $\underline{\mathring{Z}}^\gamma$ for $|\gamma|=2$, see [BDF16, 5.5, 5.6]. In case $\gamma=\alpha_i+\alpha_j$ and i,j are not connected by an edge of the Dynkin diagram of G, we have $\mathbb{C}[\underline{\mathring{Z}}^\gamma]=\mathbb{C}[w_i,w_j,y_i^{\pm1},y_j^{\pm1}]$. In case $\gamma=\alpha_i+\alpha_j$ and i,j are connected by an edge, we have $\mathbb{C}[\underline{\mathring{Z}}^\gamma]=\mathbb{C}[w_i,w_j,y_i,y_j,y_{ij}^{\pm1}]/(y_iy_j-y_{ij}(w_j-w_i))$. In case $\gamma=2\alpha_i$, we have $\mathbb{C}[\underline{\mathring{Z}}^\gamma]=\mathbb{C}[w_{i,1},w_{i,2},y_{i,1}^{\pm1},y_{i,2}^{\pm1},\xi]/(y_{i,1}-y_{i,2}-\xi(w_{i,1}-w_{i,2}))$.

2(ii). Generalized transversal slices

In this subsection λ is a dominant coweight of G, and $\mu\le\lambda$ is an *arbitrary* coweight of G, not necessarily dominant, such that $\alpha:=\lambda-\mu=\sum_{i\in Q_0}a_i\alpha_i,\ a_i\in\mathbb{N}$. We will define the analogues of slices $\overline{\mathcal{W}}{}^\lambda_{G,\mu}$ of [BF14, Section 2] and prove that they are the Coulomb branches of the corresponding quiver gauge theories.

Recall the convolution diagram $\overline{\mathrm{Gr}}{}^\lambda_G\xleftarrow{\mathbf{p}}\mathcal{GZ}_\lambda^{-\mu}\xrightarrow{\mathbf{q}}Z^{\alpha^*}$ of [FM99, 11.7]. Here $\mathcal{GZ}_\lambda^{-\mu}$ is the moduli space of the following data:

(a) a G-bundle $\mathcal{P}$ on $\mathbb{P}^1$.

(b) A trivialization $\sigma\colon \mathcal{P}_{\mathrm{triv}}|_{\mathbb{P}^1\setminus\{0\}}\xrightarrow{\sim}\mathcal{P}|_{\mathbb{P}^1\setminus\{0\}}$ having a pole of degree $\le\lambda$ at $0\in\mathbb{P}^1$. This means that for an irreducible G-module $V^{\lambda^\vee}$ and the associated vector bundle $\mathcal{V}^{\lambda^\vee}_{\mathcal{P}}$ on $\mathbb{P}^1$ we have $V^{\lambda^\vee}\otimes\mathcal{O}_{\mathbb{P}^1}(-\langle\lambda,\lambda^\vee\rangle\cdot 0)\subset\mathcal{V}^{\lambda^\vee}_{\mathcal{P}}\subset V^{\lambda^\vee}\otimes\mathcal{O}_{\mathbb{P}^1}(-\langle w_0\lambda,\lambda^\vee\rangle\cdot 0)$.

(c) a generalized B-structure ϕ on $\mathcal{P}$ of degree $w_0\mu$ having no defect at $\infty\in\mathbb{P}^1$ and having fiber $B_-\subset G$ at $\infty\in\mathbb{P}^1$ (with respect to the trivialization σ of $\mathcal{P}$ at $\infty\in\mathbb{P}^1$). This means in particular that for an irreducible G-module $V^{\lambda^\vee}$ and the associated vector bundle $\mathcal{V}^{\lambda^\vee}_{\mathcal{P}}$ on $\mathbb{P}^1$ we are given an invertible subsheaf $\mathcal{L}_{\lambda^\vee}\subset\mathcal{V}^{\lambda^\vee}_{\mathcal{P}}$ of degree $-\langle w_0\mu,\lambda^\vee\rangle$.

Now $\mathbf{p}$ forgets ϕ, while $\mathbf{q}$ sends $(\mathcal{P},\sigma,\phi)$ to a collection of invertible subsheaves $\mathcal{L}_{\lambda^\vee}(\langle w_0\lambda,\lambda^\vee\rangle\cdot 0)\subset V^{\lambda^\vee}\otimes\mathcal{O}_{\mathbb{P}^1}$. This collection will be denoted by $\sigma^{-1}\phi(w_0\lambda \cdot 0)$ for short. Clearly, $\deg \sigma^{-1}\phi(w_0\lambda \cdot 0) = \alpha^*$, i.e.

$$\deg \mathcal{L}_{\lambda^\vee}(\langle w_0\lambda, \lambda^\vee\rangle \cdot 0) = \langle w_0\lambda - w_0\mu, \lambda^\vee\rangle = -\langle \alpha^*, \lambda^\vee\rangle.$$

Note that the target of $\mathbf{q}$ in [FM99, 11.7] is erroneously claimed to be Z^α as opposed to Z^{α^*}.

We have an open subvariety $\mathring{\mathcal{G}\mathcal{Z}}{}_\lambda^{-\mu} \subset \mathcal{G}\mathcal{Z}_\lambda^{-\mu}$ formed by all the triples $(\mathcal{P}, \sigma, \phi)$ such that ϕ has no defects (i.e. is a genuine B-structure). We define the generalized slice $\overline{\mathcal{W}}{}^\lambda_\mu := \mathring{\mathcal{G}\mathcal{Z}}{}_\lambda^{-\mu}$. To avoid a misunderstanding about possible nilpotents in the structure sheaf, let us rephrase the definition. Let $'\mathrm{Bun}_G(\mathbb{P}^1)$ be the moduli stack of G-bundles on $\mathbb{P}^1$ with a B-structure at $\infty \in \mathbb{P}^1$. Let $'\overline{\mathrm{Bun}}{}_B^{w_0\mu}(\mathbb{P}^1)$ be the moduli stack of degree $w_0\mu$ generalized B-bundles on $\mathbb{P}^1$ having no defect at $\infty \in \mathbb{P}^1$. Let $\mathrm{Bun}_B^{w_0\mu}(\mathbb{P}^1)$ be its open substack formed by the genuine B-bundles. Finally, we equip $\overline{\mathrm{Gr}}{}^\lambda_G$ with the reduced scheme structure. Then $\mathcal{G}\mathcal{Z}_\lambda^{-\mu} := \overline{\mathrm{Gr}}{}^\lambda_G \times_{'\mathrm{Bun}_G(\mathbb{P}^1)} {'\overline{\mathrm{Bun}}{}_B^{w_0\mu}}(\mathbb{P}^1)$, and $\overline{\mathcal{W}}{}^\lambda_\mu = \mathring{\mathcal{G}\mathcal{Z}}{}_\lambda^{-\mu} := \overline{\mathrm{Gr}}{}^\lambda_G \times_{'\mathrm{Bun}_G(\mathbb{P}^1)} \mathrm{Bun}_B^{w_0\mu}(\mathbb{P}^1)$. Note that $\overline{\mathcal{W}}{}^\lambda_\mu$ is reduced since it is generically reduced and Cohen-Macaulay (see Lemma 2.16 below).

We denote by $s^\lambda_\mu \colon \overline{\mathcal{W}}{}^\lambda_\mu \to Z^{\alpha^*}$ the restriction of $\mathbf{q}\colon \mathcal{G}\mathcal{Z}_\lambda^{-\mu} \to Z^{\alpha^*}$ to $\overline{\mathcal{W}}{}^\lambda_\mu = \mathring{\mathcal{G}\mathcal{Z}}{}_\lambda^{-\mu} \subset \mathcal{G}\mathcal{Z}_\lambda^{-\mu}$. Note that when μ is dominant, $\mathbf{p}\colon \mathring{\mathcal{G}\mathcal{Z}}{}_\lambda^{-\mu} \to \overline{\mathrm{Gr}}{}^\lambda_G$ is a locally closed embedding [BF14, Remark 2.9], and the image coincides with the transversal slice $\overline{\mathcal{W}}{}^\lambda_{G,\mu}$ in the affine Grassmannian Gr_G [BF14, Section 2], hence the name and notation. However, when μ is nondominant, the restriction of $\mathbf{p}\colon \mathring{\mathcal{G}\mathcal{Z}}{}_\lambda^{-\mu} \to \overline{\mathrm{Gr}}{}^\lambda_G$ is not a locally closed embedding.

2(iii). Determinant line bundles and Hecke correspondences

We recall that given a family $f\colon \mathcal{X} \to S$ of smooth projective curves and two line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ on $\mathcal{X}$ Deligne defines a line bundle $\langle \mathcal{L}_1, \mathcal{L}_2\rangle$ on S [Del87, Section 7]. In terms of determinant bundles the definition is simply

$$\begin{aligned}\langle \mathcal{L}_1, \mathcal{L}_2\rangle = \det Rf_*(\mathcal{L}_1 \otimes \mathcal{L}_2) \otimes \det Rf_*(\mathcal{O}_\mathcal{X}) \\ \otimes (\det Rf_*(\mathcal{L}_1) \otimes \det Rf_*(\mathcal{L}_2))^{-1}.\end{aligned} \tag{2.3}$$

Deligne shows that the resulting pairing $\mathrm{Pic}(\mathcal{X}) \times \mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(S)$ is symmetric (obvious) and bilinear (not obvious).

Let Y (resp. $Y^\vee$) denote the coweight (resp. weight) lattice of T. Let $(\cdot,\cdot)$ be an even pairing on Y. Let also X be a smooth projective curve and let Bun_T denote the moduli stack of T-bundles on X. Then to the above data one associates a line bundle $\mathcal{L}_T$ on Bun_T in the following way. Let $e_1, \dots, e_n$ be a basis of Y and let $f_1, \dots, f_n$ be the dual basis (of the dual lattice). For every $i = 1, \ldots, n$ let $\mathcal{L}_i$ denote the line bundle on $\mathrm{Bun}_T \times X$ associated to the weight f_i. Let also $a_{ij} = (e_i, e_j) \in \mathbb{Z}$. Then we define

$$\mathcal{L}_T = \left(\bigotimes_{i=1}^{n} \langle \mathcal{L}_i, \mathcal{L}_i \rangle^{\otimes \frac{a_{ii}}{2}} \right) \otimes \left(\bigotimes_{1 \le i < j \le n} \langle \mathcal{L}_i, \mathcal{L}_j \rangle^{\otimes a_{ij}} \right). \tag{2.4}$$

It is easy to see that $\mathcal{L}_T$ does not depend on the choice of the basis (here, of course, we have to use the statement that Deligne's pairing is bilinear).

We have natural maps $\mathfrak{p}\colon \mathrm{Bun}_B \to \mathrm{Bun}_T$, $\mathfrak{q}\colon \mathrm{Bun}_B \to \mathrm{Bun}_G$. Let $(\cdot, \cdot)$ be a pairing as above. Let us in addition assume that it is W-invariant. Let $\mathcal{L}_T$ be the corresponding determinant bundle. Faltings [Fal03] shows that the pullback $\mathfrak{p}^*\mathcal{L}_T$ descends naturally to Bun_G, i.e. there exists a (canonically defined) line bundle $\mathcal{L}_G$ on Bun_G with an isomorphism $\mathfrak{p}^*\mathcal{L}_T \simeq \mathfrak{q}^*\mathcal{L}_G$. The pullback of $\mathcal{L}_G$ under the natural morphism $\mathrm{Gr}_G \to \mathrm{Bun}_G$ is the determinant line bundle on the affine Grassmannian; it will be also denoted $\mathcal{L}_G$ or even simply $\mathcal{L}$ when no confusion is likely.

2(iv). Basic properties of generalized transversal slices

The convolution diagram $\mathcal{GZ}_\lambda^{-\mu}$ is equipped with the tautological morphism $\mathbf{r}$ to the stack $\overline{\mathrm{Bun}}_B(\mathbb{P}^1)$ (see [BFGM02, Section 1] for notation). The boundary

$$\partial\overline{\mathrm{Bun}}_B(\mathbb{P}^1) := \overline{\mathrm{Bun}}_B(\mathbb{P}^1) \setminus \mathrm{Bun}_B(\mathbb{P}^1)$$

is a Cartier divisor, and $\mathcal{O}_{\mathcal{GZ}_\lambda^{-\mu}}(\mathbf{r}^{-1}(\partial\overline{\mathrm{Bun}}_B(\mathbb{P}^1))) = \mathbf{p}^*\mathcal{L}$ [BFG06, Proof of Theorem 11.6] where $\mathcal{L}$ is the very ample determinant line bundle on Gr_G.

Lemma 2.5. *$\overline{\mathcal{W}}_\mu^\lambda$ is an affine variety.*

Proof. The morphism $(\mathbf{p}, \mathbf{q})\colon \mathcal{GZ}_\lambda^{-\mu} \to \overline{\mathrm{Gr}}_G^\lambda \times Z^{\alpha^*}$ is a closed embedding. Since Z^{α^*} is affine, we conclude that $\mathbf{p}\colon \mathcal{GZ}_\lambda^{-\mu} \to \overline{\mathrm{Gr}}_G^\lambda$ is affine. The complement $\mathcal{GZ}_\lambda^{-\mu} \setminus \mathring{\mathcal{GZ}}_\lambda^{-\mu} = \mathbf{r}^{-1}(\partial\overline{\mathrm{Bun}}_B(\mathbb{P}^1))$, but $\mathcal{O}_{\mathcal{GZ}_\lambda^{-\mu}}(\mathbf{r}^{-1}(\partial\overline{\mathrm{Bun}}_B(\mathbb{P}^1))) = \mathbf{p}^*\mathcal{L}$ is the very ample determinant line bundle on $\mathcal{GZ}_\lambda^{-\mu}$ since $\mathbf{p}$ is affine. Hence the complement $\mathcal{GZ}_\lambda^{-\mu} \setminus \mathbf{r}^{-1}(\partial\overline{\mathrm{Bun}}_B(\mathbb{P}^1)) = \overline{\mathcal{W}}_\mu^\lambda$ is affine. $\square$

The proof of the following lemma is contained in a more general proof of Lemma 2.16 below:

Lemma 2.6. *$\overline{\mathcal{W}}_\mu^\lambda$ is Cohen-Macaulay.* $\square$

Lemma 2.7. *The composition* $\pi_{\alpha^*} \circ s^\lambda_\mu \colon \overline{\mathcal{W}}^\lambda_\mu \to \mathbb{A}^{\alpha^*}$ *is flat.*

Proof. Since $\overline{\mathcal{W}}^\lambda_\mu$ is Cohen-Macaulay, it suffices to prove that all the fibers of $\pi_{\alpha^*} \circ s^\lambda_\mu$ have the same dimension $|\alpha|$. To this end, for $\beta \le \alpha$, we consider a locally closed subvariety $Z^{\alpha^*}_{\beta^*} \subset Z^{\alpha^*}$ formed by all the based quasimaps whose defect at $0 \in \mathbb{A}^1$ has degree precisely β^*. Note that $Z^{\alpha^*}_{\beta^*}$ is isomorphic to an open subvariety in $Z^{\alpha^*-\beta^*}$. It is enough to prove that for $\varphi \in Z^{\alpha^*}_{\beta^*}$, we have $\dim(s^\lambda_\mu)^{-1}(\varphi) \le |\beta|$. Now the desired dimension estimate follows from the semismallness of $\mathbf{q}$ [FM99, Lemma 12.9.1]. □

2(v). A symmetric definition of generalized slices

We slightly modify our definition of the transversal slices.

Given arbitrary coweights μ_-, μ_+ such that $\mu_- + \mu_+ = \mu$ we consider the moduli space $\overline{\mathcal{W}}^\lambda_{\mu_-,\mu_+}$ of the following data: (a) G-bundles $\mathcal{P}_-, \mathcal{P}_+$ on $\mathbb{P}^1$; (b) an isomorphism $\sigma \colon \mathcal{P}_-|_{\mathbb{P}^1\setminus\{0\}} \xrightarrow{\sim} \mathcal{P}_+|_{\mathbb{P}^1\setminus\{0\}}$ having a pole of degree $\le \lambda$ at $0 \in \mathbb{P}^1$; (c) a trivialization of $\mathcal{P}_- = \mathcal{P}_+$ at $\infty \in \mathbb{P}^1$; (d) a reduction ϕ_- of $\mathcal{P}_-$ to a B_--bundle (a B_--structure on $\mathcal{P}_-$) such that the induced T-bundle has degree $-w_0\mu_-$, and the fiber of ϕ_- at $\infty \in \mathbb{P}^1$ is $B \subset G$; (e) a reduction ϕ_+ of $\mathcal{P}_+$ to a B-bundle (a B-structure on $\mathcal{P}_+$) such that the induced T-bundle has degree $w_0\mu_+$, and the fiber of ϕ_+ at $\infty \in \mathbb{P}^1$ is $B_- \subset G$.

Note that the trivial G-bundle on $\mathbb{P}^1$ has a unique B_--reduction of degree 0 with fiber B at ∞. Conversely, a G-bundle $\mathcal{P}_-$ with a B_--structure of degree 0 is necessarily trivial, and its trivialization at ∞ uniquely extends to the whole of $\mathbb{P}^1$. Hence $\overline{\mathcal{W}}^\lambda_{0,\mu} = \overline{\mathcal{W}}^\lambda_\mu$.

For arbitrary $\overline{\mathcal{W}}^\lambda_{\mu_-,\mu_+}$, the G-bundles $\mathcal{P}_-, \mathcal{P}_+$ are identified via σ on $\mathbb{P}^1 \setminus \{0\}$, so they are both equipped with B and B_--structures transversal around $\infty \in \mathbb{P}^1$, that is they are both equipped with a reduction to a T-bundle around $\infty \in \mathbb{P}^1$. So $\mathcal{P}_\pm = \mathcal{P}^T_\pm \times^T G$ for certain T-bundles $\mathcal{P}^T_\pm$ around $\infty \in \mathbb{P}^1$, trivialized at $\infty \in \mathbb{P}^1$. The modified T-bundles $'\mathcal{P}^T_\pm := \mathcal{P}^T_\pm(w_0\mu_- \cdot \infty)$ are also trivialized at $\infty \in \mathbb{P}^1$ and canonically isomorphic to $\mathcal{P}^T_\pm$ off $\infty \in \mathbb{P}^1$. We define $'\mathcal{P}_\pm$ as the result of gluing $\mathcal{P}_\pm$ and $'\mathcal{P}^T_\pm \times^T G$ in the punctured neighbourhood of $\infty \in \mathbb{P}^1$. Then the isomorphism $\sigma \colon {'\mathcal{P}_-}|_{\mathbb{P}^1\setminus\{0,\infty\}} \xrightarrow{\sim} {'\mathcal{P}_+}|_{\mathbb{P}^1\setminus\{0,\infty\}}$ extends to $\mathbb{P}^1 \setminus \{0\}$, and $\phi_\pm$ also extend from $\mathbb{P}^1 \setminus \{\infty\}$ to a B-structure $'\phi_+$ in $'\mathcal{P}_+$ of degree $w_0\mu$ (resp. a B_--structure $'\phi_-$ in $'\mathcal{P}_-$ of degree 0).

This defines an isomorphism $\overline{\mathcal{W}}^\lambda_{\mu_-,\mu_+} \simeq \overline{\mathcal{W}}^\lambda_\mu$.

2(vi). Multiplication of slices

Given $\lambda_1 \geq \mu_2$ and $\lambda_2 \geq \mu_2$ with λ_1, λ_2 dominant, we think of $\overline{\mathcal{W}}^{\lambda_1}_{\mu_1}$ (resp. $\overline{\mathcal{W}}^{\lambda_2}_{\mu_2}$) in the incarnation $\overline{\mathcal{W}}^{\lambda_1}_{\mu_1,0}$ (resp. $\overline{\mathcal{W}}^{\lambda_2}_{0,\mu_2}$). Given $(\mathcal{P}^1_\pm, \sigma_1, \phi^1_\pm) \in \overline{\mathcal{W}}^{\lambda_1}_{\mu_1,0}$ and $(\mathcal{P}^2_\pm, \sigma_2, \phi^2_\pm) \in \overline{\mathcal{W}}^{\lambda_1}_{0,\mu_2}$, we consider $(\mathcal{P}^1_-, \mathcal{P}^2_+, \sigma_2 \circ \sigma_1, \phi^1_-, \phi^2_+) \in \overline{\mathcal{W}}^{\lambda_1+\lambda_2}_{\mu_1,\mu_2} = \overline{\mathcal{W}}^{\lambda_1+\lambda_2}_{\mu_1+\mu_2}$ (note that $\mathcal{P}^2_-$ is canonically trivialized as in §2(v), and $\mathcal{P}^1_+$ is canonically trivialized for the same reason, so that $\mathcal{P}^1_+ = \mathcal{P}^2_-$). This defines a multiplication morphism $\overline{\mathcal{W}}^{\lambda_1}_{\mu_1} \times \overline{\mathcal{W}}^{\lambda_2}_{\mu_2} \to \overline{\mathcal{W}}^{\lambda_1+\lambda_2}_{\mu_1+\mu_2}$.[3]

In particular, taking $\mu_2 = \lambda_2$ so that $\overline{\mathcal{W}}^{\lambda_2}_{\lambda_2}$ is a point and $\overline{\mathcal{W}}^{\lambda_1}_{\mu_1} \times \overline{\mathcal{W}}^{\lambda_2}_{\lambda_2} = \overline{\mathcal{W}}^{\lambda_1}_{\mu_1}$, we get a stabilization morphism $\overline{\mathcal{W}}^{\lambda_1}_{\mu_1} \to \overline{\mathcal{W}}^{\lambda_1+\lambda_2}_{\mu_1+\lambda_2}$.

2(vii). Involution

For the same reason as in §2(v), $\mathcal{P}_+$ in $\overline{\mathcal{W}}^\lambda_{\mu,0}$ is canonically trivialized, so we obtain a morphism $'\mathbf{p}\colon \overline{\mathcal{W}}^\lambda_{\mu,0} \to \overline{\mathrm{Gr}}^{-w_0\lambda}_G$, sending the data of $(\mathcal{P}_\pm, \sigma, \phi_\pm)$ to $(\mathcal{P}_+ = \mathcal{P}_{\mathrm{triv}} \xrightarrow{\sigma^{-1}} \mathcal{P}_-)$. Also, recalling the "symmetric" definition of zastava [BDF16, 2.6], we obtain a morphism $'s^\lambda_\mu\colon \overline{\mathcal{W}}^\lambda_{\mu,0} \to Z^{\alpha^*}$. Namely, it takes a collection $(\mathcal{L}^+_{\lambda^\vee} \subset \mathcal{V}^{\lambda^\vee}_{\mathcal{P}_+} = V^{\lambda^\vee} \otimes \mathcal{O}_{\mathbb{P}^1})$ to a collection of invertible subsheaves $\mathcal{L}^+_{\lambda^\vee}(\langle -\lambda, \lambda^\vee\rangle \cdot 0) \subset \mathcal{V}^{\lambda^\vee}_{\mathcal{P}_-}$. This transformed generalized B-structure will be denoted by $\sigma^{-1}\phi_+(-\lambda \cdot 0)$ for short. Finally, we have an isomorphism $\iota^\lambda_\mu\colon \overline{\mathcal{W}}^\lambda_\mu = \overline{\mathcal{W}}^\lambda_{0,\mu} \xrightarrow{\sim} \overline{\mathcal{W}}^\lambda_{\mu,0}$ obtained by an application of the Cartan involution $\mathfrak{C}$ of G (interchanging B and B_-, and acting on T as $t \mapsto t^{-1}$): replacing $(\mathcal{P}_-, \mathcal{P}_+, \phi_-, \phi_+)$ by $(\mathfrak{C}\mathcal{P}_+, \mathfrak{C}\mathcal{P}_-, \mathfrak{C}\phi_+, \mathfrak{C}\phi_-)$, and σ by $\mathfrak{C}\sigma^{-1}$. Clearly, $\pi_{\alpha^*} \circ {}'s^\lambda_\mu \circ \iota^\lambda_\mu = \pi_{\alpha^*} \circ s^\lambda_\mu\colon \overline{\mathcal{W}}^\lambda_\mu \to \mathbb{A}^{\alpha^*}$. Indeed, for $(\mathcal{P}_\pm, \sigma, \phi_\pm) \in \overline{\mathcal{W}}^\lambda_{0,\mu} = \overline{\mathcal{W}}^\lambda_\mu$, the Q_0-colored divisor $\pi_{\alpha^*} \circ s^\lambda_\mu(\mathcal{P}_\pm, \sigma, \phi_\pm)$ on $\mathbb{P}^1$ measures the nontransversality of ϕ_- and $\sigma^{-1}\phi_+(-\lambda \cdot 0)$, while $\pi_{\alpha^*} \circ {}'s^\lambda_\mu \circ \iota^\lambda_\mu(\mathcal{P}_\pm, \sigma, \phi_\pm)$ measures the nontransversality of $\sigma\phi_-(-\lambda \cdot 0)$ and ϕ_+, and these two measures coincide manifestly.

Under the identification $\overline{\mathcal{W}}^\lambda_{\mu_-,\mu_+} \simeq \overline{\mathcal{W}}^\lambda_\mu$ of §2(v), the isomorphism ι^λ_μ becomes an involution[4] $\iota^\lambda_\mu\colon \overline{\mathcal{W}}^\lambda_\mu \xrightarrow{\sim} \overline{\mathcal{W}}^\lambda_\mu$.

2(viii). Divisors in the convolution diagram

For a future use we describe certain divisors in the convolution diagram. We define a divisor $\mathring{E}_i \subset \overline{\mathcal{W}}^\lambda_{0,\mu} = \mathcal{G}\mathring{\mathcal{Z}}^{-\mu}_\lambda$ as the subvariety formed by the data $(\mathcal{P}, \sigma, \phi)$ such that the transformed B-structure $\mathbf{q}(\mathcal{P}, \sigma, \phi) = \sigma^{-1}\phi(w_0\lambda \cdot 0)$

[3] We learnt of this multiplication from J. Kamnitzer, D. Gaiotto and T. Dimofte.

[4] We thank J. Kamnitzer who has convinced us such an involution should exist.

in the trivial bundle $\mathcal{P}_{\mathrm{triv}}$ acquires the defect of color i at $0 \in \mathbb{P}^1$ (the defect may be possibly bigger than $\alpha_i \cdot 0$). We define a divisor $E_i \subset \mathcal{GZ}_\lambda^{-\mu}$ as the closure of $\mathring{E}_i$. Thus $E := \bigcup_{i \in Q_0} E_i$ is the exceptional divisor of $\mathbf{q}\colon \mathcal{GZ}_\lambda^{-\mu} \to Z^{\alpha^*}$, and $s_\mu^\lambda\colon \mathring{\mathcal{GZ}}_\lambda^{-\mu} \to Z^{\alpha^*}$ restricted to $\mathring{\mathcal{GZ}}_\lambda^{-\mu} \setminus \mathring{E}$ (where $\mathring{E} := \bigcup_{i \in Q_0} \mathring{E}_i$) induces an isomorphism $\mathring{\mathcal{GZ}}_\lambda^{-\mu} \setminus \mathring{E} \xrightarrow{\sim} \mathring{Z}^{\alpha^*}$. Note that E_i can be empty if λ is nonregular.

Similarly, we define a divisor $\mathring{E}'_i \subset \overline{\mathcal{W}}_{\mu,0}^\lambda$ as the subvariety formed by the data $(\mathcal{P}_\pm, \sigma, \phi_\pm)$ such that the transformed B-structure $'s_\mu^\lambda(\mathcal{P}_\pm, \sigma, \phi_\pm) = \sigma^{-1}\phi_+(-\lambda \cdot 0)$ in $\mathcal{P}_-$ acquires the defect of color i at $0 \in \mathbb{P}^1$.

Lemma 2.8. *The full preimage* $(s_\mu^\lambda)^*(\pi_{\alpha^*}^*(\mathbb{A}_i^{\alpha^*})) = \mathring{E}_i \cup (\iota_\mu^\lambda)^{-1}(\mathring{E}'_i)$.

Proof. At a general point of $(\iota_\mu^\lambda)^{-1}(\mathring{E}'_i) \subset \overline{\mathcal{W}}_\mu^\lambda$ the transformed B-structure $\mathbf{q}(\mathcal{P}, \sigma, \phi) = \sigma^{-1}\phi(w_0\lambda \cdot 0)$ in the trivial bundle $\mathcal{P}_{\mathrm{triv}}$ has no defect but at $0 \in \mathbb{P}^1$ is not transversal to B: it lies in position s_i with respect to B. Indeed, if $\mathbb{A}_i^{\alpha^*}$ denotes the divisor formed by the configurations where at least on point of color i meets $0 \in \mathbb{A}^1$, then $\pi_{\alpha^*} \circ {}'s_\mu^\lambda(\mathring{E}'_i) \subset \mathbb{A}_i^{\alpha^*}$, and hence $\pi_{\alpha^*} \circ s_\mu^\lambda(\iota_\mu^\lambda)^{-1}(\mathring{E}'_i) \subset \mathbb{A}_i^{\alpha^*}$. Now the full preimage $(\pi_{\alpha^*} \circ s_\mu^\lambda)^*(\mathbb{A}_i^{\alpha^*})$ a priori lies in the union of the exceptional divisor $\mathring{E}$ and the strict transform $(s_\mu^\lambda)_*^{-1}(\pi_{\alpha^*}^*\mathbb{A}_i^{\alpha^*})$ of the divisor $\pi_{\alpha^*}^*\mathbb{A}_i^{\alpha^*}$. At a general point of the component $\mathring{E}_j$, $j \neq i$, the degree of the defect of $s_\mu^\lambda(\mathcal{P}, \sigma, \phi)$ at 0 is exactly α_j, hence the intersection of $\mathring{E}_j$ with the full preimage of $\mathbb{A}_i^{\alpha^*}$ is not a divisor. Thus $(\pi_{\alpha^*} \circ s_\mu^\lambda)^*(\mathbb{A}_i^{\alpha^*}) = \mathring{E}_i \cup (s_\mu^\lambda)_*^{-1}(\pi_{\alpha^*}^*\mathbb{A}_i^{\alpha^*})$, and the strict transform $(s_\mu^\lambda)_*^{-1}(\pi_{\alpha^*}^*\mathbb{A}_i^{\alpha^*})$ must coincide with $(\iota_\mu^\lambda)^{-1}(\mathring{E}'_i)$. We conclude that the full preimage $(s_\mu^\lambda)^*(\pi_{\alpha^*}^*(\mathbb{A}_i^{\alpha^*})) = \mathring{E}_i \cup (\iota_\mu^\lambda)^{-1}(\mathring{E}'_i)$. □

2(ix). Factorization

For $n \in \mathbb{N}$, let $\mathcal{S}_n$ stand for a hypersurface in $\mathbb{A}^3$ with coordinates x, y, w cut out by an equation $xy = w^n$ (in particular, $\mathcal{S}_0 \simeq \mathbb{G}_m \times \mathbb{A}^1$). Let $\Pi\colon \mathcal{S}_n \to \mathbb{A}^1$ stand for the projection onto the line with w coordinate. Given $i \in Q_0$ such that $a_i \geq 1$ (recall that $\alpha = \sum_{i \in Q_0} a_i\alpha_i$) we identify $\mathbb{A}^{\alpha_{i^*}}$ with $\mathbb{A}^1$, and we set $\beta := \alpha - \alpha_i$. We denote by $\mathbb{G}_m^{\beta^*} \subset \mathbb{A}^{\beta^*}$ the open subset formed by all the colored configurations such that none of the points equals $0 \in \mathbb{A}^1$. We denote by $(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \subset \mathbb{G}_m^{\beta^*} \times \mathbb{A}^1$ the open subset equal to the intersection $(\mathbb{A}^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \cap \mathbb{G}_m^{\beta^*} \times \mathbb{A}^1$.

Let $s_n\colon \mathcal{S}_n \to \mathbb{A}^1 \times \mathbb{A}^1 \simeq Z^{\alpha_{i^*}}$ be the birational isomorphism sending (x, y, w) to (y, w). Then s^λ_μ gives rise to the birational isomorphism

$$(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \times_{\mathbb{A}^{\alpha^*}} \overline{\mathcal{W}}^\lambda_\mu \to (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \times_{\mathbb{A}^{\alpha^*}} Z^{\alpha^*},$$

and $s_{\langle\lambda,\alpha_i^\vee\rangle}$ gives rise to the birational isomorphism

$$(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \times_{\mathbb{A}^{\alpha^*}} (\mathring{Z}^{\beta^*} \times \mathcal{S}_{\langle\lambda,\alpha_i^\vee\rangle}) \to (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \times_{\mathbb{A}^{\alpha^*}} (\mathring{Z}^{\beta^*} \times Z^{\alpha_{i^*}}).$$

Composing the above birational isomorphisms with the factorization isomorphism for zastava

$$(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \times_{\mathbb{A}^{\alpha^*}} Z^{\alpha^*} \xrightarrow{\sim} (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \times_{\mathbb{A}^{\beta^*}\times\mathbb{A}^1} (Z^{\beta^*} \times Z^{\alpha_{i^*}})$$

we obtain a birational isomorphism

$$\varphi\colon (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \times_{\mathbb{A}^{\alpha^*}} \overline{\mathcal{W}}^\lambda_\mu \dashrightarrow (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \times_{\mathbb{A}^{\beta^*}\times\mathbb{A}^1} (\mathring{Z}^{\beta^*} \times \mathcal{S}_{\langle\lambda,\alpha_i^\vee\rangle}).$$

The aim of this section is the following

Proposition 2.9. *The birational isomorphism φ extends to a regular isomorphism of the varieties over $(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}}$:*

$$(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \times_{\mathbb{A}^{\alpha^*}} \overline{\mathcal{W}}^\lambda_\mu \xrightarrow{\sim} (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \times_{\mathbb{A}^{\beta^*}\times\mathbb{A}^1} (\mathring{Z}^{\beta^*} \times \mathcal{S}_{\langle\lambda,\alpha_i^\vee\rangle}).$$

The proof will be given after a certain preparation.

Let $\dot{Z}^{\alpha^*} \subset Z^{\alpha^*}$ be an open subset formed by all the based quasimaps ϕ satisfying the following two conditions: (i) the defect $\operatorname{def}\phi$ is at most a simple coroot; (ii) the multiplicity of the origin $0 \in \mathbb{P}^1$ in the divisor $\pi_{\alpha^*}(\phi)$ is at most a simple coroot.

Note the three properties: a) The codimension of the complement $Z^{\alpha^*} \setminus \dot{Z}^{\alpha^*}$ in Z^{α^*} is 2; b) $\dot{Z}^{\alpha^*}$ inherits the factorization property from Z^{α^*}; c) $\dot{Z}^{\alpha^*}$ is smooth. We consider the open subset $\dot{\mathcal{W}}^\lambda_\mu := (s^\lambda_\mu)^{-1}(\dot{Z}^{\alpha^*}) \subset \overline{\mathcal{W}}^\lambda_\mu$, and $\dot{\mathcal{G}\mathcal{Z}}^{-\mu}_\lambda := \mathbf{q}^{-1}(\dot{Z}^{\alpha^*}) \subset \mathcal{G}\mathcal{Z}^{-\mu}_\lambda$. We set $\dot{E}_i = E_i \cap \dot{\mathcal{G}\mathcal{Z}}^{-\mu}_\lambda$. The codimension of the complement $\overline{\mathcal{W}}^\lambda_\mu \setminus \dot{\mathcal{W}}^\lambda_\mu$ in $\overline{\mathcal{W}}^\lambda_\mu$ is 2. The open embedding $(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \times_{\mathbb{A}^{\alpha^*}} \dot{\mathcal{W}}^\lambda_\mu \hookrightarrow (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \times_{\mathbb{A}^{\alpha^*}} \overline{\mathcal{W}}^\lambda_\mu$ is an isomorphism, so we have to prove that φ extends to a regular isomorphism of the varieties over $(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}}$:

$$(\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \times_{\mathbb{A}^{\alpha^*}} \dot{\mathcal{W}}^\lambda_\mu \xrightarrow{\sim} (\mathbb{G}_m^{\beta^*} \times \mathbb{A}^1)_{\mathrm{disj}} \times_{\mathbb{A}^{\beta^*}\times\mathbb{A}^1} (\mathring{Z}^{\beta^*} \times \mathcal{S}_{\langle\lambda,\alpha_i^\vee\rangle})$$

To this end we will identify $\dot{\mathcal{W}}^\lambda_\mu$ with a certain affine blowup of $\dot{Z}^{\alpha^*}$. We consider the smooth connected components $\partial_i \dot{Z}^{\alpha^*}$, $i \in Q_0$, of the boundary divisor $\dot{Z}^{\alpha^*} \setminus \mathring{Z}^{\alpha^*}$. Recall the divisors $\mathbb{A}_i^{\alpha^*} \subset \mathbb{A}^{\alpha^*}$ formed by all the colored configurations such that at least one point of color $i \in Q_0$ meets $0 \in \mathbb{A}^1$. Let $f_i \in \mathbb{C}[\mathbb{A}^{\alpha^*}]$ be an equation of $\mathbb{A}_i^{\alpha^*}$. Let $\mathcal{I}_i \subset \mathcal{O}_{\dot{Z}^{\alpha^*}}$ (resp. $\mathcal{J}_i \subset \mathcal{O}_{\dot{Z}^{\alpha^*}}$) be the ideal of functions vanishing at $\pi_{\alpha^*}^{-1}(\mathbb{A}_i^{\alpha^*})$ (resp. at $\partial_i \dot{Z}^{\alpha^*}$). We define an ideal $\mathcal{K}_i := \mathcal{I}_i^{\langle\lambda,\alpha_{i^*}^\vee\rangle} + \mathcal{J}_i$, and $\mathcal{K} := \bigcap_{i\in Q_0} \mathcal{K}_i$. We define $\mathrm{Bl}_{\mathcal{K}} \dot{Z}^{\alpha^*}$ as the blowup of $\dot{Z}^{\alpha^*}$ at the ideal $\mathcal{K}$, and $\mathrm{Bl}_{\mathcal{K}}^{\mathrm{aff}} \dot{Z}^{\alpha^*}$ as the complement in $\mathrm{Bl}_{\mathcal{K}} \dot{Z}^{\alpha^*}$ to the union of the strict transforms of the divisors $\partial_i \dot{Z}^{\alpha^*}$, $i \in Q_0$. A crucial step towards Proposition 2.9 is the following

Proposition 2.10. *The identity isomorphism over $\pi_{\alpha^*}^{-1}(\mathbb{G}_m^{\alpha^*})$ extends to a regular isomorphism of the varieties over $\dot{Z}^{\alpha^*}$: $\dot{\mathcal{W}}_\mu^\lambda \xrightarrow{\sim} \mathrm{Bl}_{\mathcal{K}}^{\mathrm{aff}} \dot{Z}^{\alpha^*}$.*

The proof will be given after a few lemmas. Recall that $\mathbf{r}^{-1}(\partial\overline{\mathrm{Bun}}_B(\mathbb{P}^1))$ is the strict transform $\sum_{i\in Q_0} \mathbf{q}_*^{-1}(\partial_i Z^{\alpha^*})$ and $\mathcal{O}_{\mathcal{GZ}_\lambda^{-\mu}}(\mathbf{r}^{-1}(\partial\overline{\mathrm{Bun}}_B(\mathbb{P}^1))) = \mathbf{p}^*\mathcal{L}$. The pullback of the zastava boundary divisor will be denoted by $\sum_{i\in Q_0} \mathbf{q}^*(\partial_i \dot{Z}^{\alpha^*})$.

Lemma 2.11. (1) $\operatorname{div} F_{\alpha^*} = \sum_{i\in Q_0} \partial_i \dot{Z}^{\alpha^*}$*;*

(2) $\operatorname{div} \mathbf{q}^* F_{\alpha^*} = \sum_{i\in Q_0} \mathbf{q}_*^{-1}(\partial_i \dot{Z}^{\alpha^*}) + \sum_{i\in Q_0} \langle\lambda,\alpha_{i^*}^\vee\rangle \dot{E}_i$*;*

(3) $\mathbf{p}^*\mathcal{L} = \mathcal{O}_{\dot{\mathcal{GZ}}_\lambda^{-\mu}}(\sum_{i\in Q_0} \mathbf{q}^*(\partial_i \dot{Z}^{\alpha^*}) - \sum_{i\in Q_0} \langle\lambda,\alpha_{i^*}^\vee\rangle \dot{E}_i)$
$\simeq \mathcal{O}_{\dot{\mathcal{GZ}}_\lambda^{-\mu}}(-\sum_{i\in Q_0} \langle\lambda,\alpha_{i^*}^\vee\rangle \dot{E}_i)$.

Proof. The first assertion is already known. Let us prove the second and third assertions that are equivalent by the remark preceding the lemma. Consider the moduli space $\mathcal{X}_\mu^\lambda$ of the following data:

1) Two G-bundles $\mathcal{P}_+, \mathcal{P}_-$ on $\mathbb{P}^1$.

2) An isomorphism $\sigma\colon \mathcal{P}_- \to \mathcal{P}_+$ away from $0 \in \mathbb{P}^1$, which lies in $G_\mathcal{O}\backslash\overline{\mathrm{Gr}}_G^\lambda$.

3) A B-structure ϕ_+ on the bundle $\mathcal{P}_+$ of degree $w_0\mu$ such that the transformed B-structure $\sigma^{-1}\phi_+(w_0\lambda\cdot 0)$ on $\mathcal{P}_-$ has no defects.

4) A trivialization of the B-bundle ϕ_+ at $\infty \in \mathbb{P}^1$.

Note that the open subspace of $\mathcal{X}_\mu^\lambda$ given by the condition of triviality of $\mathcal{P}_-$ is an open subspace of $\mathring{\mathcal{GZ}}_\lambda^{-\mu}$. We will later introduce a larger space $\widetilde{\mathcal{X}}_\mu^\lambda$ that is a $B_\mathcal{O}$-torsor over the whole of $\mathring{\mathcal{GZ}}_\lambda^{-\mu}$. We have natural maps $\pi_+, \pi_- : \mathcal{X}_\mu^\lambda \to \mathrm{Bun}_G(\mathbb{P}^1)$.

Let $\mathcal{L}_G$ denote the determinant bundle on $\mathrm{Bun}_G(\mathbb{P}^1)$. Then the pullback $\pi_-^*\mathcal{L}_G$ acquires a natural trivialization coming from the B-structure on $\mathcal{P}_-$ (note that the associated T-bundle has degree μ and is trivialized at ∞; hence it is canonically isomorphism to the T-bundle $\mathcal{O}(\mu)$. In fact, the above trivialization is well-defined up to (one) multiplicative scalar; the scalar is fixed if we trivialize the determinant of the T-bundle $\mathcal{O}(\mu)$ on $\mathbb{P}^1$).

On the other hand, consider a bigger moduli space $\widetilde{\mathcal{X}}^\lambda_\mu$ of the data 1–3 above together with a trivialization of $\mathcal{P}_+$ in the formal neighbourhood of 0 compatible with the B-structure (this is a $B_\mathcal{O}$-torsor over $\mathcal{X}^\lambda_\mu$). Then it acquires a natural map $\mathbf{p}_+$ to Gr_G; moreover, it follows easily from 2 and 3 that $\mathbf{p}_+$ actually lands in the open subvariety $\overline{\mathrm{Gr}}_G^{-w_0(\lambda)} \cap S_{-w_0(\lambda)}$ (intersection with a semiinfinite orbit). Indeed, the open subvariety $\overline{\mathrm{Gr}}_G^{-w_0(\lambda)} \cap S_{-w_0(\lambda)} \subset \overline{\mathrm{Gr}}_G^{-w_0(\lambda)}$ is the moduli space of data ($\mathcal{P}_- \xrightarrow{\sigma} \mathcal{P}_+$) where $\mathcal{P}_+$ is trivial on the formal disc, σ has a pole of degree $\leq \lambda$ at 0, and the transformation $\sigma^{-1}\phi_+(w_0\lambda \cdot 0)$ of the standard B-structure in $\mathcal{P}_+$ has no defect at 0. In effect, the latter condition is satisfied for the torus fixed point $-w_0\lambda \in \mathrm{Gr}_G$, and since the condition is $N(\mathcal{O})$-invariant, the intersection $\overline{\mathrm{Gr}}_G^{-w_0(\lambda)} \cap S_{-w_0(\lambda)}$ lies in the above moduli space. However, for the other torus fixed points $-w_0\lambda \neq \nu \in \overline{\mathrm{Gr}}_G^{-w_0(\lambda)}$ the condition is not satisfied, and hence the intersection of the above moduli space with S_ν is empty.

Let us denote by f the projection $\widetilde{\mathcal{X}}^\lambda_\mu \to \mathcal{X}^\lambda_\mu$; let $\widetilde{\pi}_- = \pi_- \circ f$. Let us also recall that we denote by $\mathcal{L}$ the determinant bundle on Gr_G. We have a canonical isomorphism $\mathbf{p}_+^*\mathcal{L} = \widetilde{\pi}_-^*\mathcal{L}_G$. This is so because $\mathbf{p}_+^*\mathcal{L}$ is naturally isomorphic to the ratio of $\widetilde{\pi}_-^*\mathcal{L}_G$ and $\widetilde{\pi}_+^*\mathcal{L}_G$ and the latter is canonically trivial, since $\mathcal{P}_+$ is equipped with a B-structure with a fixed reduction to T).[5]

Since the restriction of $\mathcal{L}$ to $\overline{\mathrm{Gr}}_G^{-w_0(\lambda)} \cap S_{-w_0(\lambda)}$ acquires a canonical trivialization, we get a trivialization of $\mathbf{p}_+^*\mathcal{L}$. This trivialization is equal to the pullback under f of the trivialization of $\pi_-^*\mathcal{L}_G$ discussed above (since both come from the same reduction of $\mathcal{P}_-$ to B).

Let us now consider a variant of this situation. Namely, we consider a moduli space $\overline{\mathcal{X}}^\lambda_\mu$ of the same data as above, except that in 3) we do not require that the transformed B-structure has no defect. Then $\mathcal{X}^\lambda_\mu$ is an open subset of $\overline{\mathcal{X}}^\lambda_\mu$.

Similarly, we have the corresponding space $\widetilde{\overline{\mathcal{X}}}^\lambda_\mu$. We will denote the extension of $\widetilde{\pi}_-$ to $\widetilde{\overline{\mathcal{X}}}^\lambda_\mu$ by $\widetilde{\overline{\pi}}_-$. Similarly, we have $\overline{\mathbf{p}}_+ \colon \widetilde{\overline{\mathcal{X}}}^\lambda_\mu \to \mathrm{Gr}_G$. The line bundles $\overline{\mathbf{p}}_+^*\mathcal{L}$ and $\widetilde{\overline{\pi}}_-^*\mathcal{L}_G$ are again canonically isomorphic, so we can regard them as the same line bundle.

[5] Of course, the fiber of the determinant bundle at $\mathcal{P}_-$ can be trivialized as well (for the same reason), but we want to ignore this here, since a little later we are going to work with a larger space where $\mathcal{P}_-$ will only be endowed with a generalized B-structure.

The above trivialization of this bundle extends to a section (without poles but with zeroes). We are interested in the divisor of this section. Namely, let $\mathcal{E}_i$ denote the divisor in $\overline{\mathcal{X}}^\lambda_\mu$ corresponding to the condition that the transformed B-structure in $\mathcal{P}_-$ acquires the defect of degree at least α_i. Then we claim that the corresponding section of $\widetilde{\overline{\pi}}^*_-\mathcal{L}_G$ vanishes to the order $\langle -w_0(\lambda), \alpha_i^\vee\rangle = \langle \lambda, \alpha_{i^*}^\vee\rangle$ on $\mathcal{E}_i$. This immediately follows from the above, since a similar statement is true on $\overline{\mathrm{Gr}}_G^{-w_0(\lambda)}$.

In effect, assume λ regular (the argument in the general case is similar but requires introducing more notations). Then we have a canonical projection $\mathrm{pr}\colon \mathrm{Gr}_G^{-w_0\lambda} \to \mathcal{B}$. The preimage under pr of the open B-orbit in $\mathcal{B}$ is nothing but $\mathrm{Gr}_G^{-w_0(\lambda)} \cap S_{-w_0(\lambda)}$. The complement to the open B-orbit in $\mathcal{B}$ is the union of Schubert divisors $D_i \subset \mathcal{B},\ i \in Q_0$, and we have $\mathcal{L}|_{\mathrm{Gr}_G^{-w_0\lambda}} \cong \mathcal{O}_{\mathrm{Gr}_G^{-w_0\lambda}}(\sum_{i\in Q_0}\langle w_0\lambda, \alpha_i^\vee\rangle\, \mathrm{pr}^* D_i)$ as can be seen by comparing the T-weights in the fibers of both sides at the T-fixed points, see e.g. the proof of [MV07, Proposition 3.1].

Finally it remains to note that when $\mathcal{P}_-$ is trivialized, its determinant is trivialized as well, and the above section of $\widetilde{\overline{\pi}}^*_-\mathcal{L}_G$ is a function which coincides with $\mathbf{q}^*F_{\alpha^*}$, by its construction in [BF14, Section 4]. $\square$

Lemma 2.12. *The divisor* $\operatorname{div}\mathbf{q}^*\pi^*_{\alpha^*} f_i$ *is the sum of* E_i *and the strict transform* $\mathbf{q}_*^{-1}(\pi^*_{\alpha^*}\mathbb{A}_i^{\alpha^*})$.

Proof. We must prove that the multiplicity of the exceptional divisor E_i in $\operatorname{div}\mathbf{q}^*\pi^*_{\alpha^*} f_i$ equals 1, or equivalently, the multiplicity of $\mathring{E}_i$ in $\operatorname{div}(s^\lambda_\mu)^*\pi^*_{\alpha^*} f_i$ equals 1. But according to Lemma 2.8 $\operatorname{div}(s^\lambda_\mu)^*\pi^*_{\alpha^*} f_i$ is a sum of multiples of $\mathring{E}_i$ and $(\iota^\lambda_\mu)^{-1}(\mathring{E}'_i)$, and the multiplicities of the summands are equal. The latter divisor coincides with the strict transform $(s^\lambda_\mu)_*^{-1}(\pi^*_{\alpha^*}\mathbb{A}_i^{\alpha^*})$ and has multiplicity one, hence the former also has multiplicity one. $\square$

Proof of Proposition 2.10. It is sufficient to prove that the identity isomorphism over $\pi^{-1}_{\alpha^*}(\mathbb{G}_m^{\alpha^*})$ extends to a regular isomorphism $\mathcal{G}\dot{\mathcal{Z}}_\lambda^{-\mu} \xrightarrow{\sim} \mathrm{Bl}_{\mathcal{K}}\dot{Z}^{\alpha^*}$ of the varieties over $\dot{Z}^{\alpha^*}$. Indeed, removing the strict transform $\mathbf{q}_*^{-1}(\partial\dot{Z}^{\alpha^*}) = \mathcal{G}\dot{\mathcal{Z}}_\lambda^{-\mu} \cap (\mathcal{G}\mathcal{Z}_\lambda^{-\mu} \setminus \mathcal{G}\mathring{\mathcal{Z}}_\lambda^{-\mu})$ we then obtain the desired isomorphism $\dot{\mathcal{W}}^\lambda_\mu \xrightarrow{\sim} \mathrm{Bl}^{\mathrm{aff}}_{\mathcal{K}}\dot{Z}^{\alpha^*}$. We first prove that $\mathbf{q}^{-1}\mathcal{K}\cdot\mathcal{O}_{\mathcal{G}\dot{\mathcal{Z}}_\lambda^{-\mu}} \subset \mathcal{O}_{\mathcal{G}\dot{\mathcal{Z}}_\lambda^{-\mu}}$ is an *invertible* sheaf of ideals. More precisely, we will prove that $\mathbf{q}^{-1}\mathcal{K}\cdot\mathcal{O}_{\mathcal{G}\dot{\mathcal{Z}}_\lambda^{-\mu}} \simeq \mathbf{p}^*\mathcal{L}$ where $\mathcal{L}$ is the very ample determinant line bundle on Gr_G. It follows from Lemma 2.11 that $\mathbf{q}^{-1}(\bigcap_{i\in Q_0}\mathcal{J}_i)\cdot\mathcal{O}_{\mathcal{G}\dot{\mathcal{Z}}_\lambda^{-\mu}}$ may be viewed as the sheaf of sections of $\mathbf{p}^*\mathcal{L}$ vanishing at the strict transform $\sum_{i\in Q_0}\mathbf{q}_*^{-1}(\partial_i\dot{Z}^{\alpha^*})$. On the other hand, it follows from Lemma 2.11(3) and Lemma 2.12 that $\mathbf{q}^{-1}(\bigcap_{i\in Q_0}\mathcal{I}_i^{\langle\lambda,\alpha_{i^*}^\vee\rangle})\cdot\mathcal{O}_{\mathcal{G}\dot{\mathcal{Z}}_\lambda^{-\mu}}$ may be viewed as the sheaf of sections of $\mathbf{p}^*\mathcal{L}(-\sum_{i\in Q_0}\langle\lambda,\alpha_{i^*}^\vee\rangle\mathbf{q}_*^{-1}(\pi_{\alpha^*}^*\mathbb{A}_i^{\alpha^*}))$. The strict transforms $\mathbf{q}_*^{-1}(\partial_i\dot{Z}^{\alpha^*})$ and $\mathbf{q}_*^{-1}(\pi_{\alpha^*}^*\mathbb{A}_j^{\alpha^*})$ do not intersect for any i,j (including the case $i=j$). Indeed, for $(\mathcal{P},\sigma,\phi)\in\mathbf{q}_*^{-1}(\partial_{i^*}\dot{Z}^{\alpha^*})$ the generalized B-structure ϕ has defect of order exactly α_{i^*}, and the saturated (non-generalized) B-structure $\widetilde{\phi}$ is well defined, so that $(\mathcal{P},\sigma,\widetilde{\phi})\in\dot{\mathcal{GZ}}_\lambda^{-\mu-\alpha_i}$. For $(\mathcal{P},\sigma,\phi)\in\mathbf{q}_*^{-1}(\partial_{i^*}\dot{Z}^{\alpha^*})\cap\mathbf{q}_*^{-1}(\pi_{\alpha^*}^*\mathbb{A}_j^{\alpha^*})$ we have $(\mathcal{P},\sigma,\widetilde{\phi})\in\mathbf{q}_*^{-1}(\pi_{\beta^*}^*\mathbb{A}_j^{\beta^*})$, and $\pi_{\alpha^*}\mathbf{q}(\mathcal{P},\sigma,\phi)$ contains the origin with multiplicity more than a simple coroot. So the above intersection must be empty. Hence, the sum of subsheaves $\mathbf{p}^*\mathcal{L}\big(-\sum_{i\in Q_0}\langle\lambda,\alpha_{i^*}^\vee\rangle\mathbf{q}_*^{-1}(\pi_{\alpha^*}^*\mathbb{A}_i^{\alpha^*})\big)$ and $\mathbf{p}^*\mathcal{L}\big(-\sum_{i\in Q_0}\mathbf{q}_*^{-1}(\partial_i\dot{Z}^{\alpha^*})\big)$ in $\mathbf{p}^*\mathcal{L}$ is the whole of $\mathbf{p}^*\mathcal{L}$.

Now by the universal property of blowup we obtain a projective morphism $\Upsilon\colon\dot{\mathcal{GZ}}_\lambda^{-\mu}\to\mathrm{Bl}_{\mathcal{K}}\dot{Z}^{\alpha^*}$. From the above, Υ is an isomorphism away from the closed subvariety of codimension 2, namely the intersection of the exceptional divisor with the strict transform of the boundary ∂Z^{α^*} and of the divisor $\prod_{i\in Q_0}f_i=0$. Hence Υ induces an isomorphism of the Picard groups, and the relative Picard group of Υ is trivial. Hence Υ is an isomorphism. Proposition 2.10 is proved. $\square$

Proof of Proposition 2.9. If $\langle\lambda,\alpha_i^\vee\rangle=0$, the desired isomorphism follows from $\mathcal{S}_0\simeq\mathbb{G}_m\times\mathbb{A}^1\simeq\mathring{Z}^{\alpha_{i^*}}$, the usual factorization for $\mathring{Z}^{\alpha^*}$, and the observation that the image of $s_\mu^\lambda\colon(\mathbb{G}_m^{\beta^*}\times\mathbb{A}^1)_{\mathrm{disj}}\times_{\mathbb{A}^{\alpha^*}}\overline{\mathcal{W}}_\mu^\lambda\to Z^{\alpha^*}$ lands into $\mathring{Z}^{\alpha^*}\subset Z^{\alpha^*}$.

For arbitrary λ, we have the isomorphisms

$$
\begin{aligned}
&(\mathbb{G}_m^{\beta^*}\times\mathbb{A}^1)_{\mathrm{disj}}\times_{\mathbb{A}^{\alpha^*}}\dot{\mathcal{W}}_\mu^\lambda\\
&\xrightarrow{\sim}(\mathbb{G}_m^{\beta^*}\times\mathbb{A}^1)_{\mathrm{disj}}\times_{\mathbb{A}^{\alpha^*}}\mathrm{Bl}_{\mathcal{K}}^{\mathrm{aff}}\dot{Z}^{\alpha^*}\\
&\xrightarrow{\sim}(\mathbb{G}_m^{\beta^*}\times\mathbb{A}^1)_{\mathrm{disj}}\times_{\mathbb{A}^{\beta^*}\times\mathbb{A}^1}\mathrm{Bl}_{\mathcal{K}}^{\mathrm{aff}}(\mathring{Z}^{\beta^*}\times Z^{\alpha_{i^*}})\\
&\xrightarrow{\sim}(\mathbb{G}_m^{\beta^*}\times\mathbb{A}^1)_{\mathrm{disj}}\times_{\mathbb{A}^{\beta^*}\times\mathbb{A}^1}(\mathring{Z}^{\beta^*}\times\mathcal{S}_{\langle\lambda,\alpha_i^\vee\rangle})
\end{aligned}
\tag{2.13}
$$

Proposition 2.9 is proved. $\square$

Remark 2.14. Proposition 2.9 along with its proof holds for an arbitrary almost simple simply connected complex algebraic group G, not necessarily simply laced.

2(x). BD slices

Recall the definition of Beilinson-Drinfeld slices $\overline{\mathcal{W}}_\mu^{\underline{\lambda}}$ from [KWWY14, 2.4]. Here $\lambda\geq\mu$ are dominant coweights of G, and $\underline{\lambda}=(\omega_{i_1},\dots,\omega_{i_N})$ is a sequence of fundamental coweights of G such that $\sum_{s=1}^N\omega_{i_s}=\lambda$. Namely, $\overline{\mathcal{W}}_\mu^{\underline{\lambda}}$ is the moduli space of the following data:

(a) a collection of points $(z_1, \dots, z_N) \in \mathbb{A}^N$;

(b) a G-bundle $\mathcal{P}$ on $\mathbb{P}^1$ of isomorphism type μ;

(c) a trivialization (a section) σ of $\mathcal{P}$ on $\mathbb{P}^1 \setminus \{z_1, \dots, z_N\}$ with a pole of degree $\leq \sum_{s=1}^N \omega_{i_s} \cdot z_s$ on the complement, such that the value of the Harder-Narasimhan flag of $\mathcal{P}$ at $\infty \in \mathbb{P}^1$ (where $\mathcal{P}$ is trivialized via σ) is compatible with $B_- \subset G$.

Note that the Harder-Narasimhan flag above can be uniquely refined to a full flag of degree $w_0\mu$ with value $B_- \subset G$ at $\infty \in \mathbb{P}^1$, and this flag is the unique flag of degree $w_0\mu$ with the prescribed value at ∞. Hence the above definition of $\overline{\mathcal{W}}{}^{\underline{\lambda}}_\mu$ can be extended to the case when $\mu \leq \lambda$ is not necessarily dominant (but $\lambda = \sum_{s=1}^N \omega_{i_s}$ is still dominant) as follows: $\overline{\mathcal{W}}{}^{\underline{\lambda}}_\mu$ is the moduli space of the following data:

(a) a collection of points $(z_1, \dots, z_N) \in \mathbb{A}^N$;

(b) a G-bundle $\mathcal{P}$ on $\mathbb{P}^1$;

(c) a trivialization (a section) σ of $\mathcal{P}$ on $\mathbb{P}^1 \setminus \{z_1, \dots, z_N\}$ with a pole of degree $\leq \sum_{s=1}^N \omega_{i_s} \cdot z_s$ on the complement;

(d) a B-structure ϕ on $\mathcal{P}$ of degree $w_0\mu$ having fiber $B_- \subset G$ at $\infty \in \mathbb{P}^1$ (with respect to the trivialization σ).

If in this definition we allow B-structure in (d) to be generalized (but with no defects at $\infty \in \mathbb{P}^1$), then we obtain a partial compactification $\mathcal{GZ}^{-\mu}_{\underline{\lambda}} \supset \overline{\mathcal{W}}{}^{\underline{\lambda}}_\mu$. As in §2(ii), let us rephrase the definition to avoid a possible misunderstanding about nilpotents in the structure sheaf. We equip $\overline{\mathrm{Gr}}{}^{\underline{\lambda}}_{G,BD}$ with the reduced scheme structure. Then $\mathcal{GZ}^{-\mu}_{\underline{\lambda}} := \overline{\mathrm{Gr}}{}^{\underline{\lambda}}_{G,BD} \times_{'\mathrm{Bun}_G(\mathbb{P}^1)} {}'\overline{\mathrm{Bun}}{}^{w_0\mu}_B(\mathbb{P}^1)$, and $\overline{\mathcal{W}}{}^{\underline{\lambda}}_\mu := \overline{\mathrm{Gr}}{}^{\underline{\lambda}}_{G,BD} \times_{'\mathrm{Bun}_G(\mathbb{P}^1)} \mathrm{Bun}^{w_0\mu}_B(\mathbb{P}^1)$.

As in Lemma 2.5 one can prove that $\overline{\mathcal{W}}{}^{\underline{\lambda}}_\mu$ is an affine algebraic variety. For $\alpha = \lambda - \mu$, we have the convolution diagram $\overline{\mathrm{Gr}}{}^{\underline{\lambda}}_{G,BD} \xleftarrow{\mathbf{p}} \mathcal{GZ}^{-\mu}_{\underline{\lambda}} \xrightarrow{\mathbf{q}} Z^{\alpha^*} \times \mathbb{A}^N$ defined similarly to §2(ii). In particular, $\mathbf{q}$ sends $(z_1, \dots, z_N, \mathcal{P}, \sigma, \phi)$ to a collection of invertible subsheaves $\mathcal{L}_{\lambda^\vee}(\sum_{1\leq s\leq N} \langle w_0\omega_{i_s}, \lambda^\vee\rangle \cdot z_s) \subset V^{\lambda^\vee} \otimes \mathcal{O}_{\mathbb{P}^1}$. The restriction of $\mathbf{q}$ to $\overline{\mathcal{W}}{}^{\underline{\lambda}}_\mu \subset \mathcal{GZ}^{-\mu}_{\underline{\lambda}}$ is denoted by $s^{\underline{\lambda}}_\mu \colon \overline{\mathcal{W}}{}^{\underline{\lambda}}_\mu \to Z^{\alpha^*} \times \mathbb{A}^N$. We also have a morphism $\mathbf{r} \colon \mathcal{GZ}^{-\mu}_{\underline{\lambda}} \to \overline{\mathrm{Bun}}_B(\mathbb{P}^1)$ forgetting the data (a,c) above.

Let $f_{i,\underline{\lambda}} \in \mathbb{C}[\mathbb{A}^{\alpha^*} \times \mathbb{A}^N]$ be defined as

$$f_{i,\underline{\lambda}}(\underline{w}, \underline{z}) = \prod_{\substack{1\leq r\leq a_i \\ 1\leq s\leq N\colon i_s = i^*}} (w_{i,r} - z_s).$$

By an abuse of notation we will keep the name $f_{i,\underline{\lambda}}$ for $\pi^*_{\alpha^*} f_{i,\underline{\lambda}} \in \mathbb{C}[Z^{\alpha^*} \times \mathbb{A}^N]$. Let $Z^{\alpha^*} \dot{\times} \mathbb{A}^N \subset Z^{\alpha^*} \times \mathbb{A}^N$ be an open subset formed by all the pairs

$(\phi, \underline{z})$ of the based quasimaps and configurations satisfying the following two conditions: (i) the defect of ϕ is at most a simple coroot; (ii) the multiplicity of z_i in the divisor $\pi_{\alpha^*}(\phi)$ is at most a simple coroot for any $i = 1, \dots, N$. We define an open subvariety $\dot{\mathcal{W}}^{\underline{\lambda}}_{\mu} \subset \overline{\mathcal{W}}^{\underline{\lambda}}_{\mu}$ (resp. $\dot{\mathcal{G}}\mathcal{Z}^{-\mu}_{\underline{\lambda}} \subset \mathcal{G}\mathcal{Z}^{-\mu}_{\underline{\lambda}}$) as $(s^{\underline{\lambda}}_{\mu})^{-1}(Z^{\alpha^*} \dot{\times} \mathbb{A}^N)$ (resp. $\mathbf{q}^{-1}(Z^{\alpha^*} \dot{\times} \mathbb{A}^N)$)

Let $\mathcal{I}^{\underline{\lambda}}_i \subset \mathcal{O}_{Z^{\alpha^*} \dot{\times} \mathbb{A}^N}$ (resp. $\mathcal{J}_i \subset \mathcal{O}_{Z^{\alpha^*} \dot{\times} \mathbb{A}^N}$) be the ideal generated by $f_{i,\underline{\lambda}}$ (resp. the ideal of functions vanishing at $\partial_i Z^{\alpha^*} \dot{\times} \mathbb{A}^N$). We define the ideals $\mathcal{K}_i := \mathcal{I}^{\underline{\lambda}}_i + \mathcal{J}_i$ and $\mathcal{K} := \bigcap_{i \in Q_0} \mathcal{K}_i$. Finally, we define $\mathrm{Bl}_{\mathcal{K}}(Z^{\alpha^*} \dot{\times} \mathbb{A}^N)$ as the blowup of $Z^{\alpha^*} \dot{\times} \mathbb{A}^N$ at the ideal $\mathcal{K}$, and $\mathrm{Bl}^{\mathrm{aff}}_{\mathcal{K}}(Z^{\alpha^*} \dot{\times} \mathbb{A}^N)$ as the complement in $\mathrm{Bl}_{\mathcal{K}}(Z^{\alpha^*} \dot{\times} \mathbb{A}^N)$ to the union of the strict transforms of the divisors $\partial_i Z^{\alpha^*} \dot{\times} \mathbb{A}^N,\ i \in Q_0$.

The proof of the following proposition is parallel to the one of Proposition 2.10.

Proposition 2.15. *The identity isomorphism over $\bigcap_{i \in Q_0} f^{-1}_{i,\underline{\lambda}}(\mathbb{G}_m)$ extends to the isomorphisms $\dot{\mathcal{G}}\mathcal{Z}^{-\mu}_{\underline{\lambda}} \xrightarrow{\sim} \mathrm{Bl}_{\mathcal{K}}(Z^{\alpha^*} \dot{\times} \mathbb{A}^N)$ and $\dot{\mathcal{W}}^{\underline{\lambda}}_{\mu} \xrightarrow{\sim} \mathrm{Bl}^{\mathrm{aff}}_{\mathcal{K}}(Z^{\alpha^*} \dot{\times} \mathbb{A}^N)$ of the varieties over $Z^{\alpha^*} \dot{\times} \mathbb{A}^N$.*

Lemma 2.16. *$\overline{\mathcal{W}}^{\underline{\lambda}}_{\mu}$ is Cohen-Macaulay.*

Proof. The natural morphism $\overline{\mathrm{Gr}}^{\underline{\lambda}}_{G,BD} \to \mathbb{A}^N$ is flat with fibers isomorphic to the products of Schubert varieties in Gr_G, as a consequence of [FL06, Theorem 1]. Since these Schubert varieties are Cohen-Macaulay, we deduce from [Mat86, Corollary of Theorem 23.3] that $\overline{\mathrm{Gr}}^{\underline{\lambda}}_{G,BD}$ is Cohen-Macaulay as well.

The morphisms

$$\overline{\mathrm{Gr}}^{\underline{\lambda}}_{G,BD} \xrightarrow{p} {}'\mathrm{Bun}_G(\mathbb{P}^1) \leftarrow \mathrm{Bun}^{w_0\mu}_B(\mathbb{P}^1)$$

are Tor-independent since the left morphism is a product locally in the smooth topology. In effect, let ${}'\mathrm{Bun}_G(\mathbb{P}^1) \leftarrow \mathcal{H}^{\underline{\lambda}}_{G,BD} \to {}'\mathrm{Bun}_G(\mathbb{P}^1)$ be the (Beilinson-Drinfeld-)Hecke correspondence. Then the right projection is a product locally in the smooth topology. Let ${}'\mathrm{Bun}^{\mathrm{triv}}_G(\mathbb{P}^1) \subset {}'\mathrm{Bun}_G(\mathbb{P}^1)$ be the open substack of trivial G-bundles. Then its preimage in $\mathcal{H}^{\underline{\lambda}}_{G,BD}$ under the left projection to ${}'\mathrm{Bun}_G(\mathbb{P}^1)$ is $G\backslash\overline{\mathrm{Gr}}^{\underline{\lambda}}_{G,BD}$, and the restriction of the right projection to the preimage is $G\backslash p$.

Now the morphism ${}'\mathrm{Bun}_G(\mathbb{P}^1) \leftarrow \mathrm{Bun}^{w_0\mu}_B(\mathbb{P}^1)$ is locally complete intersection (lci) since both the target and the source are smooth. Hence its base change $\overline{\mathrm{Gr}}^{\underline{\lambda}}_{G,BD} \leftarrow \overline{\mathcal{W}}^{\underline{\lambda}}_{\mu}$ is also lci (see [Ill71, Corollary 2.2.3(i)]). Hence the

Cohen-Macaulay property of $\overline{\mathrm{Gr}}^\lambda_{G,BD}$ implies the one of the fiber product $\overline{\mathrm{Gr}}^\lambda_{G,BD} \times_{'\mathrm{Bun}_G(\mathbb{P}^1)} \mathrm{Bun}_B^{w_0\mu}(\mathbb{P}^1) = \overline{\mathcal{W}}^\lambda_\mu$.[6] $\square$

2(xi). An embedding into $G(z)$

Given a collection $(z_1,\dots,z_N) \in \mathbb{A}^N$ we define $P_{\underline{z}}(z) := \prod_{s=1}^N (z-z_s) \in \mathbb{C}[z]$. We also define a closed subvariety $\overline{\mathcal{W}}^{\lambda,\underline{z}}_\mu \subset \overline{\mathcal{W}}^{\underline{\lambda}}_\mu$ as the fiber of the latter over $\underline{z} = (z_1,\dots,z_N)$. We construct a locally closed embedding $\Psi\colon \overline{\mathcal{W}}^{\lambda,\underline{z}}_\mu \hookrightarrow G[z,P^{-1}]$ into an ind-affine scheme as follows. Similarly to §2(v), we have a symmetric definition of BD slices and an isomorphism $\zeta\colon \overline{\mathcal{W}}^{\underline{\lambda},\underline{z}}_\mu = \overline{\mathcal{W}}^{\underline{\lambda},\underline{z}}_{0,\mu} \xrightarrow{\sim} \overline{\mathcal{W}}^{\underline{\lambda},\underline{z}}_{\mu,0}$. We denote $\zeta(\mathcal{P}_\pm,\sigma,\phi_\pm)$ by $(\mathcal{P}'_\pm,\sigma',\phi'_\pm)$. Note that $\mathcal{P}_-$ and $\mathcal{P}'_+$ are trivialized, and $\mathcal{P}'_+$ is obtained from $\mathcal{P}_+$ by an application of a certain Hecke transformation at $\infty \in \mathbb{P}^1$. In particular, we obtain an isomorphism $\mathcal{P}_+|_{\mathbb{A}^1} \xrightarrow{\sim} \mathcal{P}'_+|_{\mathbb{A}^1} = \mathcal{P}_{\mathrm{triv}}|_{\mathbb{A}^1}$. Composing it with $\sigma\colon \mathcal{P}_{\mathrm{triv}}|_{\mathbb{A}^1\setminus\underline{z}} = \mathcal{P}_-|_{\mathbb{A}^1\setminus\underline{z}} \xrightarrow{\sim} \mathcal{P}_+|_{\mathbb{A}^1\setminus\underline{z}}$ we obtain an isomorphism $\mathcal{P}_{\mathrm{triv}}|_{\mathbb{A}^1\setminus\underline{z}} \xrightarrow{\sim} \mathcal{P}_{\mathrm{triv}}|_{\mathbb{A}^1\setminus\underline{z}}$ i.e. an element of $G[z,P_{\underline{z}}^{-1}]$.

Note that if $N=0$, then $P_{\underline{z}} = 1$, and $\overline{\mathcal{W}}^{\lambda,\underline{z}}_\mu = \mathring{Z}^\alpha$ where $\alpha = w_0\mu$. Thus we obtain an embedding $\Psi\colon \mathring{Z}^\alpha \hookrightarrow G[z]$, which should be the same as the one in [Jar98, 4.2].

Here is an equivalent construction of the above embedding due to J. Kamnitzer. Given $(\mathcal{P}_\pm,\sigma,\phi_\pm) \in \overline{\mathcal{W}}^{\underline{\lambda},\underline{z}}_{\mu_-,\mu_+}$, we choose a trivialization of the B-bundle $\phi_+|_{\mathbb{A}^1}$ (resp. of the B_--bundle $\phi_-|_{\mathbb{A}^1}$); two choices of such a trivialization differ by the action of an element of $B[z]$ (resp. $B_-[z]$). This trivialization gives rise to a trivialization of the G-bundle $\mathcal{P}_+|_{\mathbb{A}^1}$ (resp. of $\mathcal{P}_-|_{\mathbb{A}^1}$), so that σ becomes an element of $G(z)$ well-defined up to the left multiplication by an element of $B[z]$ and the right multiplication by an element of $B_-[z]$, i.e. a well defined element of $B[z]\backslash G(z)/B_-[z]$. Clearly, this element of $G(z)$ lies in the closure of the double coset $\overline{G[z]z^{\underline{\lambda},\underline{z}}G[z]}$ where $z^{\underline{\lambda},\underline{z}} := \prod_{s=1}^N (z-z_s)^{\omega_{i_s}}$. Thus we have constructed an embedding $\Psi'\colon \overline{\mathcal{W}}^{\underline{\lambda},\underline{z}}_{\mu_-,\mu_+} \to B[z]\backslash\overline{G[z]z^{\underline{\lambda},\underline{z}}G[z]}/B_-[z]$. If we compose with an embedding $G(z) \hookrightarrow G((z^{-1}))$, then the image of Ψ' lies in

$$B[z]\backslash B_1[[z^{-1}]]z^\mu B_{-,1}[[z^{-1}]]/B_-[z]$$

where $B_1[[z^{-1}]] \subset B[[z^{-1}]]$ (resp. $B_{-,1}[[z^{-1}]] \subset B_-[[z^{-1}]]$) stands for the kernel of evaluation at $\infty \in \mathbb{P}^1$. However, the projection

$$B_1[[z^{-1}]]z^\mu B_{-,1}[[z^{-1}]] \to B[z]\backslash B_1[[z^{-1}]]z^\mu B_{-,1}[[z^{-1}]]/B_-[z]$$

[6] The last part of the proof is due to M. Temkin.

is clearly one-to-one. Summing up, we obtain an embedding

$$\Psi\colon \overline{\mathcal{W}}^{\underline{\lambda},\underline{z}}_{\mu_-,\mu_+} \to B_1[[z^{-1}]]z^\mu B_{-,1}[[z^{-1}]] \bigcap \overline{G[z]z^{\underline{\lambda},\underline{z}}G[z]}.$$

We claim that Ψ is an isomorphism. To see it, we construct the inverse map to $\overline{\mathcal{W}}^{\underline{\lambda},\underline{z}}_{0,\mu}$: given $g(z) \in B_1[[z^{-1}]]z^\mu B_{-,1}[[z^{-1}]] \bigcap \overline{G[z]z^{\underline{\lambda},\underline{z}}G[z]}$, we use it to glue $\mathcal{P}_+$ together with a rational isomorphism $\sigma\colon \mathcal{P}_{\mathrm{triv}} = \mathcal{P}_- \to \mathcal{P}_+$, and define ϕ_+ as the image of the standard B-structure in $\mathcal{P}_{\mathrm{triv}}$ under σ.

Note that the same space of scattering matrices appears in [BDG17, 6.4.1].

2(xii). An example

Let $G = \mathrm{GL}(2) = \mathrm{GL}(V)$ where $V = \mathbb{C}e_1 \oplus \mathbb{C}e_2$. Let $N, m \in \mathbb{N}$; $\underline{\lambda}$ be an N-tuple of fundamental coweights $(1,0)$, and $\mu = (N-m, m)$, so that $w_0\mu = (m, N-m)$. Let $\mathcal{O} := \mathcal{O}_{\mathbb{P}^1}$. We fix a collection $(z_1, \ldots, z_N) \in \mathbb{A}^N$ and define $P_{\underline{z}}(z) := \prod_{s=1}^N (z - z_s) \in \mathbb{C}[z]$. Then $\overline{\mathcal{W}}^{\underline{\lambda},\underline{z}}_\mu$ is the moduli space of flags $(\mathcal{O} \otimes V \supset \mathcal{V} \supset \mathcal{L})$, where

(a) $\mathcal{V}$ is a 2-dimensional locally free subsheaf in $\mathcal{O} \otimes V$ coinciding with $\mathcal{O} \otimes V$ around $\infty \in \mathbb{P}^1$ and such that on $\mathbb{A}^1 \subset \mathbb{P}^1$ the global sections of $\det \mathcal{V}$ coincide with $P_{\underline{z}}\mathbb{C}[z]e_1 \wedge e_2$ as a $\mathbb{C}[z]$-submodule of $\Gamma(\mathbb{A}^1, \det(\mathcal{O}_{\mathbb{A}^1} \otimes V)) = \mathbb{C}[z]e_1 \wedge e_2$.

(b) $\mathcal{L}$ is a line subbundle in $\mathcal{V}$ of degree $-m$, assuming the value $\mathbb{C}e_1$ at $\infty \in \mathbb{P}^1$. In particular, $\deg \mathcal{V}/\mathcal{L} = m - N$.

On the other hand, let us introduce a closed subvariety $\mathcal{M}^{\underline{\lambda},\underline{z}}_\mu$ in $\mathrm{Mat}_2[z]$ formed by all the matrices $\mathsf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that A is a monic polynomial of degree m, while the degrees of B and C are strictly less than m, and $\det \mathsf{M} = P_{\underline{z}}(z)$.

Finally, let $\mathrm{inv}\colon \mathrm{Mat}_2^*(z) \to \mathrm{Mat}_2^*(z)$ denote the inversion operation on matrices with nonzero determinant.

Proposition 2.17 (J. Kamnitzer). *The composition $\Phi := \mathrm{inv} \circ \Psi$ establishes an isomorphism $\overline{\mathcal{W}}^{\underline{\lambda},\underline{z}}_\mu \xrightarrow{\sim} \mathcal{M}^{\underline{\lambda},\underline{z}}_\mu$.*

Proof. First note that a morphism between two line bundles on $\mathbb{P}^1$ trivialized at $\infty \in \mathbb{P}^1$, viewed as a polynomial in z, has a leading term 1 if and only if the morphism preserves the trivializations at ∞.

Let us denote $\Phi(\mathcal{O} \otimes V \supset \mathcal{V} \supset \mathcal{L})$ by $\mathsf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Mat}_2[z]$. By construction $\det \mathsf{M}$ is proportional to $P_{\underline{z}}(z)$. If we view $\det \mathsf{M}$ as a rational morphism from $\det \mathcal{V}$ to $\mathcal{O}$ compatible with trivializations at ∞, we deduce that the leading coefficient of $\det \mathsf{M} = 1$, i.e. $\det \mathsf{M} = P_{\underline{z}}(z)$.

Furthermore, the pole of the first column of M at $\infty \in \mathbb{P}^1$ has order exactly m; more precisely, the leading term of A is az^m, $a \in \mathbb{C}^\times$, while C has a smaller degree. If we view A as a morphism $\mathcal{L} \to \mathcal{O}$ compatible with trivializations at ∞, we obtain $a = 1$.

Let us consider the involution $\iota_\mu^{\lambda,\underline{z}} \colon \overline{\mathcal{W}}_\mu^{\lambda,\underline{z}} \xrightarrow{\sim} \overline{\mathcal{W}}_\mu^{\lambda,\underline{z}}$ defined as in §2(vii). Then by construction, $\Phi \circ \iota_\mu^{\lambda,\underline{z}}$ equals the composition of transposition and Φ. Hence we obtain that $\deg B < m$ as well, so that the image of Φ lies in $\mathcal{M}_\mu^{\lambda,\underline{z}}$.

Now let us describe the inverse morphism $\mho \colon \mathcal{M}_\mu^{\lambda,\underline{z}} \xrightarrow{\sim} \overline{\mathcal{W}}_\mu^{\lambda,\underline{z}}$. Given $\mathsf{M} \in \mathcal{M}_\mu^{\lambda,\underline{z}}$ we view it as a transition matrix in a punctured neighbourhood of $\infty \in \mathbb{P}^1$ to glue a vector bundle $\mathcal{V}$ which embeds, by construction, as a locally free subsheaf into $\mathcal{O} \otimes V$. The morphism $\mathsf{M}\mathcal{O}_{\mathbb{A}^1} e_1 \hookrightarrow \mathcal{O}_{\mathbb{A}^1} \otimes V$ naturally extends to $\infty \in \mathbb{P}^1$ with a pole of degree m, hence it extends to an embedding of $\mathcal{O}(-m \cdot \infty)$ into $\mathcal{V} \subset \mathcal{O} \otimes V$. The image of this embedding is the desired line subbundle $\mathcal{L} \subset \mathcal{V}$.

Finally, one can check that Φ and $\mho$ are inverse to each other. $\square$

Note that this argument is just a special case of the one in §2(xi). Indeed, $z^\mu = \operatorname{diag}(z^{N-m}, z^m)$, and

$$B_1[[z^{-1}]]z^\mu B_{-,1}[[z^{-1}]] = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \deg(a_{22}) = m > \deg(a_{21}), \deg(a_{12}) \right\}.$$

Furthermore, $z^{\lambda,\underline{z}} = \operatorname{diag}(P(z)^{-1}, 1)$, so that $\operatorname{inv}\left(\overline{G[z]z^{\lambda,\underline{z}}G[z]}\right)$ consists of matrices with entries in $\mathbb{C}[z]$ and determinant $P(z)$ up to a scalar multiple.

2(xiii). Scattering matrix

The isomorphism between moduli spaces of G_c-monopoles and rational maps is given by the scattering matrix [Hur85, Jar98]. Although we do not use this fact, let us briefly review it following [AH88], as it seems closely related to a version of definition of the zastava due to Drinfeld (see [BFG06, 2.12]).

Let (A, Φ) be a monopole on a G_c-bundle P over $\mathbb{R}^3$. Let us assume $G_c = \mathrm{SU}(2)$ for brevity. Let k be the monopole charge. Therefore the Higgs field has the asymptotic behaviour

$$\Phi = \sqrt{-1}\operatorname{diag}(1, -1) - \frac{\sqrt{-1}}{2r}\operatorname{diag}(k, -k) + O(r^{-2}).$$

We fix an isomorphism $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$ and consider rays (t, z) $(t \to \pm\infty)$. We solve $(\nabla_A - \sqrt{-1}\Phi)s = 0$ along rays for the associated rank 2 vector bundle $P \times_{\mathrm{SU}(2)} \mathbb{C}^2$. We have two sections s_0, s_1 along $t \to \infty$ and s_0', s_1' along $t \to -\infty$. Here s_0 and s_0' are exponentially decaying while s_1 and s_1' are exponentially growing. The scattering matrix is defined as the transition between (s_0, s_1) and (s_0', s_1'). We consider the framed moduli space, i.e., we choose an eigenvector for Φ at $+\infty$ with eigenvalue $\sqrt{-1}$. Then s_0 is uniquely determined, while s_1 is well-defined up to the addition of a multiple of s_0. On the other hand, s_0' is well-defined up to a multiple of scalar as we do not take the framing at $-\infty$. Therefore the scattering matrix is naturally a map from $\mathbb{C}$ to the quotient stack $B\backslash G/U$ where $G = \mathrm{SL}(2)$, B (resp. U) is the group of upper triangular (resp. uni-triangular) matrices in G. This is nothing but a description of the zastava due to Drinfeld, as explained in [BFG06, §2.12]. Moreover these maps make sense for a Riemann surface X, not only $\mathbb{C}$. As we have learned from Gaiotto, we expect that the scattering matrix for a monopole on $\mathbb{R} \times X$ is a map from X to $B\backslash G/U$.

We write $s_0'(z) = g(z)s_0(z) + f(z)s_1(z)$. Then f and g have no common zeroes and they are well-defined up to

1) multiplying both by an invertible function on X,
2) adding a multiple of f to g.

For $X = \mathbb{C}$, we can uniquely bring it to a pair (f, g) such that

(a) g is a monic polynomial of some degree k,
(b) f is a polynomial of degree $< k$.

Thus we have a based map $z \mapsto g(z)/f(z)$.

See [BDG17, App. A] for the consideration for singular monopoles.

3. Quiver gauge theories

We choose an orientation of the Dynkin graph of G, and denote the set of oriented arrows by Q_1.

3(i). $W = 0$ cases

We set $V_i = \mathbb{C}^{a_i}$, and $\mathrm{GL}(V) := \prod_{i \in Q_0} \mathrm{GL}(V_i)$. The group $\mathrm{GL}(V)$ acts naturally on $\mathbf{N} = \mathbf{N}^\alpha := \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{\mathrm{o}(h)}, V_{\mathrm{i}(h)})$. This representation gives rise to the variety of triples $\mathcal{R} \to \mathrm{Gr} = \mathrm{Gr}_{\mathrm{GL}(V)}$. The equivariant Borel-Moore homology $H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R})$ equipped the convolution product forms a commutative algebra, and its spectrum is the Coulomb branch $\mathcal{M}_C = \mathcal{M}_C(\mathrm{GL}(V), \mathbf{N})$. We choose a maximal torus $T(V) \subset \mathrm{GL}(V)$ and its identification with $\prod_{i\in Q_0} \mathbb{G}_m^{a_i}$. The basic characters of $T(V)$ are denoted $\mathsf{w}_{i,r},\ i \in Q_0,\ 1 \leq r \leq a_i$; their differentials are $w_{i,r} \in \mathfrak{t}^\vee(V)$. The generalized roots are $w_{i,r} - w_{i,s},\ r \neq s$, and $w_{i,r} - w_{j,s}$ for $i \neq j$ vertices connected in the Dynkin diagram.

We consider the algebra homomorphism

$$\iota_*\colon H_*^{T(V)_\mathcal{O}}(\mathcal{R}_{T(V),\mathbf{N}_{T(V)}}) \to H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}) \otimes_{H^*_{\mathrm{GL}(V)}(\mathrm{pt})} H^*_{T(V)}(\mathrm{pt})$$

of [Part II, Lemma 5.17]. According to *loc. cit.*, ι_* becomes an isomorphism over $\mathfrak{t}^\circ(V)$ (note that $\mathfrak{t}^\circ(V)/S_\alpha = \mathring{\mathbb{A}}^\alpha$). We denote by

$$\mathsf{y}_{i,r} \in H_*^{T(V)_\mathcal{O}}(\mathcal{R}_{T(V),\mathbf{N}_{T(V)}})$$

the fundamental class of the fiber of $\mathcal{R}_{T(V),\mathbf{N}_{T(V)}}$ over the point of $\mathrm{Gr}_{T(V)}$ equal to the cocharacter $w^*_{i,r}$ of $T(V)$: an element of the dual basis to $\{w_{i,r}\}_{i\in Q_0,\ 1\leq r\leq a_i}$. Finally, we denote $\mathsf{u}_{i,r} \in H_*^{T(V)_\mathcal{O}}(\mathrm{Gr}_{T(V)})$ the fundamental class of the point $w^*_{i,r}$. According to [Part II, Proposition 5.19],

$$H_*^{T(V)_\mathcal{O}}(\mathrm{Gr}_{T(V)}) \cong \mathbb{C}[\mathfrak{t}(V) \times T^\vee(V)] = \mathbb{C}[w_{i,r}, \mathsf{u}_{i,r}^{\pm 1}\colon i \in Q_0,\ 1 \leq r \leq a_i].$$

We define an isomorphism

$$\Xi\colon \mathbb{C}[\mathring{Z}^\alpha] \otimes_{\mathbb{C}[\mathbb{A}^\alpha]} \mathbb{C}[\mathring{\mathbb{A}}^\alpha] \xrightarrow{\sim} \mathbb{C}[\mathfrak{t}(V) \times T^\vee(V)] \otimes_{\mathbb{C}[\mathbb{A}^\alpha]} \mathbb{C}[\mathring{\mathbb{A}}^\alpha]$$

identical on $w_{i,r}$ and sending $y_{i,r}$ to $\mathsf{u}_{i,r} \cdot \prod_{h\in Q_1:\mathrm{o}(h)=i} \prod_{1\leq s\leq a_{\mathrm{i}(h)}} (w_{\mathrm{i}(h),s} - w_{i,r})$. In notations of [Part II, Theorem 5.26] this defines a generic isomorphism $\Xi^\circ\colon \mathbb{C}[\mathring{Z}^\alpha] \otimes_{\mathbb{C}[\mathbb{A}^\alpha]} \mathbb{C}[\mathring{\mathbb{A}}^\alpha] \xrightarrow{\sim} H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}) \otimes_{\mathbb{C}[\mathbb{A}^\alpha]} \mathbb{C}[\mathring{\mathbb{A}}^\alpha]$. According to [Part II, §4(vi)], the homomorphism

$$\mathbf{z}^*\colon H_*^{T(V)_\mathcal{O}}(\mathcal{R}_{T(V),\mathbf{N}_{T(V)}}) \to H_*^{T(V)_\mathcal{O}}(\mathrm{Gr}_{T(V)})$$

takes $\mathsf{y}_{i,r}$ to $\mathsf{u}_{i,r} \cdot \prod_{h\in Q_1:\mathrm{o}(h)=i} \prod_{1\leq s\leq a_{\mathrm{i}(h)}} (w_{\mathrm{i}(h),s} - w_{i,r})$. Thus in notations of [Part II, Theorem 5.26], Ξ° takes $y_{i,r}$ to $\iota_*\mathsf{y}_{i,r}$.

Theorem 3.1. *The isomorphism*

$$\Xi^\circ\colon \mathbb{C}[\mathring{Z}^\alpha] \otimes_{\mathbb{C}[\mathbb{A}^\alpha]} \mathbb{C}[\mathring{\mathbb{A}}^\alpha] \xrightarrow{\sim} H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}) \otimes_{\mathbb{C}[\mathbb{A}^\alpha]} \mathbb{C}[\mathring{\mathbb{A}}^\alpha]$$

extends to a biregular isomorphism $\mathbb{C}[\mathring{Z}^\alpha] \simeq H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R})$.

Proof. Recall the coordinates $\mathsf{w}_{i,r},\ i \in Q_0,\ 1 \leq r \leq a_i$ (the characters) on $T(V)$ whose differentials are $w_{i,r} \in \mathfrak{t}^\vee(V)$. Let $t \in \mathring{\mathbb{A}}^{|\alpha|} \setminus \mathring{\mathbb{A}}^{|\alpha|} \subset \mathfrak{t}(V)$. According to [Part II, Theorem 5.26] and [Part II, Remark 5.27], it suffices to identify the localizations $\mathbb{C}[\underline{\mathring{Z}}^\alpha]_t$ and $\left(H_*^{T(V)_\mathcal{O}}(\mathcal{R})\right)_t$ as $\mathbb{C}[\mathbb{A}^{|\alpha|}]_t = \mathbb{C}[\mathfrak{t}(V)]_t$-modules. Our t lies on a diagonal divisor. We will consider two possibilities.

First we can have $(w_{i,r} - w_{j,s})(t) = 0$ for $i \neq j$. Then the fixed point set $\mathcal{R}^t_{\mathbf{N}}$ is isomorphic to the product $\mathrm{Gr}_{T_1} \times \mathcal{R}_{T_2,\mathbf{N}'}$. Here T_2 is a 2-dimensional torus with coordinates $\mathsf{w}_{i,r}, \mathsf{w}_{j,s}$, and T_1 is an $(|\alpha| - 2)$-dimensional torus with coordinates $\{\mathsf{w}_{k,p} \colon (i,r) \neq (k,p) \neq (j,s)\}$, so that $T(V) = T_1 \times T_2$. Furthermore, $\mathbf{N}'$ is the following representation of T_2: if i,j are not connected by an edge of the Dynkin diagram, then $\mathbf{N}' = 0$; and if there is an arrow h from i to j in our orientation Ω, then $\mathbf{N}'$ is a character $\mathsf{w}_{i,r}^{-1}\mathsf{w}_{j,s}$ with differential $w_{j,s} - w_{i,r}$. In case i,j are not connected we conclude that in notations of [Part II, Theorem 5.26(2)], $G' = T(V)$, and

$$\begin{aligned}\left(H_*^{G'_\mathcal{O}}(\mathcal{R}_{G',\mathbf{N}'})\right)_t &= \left(H_*^{T(V)_\mathcal{O}}(\mathrm{Gr}_{T(V)})\right)_t \\ &= (\mathbb{C}[\mathfrak{t}(V) \times T^\vee(V)])_t = (\mathbb{C}[\mathbb{A}^{|\alpha|} \times \mathbb{G}_m^{|\alpha|}])_t.\end{aligned}$$

We define $\Xi^t \colon (\mathbb{C}[\mathring{Z}^\alpha] \otimes_{\mathbb{C}[\mathbb{A}^\alpha]} \mathbb{C}[\mathbb{A}^{|\alpha|}])_t \to \left(H_*^{G'_\mathcal{O}}(\mathcal{R}_{G',\mathbf{N}'})\right)_t$ identical on $w_{k,p}$ and sending $y_{k,p}$ to $\mathsf{u}_{k,p} \cdot \prod_{h \in Q_1 : \mathrm{o}(h) = k} \prod_{1 \leq q \leq a_{\mathrm{i}(h)}} (w_{\mathrm{i}(h),q} - w_{k,p})$. Note that at the moment Ξ^t is defined only as a rational morphism. The condition of [Part II, Theorem 5.26(2)] is trivially satisfied. Also, Ξ^t is a regular isomorphism due to the factorization property of zastava (2.1) and e.g. [BDF16, 5.5], and the fact that the factors $(w_{\mathrm{i}(h),q} - w_{k,p})$ in the formula for Ξ^t are all invertible at t.

In case $h \in Q_1$ with $\mathrm{o}(h) = i$, $\mathrm{i}(h) = j$, in notations of [Part II, Theorem 5.26(2)], $G' = T(V)$, and

$$\left(H_*^{G'_\mathcal{O}}(\mathcal{R}_{G',\mathbf{N}'})\right)_t = \left(H_*^{T(V)_\mathcal{O}}(\mathrm{Gr}_{T_1} \times \mathcal{R}_{T_2,\mathbf{N}'})\right)_t = (\mathbb{C}[\mathfrak{t}_1 \times T_1^\vee] \otimes A)_t$$

where A is generated by $w_{i,r}, w_{j,s}$ and the fundamental classes $\mathsf{y}'_{i,r}, \mathsf{y}'_{j,s}, \mathsf{y}'_{irjs}, \mathsf{y}'^{-1}_{irjs}$ of the fibers of $\mathcal{R}_{T_2,\mathbf{N}'}$ over the points $w^*_{i,r}, w^*_{j,s}, w^*_{i,r} + w^*_{j,s}, -w^*_{i,r} - w^*_{j,s} \in \mathrm{Gr}_{T_2}$ respectively. According to [Part II, Theorem 4.1], the relations in A are as follows: $\mathsf{y}'_{irjs} \cdot \mathsf{y}'^{-1}_{irjs} = 1;\ \mathsf{y}'_{i,r} \cdot \mathsf{y}'_{j,s} = \mathsf{y}'_{irjs} \cdot (w_{j,s} - w_{i,r})$.

According to [Part II, §4(vi)], the homomorphism

$$\mathbf{z}'^* \colon H_*^{T(V)_\mathcal{O}}(\mathrm{Gr}_{T_1} \times \mathcal{R}_{T_2,\mathbf{N}'}) \to H_*^{T(V)_\mathcal{O}}(\mathrm{Gr}_{T(V)})$$

takes $\mathsf{y}'_{i,r}$ to $(w_{j,s} - w_{i,r})\mathsf{u}_{i,r}$, while $\mathbf{z}'^*\mathsf{y}'_{j,s} = \mathsf{u}_{j,s}$, $\mathbf{z}'^*\mathsf{y}'_{irjs} = \mathsf{u}_{i,r}\mathsf{u}_{j,s}$, $\mathbf{z}'^*\mathsf{y}'^{-1}_{irjs} = \mathsf{u}^{-1}_{i,r}\mathsf{u}^{-1}_{j,s}$. We define $\Xi^t\colon (\mathbb{C}[\mathring{Z}^\alpha] \otimes_{\mathbb{C}[\mathbb{A}^\alpha]} \mathbb{C}[\mathbb{A}^{|\alpha|}])_t \to \left(H^{G'_\mathcal{O}}_*(\mathcal{R}_{G',\mathbf{N}'})\right)_t$ identical on $w_{k,p}$ and sending $y_{k,p}$ to $\mathsf{u}_{k,p} \cdot \prod_{h\in Q_1 : \mathrm{o}(h)=k} \prod_{1\le q\le a_{\mathrm{i}(h)}} (w_{\mathrm{i}(h),q} - w_{k,p})$. In particular, $y_{i,r}$ goes to $\mathsf{y}'_{i,r} \cdot (w_{j,s} - w_{i,r})^{-1} \prod_{h\in Q_1 : \mathrm{o}(h)=i} \prod_{1\le q\le a_{\mathrm{i}(h)}} (w_{\mathrm{i}(h),q} - w_{i,r})$. Note that at the moment Ξ^t is defined only as a rational morphism. The condition of [Part II, Theorem 5.26(2)] is trivially satisfied. Also, Ξ^t is a regular isomorphism due to the factorization property of zastava and e.g. [BDF16, 5.6], and the fact that the factors $(w_{\mathrm{i}(h),q} - w_{k,p})$ in the formula for Ξ^t (note that the factor $(w_{j,s} - w_{i,r})$ is excluded) are all invertible at t. In particular, Ξ^t sends y_{irjs} of Remark 2.2 to y'_{irjs} up to a product of invertible factors $(w_{\mathrm{i}(h),q} - w_{k,p})$.

The second possibility is $(w_{i,r} - w_{i,s})(t) = 0$. Then the fixed point set $\mathcal{R}^t$ is isomorphic to the product $\mathrm{Gr}_{T_1} \times \mathrm{Gr}_{\mathrm{GL}(V'')}$. Here $V'' \subset V_i$ is a 2-dimensional subspace whose $T(V)$-weights are $\mathsf{w}_{i,r}, \mathsf{w}_{i,s}$, and T_1 is an $(|\alpha| - 2)$-dimensional torus with coordinates $\{\mathsf{w}_{k,p}\colon (i,r) \ne (k,p) \ne (i,s)\}$, and T_2 is a 2-dimensional torus with coordinates $\mathsf{w}_{i,r}, \mathsf{w}_{i,s}$. Hence in notations of [Part II, Theorem 5.26(2)], $G' = T_1 \times \mathrm{GL}(V'')$, $\mathbf{N}' = 0$, and

$$\begin{aligned} H^{G'_\mathcal{O}}_*(\mathcal{R}_{G',\mathbf{N}'}) \otimes_{H^*_{G'}(\mathrm{pt})} \mathbb{C}[\mathfrak{t}(V)] &= H^{T_{1,\mathcal{O}}}_*(\mathrm{Gr}_{T_1}) \otimes H^{T_{2,\mathcal{O}}}_*(\mathrm{Gr}_{\mathrm{GL}(V'')}) \\ &= \mathbb{C}[\mathfrak{t}_1 \times T_1^\vee] \otimes B \end{aligned}$$

where B is the following algebra. It has generators $\iota'_*\mathsf{u}_{i,r}, \iota'_*\mathsf{u}_{i,s}, w_{i,r}, w_{i,s}, \eta$ and relation $\iota'_*\mathsf{u}_{i,r} - \iota'_*\mathsf{u}_{i,s} = (w_{i,r} - w_{i,s})\eta$, subject to the condition that $\iota'_*\mathsf{u}_{i,r}, \iota'_*\mathsf{u}_{i,s}$ are invertible. In effect, the isomorphism $B \xrightarrow{\sim} H^{T_2}_*(\mathrm{Gr}_{\mathrm{GL}(V'')})$ takes η to the fundamental cycle $[\mathbb{P}^1_1] \in H^{T_2}_2(\mathrm{Gr}_{\mathrm{GL}(V'')})$ where $\mathbb{P}^1_1$ is the 1-dimensional $\mathrm{GL}(V'')$-orbit containing $w^*_{i,r}$ and $w^*_{i,s}$ (use the argument in [BFM05, 3.10]).

We define $\Xi^t\colon (\mathbb{C}[\mathring{Z}^\alpha] \otimes_{\mathbb{C}[\mathbb{A}^\alpha]} \mathbb{C}[\mathbb{A}^{|\alpha|}])_t \to \left(H^{T(V)_\mathcal{O}}_*(\mathcal{R}_{G',\mathbf{N}'})\right)_t$ identical on $w_{k,p}$ and sending $y_{k,p}$ to $\iota'_*\mathsf{u}_{k,p} \cdot \prod_{h\in Q_1 : \mathrm{o}(h)=k} \prod_{1\le q\le a_{\mathrm{i}(h)}} (w_{\mathrm{i}(h),q} - w_{k,p})$. Note that at the moment Ξ^t is defined only as a rational morphism. The condition of [Part II, Theorem 5.26(2)] is trivially satisfied. Also, Ξ^t is a regular isomorphism due to the factorization property of zastava and e.g. [BDF16, 5.5], and the fact that the factors $(w_{l,q} - w_{k,p})$ in the formula for Ξ^t are all invertible at t.

The theorem is proved. $\square$

Remark 3.2. $H^{\mathrm{GL}(V)_\mathcal{O}}_*(\mathcal{R})$ is naturally graded by $\pi_1\,\mathrm{GL}(V) = \mathbb{Z}^{Q_0}$. Under the isomorphism of Theorem 3.1 this grading becomes the grading of $\mathbb{C}[\mathring{Z}^\alpha]$

by the root lattice of the Cartan torus $T \subset G$: $\mathbb{Z}^{Q_0} = \mathbb{Z}\langle\alpha_i^\vee\rangle_{i\in Q_0}$ corresponding to the natural action of T on $\mathring{Z}^\alpha$. Indeed, the weight of $w_{i,r}$ is 0, while the weight of $y_{i,r}$ is $\alpha_i^\vee$.

Remark 3.3. The LHS of Theorem 3.1 is naturally graded by half the homological degree $\deg_h$, while the RHS is naturally graded by the action of loop rotations, $\deg_r$. These gradings are different. Let x be a homogeneous homology class supported at the connected component $\nu = (n_i) \in \mathbb{Z}^{Q_0} = \pi_0\mathrm{Gr}_{\mathrm{GL}(V)}$. Then one can check that $\deg_r(x) = \deg_h(x) - \nu^t \cdot \sqrt{\det \mathbf{N}} + \frac{1}{2}\nu^t \cdot \mathbf{C}\cdot\alpha$. Here $\mathbf{C}$ is the Cartan matrix of G, and we view $\sqrt{\det\mathbf{N}}$ as a (rational) character of $\mathrm{GL}(V)$, i.e. an element of $\frac{1}{2}\mathbb{Z}^{Q_0}$. Note that $\deg_h(x) - \nu^t\cdot\sqrt{\det\mathbf{N}}$ coincides with the monopole formula exponent $\Delta(x)$ of [Part II, (2.10)], see [Part II, Remark 2.8(2)].

3(ii). Positive part of an affine Grassmannian

Given a vector space U we define[7] $\mathrm{Gr}^+_{\mathrm{GL}(U)} \subset \mathrm{Gr}_{\mathrm{GL}(U)}$ as the moduli space of vector bundles $\mathcal{U}$ on the formal disc D equipped with trivialization $\sigma : \mathcal{U}|_{D^*} \xrightarrow{\sim} U \otimes \mathcal{O}_{D^*}$ on the punctured disc such that σ extends through the puncture as an embedding $\sigma : \mathcal{U} \hookrightarrow U\otimes\mathcal{O}_D$.

Now since $\mathrm{GL}(V) = \prod_{i\in Q_0}\mathrm{GL}(V_i)$ (notations of §3(i)), $\mathrm{Gr}_{\mathrm{GL}(V)} = \prod_{i\in Q_0}\mathrm{Gr}_{\mathrm{GL}(V_i)}$, and we define $\mathrm{Gr}^+_{\mathrm{GL}(V)} = \prod_{i\in Q_0}\mathrm{Gr}^+_{\mathrm{GL}(V_i)}$. We define $\mathcal{R}^+$ as the preimage of $\mathrm{Gr}^+_{\mathrm{GL}(V)} \subset \mathrm{Gr}_{\mathrm{GL}(V)}$ under $\mathcal{R}\to\mathrm{Gr}_{\mathrm{GL}(V)}$. Then $H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+)$ forms a convolution subalgebra of $H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R})$. Note that $\mathcal{R}_0^+$ (the preimage in $\mathcal{R}$ of the base point in $\mathrm{Gr}_{\mathrm{GL}(V)}$) is a connected component of $\mathcal{R}^+$, and the union of the remaining connected components supports an "augmentation" ideal of $H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+)$. Hence we have an algebra homomorphism $H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+) \to H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}_0^+) = H^*_{\mathrm{GL}(V)}(\mathrm{pt})$. The proof of Theorem 3.1 repeated essentially word for word gives a proof (using the fact that Z^α is normal) of the following

Corollary 3.4. *The pushforward with respect to the closed embedding $\mathcal{R}^+ \hookrightarrow \mathcal{R}$ induces an injective algebra homomorphism $H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+) \hookrightarrow H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R})$. The isomorphism $\mathbb{C}[\mathring{Z}^\alpha] \xrightarrow{\sim} H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R})$ of Theorem 3.1 takes $\mathbb{C}[Z^\alpha] \subset \mathbb{C}[\mathring{Z}^\alpha]$ onto $H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+) \subset H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R})$, and so induces an isomorphism*

[7] The second named author thanks Joel Kamnitzer for correcting his mistake.

$\mathbb{C}[Z^\alpha] \xrightarrow{\sim} H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+)$. *The above homomorphism*

$$\mathbb{C}[Z^\alpha] = H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+) \to H^*_{\mathrm{GL}(V)}(\mathrm{pt}) = \mathbb{C}[\mathbb{A}^\alpha]$$

corresponds to the section $s_\alpha\colon \mathbb{A}^\alpha \hookrightarrow Z^\alpha$ *of §2(i).* □

Remark 3.5. The center of $\mathrm{GL}(V)$ is canonically identified with $\mathbb{G}_m^{Q_0}$, and we have the diagonal embedding $\Delta\colon \mathbb{G}_m \hookrightarrow \mathbb{G}_m^{Q_0} \hookrightarrow \mathrm{GL}(V)$. We can view Δ as a cocharacter of $\mathrm{GL}(V)$, and hence a point of $\mathrm{Gr}^+_{\mathrm{GL}(V)} \subset \mathrm{Gr}_{\mathrm{GL}(V)}$. Note that this point is a $\mathrm{GL}(V)_\mathcal{O}$-orbit. We denote the fundamental class of its preimage in $\mathcal{R}^+$ by $F_\Delta \in H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+)$. Then under the isomorphism $H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+) \simeq \mathbb{C}[Z^\alpha]$ the class F_Δ goes to the boundary equation F_α of [BF14, Section 4]. Indeed, F_Δ viewed as an element in $H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}) = \mathbb{C}[\mathring{Z}^\alpha]$ is clearly invertible, but all the invertible regular functions on $\mathring{Z}^\alpha$ are of the form $cF_\alpha^k,\ k \in \mathbb{Z},\ c \in \mathbb{C}^\times$ [BDF16, Lemma 5.4]. Now for degree reasons, F_Δ must coincide with cF_α.

Remark 3.6. We consider the following isomorphism $\mathfrak{i}\colon \mathrm{Gr}_{\mathrm{GL}(V)} \xrightarrow{\sim} \mathrm{Gr}_{\mathrm{GL}(V^*)}$: it takes $(\mathcal{P}, \sigma)$ to $(\mathcal{P}^\vee,\ {}^t\sigma^{-1})$ where $\mathcal{P}$ is a $\mathrm{GL}(V)$-bundle on the formal disc, σ is a morphism from the trivial $\mathrm{GL}(V)$-bundle on the punctured disc, $\mathcal{P}^\vee$ is the dual $\mathrm{GL}(V^*)$-bundle, and ${}^t\sigma$ is the transposed morphism from $\mathcal{P}^\vee$ to the (dual) trivial bundle on the punctured disc. Let $\overline{Q_1}$ be the opposite orientation of our quiver, and let $\mathbf{N}$ be the corresponding representation of $\mathrm{GL}(V^*)$ (note that $\mathrm{Hom}(V_i, V_j) = \mathrm{Hom}(V_j^*, V_i^*)$). Then $\mathfrak{i}$ lifts to the same named isomorphism $\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}} \xrightarrow{\sim} \mathcal{R}_{\mathrm{GL}(V^*),\mathbf{N}}$. Together with an isomorphism $\mathrm{GL}(V) \xrightarrow{\sim} \mathrm{GL}(V^*)$, $g \mapsto {}^tg^{-1}$, it gives rise to a convolution algebra isomorphism $\mathfrak{i}_*\colon H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}}) \xrightarrow{\sim} H_*^{\mathrm{GL}(V^*)_\mathcal{O}}(\mathcal{R}_{\mathrm{GL}(V^*),\mathbf{N}})$. The composition

$$\mathbb{C}[\mathring{Z}^\alpha] \simeq H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}}) \xrightarrow{\mathfrak{i}_*} H_*^{\mathrm{GL}(V^*)_\mathcal{O}}(\mathcal{R}_{\mathrm{GL}(V^*),\mathbf{N}}) \simeq \mathbb{C}[\mathring{Z}^\alpha]$$

is an involution of the algebra $\mathbb{C}[\mathring{Z}^\alpha]$. This involution arises from the Cartan involution ι of $\mathring{Z}^\alpha$ [BDF16, Section 4] composed with the involution $\varkappa_{-1}$ of $\mathring{Z}^\alpha$ induced by an automorphism $z \mapsto -z\colon \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1$ and finally composed with the action $a(h)$ of a certain element of the Cartan torus $h = \beta(-1) \in T$.[8] Here β is a cocharacter of T equal to $\sum_{i \in Q_0} b_i \omega_i$ where $b_i = a_i - \sum_{h \in Q_1 : \mathrm{o}(h) = i} a_{\mathrm{i}(h)}$.

[8]The second named author thanks Joel Kamnitzer for correcting his mistake.

In effect,

$$\iota(w_{i,r}, y_{i,r}) = \left(w_{i,r}, y_{i,r}^{-1} \prod_{\substack{h \in Q_1 \sqcup \overline{Q_1} \\ \mathrm{o}(h)=i}}^{1 \le s \le a_{\mathrm{i}(h)}} (w_{i,r} - w_{\mathrm{i}(h),s}) \right)$$

[BDF16, Proposition 4.2] (product over *unoriented* edges of the Dynkin diagram connected to i),

$$\varkappa_{-1}(w_{i,r}, y_{i,r}) = (-w_{i,r}, (-1)^{a_i} y_{i,r}),$$

and

$$a(\beta(-1))(w_{i,r}, y_{i,r}) = (w_{i,r}, (-1)^{a_i - \sum_{h \in Q_1 : \mathrm{o}(h)=i} a_{\mathrm{i}(h)}} y_{i,r}).$$

One checks explicitly that Ξ° intertwines $a(\beta(-1)) \circ \varkappa_{-1} \circ \iota$ and $\mathfrak{i}_*$. The desired claim follows.

Remark 3.7. The $\mathrm{GL}(V)_\mathcal{O}$-orbits in $\mathrm{Gr}^+_{\mathrm{GL}(V)}$ are numbered by Q_0-multipartitions $(\lambda^{(i)})_{i \in Q_0}$, $\lambda^{(i)} = (\lambda_1^{(i)} \ge \lambda_2^{(i)} \ge \cdots)$, such that the number of parts $l(\lambda^{(i)}) \le a_i$. Given a positive roots combination $\alpha^\vee = \sum_{i \in Q_0} m_i \alpha_i^\vee$, we define a closed $\mathrm{GL}(V)_\mathcal{O}$-invariant subvariety $\overline{\mathrm{Gr}}^{+,\alpha^\vee}_{\mathrm{GL}(V)} \subset \mathrm{Gr}^+_{\mathrm{GL}(V)}$ as the union of orbits $\mathrm{Gr}^{(\lambda^{(i)})}_{\mathrm{GL}(V)}$ such that $\lambda_1^{(i)} \le m_i \ \forall i \in Q_0$. We define $\mathcal{R}^+_{\le \alpha^\vee} \subset \mathcal{R}^+$ as the preimage of $\overline{\mathrm{Gr}}^{+,\alpha^\vee}_{\mathrm{GL}(V)} \subset \mathrm{Gr}^+_{\mathrm{GL}(V)}$ under $\mathcal{R}^+ \to \mathrm{Gr}^+_{\mathrm{GL}(V)}$. This filtration is the intersection of a certain coarsening of the one of [Part II, §2(ii)] and [Part II, §6] with $\mathcal{R}^+ \subset \mathcal{R}$. We consider an increasing multifiltration

$$H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+) = \bigcup_{\alpha^\vee} H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+_{\le \alpha^\vee}),$$

and its Rees algebra $\mathsf{Rees} F_\bullet H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+)$. This is a multigraded algebra, and we take its multiprojective spectrum $\operatorname{Proj} \mathsf{Rees} F_\bullet H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+)$. It contains $\operatorname{Spec} H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+) \simeq Z^\alpha$ as an open dense subvariety. The relative compactification $\operatorname{Proj} \mathsf{Rees} F_\bullet H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+) \to \mathbb{A}^\alpha$ of $\operatorname{Spec} H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+) \simeq Z^\alpha \to \mathbb{A}^\alpha$ is nothing but the "two-sided" compactified zastava $\overline{Z}^\alpha \to \mathbb{A}^\alpha$ [Gai08, 7.2] where e.g. in the "symmetric" definition of [BDF16, 2.6] we allow both B and B_--structures to be generalized (cf. [Mir14, 1.4]). Note that the Cartan involution ι of $\mathring{Z}^\alpha$ (Remark 3.6) extends to the same named involution of $\overline{Z}^\alpha$.

In effect, it suffices to check that the multifiltration $F_\bullet H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+)$ of $H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+) \simeq \mathbb{C}[Z^\alpha]$ coincides with the multifiltration of $\mathbb{C}[Z^\alpha]$ by the order of the pole at the components of the boundary $\overline{Z}^\alpha \setminus Z^\alpha$. Due to Remark 3.6, it suffices to check that the multifiltration of $\mathbb{C}[\mathring{Z}^\alpha] \simeq H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R})$ by the order of the pole at the components of the boundary ∂Z^α coincides with the following multifiltration $\mathbb{F}_\bullet H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R})$. The $\mathrm{GL}(V)_\mathcal{O}$-orbits in $\mathrm{Gr}_{\mathrm{GL}(V)}$ are numbered by generalized Q_0-multipartitions $(\lambda^{(i)})_{i\in Q_0}$, $\lambda^{(i)} = (\lambda_1^{(i)} \geq \lambda_2^{(i)} \geq \cdots \geq \lambda_{a_i}^{(i)})$, $\lambda_r^{(i)} \in \mathbb{Z}$. Given a positive roots combination $\alpha^\vee = \sum_{i\in Q_0} m_i\alpha_i^\vee$, we define a closed $\mathrm{GL}(V)_\mathcal{O}$-invariant ind-subvariety $\mathrm{Gr}^+_{\mathrm{GL}(V)} \subset \mathrm{Gr}^{\geq -\alpha^\vee}_{\mathrm{GL}(V)} \subset \mathrm{Gr}_{\mathrm{GL}(V)}$ as the union of orbits $\mathrm{Gr}^{(\lambda^{(i)})}_{\mathrm{GL}(V)}$ such that $\lambda_{a_i}^{(i)} \geq -m_i\ \forall i \in Q_0$. In particular, $\mathrm{Gr}^{\geq -0}_{\mathrm{GL}(V)} = \mathrm{Gr}^+_{\mathrm{GL}(V)}$. We define $\mathcal{R}_{\geq -\alpha^\vee} \subset \mathcal{R}$ as the preimage of $\mathrm{Gr}^{\geq -\alpha^\vee}_{\mathrm{GL}(V)} \subset \mathrm{Gr}_{\mathrm{GL}(V)}$ under $\mathcal{R} \to \mathrm{Gr}_{\mathrm{GL}(V)}$. The desired increasing multifiltration is $\mathbb{F}_{\alpha^\vee} H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}) = H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}_{\geq -\alpha^\vee})$. It coincides with the multifiltration by the order of the pole at ∂Z^α by Remark 3.5.

Remark 3.8. We consider the Rees algebra $\mathsf{Rees}\mathbb{F}_\bullet H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R})$ of the multifiltration $\mathbb{F}_\bullet$ of Remark 3.7. This is an algebra over $\mathbb{C}[\overline{T}_{\mathrm{ad}}]$ where $\overline{T}_{\mathrm{ad}} \supset T_{\mathrm{ad}}$ is a partial closure of the adjoint Cartan torus T_{ad} determined by the cone of positive combinations of simple roots in the weight lattice of T. It seems likely that the spectrum of the Rees algebra $\widetilde{Z}^\alpha \to \overline{T}_{\mathrm{ad}}$ is nothing but the *local model* of the Drinfeld-Lafforgue-Vinberg degeneration of [Sch16, 6.1].

In view of these Remarks, it is interesting to consider the Rees algebra of the filtration in [Part II, §6] in general. We do not know what it is in general.

Remarks 3.9. (1) According to [Part II, Remark 3.9(3)], we can consider the convolution algebra $K^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R})$. Then, similarly to Theorem 3.1, one can construct an isomorphism $\mathbb{C}[{}^\dagger\mathring{Z}^\alpha] \simeq K^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R})$ where ${}^\dagger\mathring{Z}^\alpha \subset \mathring{Z}^\alpha$ stands for the trigonometric open zastava of [FKR18], that is the preimage of $\mathbb{G}_m^\alpha \subset \mathbb{A}^\alpha$ under the factorization projection $\mathring{Z}^\alpha \to \mathbb{A}^\alpha$. Note that the embedding ${}^\dagger\mathring{Z}^\alpha \subset \mathring{Z}^\alpha$ is *not* compatible with the symplectic structure by the argument in §3(iv).

(2) Here is what we have learned from Gaiotto:

Let us consider the K-theoretic Coulomb branch $\mathcal{M}_C^K = \operatorname{Spec} K^{G_\mathcal{O}}(\mathcal{R})$. Then $\mathcal{M}_C^K$ is supposed to be isomorphic to the Coulomb branch of the corresponding 4-dimensional gauge theory with a generic complex structure. (Recall that the latter is a hyper-Kähler manifold, which shares many common properties with Hitchin's moduli spaces of solutions of the self-duality equation over a Riemann surface. Among the S^2-family of complex structures, two are special and others are isomorphic.) For an unframed quiver gauge theory, the latter is known to be the moduli space of $G_{ADE,c}$-monopoles on $\mathbb{R}^2 \times S^1$ [NP12]. Moreover the isomorphism is given by the scattering matrix: we identify $\mathbb{R}^2 \times S^1$ with $\mathbb{R} \times \mathbb{C}^\times$ and consider scattering at $t \to \pm\infty$ in the first factor. Then for $G_c = \mathrm{SU}(2)$ as in §2(xiii), we transform f, g uniquely to polynomials (instead of Laurent polynomials) such that

(a) g is a monic polynomial of degree k,

(b) f is a polynomial of degree $< k$.

(c) $g(0) \neq 0$.

Thus we recover the trigonometric open zastava.

We do not have further evidences of this conjecture. For example, we cannot see the remaining two special complex structures. We also remark that our definition of K-theoretic Coulomb branches makes sense for any $(G, \mathbf{N})$, while 4-dimensional Coulomb branches are usually considered only when $\mathbf{N}$ is 'smaller' than G (conformal or asymptotically free in physics terminology).

3(iii). General cases

Recall the setup of §3(i). We write down a dominant coweight λ of G as a linear combination of fundamental coweights $\lambda = \sum_{i \in Q_0} l_i \omega_i$. Given another coweight $\mu \leq \lambda$ such that $\lambda - \mu = \alpha = \sum_{i \in Q_0} a_i \alpha_i$, we set $W_i = \mathbb{C}^{l_i}$, and consider the natural action of $\mathrm{GL}(V)$ on $\mathbf{N} = \mathbf{N}^\lambda_\mu := \bigoplus_{h \in Q_1} \mathrm{Hom}(V_{\mathrm{o}(h)}, V_{\mathrm{i}(h)}) \oplus \bigoplus_{i \in Q_0} \mathrm{Hom}(W_i, V_i)$. The corresponding variety of triples will be denoted $\mathcal{R}^\lambda_\mu$. Our goal is to describe the convolution algebra $H^{\mathrm{GL}(V)_\mathcal{O}}_*(\mathcal{R}^\lambda_\mu)$. To this end we introduce $\lambda^* := -w_0\lambda$, $\mu^* := -w_0\mu$. Note that λ^* is dominant, and $(\lambda^* - \mu^*)^* = \alpha$. We consider an open subset $\mathring{\mathbb{G}}_m^{|\alpha|} \subset \mathring{\mathbb{A}}^{|\alpha|}$ defined as the complement in $\mathbb{G}_m^{|\alpha|}$ to all the diagonals $w_{i,r} = w_{j,s}$, and also $\mathring{\mathbb{G}}_m^\alpha := \mathring{\mathbb{G}}_m^{|\alpha|}/S_\alpha \subset \mathbb{A}^\alpha$ (notations of §2(i)). The generalized roots are $w_{i,r} - w_{j,s}$ for $i \neq j$ connected in the Dynkin diagram; $w_{i,r} - w_{i,s}$, and finally $w_{i,r}$. The isomorphism of [Part II, (5.18) and Proposition 5.19] identifies $H^{\mathrm{GL}(V)_\mathcal{O}}_*(\mathcal{R}^\lambda_\mu)|_{\mathring{\mathbb{G}}_m^\alpha}$ and $H^{\mathrm{GL}(V)_\mathcal{O}}_*(\mathcal{R})|_{\mathring{\mathbb{G}}_m^\alpha}$ (with $(\mathbb{C}[\mathbb{G}_m^{|\alpha|} \times \mathbb{G}_m^{|\alpha|}])^{S_\alpha}|_{\mathring{\mathbb{G}}_m^\alpha}$). Furthermore, $H^{\mathrm{GL}(V)_\mathcal{O}}_*(\mathcal{R})$ is identified with $\mathbb{C}[Z^\alpha]$ by Theorem 3.1, and $\mathbb{C}[Z^\alpha]|_{\mathring{\mathbb{G}}_m^\alpha}$ is identified with $\mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}]|_{\mathring{\mathbb{G}}_m^\alpha}$ via $s^{\lambda^*}_{\mu^*}$.

The composition of the above identifications gives us a generic isomorphism

$$\Xi^\circ\colon \mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}]|_{\mathring{\mathbb{G}}_m^\alpha} \xrightarrow{\sim} H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^\lambda_\mu)|_{\mathring{\mathbb{G}}_m^\alpha}.$$

Equivalently, as in §3(i), we denote by $\bar{\mathsf{y}}_{i,r} \in H_*^{T(V)_\mathcal{O}}(\mathcal{R}^\lambda_{\mu,T(V),\mathbf{N}_{T(V)}})$ the fundamental class of the fiber of $\mathcal{R}^\lambda_{\mu,T(V),\mathbf{N}_{T(V)}}$ over the point $w^*_{i,r} \in \mathrm{Gr}_{T(V)}$. We have the algebra homomorphism

$$\iota_*\colon H_*^{T(V)_\mathcal{O}}(\mathcal{R}^\lambda_{\mu,T(V),\mathbf{N}_{T(V)}}) \to H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^\lambda_\mu) \otimes_{H^*_{\mathrm{GL}(V)}(\mathrm{pt})} H^*_{T(V)}(\mathrm{pt}),$$

and Ξ° sends $y_{i,r}$ to $\iota_*\bar{\mathsf{y}}_{i,r}$ (and is identical on $w_{i,r}$).

Theorem 3.10. *The isomorphism* $\Xi^\circ\colon \mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}]|_{\mathring{\mathbb{G}}_m^\alpha} \xrightarrow{\sim} H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^\lambda_\mu)|_{\mathring{\mathbb{G}}_m^\alpha}$ *extends to a biregular isomorphism* $\mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}] \xrightarrow{\sim} H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^\lambda_\mu)$.

Proof. We repeat the argument in the proof of Theorem 3.1, and use its notations. We introduce $\underline{\mathcal{W}}^{\lambda^*}_{\mu^*} := \overline{\mathcal{W}}^{\lambda^*}_{\mu^*} \times_{\mathbb{A}^\alpha} \mathbb{A}^{|\alpha|}$. We consider a general $t \in \mathring{\mathbb{A}}^{|\alpha|} \setminus \mathring{\mathbb{G}}_m^{|\alpha|} \subset \mathfrak{t}(V)$. According to [Part II, Theorem 5.26] and [Part II, Remark 5.27], we must identify the localizations $\left(\mathbb{C}[\underline{\mathcal{W}}^{\lambda^*}_{\mu^*}]\right)_t$ and $\left(H_*^{T(V)_\mathcal{O}}(\mathcal{R}^\lambda_\mu)\right)_t$ as $\mathbb{C}[\mathbb{A}^{|\alpha|}]_t = \mathbb{C}[\mathfrak{t}(V)]_t$-modules. There are two possibilities: either t lies on a diagonal divisor, or t is a general point of a coordinate hyperplane $w_{i,r}(t) = 0$. The former case having been dealt with in the proof of Theorem 3.1, it is enough to treat the latter case.

Then the fixed point set $(\mathcal{R}^\lambda_\mu)^t$ is isomorphic to the product $\mathrm{Gr}_{T_1} \times \mathcal{R}_{T_2,\mathbf{N}'}$. Here T_2 is a 1-dimensional torus with coordinate $\mathsf{w}_{i,r}$, and T_1 is an $(|\alpha|-1)$-dimensional torus with coordinates $\{\mathsf{w}_{j,s}\colon (j,s) \neq (i,r)\}$, so that $T(V) = T_1 \times T_2$. Furthermore, $\mathbf{N}'$ is an l_i-dimensional representation of T_2 equal to the direct sum of l_i copies of the character $\mathsf{w}_{i,r}$ with differential $w_{i,r}$. In notations of [Part II, Theorem 5.26(2)], $G' = T(V)$, and

$$\left(H_*^{G'_\mathcal{O}}(\mathcal{R}_{G',\mathbf{N}'})\right)_t = \left(H_*^{T(V)_\mathcal{O}}(\mathrm{Gr}_{T_1} \times \mathcal{R}_{T_2,\mathbf{N}'})\right)_t = (\mathbb{C}[\mathfrak{t}_1 \times T_1^\vee] \otimes C)_t$$

where C is generated by $w_{i,r}$ and the fundamental classes $\mathsf{x}_{i,r}, \bar{\mathsf{y}}'_{i,r}$ of the fibers of $\mathcal{R}_{T_2,\mathbf{N}'}$ over the points $-w^*_{i,r}, w^*_{i,r} \in \mathrm{Gr}_{T_2}$ respectively. According to [Part II, Theorem 4.1], the relations in C are as follows: $\mathsf{x}_{i,r}\bar{\mathsf{y}}'_{i,r} = w_{i,r}^{l_i}$.

According to [Part II, §4(vi)], the homomorphism $\mathbf{z}'^*\colon H_*^{T(V)_\mathcal{O}}(\mathrm{Gr}_{T_1} \times \mathcal{R}_{T_2,\mathbf{N}'}) \to H_*^{T(V)_\mathcal{O}}(\mathrm{Gr}_{T(V)})$ takes $\bar{\mathsf{y}}'_{i,r}$ to $\mathsf{u}_{i,r}$, while $\mathbf{z}'^*\mathsf{x}_{i,r} = w_{i,r}^{l_i}\mathsf{u}_{i,r}^{-1}$. We define $\Xi^t\colon \left(\mathbb{C}[\underline{\mathcal{W}}^\lambda_\mu]\right)_{t^*} \to \left(H_*^{G'_\mathcal{O}}(\mathcal{R}_{G',\mathbf{N}'})\right)_t$ identical on $w_{k,p}$ and sending $y_{k,p}$

to $\mathsf{u}_{k,p} \cdot \prod_{h\in Q_1:\mathrm{o}(h)=k} \prod_{1\le q\le a_{\mathrm{i}(h)}} (w_{\mathrm{i}(h),q} - w_{k,p})$. In particular, $x_{i,r} = y_{i,r}^{-1} w_{i,r}^{l_i}$ goes to $\mathsf{x}_{i,r} \prod_{h\in Q_1:\mathrm{o}(h)=i} \prod_{1\le q\le a_{\mathrm{i}(h)}} (w_{\mathrm{i}(h),q} - w_{i,r})$. Note that at the moment Ξ^t is defined only as a rational morphism. The condition of [Part II, Theorem 5.26(2)] is trivially satisfied. Also, Ξ^t is a regular isomorphism due to the factorization property of $\overline{\mathcal{W}}^\lambda_\mu$ and the fact that the factors $(w_{\mathrm{i}(h),q} - w_{k,p})$ in the formula for Ξ^t are all invertible at t. The theorem is proved. $\square$

Remark 3.11. It follows from Proposition 2.10 that the restriction of $s^{\lambda^*}_{\mu^*}\colon \overline{\mathcal{W}}^{\lambda^*}_{\mu^*} \to Z^\alpha$ to $\mathring{Z}^\alpha \subset Z^\alpha$ is an isomorphism $(s^{\lambda^*}_{\mu^*})^{-1}(\mathring{Z}^\alpha) \xrightarrow{\sim} \mathring{Z}^\alpha$, and thus we have a canonical localization embedding $\mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}] \hookrightarrow \mathbb{C}[\mathring{Z}^\alpha]$. Under the isomorphisms of Theorem 3.10 and Theorem 3.1 this embedding is nothing but the one of [Part II, Remark 5.14] corresponding to

$$\mathbf{N}_{\mathrm{hor}} \hookrightarrow \mathbf{N}_{\mathrm{hor}} \oplus \mathbf{N}_{\mathrm{vert}}.$$

Here $\mathbf{N}_{\mathrm{hor}} := \bigoplus_{h\in Q_1} \operatorname{Hom}(V_{\mathrm{o}(h)}, V_{\mathrm{i}(h)})$ (resp. $\mathbf{N}_{\mathrm{vert}} := \bigoplus_{i\in Q_0} \operatorname{Hom}(W_i, V_i)$) is a direct summand of $\mathbf{N}^\lambda_\mu = \bigoplus_{h\in Q_1} \operatorname{Hom}(V_{\mathrm{o}(h)}, V_{\mathrm{i}(h)}) \oplus \bigoplus_{i\in Q_0} \operatorname{Hom}(W_i, V_i)$.

In the same way, we have $\mathbb{C}[\overline{\mathcal{W}}^{\lambda^*+\nu^*}_{\mu^*+\nu^*}] \to \mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}]$ by adding $\mathbb{C}^{n_i}$ to W_i for $\nu = \sum \nu_i \omega_i$. The corresponding birational morphism $\overline{\mathcal{W}}^{\lambda^*}_{\mu^*} \to \overline{\mathcal{W}}^{\lambda^*+\nu^*}_{\mu^*+\nu^*}$ was constructed in §2(vi).

Remark 3.12. $H^{\mathrm{GL}(V)_\mathcal{O}}_*(\mathcal{R}^\lambda_\mu)$ is naturally graded by $\pi_1 \operatorname{GL}(V) = \mathbb{Z}^{Q_0}$. Under the isomorphism of Theorem 3.10 this grading becomes the grading of $\mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}]$ by the root lattice of the Cartan torus $T \subset G$: $\mathbb{Z}^{Q_0} = \mathbb{Z}\langle \alpha_i^\vee \rangle_{i\in Q_0}$ corresponding to the natural action of T on $\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}$.

This abelian group action extends to an action of $\operatorname{Stab}_G(\mu^*)$, the stabilizer of μ^* in G. This is the expected property [Nak16, §4(iv)(d)].

Similarly, if α happens to be a dominant coweight $\alpha = \lambda$, the action of the Cartan torus T on $\mathring{Z}^\alpha$ of Remark 3.2 extends to the action of $\operatorname{Stab}_G(\lambda)$. Indeed, the morphism $s^{\lambda^*}_0 : \overline{\mathcal{W}}^{\lambda^*}_0 \to Z^\alpha$ restricts to an isomorphism of the open subvarieties $\overline{\mathcal{W}}^{\lambda^*}_0 \setminus \bigcup_i \mathring{E}_i \xrightarrow{\sim} \mathring{Z}^\alpha$ (see §2(viii)). The isomorphism $\iota^{\lambda^*}_0$ restricts to $\overline{\mathcal{W}}^{\lambda^*}_0 \setminus \bigcup_i \mathring{E}_i \xrightarrow{\sim} S_\lambda \cap \overline{\mathcal{W}}^\lambda_0$ (the open intersection with a semiinfinite orbit). The composition of the above isomorphisms gives an identification $\mathring{Z}^\alpha \xrightarrow{\sim} S_\lambda \cap \overline{\mathcal{W}}^\lambda_0$, and the latter intersection is naturally acted upon by $\operatorname{Stab}_G(\lambda)$.

These actions will be realized directly in terms of Coulomb branches in [Affine, App. A].

Remark 3.13. The LHS of Theorem 3.10 is naturally graded by half the homological degree $\deg_h$, while the RHS is naturally graded by the action of loop rotations, $\deg_r$. These gradings are different. Let x be a homogeneous homology class supported at the connected component $\nu = (n_i) \in \mathbb{Z}^{Q_0} = \pi_0 \mathrm{Gr}_{\mathrm{GL}(V)}$. Then one can check that $\deg_r(x) = \deg_h(x) - \nu^t \cdot \sqrt{\det \mathbf{N}_{\mathrm{hor}}} + \frac{1}{2}\nu^t \cdot \mathbf{C} \cdot \alpha = \Delta(x) + \nu^t \cdot \sqrt{\det \mathbf{N}_{\mathrm{vert}}} + \frac{1}{2}\nu^t \cdot \mathbf{C} \cdot \alpha$ (cf. Remark 3.3).

Remark 3.14. Let Q be a folding of the Dynkin diagram of G with the corresponding nonsymmetric Cartan matrix $\mathbf{C}_Q$, and the corresponding nonsimply laced group G_Q. Let $\lambda_Q \geq \mu_Q$ be dominant coweights of G_Q, and $\overline{\mathcal{W}}^{\lambda_Q}_{G_Q,\mu_Q}$ the corresponding slice. Then the Hilbert series of $\mathbb{C}[\overline{\mathcal{W}}^{\lambda_Q}_{G_Q,\mu_Q}]$ graded by the loop rotations equals

$$\sum_\theta t^{2\Delta(\theta)+\nu^t\cdot\det \mathbf{N}_{\mathrm{vert}}+\nu^t\cdot\mathbf{C}_Q\cdot\alpha_Q} P_{\mathrm{GL}(V_Q)}(t;\theta).$$

Here θ runs through the set of dominant coweights of $\mathrm{GL}(V_Q)$, and ν stands for the connected component of $\mathrm{Gr}_{\mathrm{GL}(V_Q)}$ containing θ, and $\Delta(\theta)$ is defined in [CFHM14, (3.2)–(3.4)] (see also §4 below). A proof follows from the realization of $\overline{\mathcal{W}}^{\lambda_Q}_{G_Q,\mu_Q}$ as the folding of an appropriate $\overline{\mathcal{W}}^\lambda_\mu$ and Remark 3.13.

Remark 3.15. Similarly to §3(ii), we define $\mathcal{R}^{\lambda+}_\mu$ as the preimage of $\mathrm{Gr}^+_{\mathrm{GL}(V)} \subset \mathrm{Gr}_{\mathrm{GL}(V)}$ under $\mathcal{R}^\lambda_\mu \to \mathrm{Gr}_{\mathrm{GL}(V)}$. Then $H^{\mathrm{GL}(V)_\mathcal{O}}_*(\mathcal{R}^{\lambda+}_\mu)$ forms a convolution subalgebra of $H^{\mathrm{GL}(V)_\mathcal{O}}_*(\mathcal{R}^\lambda_\mu)$. On the other hand, the pullback under $s^{\lambda^*}_{\mu^*}\colon \overline{\mathcal{W}}^{\lambda^*}_{\mu^*} \to Z^\alpha$ realizes $\mathbb{C}[Z^\alpha]$ as a subalgebra of $\mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}]$. The proof of Theorem 3.10 repeated essentially word for word shows (cf. Corollary 3.4) that the isomorphism $\mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}] \xrightarrow{\sim} H^{\mathrm{GL}(V)_\mathcal{O}}_*(\mathcal{R}^\lambda_\mu)$ takes $\mathbb{C}[Z^\alpha] \subset \mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}]$ onto $H^{\mathrm{GL}(V)_\mathcal{O}}_*(\mathcal{R}^{\lambda+}_\mu) \subset H^{\mathrm{GL}(V)_\mathcal{O}}_*(\mathcal{R}^\lambda_\mu)$, and in particular induces an isomorphism $\mathbb{C}[Z^\alpha] \xrightarrow{\sim} H^{\mathrm{GL}(V)_\mathcal{O}}_*(\mathcal{R}^{\lambda+}_\mu)$.

Remark 3.16. Let us consider the opposite orientation $\overline{Q_1}$ of our quiver and the representation $\mathbf{N}^\lambda_\mu = \bigoplus_{h\in\overline{Q_1}} \mathrm{Hom}(V_i^*, V_j^*) \oplus \bigoplus_{i\in Q_0} \mathrm{Hom}(V_i^*, W_i^*)$ of $\mathrm{GL}(V^*)$. Similarly to Theorem 3.10, we have an isomorphism $\mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}] \xrightarrow{\sim} H^{\mathrm{GL}(V^*)_\mathcal{O}}_*(\mathcal{R}_{\mathrm{GL}(V^*),\mathbf{N}^\lambda_\mu})$. Similarly to Remark 3.6, we have a convolution algebra isomorphism

$$\mathfrak{i}^\lambda_{\mu*}\colon H^{\mathrm{GL}(V)_\mathcal{O}}_*(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}^\lambda_\mu}) \xrightarrow{\sim} H^{\mathrm{GL}(V^*)_\mathcal{O}}_*(\mathcal{R}_{\mathrm{GL}(V^*),\mathbf{N}^\lambda_\mu}).$$

The composition

$$\mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}] \simeq H^{\mathrm{GL}(V)_\mathcal{O}}_*(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}^\lambda_\mu}) \xrightarrow{\mathfrak{i}^\lambda_{\mu*}} H^{\mathrm{GL}(V^*)_\mathcal{O}}_*(\mathcal{R}_{\mathrm{GL}(V^*),\mathbf{N}^\lambda_\mu}) \simeq \mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}]$$

is an involution of the algebra $\mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}]$. Similarly to Remark 3.6, this involution arises from the involution $\iota^{\lambda^*}_{\mu^*}$ of $\mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}]$ (see §2(vii)) composed with the involution $\varkappa_{-1}$ of $\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}$ induced by an automorphism $z \mapsto -z\colon \mathbb{P}^1 \xrightarrow{\sim} \mathbb{P}^1$ and finally composed with the action $a(h)$ of a certain element of the Cartan torus $h = \beta(-1) \in T$. Here β is a cocharacter of T equal to $\sum_{i\in Q_0} b_i\omega_i$ where $b_i = a_i - \sum_{h\in Q_1:\mathrm{o}(h)=i} a_{\mathrm{i}(h)}$.

Remark 3.17. According to [Part II, Remark 3.9(3)], we can consider the convolution algebra $K^{\mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}^\lambda_\mu)$. Then, similarly to Theorem 3.10, one can construct an isomorphism $\mathbb{C}[{}^\dagger\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}] \simeq K^{\mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}^\lambda_\mu)$ where ${}^\dagger\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}$ stands for the moduli space of the triples $(\mathcal{P}, \sigma, \phi)$ where $\mathcal{P}$ is a G-bundle on $\mathbb{P}^1$; σ is a trivialization of $\mathcal{P}$ off $1 \in \mathbb{P}^1$ having a pole of degree $\le \lambda^*$ at $1 \in \mathbb{P}^1$, and ϕ is a B-structure on $\mathcal{P}$ of degree $-\mu$ having the fiber B_- at $\infty \in \mathbb{P}^1$ and transversal to B at $0 \in \mathbb{P}^1$ (a trigonometric slice), cf. [FKR18, 1.5].

3(iv). Poisson structures

The convolution algebra $H^{\mathrm{GL}(V)_{\mathcal{O}}}_*(\mathcal{R})$ (resp. $H^{\mathrm{GL}(V)_{\mathcal{O}}}_*(\mathcal{R}^\lambda_\mu)$) carries a Poisson structure $\{,\}_C$ because of the deformation $H^{\mathrm{GL}(V)_{\mathcal{O}}\rtimes\mathbb{C}^\times}_*(\mathcal{R})$ (resp. $H^{\mathrm{GL}(V)_{\mathcal{O}}\rtimes\mathbb{C}^\times}_*(\mathcal{R}^\lambda_\mu)$). The algebra $\mathbb{C}[\mathring{Z}^\alpha]$ carries (a nondegenerate, i.e. symplectic) Poisson structure $\{,\}_Z$ defined in [FKMM99]. In case μ is dominant, the algebra $\mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}]$ carries a Poisson structure $\{,\}_{\mathcal{W}}$ defined in [KWWY14].

Proposition 3.18. (1) *The isomorphism of Theorem 3.1 takes* $\{,\}_Z$ *to* $-\{,\}_C$.

(2) *The isomorphism of Theorem 3.10 takes* $\{,\}_{\mathcal{W}}$ *to* $-\{,\}_C$.

Proof. (1) It is enough to check the claim generically, over $\mathring{\mathbb{A}}^\alpha$. We consider the new coordinates $u_{i,r} := y_{i,r}\cdot\prod_{j\leftarrow i}\prod_{1\le s\le a_j}(w_{j,s}-w_{i,r})^{-1}$ on $\mathring{\underline{Z}}^\alpha$ (note that the new coordinates depend on the choice of orientation). It is easy to check that the only nonvanishing Poisson brackets on $\mathring{\underline{Z}}^\alpha$ are $\{w_{i,r}, u_{j,s}\} = \delta_{i,j}\delta_{r,s}u_{j,s}$ (see [FKMM99, Proposition 2]). Hence the generic isomorphism $\Xi\colon \mathbb{C}[\mathring{Z}^\alpha]\otimes_{\mathbb{C}[\mathbb{A}^\alpha]}\mathbb{C}[\mathring{\mathbb{A}}^\alpha] \xrightarrow{\sim} \mathbb{C}[\mathfrak{t}(V)\times T^\vee(V)]\otimes_{\mathbb{C}[\mathbb{A}^\alpha]}\mathbb{C}[\mathring{\mathbb{A}}^\alpha]$ of §3(i) takes the Poisson structure on $\mathring{Z}^\alpha$ to the negative of the standard Poisson structure on $\mathfrak{t}(V)\times T^\vee(V)$. According to [Part II, Corollary 5.21], $\mathbf{z}^*\iota_*^{-1}$ takes the latter structure to the one on $H^{\mathrm{GL}(V)_{\mathcal{O}}}_*(\mathcal{R})$.

(2) Again it suffices to check the claim generically where it follows from part (1). In effect, $s^{\lambda^*}_{\mu^*}$ generically is a symplectomorphism according

to [FKR18, Theorem 4.9], while the identification of $H_*^{\mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R}_\mu^\lambda)|_{\mathring{\mathbb{G}}_m^\alpha}$ and $H_*^{\mathrm{GL}(V)_{\mathcal{O}}}(\mathcal{R})|_{\mathring{\mathbb{G}}_m^\alpha}$ is symplectic by [Part II, Lemma 5.11]. □

Remark 3.19. Let $\mathcal{W}_{\mu^*}^{\lambda^*}$ denote the open subvariety of $\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}$ consisting of triples $(\mathcal{P}, \sigma, \phi)$ such that σ has a pole exactly of order λ^*. It is the preimage of the orbit $\mathrm{Gr}_G^{\lambda^*}$ under $\mathbf{p}$. We have the decomposition

$$\overline{\mathcal{W}}_{\mu^*}^{\lambda^*} = \bigsqcup_{\substack{\mu\le\nu\le\lambda \\ \nu\,:\,\text{dominant}}} \mathcal{W}_{\mu^*}^{\nu^*}.$$

In [Nak15, §5(iii)] the third named author showed that this is the decomposition into symplectic leaves. However the argument is based on the description of $\mathcal{W}_{\mu^*}^{\lambda^*}$ as the moduli space of singular monopoles, which is not justified yet. In particular, we do not know how to show that $\mathcal{W}_{\mu^*}^{\nu^*}$ is smooth in our current definition. In fact, this is the only problem: (1) We know that the Poisson structure on the Coulomb branch is symplectic on its smooth locus by [Part II, Proposition 6.15]. (2) The embedding $\overline{\mathcal{W}}_{\mu^*}^{\nu^*} \to \overline{\mathcal{W}}_{\mu^*}^{\lambda^*}$ is Poisson, as it is so when μ is dominant by [KWWY14, Th. 2.5] and the birational isomorphisms $\overline{\mathcal{W}}_{\mu^*}^{\lambda^*} \to \overline{\mathcal{W}}_{\mu^*+\varsigma^*}^{\lambda^*+\varsigma^*}$, $\overline{\mathcal{W}}_{\mu^*}^{\nu^*} \to \overline{\mathcal{W}}_{\mu^*+\varsigma^*}^{\nu^*+\varsigma^*}$ in Remark 3.11 is Poisson by [Part II, Lemma 5.11].

It was also shown that a transversal slice to $\mathcal{W}_{\mu^*}^{\nu^*}$ in $\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}$ is isomorphic to $\overline{\mathcal{W}}_{\nu^*}^{\lambda^*}$ in [Nak15, §5(iii)]. We do not know how to justify this either.

These two assertions are true if ν is dominant, as $\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}$ is a transversal slice to $\mathrm{Gr}_G^{\mu^*}$ in $\overline{\mathrm{Gr}}_G^{\lambda^*}$ in this case.

We also know these under the following condition: Let $\mu \le w_0\lambda$ be antidominant. Then the projection $\mathbf{p}\colon \overline{\mathcal{W}}_{\mu^*}^{\lambda^*} \to \overline{\mathrm{Gr}}_G^{\lambda^*}$ is smooth and its image intersects nontrivially all the $G_{\mathcal{O}}$-orbits $\mathrm{Gr}_G^{\nu^*} \subset \overline{\mathrm{Gr}}_G^{\lambda^*}$. Indeed, the smoothness of $\mathbf{p}$ for antidominant μ follows by the base change from the smoothness of $\mathrm{Bun}_B^{-\mu}(\mathbb{P}^1) \to {}'\mathrm{Bun}_G(\mathbb{P}^1)$ (see §2(ii)). To check the latter smoothness at a point $(\mathcal{P}, \phi) \in \mathrm{Bun}_B^{-\mu}(\mathbb{P}^1)$ we have to prove the surjectivity at the level of tangent spaces, and this follows from the long exact sequence of cohomology and the vanishing of $H^1(\mathbb{P}^1, (\mathfrak{g}/\mathfrak{b})_\phi(-1))$. The latter vanishing holds since the vector bundle $(\mathfrak{g}/\mathfrak{b})_\phi$ is filtered with the associated graded $\bigoplus_{\alpha^\vee \in R_+^\vee} \mathcal{O}_{\mathbb{P}^1}(-\langle \mu, \alpha^\vee \rangle)$: a direct sum of ample line bundles. To see that $\mathbf{p}(\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}) \cap \mathrm{Gr}_G^{\nu^*} \neq \emptyset$ for $\mu \le w_0\lambda,\ \nu \le \lambda$, recall that

$$ {}'\mathrm{Bun}_G(\mathbb{P}^1) = \bigsqcup_{\chi\,:\,\text{dominant}} {}'\mathrm{Bun}_G(\mathbb{P}^1)_\chi$$

is stratified according to the isomorphism types of G-bundles. The image of $\overline{\mathrm{Gr}}_G^{\lambda^*}$ in $'\mathrm{Bun}_G(\mathbb{P}^1)$ lies in the open substack

$$'\mathrm{Bun}_G(\mathbb{P}^1)_{\leq\lambda^*} := \bigsqcup_{\chi\leq\lambda^*} {'\mathrm{Bun}_G(\mathbb{P}^1)_\chi}.$$

Finally, the image of $\mathrm{Bun}_B^{-\mu}(\mathbb{P}^1) \to {'\mathrm{Bun}_G(\mathbb{P}^1)}$ contains the open substack $'\mathrm{Bun}_G(\mathbb{P}^1)_{\leq\lambda^*}$ if $\mu \leq w_0\lambda$.

Hence $\mathcal{W}_{\mu^*}^{\nu^*}$ is smooth and its transversal slice is isomorphic to $\overline{\mathcal{W}}_{\nu^*}^{\lambda^*}$.

This assumption is not artificial. The above stratification is *dual* to one for the corresponding Higgs branches (quiver varieties $\mathfrak{M}_0(V,W)$). It is known that $\mathfrak{M}_0(V,W)$ stabilizes for sufficiently large V, more precisely if $\mu \leq \overline{\lambda}$, where $\overline{\lambda}$ is the minimal dominant weight $\leq \lambda$. (See e.g., [Nak98, Rem. 3.28].) This condition is weaker than $\mu \leq w_0\lambda$ and antidominant, but we at least see that singularities of both Higgs and Coulomb branches do not change if μ is sufficiently antidominant.

3(v). Deformations

Recall the setup of [Part II, §3(viii)]. We choose a Cartan torus $T(W) = \prod_{i\in Q_0} T(W_i) \subset \prod_{i\in Q_0} \mathrm{GL}(W_i) = \mathrm{GL}(W)$ (notations of §3(iii)). We consider the extended group $1 \to \mathrm{GL}(V) \to \mathrm{GL}(V)\times T(W) \to T(W) \to 1$ acting on $\mathbf{N}_\mu^\lambda$, so that $T(W)$ is the flavor symmetry group. We choose a basis $z_1,\dots,z_N$ of the character lattice of $T(W)$ (compatible with the product decomposition $T(W) = \prod_{i\in Q_0} T(W_i)$), and view it as a basis of $\mathfrak{t}^*(W)$, i.e. the coordinate functions on $\mathfrak{t}(W) = \mathbb{A}^N$. According to [Part II, §3(viii)],

$$H_*^{\mathrm{GL}(V)_\mathcal{O}\times T(W)_\mathcal{O}}(\mathcal{R}_\mu^\lambda)$$

is a deformation of $H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}_\mu^\lambda)$ over the base $\mathrm{Spec}(H^*_{T(W)}(\mathrm{pt})) = \mathfrak{t}(W) = \mathbb{A}^N$.

We denote the intersection of the open subsets $\mathring{\mathbb{A}}^\alpha \times \mathbb{A}^N \subset \mathbb{A}^\alpha\times\mathbb{A}^N$ and $\bigcap_{i\in Q_0} f_{i,\underline{\lambda}^*}^{-1}(\mathbb{G}_m) \subset \mathbb{A}^\alpha\times\mathbb{A}^N$ by $\mathring{\mathbb{A}}^{\alpha,N}$. We define a generic isomorphism $\Xi^\circ\colon \mathbb{C}[\overline{\mathcal{W}}_{\mu^*}^{\underline{\lambda}^*}]|_{\mathring{\mathbb{A}}^{\alpha,N}} \xrightarrow{\sim} H_*^{\mathrm{GL}(V)_\mathcal{O}\times T(W)_\mathcal{O}}(\mathcal{R}_\mu^\lambda)|_{\mathring{\mathbb{A}}^{\alpha,N}}$ as in §3(iii): identical on z_s and $w_{i,r}$, and sending $y_{i,r}$ to $\iota_*\bar{\mathsf{y}}_{i,r}$. Here we view the fundamental class of the fiber of $\mathcal{R}_{\mu,T(V),\mathbf{N}_{T(V)}}^\lambda$ over the point $w_{i,r}^* \in \mathrm{Gr}_{T(V)}$ as an element $\bar{\mathsf{y}}_{i,r} \in H_*^{T(V)_\mathcal{O}\times T(W)_\mathcal{O}}(\mathcal{R}_{\mu,T(V),\mathbf{N}_{T(V)}}^\lambda)$.

Theorem 3.20. *The isomorphism*

$$\Xi^\circ\colon \mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}]|_{\mathring{\mathbb{A}}^{\alpha,N}} \xrightarrow{\sim} H^{\mathrm{GL}(V)_{\mathcal{O}}\times T(W)_{\mathcal{O}}}_*(\mathcal{R}^\lambda_\mu)|_{\mathring{\mathbb{A}}^{\alpha,N}}$$

extends to a biregular isomorphism $\mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}] \xrightarrow{\sim} H^{\mathrm{GL}(V)_{\mathcal{O}}\times T(W)_{\mathcal{O}}}_*(\mathcal{R}^\lambda_\mu)$.

Proof. Same as the proof of Theorem 3.10. □

Remark 3.21. The Poisson structure of $H^{\mathrm{GL}(V)_{\mathcal{O}}\times T(W)_{\mathcal{O}}}_*(\mathcal{R}^\lambda_\mu)$ transferred to $\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}$ via Ξ° is still given by the formulas of Proposition 3.18. But we do not know its modular definition, and we cannot see a priori that these generic formulas extend regularly to the whole of $\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}$.

3(vi). Affine case

We change the setup of §3(i). The Dynkin graph of G is replaced by its *affinization*, so that Ω is an orientation of this affinization; $\mathbf{N}$ is a representation space of Ω in the new sense, and so on. We change the setup of §2(i) accordingly: now Z^α stands for the zastava space of the affine group G_{aff}, denoted by $\mathfrak{U}^\alpha_{G;B}$ in [BFG06, 9.2], and $\mathring{Z}^\alpha \subset Z^\alpha$ stands for its open subscheme denoted by $\mathring{\mathfrak{U}}^\alpha_{G,B}$ in [BFG06, 11.8]: it is formed by all the points of Z^α with defects allowed only in the open subset $\mathbb{A}^1_{\mathrm{horizontal}} \times (\mathbb{A}^1_{\mathrm{vertical}} \setminus \{0\}) \subset \mathbb{A}^1_{\mathrm{horizontal}} \times \mathbb{A}^1_{\mathrm{vertical}} = \mathbb{A}^2$.

Similarly to Theorem 3.1 and Corollary 3.4, we have the following conditional

Theorem 3.22. *Assume $\mathring{Z}^\alpha$ is normal. The isomorphism*

$$\Xi^\circ\colon \mathbb{C}[\mathring{Z}^\alpha]|_{\mathring{\mathbb{A}}^\alpha_{\mathrm{horizontal}}} \xrightarrow{\sim} H^{\mathrm{GL}(V)_{\mathcal{O}}}_*(\mathcal{R})|_{\mathring{\mathbb{A}}^\alpha}$$

defined as in §3(i) extends to a biregular isomorphism $\mathbb{C}[\mathring{Z}^\alpha] \simeq H^{\mathrm{GL}(V)_{\mathcal{O}}}_*(\mathcal{R})$ *and to* $\mathbb{C}[Z^\alpha] \simeq H^{\mathrm{GL}(V)_{\mathcal{O}}}_*(\mathcal{R}^+)$.

Proof. We essentially repeat the proof of Theorem 3.1. The only problem arises with the application of [Part II, Theorem 5.26] whose assumptions can be verified only conditionally. Namely, we do not know if $Z^\alpha, \mathring{Z}^\alpha$ are Cohen-Macaulay. We do know that all the fibers of horizontal factorization $\pi_\alpha\colon Z^\alpha \to \mathbb{A}^\alpha_{\mathrm{horizontal}}$ are of the same dimension $|\alpha|$ ([BFG06, Corollary 15.4 of Conjecture 15.3 proved in 15.6]). Note that for the condition $\pi_*\mathcal{O}_{\mathcal{M}} \xrightarrow{\sim} j_*\pi_*\mathcal{O}_{\mathcal{M}^\bullet}$ i.e. $\mathcal{O}_{\mathcal{M}} \xrightarrow{\sim} j_*\mathcal{O}_{\mathcal{M}^\bullet}$ of [Part II, Theorem 5.26] it suffices to use the

S_2-property i.e. the normality of $Z^\alpha, \mathring{Z}^\alpha$ (see [Part II, Remark 5.27]). The normality of Z^α (and in fact the Cohen-Macaulay property and even the Gorenstein property) is proved in type A in [BF14, Corollary 3.6].

Note that if we redefine $\mathring{Z}^\alpha$ as the affinization of the space of degree α based maps from $\mathbb{P}^1$ to the Kashiwara flag scheme of G_{aff}, then the normality (and hence the first part of the theorem) would follow unconditionally.

Note also that in the affine case one more possibility for $\mathring{\underline{Z}}^\gamma,\ |\gamma| = 2$, arises; namely, $G_{\mathrm{aff}} = \mathrm{SL}(2)_{\mathrm{aff}},\ \gamma = \alpha_i + \alpha_j$. Then according to [FR14, Example 2.8.3], $\mathbb{C}[\mathring{\underline{Z}}^\gamma] = \mathbb{C}[w_i, w_j, y_i, y_j, y_{ij}^{\pm 1}]/(y_i y_j - y_{ij}(w_i - w_j)^2)$.

If [BFG06, Conjecture 15.3] holds for a symmetric Kac-Moody Lie algebra $\mathfrak{g}$, then the above argument shows that the spectrum of $H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R})$ is isomorphic to the affinization of $\mathring{Z}^\alpha$. $\square$

3(vii). Jordan quiver

We start with a general result. For a reductive group G and its adjoint representation $\mathbf{N} = \mathfrak{g}$ we consider the variety of triples $\mathcal{R} \to \mathrm{Gr}_G$. Its equivariant Borel-Moore homology $H_*^{G_\mathcal{O}}(\mathcal{R})$ equipped with the convolution product forms a commutative algebra, and its spectrum is the Coulomb branch $\mathcal{M}_C(G, \mathfrak{g})$.

Proposition 3.23. *The birational isomorphism*

$$(\mathfrak{t}^\circ \times T^\vee)/\mathbb{W} \simeq \mathcal{M}_C(G, \mathfrak{g})|_{\Phi^{-1}(\mathfrak{t}^\circ/\mathbb{W})}$$

of [Part II, Corollary 5.21] extends to a biregular isomorphism

$$(\mathfrak{t} \times T^\vee)/\mathbb{W} \xrightarrow{\sim} \mathcal{M}_C(G, \mathfrak{g}).$$

Here we denote the Weyl group by $\mathbb{W}$ in order to avoid a conflict with the vector space W.

This is nothing but [Part II, Proposition 6.14]. We give another proof.

Proof. It is but a slight variation of the proof of [BFM05, Theorem 7.3]. We have to replace the equivariant K-theory in [BFM05, Theorem 7.3] by the equivariant Borel-Moore homology. More precisely, in notations of [BFM05, Theorem 7.3], replacing the torus H by our Cartan torus T, we have to prove the following analogue of [BFM05, Lemma 7.6]: In $H_n^T(M)$ we have an equality $\imath^* \jmath_* (H^T_{n+2\dim M'}(\mathcal{L}')) = \bigoplus_\mu i_{\mu *}(H_n^T(\mu))$ where $\imath$ stands for the closed embedding $M \hookrightarrow T^*M$, while $\jmath$ stands for the closed embedding $\mathcal{L}' \hookrightarrow T^*M$ Recall that the proof of [BFM05, Lemma 7.6] used a homomorphism

$SS\colon K^T(D_M) \to K^T(T^*M)$ from the Grothendieck group of weakly T-equivariant holonomic D-modules on M. By definition, $SS = \jmath_* \circ \mathrm{gr}$ where gr stands for the associated graded with respect to a good filtration. Instead, we will use a homomorphism $SS'\colon K^T(D_M) \to H^T_{2\dim M'}(T^*M)$ associating to a weakly T-equivariant D-module $\mathcal{F}$ the pushforward $\jmath_*$ of the fundamental class $CC(\mathcal{F}) \in H^T_{2\dim M'}(\mathcal{L}')$ of its characteristic cycle. Note that $CC = \mathrm{symb} \circ \mathrm{gr}$ where symb stands for the $(2\dim M')$-th (top) graded component of the Chern character (in the *homological* grading). With this replacement, the proof of [BFM05, Lemma 7.6] carries over to our homological situation. E.g. an equality $i^*_\nu SS'(j_{\mu!}\mathcal{O}_{M_\mu}) = 0 \in H^T_0(T^*_\nu M')$ follows from the similar one in K-theory of the above cited proof and the fact that i^*_ν commutes with the Chern character (defined with respect to the smooth ambient variety T^*M') and shifts the homological grading by $2\dim M'$, while $\jmath_*$ commutes with the top part of the Chern character.

Also, the proof of [BFM05, Lemma 7.8] carries over to our homological situation essentially word for word. The proposition is proved. □

Now let V be an n-dimensional vector space. We consider the adjoint action of $G = \mathrm{GL}(V)$ on $\mathrm{End}(V) = \mathfrak{g}$. We choose a base in V; it gives rise to a Cartan torus $T(V)$ along with an identification $T^\vee(V) \simeq \mathbb{G}_m^n$, $\mathfrak{t}(V) \simeq \mathbb{A}^n$. From Proposition 3.23 we obtain an isomorphism

$$\mathrm{Sym}^n \mathcal{S}_0 \xrightarrow{\sim} \mathcal{M}_C(\mathrm{GL}(V), \mathrm{End}(V))$$

where $\mathcal{S}_0 = \mathbb{G}_m \times \mathbb{A}^1$ (see §2(ix)). This is the Coulomb branch of the pure quiver gauge theory for the Jordan quiver. Now we consider the Coulomb branch of the Jordan quiver gauge theory with framing $W = \mathbb{C}^l$. Recall that $\mathcal{S}_l$ is the hypersurface in $\mathbb{A}^3$ given by the equation $xy = w^l$.

Proposition 3.24. *The birational isomorphism*

$$(\mathfrak{t}^\circ(V) \times T^\vee(V))/S_n \simeq \mathcal{M}_C(\mathrm{GL}(V), \mathrm{End}(V) \oplus V \otimes \mathbb{C}^l)|_{\Phi^{-1}(\mathfrak{t}^\circ(V)/S_n)}$$

of [Part II, Corollary 5.21] extends to a biregular isomorphism $\mathrm{Sym}^n \mathcal{S}_l \xrightarrow{\sim} \mathcal{M}_C(\mathrm{GL}(V), \mathrm{End}(V) \oplus V \otimes \mathbb{C}^l)$. *The projection* $\mathcal{M}_C \to \mathfrak{t}(V)/S_n = \mathrm{Sym}^n \mathbb{A}^1$ *is the n-th symmetric power of the projection* $\mathcal{S}_l \to \mathbb{A}^1\colon (x, y, w) \mapsto w$.

Proof. Similar to the one of Theorem 3.10. More precisely, the proof is reduced to a consideration at the generic points of the generalized root hyperplanes. If a generalized root is a root $w_i - w_j$ of $\mathrm{GL}(V)$, then we are reduced to the $G = \mathrm{GL}(2)$ case of Proposition 3.23. If a generalized root is w_i, we are in the abelian case. □

3(viii). Towards geometric Satake correspondence for Kac-Moody Lie algebras

In this subsection we formulate conjectural geometric Satake correspondence for Kac-Moody Lie algebras using Coulomb branches.[9] See [Fin18] for more thorough historical accounts.

Let us assume that a quiver $Q = (Q_0, Q_1)$ has no edge loops, but is not necessarily of finite nor affine type. We have the associated symmetric Kac-Moody Lie algebra $\mathfrak{g}_{\mathrm{KM}}$.

Taking Q_0-graded vector spaces $V = \bigoplus_i V_i$, $W = \bigoplus_i W_i$, we define $\mathbf{N}$ as above. Taking simple roots α_i and fundamental weights ω_i ($i \in Q_0$), we assign two weights $\lambda = \sum_{i\in Q_0} \dim W_i \omega_i$, $\mu = \lambda - \sum_{i\in Q_0} \dim V_i \alpha_i$. Let $\mathbf{M} = \mathbf{N} \oplus \mathbf{N}^*$, and $\boldsymbol{\mu}\colon \mathbf{M} \to \operatorname{Lie} \mathrm{GL}(V_Q)$ be the moment map with respect to the natural $\mathrm{GL}(V_Q)$-action on $\mathbf{M}$. Let $\mathfrak{M}(\lambda,\mu)$, $\mathfrak{M}_0(\lambda,\mu)$ be quiver varieties defined by the third-named author [Nak94, Nak98]:

$$\mathfrak{M}_0(\lambda,\mu) = \boldsymbol{\mu}^{-1}(0) /\!\!/ \mathrm{GL}(V_Q), \quad \mathfrak{M}(\lambda,\mu) = \boldsymbol{\mu}^{-1}(0) /\!\!/_\chi \mathrm{GL}(V_Q),$$

where $\chi\colon \mathrm{GL}(V_Q) \to \mathbb{C}^\times$ is the character given by $\chi(g) = \prod_{i\in Q_0} \det g_i$, and $/\!\!/_\chi$ is the geometric invariant theory quotient with respect to the polarization χ. By its construction $\mathfrak{M}_0(\lambda,\mu)$ is an affine variety, and we have a projective morphism $\pi\colon \mathfrak{M}(\lambda,\mu) \to \mathfrak{M}_0(\lambda,\mu)$. It is known that $\mathfrak{M}(\lambda,\mu)$ is nonsingular and π is semi-small. Let $\mathfrak{L}(\lambda,\mu)$ be the inverse image of 0 under π. It is known to be a half-dimensional subvariety in $\mathfrak{M}(\lambda,\mu)$.

Then the main result in [Nak94, Nak98] says that

$$\bigoplus_\mu H_{\mathrm{top}}(\mathfrak{L}(\lambda,\mu))$$

has a structure of an integrable highest weight representation $V(\lambda)$ of $\mathfrak{g}_{\mathrm{KM}}$ with highest weight λ. Moreover the summand $H_{\mathrm{top}}(\mathfrak{L}(\lambda,\mu))$ corresponds to the weight space of $V(\lambda)_\mu$ with weight μ. Here 'top' denotes the top degree homology group, i.e. of degree $2\dim \mathfrak{L}(\lambda,\mu)$. Since π is semi-small, we can identify $H_{\mathrm{top}}(\mathfrak{L}(\lambda,\mu))$ with the isotypical component of $\mathrm{IC}(\{0\})$ in the direct image $\pi_*(\mathbb{C}_{\mathfrak{M}(\lambda,\mu)}[\dim \mathfrak{M}(\lambda,\mu)])$.

---

[9] When the quiver is of affine type A, the conjecture was given in [NT17, §7.8] in terms of bow varieties. This subsection is written afterwards, but the origin of the conjecture in [NT17] is what we explain here.

Let us turn to the corresponding Coulomb branch

$$\mathcal{M}_C(\lambda,\mu) = \mathcal{M}_C(\mathrm{GL}(V),\mathbf{N}).$$

By [Nak16, Remark 4.5] and references therein, we identify $\chi\colon \mathrm{GL}(V_Q) \to \mathbb{C}^\times$ with the cocharacter $\chi = \pi_1(\chi)^\wedge\colon \mathbb{C}^\times = \pi_1(\mathbb{C}^\times)^\wedge \to \pi_1(\mathrm{GL}(V_Q))^\wedge$, where $(\ )^\wedge$ denotes the Pontryagin dual. Recall that $\pi_1(\mathrm{GL}(V_Q))^\wedge$, which is a torus of dimension $\#Q_0$, acts naturally on $\mathcal{M}_C(\lambda,\mu)$ as in Remarks 3.2, 3.12. In physics terminology a Kähler parameter for Higgs branch is an equivariant parameter for Coulomb branch.

Let us define the corresponding attracting set

$$\mathfrak{A}_\chi(\lambda,\mu) \overset{\text{def.}}{=} \left\{ x \in \mathcal{M}_C(\lambda,\mu) \,\middle|\, \exists \lim_{t\to 0} \chi(t)x \right\},$$

which is a closed subvariety in $\mathcal{M}_C(\lambda,\mu)$, possibly empty in general.

These $\mathcal{M}_C(\lambda,\mu)$, $\mathfrak{A}_\chi(\lambda,\mu)$ are related to representation theory in many situations:

(a) Recall that $\mathcal{M}_C(\lambda,\mu)$ is the (usual) transversal slice $\overline{\mathcal{W}}^\lambda_\mu$ in the affine Grassmannian Gr_G if Q is of type ADE and μ is dominant. (We ignore the diagram automorphism $*$.) Then $\mathfrak{A}_\chi(\lambda,\mu)$ was studied in [MV07]. It is nonempty if and only if the corresponding weight space $V(\lambda)_\mu$ is nonzero, it is of pure dimension $2|\lambda-\mu|$, and $H_{\mathrm{top}}(\mathfrak{A}_\chi(\lambda,\mu))$ is naturally isomorphic to $V(\lambda)_\mu$. It can be also considered as a stalk of the hyperbolic restriction [Bra03] of the intersection cohomology complex $\mathrm{IC}(\overline{\mathcal{W}}^\lambda_\mu)$ of $\overline{\mathcal{W}}^\lambda_\mu$ with respect to χ. Moreover the stalk of $\mathrm{IC}(\overline{\mathcal{W}}^\lambda_\mu)$ at μ is the associated graded $\operatorname{gr} V(\lambda)_\mu$ with respect to the Brylinski-Kostant filtration up to shift [Lus83, Gin95].

(b) Suppose that Q is of affine type. Then $\mathcal{M}_C(\lambda,\mu)$ is conjecturally the Uhlenbeck partial compactification of a moduli space of instantons on the Taub-NUT space invariant under a cyclic group action, which is proved in affine type A [NT17]. When μ is dominant, it is conjecturally isomorphic to the Uhlenbeck partial compactification of a moduli space of instantons on $\mathbb{R}^4$ invariant under a cyclic group action, which is again proved in affine type A [NT17]. In [BF10, BF12, BF13] the first and second-named authors conjectured that statements as in (a) hold for affine Kac-Moody Lie algebras,[10] where the definition of the affine Brylinski-Kostant filtration was later corrected in [Slo12].

(c) When $\mathcal{M}_C(\lambda,\mu)$ is isomorphic to a quiver variety (for different quiver, and V, W), the attracting set $\mathfrak{A}_\chi(\lambda,\mu)$ is the tensor product variety in

[10]Strictly speaking, only instantons for simply-connected groups are considered. Correspondingly representations descend to the adjoint group.

[Nak01]. In particular, the number of its irreducible components is given by the tensor product multiplicities. In this way, some of conjectures in [BF10, BF12, BF13] can be proved for affine type A by being combined with I. Frenkel's level-rank duality.

(d) Hyperbolic restrictions of the intersection cohomology complex IC of the Uhlenbeck partial compactification of a moduli space of instantons on $\mathbb{R}^4$ was studied in [BFN16]. This Uhlenbeck space is conjecturally isomorphic to $\mathcal{M}_C(\omega_0, \omega_0 - n\delta)$ $(n \in \mathbb{Z}_{\geq 0})$ for an affine quiver, where ω_0 is the fundamental weight corresponding to the special vertex 0, and δ is the primitive positive imaginary root. It follows that the direct sum (over n) of hyperbolic restriction of IC is isomorphic to $\mathrm{Sym}(\bigoplus_{d>0} z^{-d} \otimes \mathfrak{h})$, where $\mathfrak{h}$ is the Cartan subalgebra of the underlying finite dimensional Lie algebra. (See also [Nak17, §7.5].)

In above examples, we assume that μ is dominant. (It is so in all known examples in (c).) For the original geometric Satake correspondence, we *do* understand all weight spaces $V(\lambda)_\mu$ not necessarily dominant. In (b) there is a candidate for a space which we should consider when μ is not necessarily dominant in [BF13], which was later found out to be close to the quiver description of bow varieties in [NT17], but the conjecture there was not checked even for affine type A.

Since Coulomb branches are defined for any quiver without assumption μ dominant, not necessarily of finite or affine types, we propose the following conjecture, which makes the situation much simpler:

Conjecture 3.25. (1) $\mathfrak{A}_\chi(\lambda,\mu)$ is empty if and only if $V(\lambda)_\mu = 0$. Moreover $\mathcal{M}_C(\lambda,\mu)^{\chi(\mathbb{C}^\times)}$ is a single point if it is nonempty.

(2) The intersection of $\mathfrak{A}_\chi(\lambda,\mu)$ with symplectic leaves of $\mathcal{M}_C(\lambda,\mu)$ are lagrangian. Hence the hyperbolic restriction functor Φ for χ [Bra03] is hyperbolic semi-small in the sense of [BFN16, 3.5.1]. In particular, $\Phi(\mathrm{IC}(\mathcal{M}_C(\lambda,\mu)))$ remains perverse, and is isomorphic to $H_{\mathrm{top}}(\mathcal{A}_\chi(\lambda,\mu))$.

(3) The direct sum

$$\bigoplus_\mu \Phi(\mathrm{IC}(\mathcal{M}_C(\lambda,\mu))) = \bigoplus_\mu H_{\mathrm{top}}(\mathcal{A}_\chi(\lambda,\mu))$$

has a structure of a $\mathfrak{g}_{\mathrm{KM}}$-module, isomorphic to $V(\lambda)$ so that each summand is isomorphic to $V(\lambda)_\mu$.

We naively expect that the usual stalk $\mathrm{IC}(\mathcal{M}_C(\lambda,\mu))$ at the fixed point is 'naturally' isomorphic to the associated graded of $V(\lambda)_\mu$ with respect to a certain filtration which has a representation theoretic origin. But we do not know what we mean 'natural' nor how we define the filtration in general. Also we could take another generic cocharacter χ of $\pi_1(\mathrm{GL}(V_Q))^\wedge$, but we do not know how to relate it to a representation theoretic object.

This conjecture is checked (except (2)) for finite type [Kry18].

As another evidence, we consider the following example, which is not necessarily finite or affine. Let us suppose $\dim V_i = 1$ for any $i \in Q_0$. The Higgs branch $\mathfrak{M}_0(\lambda,\mu)$ is a quiver variety, but it is also an example of a Goto-Bielawski-Dancer toric hyper-Kähler manifold. The Coulomb branch $\mathcal{M}_C(\lambda,\mu)$ is also. By a recent work of Braden-Mautner [BM19],[11] we have a Ringel duality between perverse sheaves on $\mathcal{M}_C(\lambda,\mu)$ and those of $\mathfrak{M}_0(\lambda,\mu)$. In particular, $\Phi(\mathrm{IC}(\mathcal{M}_C(\lambda,\mu)))$ is isomorphic to $H_{\mathrm{top}}(\mathfrak{L}(\lambda,\mu))$, hence is isomorphic to the weight space $V(\lambda)_\mu$.

In fact, [BM19] and the above conjecture both come from a 'meta conjecture' saying the category of perverse sheaves on a Higgs branch (e.g. $\mathfrak{M}_0(\lambda,\mu)$ and one on the corresponding Coulomb branch (e.g. $\mathcal{M}_C(\lambda,\mu)$) should be dual in an appropriate way. We do not know how strata of $\mathcal{M}_C(\lambda,\mu)$ look like in general, and the category is probably not highest weight as studied briefly in [Nak15]. Nevertheless it is expected that the pushforward from $\mathfrak{M}(\lambda,\mu)$ and the hyperbolic restriction for χ are exchanged under the duality. It should be also related to the symplectic duality [BLPW16].

Remarks 3.26. (1) Let us take just $V = \bigoplus_i V_i$ and consider the corresponding Coulomb branch

$$\mathcal{M}_C(\alpha) = \mathcal{M}_C(\mathrm{GL}(V), \mathbf{N})$$

with $\alpha = \sum_{i\in Q_0} \dim V_i \alpha_i$. It is expected that $\mathcal{M}_C(\alpha)$ has no fixed point with respect to $\chi(\mathbb{C}^\times)$, hence the above construction does not work. Instead we consider $\mathcal{M}_C^+(\alpha) \overset{\text{def.}}{=} H_*^{\mathrm{GL}(V)_\mathcal{O}}(\mathcal{R}^+)$ as in §3(ii). This is supposed to be a Kac-Moody generalization of the zastava space. The same construction with W gives the same space $\mathcal{M}_C^+(\alpha)$, hence we have a morphism $\mathcal{M}_C(\lambda,\mu) \to \mathcal{M}_C^+(\alpha)$ as in Remark 3.15. It is expected that $\mathcal{M}_C^+(\alpha)$ is a limit of $\mathcal{M}_C(\lambda,\mu)$ when λ, $\mu \to \infty$ keeping $\lambda - \mu = \alpha$.

We define the attracting set $\mathfrak{A}_\chi^+(\alpha)$ as the set of points contracted to $s_\alpha(0)$ by the action of χ, where $s_\alpha\colon \mathbb{A}^\alpha \hookrightarrow \mathcal{M}_C^+(\alpha)$ is the section as in Corollary 3.4. Note that the action of χ contracts the whole of $\mathcal{M}_C^+(\alpha)$ to

[11]The authors of [BM19] call Goto-Bielawski-Dancer toric hyper-Kähler manifolds as hypertoric manifolds.

$s_\alpha(\mathbb{A}^\alpha) = \mathcal{M}_C^+(\alpha)^{\chi(\mathbb{C}^\times)}$, cf. Remark 3.2, so that for any $\phi \in \mathcal{M}_C^+(\alpha)$, there exists $\lim_{t\to 0} \chi(t)$. The integrable system $\varpi_\alpha^+ \colon \mathcal{M}_C^+(\alpha) \to \mathbb{A}^\alpha$ is $\chi(\mathbb{C}^\times)$-equivariant, so $\mathfrak{A}_\chi^+(\alpha)$ coincides with the fiber $(\varpi_\alpha^+)^{-1}(0)$ over $0 \in \mathbb{A}^\alpha$. Furthermore, since we expect $\mathcal{M}_C(\lambda,\mu)^{\chi(\mathbb{C}^\times)}$ to consist of one point if nonempty, this point must be fixed with respect to another action of $\mathbb{C}^\times$ corresponding to the cohomological grading of the Coulomb branch. Thus the image of the fixed point under the morphism $\mathcal{M}_C(\lambda,\mu) \to \mathcal{M}_C^+(\alpha)$ must be $s_\alpha(0) \in s_\alpha(\mathbb{A}^\alpha) \subset \mathcal{M}_C^+(\alpha)$. It follows that the image of $\mathfrak{A}_\chi(\lambda,\mu)$ lies in $\mathfrak{A}_\chi^+(\alpha)$.

Then we expect that the corresponding statements in Conjecture 3.25 are true. In particular, the direct sum

$$\bigoplus_\alpha \Phi(\mathrm{IC}(\mathcal{M}_C^+(\alpha)) = \bigoplus_\alpha H_{\mathrm{top}}(\mathfrak{A}_\chi^+(\alpha))$$

is isomorphic to $U(\mathfrak{n}_-)$ where $\mathfrak{n}_-$ is the negative half of $\mathfrak{g}_{\mathrm{KM}}$. Moreover the pull-back homomorphism $H_{\mathrm{top}}(\mathfrak{A}_\chi^+(\alpha)) \to H_{\mathrm{top}}(\mathfrak{A}_\chi(\lambda,\mu))$ corresponds to the quotient map $U(\mathfrak{n}_-) \to V(\lambda)$. When Q is of finite type, these statements will be proved in a forthcoming paper by J. Kamnitzer, P. Baumann and A. Knutson.

(2) Let $\mathrm{Irr}(\mathfrak{A}_\chi(\lambda,\mu))$, $\mathrm{Irr}(\mathfrak{A}_\chi^+(\alpha))$ be the set of irreducible components of $\mathfrak{A}_\chi(\lambda,\mu)$, $\mathfrak{A}_\chi^+(\alpha)$ respectively. Then we conjecture that

$$\bigsqcup_\mu \mathrm{Irr}(\mathfrak{A}_\chi(\lambda,\mu)), \qquad \bigsqcup_\alpha \mathrm{Irr}(\mathfrak{A}_\chi^+(\alpha))$$

have structures of Kashiwara crystal, isomorphic to crystals $B(\infty)$, $B(\lambda)$ of $U_q(\mathfrak{n}_-)$, $V_q(\lambda)$ of the quantized enveloping algebra $U_q(\mathfrak{g}_{\mathrm{KM}})$ respectively. Moreover the inclusion $\mathrm{Irr}(\mathfrak{A}_\chi(\lambda,\mu)) \subset \mathrm{Irr}(\mathfrak{A}_\chi^+(\alpha))$ corresponds to the embedding $B(\lambda) \subset B(\infty)$. When Q is of finite type, these statements follow from the comparison of the crystal structures defined in [BFG06, Section 13] and in [BG01, Kry18], cf. [BG08, Proposition 4.3].

Furthermore, we expect that the zero level $\varpi_\alpha^{-1}(0)$ of the integrable system $\varpi_\alpha \colon \mathcal{M}_C(\alpha) \to \mathbb{A}^\alpha$ is a dense open subset of $\mathfrak{A}_\chi^+(\alpha)$, so we have a canonical bijection $\mathrm{Irr}(\mathfrak{A}_\chi^+(\alpha)) = \mathrm{Irr}(\varpi_\alpha^{-1}(0))$. Now the Cartan involution of $\mathcal{M}_C(\alpha)$ described in 3.6 induces an involution of $\mathrm{Irr}(\varpi_\alpha^{-1}(0))$ and we conjecture that the latter involution corresponds to Kashiwara's involution $*\colon B(\infty) \to B(\infty)$ [Kas95, 8.3]. If Q is of finite type, this conjecture follows from the definition of crystal structure in [BFG06, Section 13.5], cf. [BDF16, Remark 1.7].

(3) It is conjectured that there is a natural bijection between symplectic leaves of $\mathcal{M}_C(\lambda,\mu)$ and $\mathfrak{M}_0(\lambda,\mu)$ [Nak15]. When Q is of finite type, closures

of strata are of the forms $\mathcal{M}_C(\nu,\mu)$ and $\mathfrak{M}_0(\lambda,\nu)$ respectively, where ν runs through dominant weights between μ and λ. This is known for quiver varieties [Nak94, Prop. 6.7], while it is only conjectural for Coulomb branches. See Remark 3.19. In this case the bijection is given by $\mathcal{M}_C(\nu,\mu) \leftrightarrow \mathfrak{M}_0(\lambda,\nu)$. In particular, $\mathcal{M}_C(\lambda,\mu)$ corresponds to $\mathfrak{M}_0(\lambda,\lambda)$, which is a point. For more general Q, the description of the strata of $\mathcal{M}_0(\lambda,\mu)$ are given in [Nak94, §6], by being combined with [CB01]. For an affine type Q, extra strata come from symmetric products of simple singularities, which can be checked easily. We do not have any description of strata of $\mathcal{M}_C(\lambda,\mu)$ if Q is neither finite nor affine.

This bijection should be upgraded to a bijection between pairs of strata and simple local systems on them, but it becomes even more speculative. Assuming this bijection, we conjecture the following: Suppose (S_C,ϕ_C) and (S_H,ϕ_H) are strata of $\mathcal{M}_C(\lambda,\mu)$ and $\mathfrak{M}_0(\lambda,\mu)$ and simple local system on them respectively, corresponding under the conjectural bijection. Then the isotypical component of $\mathrm{IC}(S_H,\phi_H)$ in $\pi_*(\mathbb{C}_{\mathfrak{M}(\lambda,\mu)}[\dim\mathfrak{M}(\lambda,\mu)])$ is isomorphic to $\Phi(\mathrm{IC}(S_C,\phi_C))$. The above conjecture studies the case $(S_C,\phi_C) = (\mathcal{M}_C(\lambda,\mu),\mathrm{triv})$, $(S_H,\phi_H) = (\mathfrak{M}_0(\lambda,\lambda),\mathrm{triv})$, where triv denotes the trivial local system.

Next we consider a structure giving tensor products of integrable modules. For quiver varieties it is a tensor product variety $\mathfrak{Z}(\lambda^1;\lambda^2)$ corresponding to a decomposition $W = W^1 \oplus W^2$ with $\lambda^a = \sum_i \dim W_i^a \omega_i$ ($a = 1,2$). It is defined as an attracting set in $\bigsqcup_\mu \mathfrak{M}(\lambda,\mu)$ with respect to the cocharacter $\rho\colon \mathbb{C}^\times \to \mathrm{GL}(W)$ given by $\rho(t) = \mathrm{id}_{W^1} \oplus t\,\mathrm{id}_{W^2}$. We introduce a smaller subvariety $\widetilde{\mathfrak{Z}}(\lambda^1;\lambda^2)$ requiring the limit $\lim_{t\to 0}$ lies in the lagrangian $\bigsqcup_{\mu^1,\mu^2} \mathfrak{L}(\lambda^1,\mu^1)\times\mathfrak{L}(\lambda^2,\mu^2)$. Then [Nak01] says

$$H_{\mathrm{top}}(\widetilde{\mathfrak{Z}}(\lambda^1;\lambda^2))$$

is isomorphic to the tensor product $V(\lambda^1)\otimes V(\lambda^2)$ under the convolution product. (See [MO19] for a better conceptual construction.) For tensor products $V(\lambda^1)\otimes\cdots\otimes V(\lambda^N)$, we just take $W = W^1\oplus\cdots\oplus W^N$ and repeat the same construction.

Let us turn to the Coulomb branch side. We take a maximal torus $T(W)$ of $\mathrm{GL}(W)$ and regard $\mathbf{N}$ as a representation of $\tilde{G} \overset{\text{def.}}{=} \mathrm{GL}(V)\times T(W)$. This gives a deformation of $\mathcal{M}_C(\lambda,\mu)$ parametrized by $\mathrm{Lie}(T(W))$ as $H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R}_{G,\mathbf{N}})$ as in §3(v). We restrict it to the direction of $d\rho$, that is $H^{G_\mathcal{O}\times\rho(\mathbb{C}^\times)}(\mathcal{R}_{G,\mathbf{N}})$, and denote it by $\underline{\mathcal{M}}_C(\lambda,\mu)$. Thus we have a morphism $\underline{\mathcal{M}}_C(\lambda,\mu)\to\mathbb{C}$. We also consider the variety of triples $\mathcal{R}_{\tilde{G},\mathbf{N}}$ for the larger group $\tilde{G}$ and the corresponding Coulomb branch $\mathcal{M}_C(\tilde{G},\mathbf{N}) = H_*^{\tilde{G}_\mathcal{O}}(\mathcal{R}_{\tilde{G},\mathbf{N}})$. By [Part II, Proposition 3.18] the original $\mathcal{M}_C(\lambda,\mu)$ is the Hamiltonian reduction of $\mathcal{M}_C(\tilde{G},\mathbf{N})$ by $\pi_1(\mathbb{C}^\times)^\wedge$. Note that $\pi_1(T(W))^\wedge$ is the dual torus $T(W)^\vee$ of the original torus $T(W)$. Therefore the cocharacter $\rho\colon \mathbb{C}^\times \to T(W)$ can be regarded as a character $T(W)^\vee \to \mathbb{C}^\times$. Therefore we can consider the corresponding geometric invariant theory quotient

$$\widetilde{\mathcal{M}}_C(\lambda,\mu) \overset{\text{def.}}{=} \boldsymbol{\mu}^{-1}(0)/\!\!/_\rho T(W)^\vee,$$

as in [Part II, Proposition 3.25], where $\boldsymbol{\mu}$ denotes the moment map

$$\mathcal{M}_C(\tilde{G},\mathbf{N}) \to \operatorname{Lie} T(W) = \operatorname{Spec} H^*_{T(W)}(\mathrm{pt})$$

for the $T(W)^\vee$ action. It is equipped with a projective morphism $\pi_C : \widetilde{\mathcal{M}}_C(\lambda,\mu) \to \mathcal{M}_C(\lambda,\mu)$. If we replace the equation $\boldsymbol{\mu} = 0$ by $\boldsymbol{\mu} \in \mathbb{C}d\rho$, we have a family version $\widetilde{\underline{\mathcal{M}}}_C(\lambda,\mu)$ equipped with a projective morphism $\widetilde{\underline{\mathcal{M}}}_C(\lambda,\mu) \to \underline{\mathcal{M}}_C(\lambda,\mu)$. We conjecture that this is a small birational morphism and $\widetilde{\underline{\mathcal{M}}}_C(\lambda,\mu)$ is a topologically trivial family, as for quiver varieties. Therefore $\psi(\mathrm{IC}(\underline{\mathcal{M}}_C(\lambda,\mu))) = \pi_{C,*}(\mathrm{IC}(\widetilde{\mathcal{M}}_C(\lambda,\mu)))$, where ψ is the nearby cycle functor for $\underline{\mathcal{M}}_C(\lambda,\mu) \to \mathbb{C}$ [KS90, §8.6]. Moreover it contains $\mathrm{IC}(\mathcal{M}_C(\lambda,\mu))$ with multiplicity one.

Conjecture 3.27. (1) $\widetilde{\underline{\mathcal{M}}}_C(\lambda,\mu)^{\chi(\mathbb{C}^\times)}$ is a disjoint union of finitely many copies of $\mathbb{C}$ such that the restriction of the morphism $\widetilde{\underline{\mathcal{M}}}_C(\lambda,\mu)^{\chi(\mathbb{C}^\times)} \to \mathbb{C}$ to each summand is the identity map. And $\underline{\mathcal{M}}_C(\lambda,\mu)^{\chi(\mathbb{C}^\times)}$ is obtained from $\widetilde{\underline{\mathcal{M}}}_C(\lambda,\mu)^{\chi(\mathbb{C}^\times)}$ by identifying the origin of each summand.

(2) A summand in (1) corresponds, in bijection, to a decomposition $\mu = \mu^1 + \mu^2$ with $V(\lambda^1)_{\mu^1}$, $V(\lambda^2)_{\mu^2} \neq 0$. The hyperbolic restriction of $\mathrm{IC}(\widetilde{\underline{\mathcal{M}}}_C(\lambda,\mu))$ is the direct sum $\bigoplus \Phi(\mathrm{IC}(\mathcal{M}_C(\lambda^1,\mu^1)) \otimes \Phi(\mathrm{IC}(\mathcal{M}_C(\lambda^2,\mu^2))$, where each summand is considered as a trivial local system on $\mathbb{C}$. Hence

$$\begin{aligned}\psi \circ \Phi(\mathrm{IC}(\underline{\mathcal{M}}_C(\lambda,\mu))) &= \Phi\pi_{C,*}(\mathrm{IC}(\widetilde{\mathcal{M}}_C(\lambda,\mu))) \\ &\cong \bigoplus_{\mu=\mu^1+\mu^2} V(\lambda^1)_{\mu^1} \otimes V(\lambda^2)_{\mu^2}.\end{aligned}$$

In the first equality we use the commutativity of the nearby cycle and hyperbolic restriction functors (see e.g., [Nak17, Prop. 5.4.1]).

(3) The sum of homomorphisms

$$\Phi(\mathrm{IC}(\mathcal{M}_C(\lambda,\mu))) \to \Phi(\pi_{C,*}(\mathrm{IC}(\widetilde{\mathcal{M}}_C(\lambda,\mu))))$$

over μ is the homomorphism $V(\lambda) \to V(\lambda^1) \otimes V(\lambda^2)$ of $\mathfrak{g}_{\mathrm{KM}}$-modules, sending v_λ to $v_{\lambda^1} \otimes v_{\lambda^2}$, where v_λ is the highest weight vector corresponding to the fundamental class of the point $\mathcal{M}_C(\lambda,\lambda)$.

4. Non-simply-laced case

In order to describe instanton moduli spaces for non-simply-laced groups as Coulomb branches, Cremonesi, Ferlito, Hanany and Mekareeya have introduced a modification of the monopole formula [CFHM14]. See also [Mek15] for more examples.

Let us consider the case of G_2 k-instantons on the Taub-NUT space for brevity. (See [CFHM14, §4].) We suppose that we already know that a quiver gauge theory associated with a symmetric affine Dynkin diagram ($D_4^{(1)}$ in this case) has the Coulomb branch isomorphic to an instanton moduli space of the corresponding group. This is a special case of the conjecture mentioned in the introduction. Moreover, it is also conjectured that moduli spaces of instantons on $\mathbb{R}^4$ and on the Taub-NUT spaces are isomorphic as affine algebraic varieties. We do not have a proof of this assertion either for $\mathbb{R}^4$ of the Taub-NUT space, but the following argument works more generally.

As for simply-laced cases, the mirror of an instanton moduli space is, roughly, a quiver gauge theory associated with the corresponding affine Dynkin diagram of type $G_2^{(1)}$ with dimension vectors $\mathbf{v} = k\delta$, $\mathbf{w} = \Lambda_0$. See Figure 1 left, where we put the numbering 0, 1, 2 on vertices.

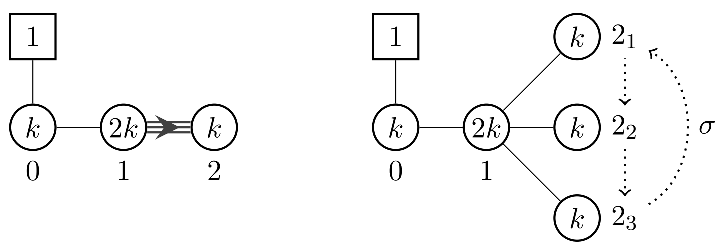

Figure 1: $\mathcal{M}_C$: G_2, D_4 k-instantons on $\mathbb{R}^4$ and folding.

Let $G = \mathrm{GL}(k) \times \mathrm{GL}(2k) \times \mathrm{GL}(k)$, product of general linear groups for circled vertices as usual. We take a triple $(\lambda^0, \lambda^1, \lambda^2)$ of coweights of $\mathrm{GL}(k)$, $\mathrm{GL}(2k)$, $\mathrm{GL}(k)$. Let us denote a triple by λ, considered as a coweight of

G. Let Y be the coweight lattice of G, and W the Weyl group of G. The monopole formula in [Part II, (2.9)] says the Hilbert series of the Coulomb branch is

$$\sum_{\lambda \in Y/W} t^{2\Delta(\lambda)} P_G(t;\lambda).$$

The definition of $P_G(t;\lambda) = P_{\mathrm{GL}(k)}(t;\lambda^0) P_{\mathrm{GL}(2k)}(t;\lambda^1) P_{\mathrm{GL}(k)}(t;\lambda^2)$ is the same as usual. The term $\Delta(\lambda)$ has two parts (see [Part II, (2.10)]. The first part is the pairing between λ and positive roots of G. This needs no modification. The second part, in this example, comes from bi-fundamental representations on edges. For a usual edge, the contribution is given by the pairing between its weight with coweights of groups at two ends. Concretely we write $\lambda^i = (\lambda^i_1, \dots, \lambda^i_k)$ $(i = 0, 2)$, $\lambda^1 = (\lambda^1_1, \dots, \lambda^1_{2k})$, the edge between vertices 0 and 1 gives the contribution $|\lambda^0_a - \lambda^1_b|$ for $a = 1, \dots, k$, $b = 1, \dots, 2k$. On the squared vertex, one should put the coweight 0, hence there is also $|\lambda^0_a|$ for $a = 1, \dots, k$.

A modification of the rule is required only for the edge between 1 and 2. The rule introduced in [CFHM14] is $|3\lambda^1_b - \lambda^2_c|$ for $b = 1, \dots, 2k$, $c = 1, \dots, k$. Thus

$$\begin{aligned} 2\Delta(\lambda) = & -2\sum_{a \neq a'} |\lambda^0_a - \lambda^0_{a'}| - 2\sum_{b \neq b'} |\lambda^1_b - \lambda^1_{b'}| - 2\sum_{c \neq c'} |\lambda^2_c - \lambda^2_{c'}| \\ & + \sum_{a=1}^{k} |\lambda^0_a| + \sum_{a=1}^{k}\sum_{b=1}^{2k} |\lambda^0_a - \lambda^1_b| + \sum_{b=1}^{2k}\sum_{c=1}^{k} |3\lambda^1_b - \lambda^2_c|. \end{aligned}$$

Now let us explain how to modify our definition of the Coulomb branch to recover this *twisted monopole formula*.

We consider the unfolding of our affine Dynkin diagram as in [Lus93, 14.1.5(f)].[12] It is a $D_4^{(1)}$ affine Dynkin graph with circled vertices $0, 1, 2_1, 2_2, 2_3$ (and 0 is connected to a squared vertex). See Figure 1 right. The corresponding vector spaces are of dimensions $k, 2k, k, k, k$ (and 1). We orient all the edges from the vertex 1 (and from the squared vertex). We consider an automorphism σ rotating cyclically the vertices $2_1, 2_2, 2_3$. We set $\mathbf{N} = V_0 \oplus \operatorname{Hom}(V_1, V_0) \oplus \operatorname{Hom}(V_1, V_{2_1}) \oplus \operatorname{Hom}(V_1, V_{2_2}) \oplus \operatorname{Hom}(V_1, V_{2_3})$ (a representation of $\hat{G} := \mathrm{GL}(V_0) \times \mathrm{GL}(V_1) \times \mathrm{GL}(V_{2_1}) \times \mathrm{GL}(V_{2_2}) \times \mathrm{GL}(V_{2_3})$). Then σ acts naturally on $\hat{G}$ and on $\mathbf{N}$, hence on $\mathcal{M}_C(\hat{G}, \mathbf{N})$. We consider the fixed

[12] It should be noted that the non simply-laced Lie algebra obtained by the folding in [Lus93] is the Langlands dual of what we get.

point set $\mathcal{M}_C(\hat{G}, \mathbf{N})^\sigma$. We have a surjection

$$\varphi\colon H^{\hat{G}_\mathcal{O}}(\mathcal{R}_{\hat{G},\mathbf{N}}) \twoheadrightarrow \mathbb{C}[\mathcal{M}_C(\hat{G}, \mathbf{N})^\sigma].$$

Thus the grading of $H^{\hat{G}_\mathcal{O}}(\mathcal{R}_{\hat{G},\mathbf{N}})$ whose Hilbert series is given by the monopole formula induces a grading of $\mathbb{C}[\mathcal{M}_C(\hat{G}, \mathbf{N})^\sigma]$.

The above formulation and the following proposition work for any quiver gauge theory with diagram automorphisms. In particular, they work for quiver gauge theories studied in §3, where their Coulomb branches are moduli spaces of bundles of an ADE group over $\mathbb{P}^1$ with additional structures. The fixed point subscheme $\mathcal{M}_C(\hat{G}, \mathbf{N})^\sigma$ is identified with a moduli space of bundles of a non simply-laced group.

Proposition 4.1. *The Hilbert series of the induced grading on*

$$\mathbb{C}[\mathcal{M}_C(\hat{G}, \mathbf{N})^\sigma]$$

is given by the twisted monopole formula.

Proof. Recall the multifiltration on $\mathbb{C}[\mathcal{M}_C(\hat{G}, \mathbf{N})]$ introduced in [Part II, §6(i)]. Let us denote the spectrum of the associated graded algebra by $\overline{\mathcal{M}}_C(\hat{G}, \mathbf{N})$. The filtration is σ-invariant, so we have the induced automorphism σ of $\overline{\mathcal{M}}_C(\hat{G}, \mathbf{N})$. Moreover, the associated graded of the ideal $I_\sigma \subset \mathbb{C}[\mathcal{M}_C(\hat{G}, \mathbf{N})]$ of functions vanishing on $\mathcal{M}_C(\hat{G}, \mathbf{N})^\sigma$ is the ideal $\overline{I}_\sigma \subset \operatorname{gr}\mathbb{C}[\mathcal{M}_C(\hat{G}, \mathbf{N})]$ of functions vanishing on $\overline{\mathcal{M}}_C(\hat{G}, \mathbf{N})^\sigma$. Hence it suffices to prove that the Hilbert series of the induced monopole grading on $\mathbb{C}[\overline{\mathcal{M}}_C(\hat{G}, \mathbf{N})^\sigma]$ is given by the twisted monopole formula.

We fix a σ-invariant Cartan torus $\hat{T} \subset \hat{G}$ corresponding to a σ-invariant decomposition of V_i into a direct sum of lines. We have $\hat{\mathfrak{t}}^\sigma = \mathfrak{t}$ (the Lie algebra of the Cartan torus $T = \hat{T}^\sigma \subset \hat{G}^\sigma = G$). Let us specify a vector subspace E of $\operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathcal{R}_{\hat{G},\mathbf{N}})$ such that the restriction $\overline{\varphi}|_E$ is an isomorphism onto $\mathbb{C}[\overline{\mathcal{M}}_C(\hat{G}, \mathbf{N})^\sigma]$. Recall from [Part II, §6(i)] that

$$\operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathcal{R}_{\hat{G},\mathbf{N}}) = \bigoplus_{\hat{\lambda}\in\hat{Y}^+} \mathbb{C}[\hat{\mathfrak{t}}]^{W_{\hat{\lambda}}}[\mathcal{R}_{\hat{\lambda}}].$$

Here $\hat{Y}^+ \subset \hat{Y}$ is the cone of dominant coweights of $\hat{T}$, $\hat{Y}^+ \xrightarrow{\sim} \hat{Y}/\hat{W}$. We define $\hat{Y}' \subset \hat{Y}^+$ as the set of collections $(\hat{\lambda}^0, \hat{\lambda}^1, \hat{\lambda}^{2_1}, \hat{\lambda}^{2_2}, \hat{\lambda}^{2_3})$ such that $\hat{\lambda}^{2_1}_c \geq \hat{\lambda}^{2_2}_c \geq \hat{\lambda}^{2_3}_c \geq \hat{\lambda}^{2_1}_c - 1$ for any $c = 1, \dots, k$. There is a bijection $\psi\colon \hat{Y}' \xrightarrow{\sim} Y^+$

(the dominant weights of T):

$$(\hat{\lambda}^0, \hat{\lambda}^1, \hat{\lambda}^{2_1}, \hat{\lambda}^{2_2}, \hat{\lambda}^{2_3}) \mapsto (\lambda^0, \lambda^1, \lambda^2) := (\hat{\lambda}^0, \hat{\lambda}^1, \hat{\lambda}^{2_1} + \hat{\lambda}^{2_2} + \hat{\lambda}^{2_3}).$$

Note that for $\hat{\lambda} \in \hat{Y}'$ we have $\Delta(\hat{\lambda}) = \Delta(\psi\hat{\lambda})$ (the RHS Δ is the twisted one). Finally note that $W_{\psi\hat{\lambda}} = W_{\lambda^0} \times W_{\lambda^1} \times W_{\lambda^2}$, and $W_{\lambda^0} = W_{\hat{\lambda}^0}$, $W_{\lambda^1} = W_{\hat{\lambda}^1}$, $W_{\lambda^2} = W_{\hat{\lambda}^{2_1}} \cap W_{\hat{\lambda}^{2_2}} \cap W_{\hat{\lambda}^{2_3}}$. The diagonal embedding $\mathfrak{t}^2 \hookrightarrow \hat{\mathfrak{t}}^{2_1} \oplus \hat{\mathfrak{t}}^{2_2} \oplus \hat{\mathfrak{t}}^{2_3}$ induces a surjection $\mathbb{C}[\hat{\mathfrak{t}}^{2_1}]^{W_{\hat{\lambda}^{2_1}}} \otimes \mathbb{C}[\hat{\mathfrak{t}}^{2_2}]^{W_{\hat{\lambda}^{2_2}}} \otimes \mathbb{C}[\hat{\mathfrak{t}}^{2_3}]^{W_{\hat{\lambda}^{2_3}}} \twoheadrightarrow \mathbb{C}[\mathfrak{t}^2]^{W_{\lambda^2}}$. We choose a homogeneous section ε of this surjection, and denote by $E_{\lambda^2} \subset \mathbb{C}[\hat{\mathfrak{t}}^{2_1}]^{W_{\hat{\lambda}^{2_1}}} \otimes \mathbb{C}[\hat{\mathfrak{t}}^{2_2}]^{W_{\hat{\lambda}^{2_2}}} \otimes \mathbb{C}[\hat{\mathfrak{t}}^{2_3}]^{W_{\hat{\lambda}^{2_3}}}$ the image of ε. Now we define $E := \bigoplus_{\hat{\lambda} \in \hat{Y}'} \mathbb{C}[\mathfrak{t}]^{W_{\psi\hat{\lambda}}}[\mathcal{R}_{\hat{\lambda}}]$ where $\mathbb{C}[\mathfrak{t}]^{W_{\psi\hat{\lambda}}} = \mathbb{C}[\mathfrak{t}^0]^{W_{\lambda^0}} \otimes \mathbb{C}[\mathfrak{t}^1]^{W_{\lambda^1}} \otimes E_{\lambda^2}$ is embedded into

$$\mathbb{C}[\hat{\mathfrak{t}}]^{W_{\hat{\lambda}}} = \mathbb{C}[\hat{\mathfrak{t}}^0]^{W_{\hat{\lambda}^0}} \otimes \mathbb{C}[\hat{\mathfrak{t}}^1]^{W_{\hat{\lambda}^1}} \otimes \mathbb{C}[\hat{\mathfrak{t}}^{2_1}]^{W_{\hat{\lambda}^{2_1}}} \otimes \mathbb{C}[\hat{\mathfrak{t}}^{2_2}]^{W_{\hat{\lambda}^{2_2}}} \otimes \mathbb{C}[\hat{\mathfrak{t}}^{2_3}]^{W_{\hat{\lambda}^{2_3}}}.$$

The character of E is given by the twisted monopole formula.

It remains to check that $\overline{\varphi}\colon E \xrightarrow{\sim} \mathbb{C}[\overline{\mathcal{M}}_C(\hat{G}, \mathbf{N})^\sigma]$. First we consider the similar problem for $\mathbf{N} = 0$. Namely, let $\widetilde{I}_\sigma \subset \operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathrm{Gr}_{\hat{G}})$ be the ideal generated by the expressions $f - \sigma^* f$, and let $\widetilde{E} \subset \operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathrm{Gr}_{\hat{G}})$ be defined the same way as E. We will prove $\operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathrm{Gr}_{\hat{G}}) = \widetilde{E} \oplus \widetilde{I}_\sigma$. To check the surjectivity of $\widetilde{E} \to \operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathrm{Gr}_{\hat{G}})/\widetilde{I}_\sigma$ we will find for any $\hat{\lambda} \in \hat{Y}^+$ a coweight $\hat{\mu} \in \hat{Y}'$ such that $[\mathrm{Gr}^{\hat{\lambda}}_{\hat{G}}] - [\mathrm{Gr}^{\hat{\mu}}_{\hat{G}}] \in \widetilde{I}_\sigma$. In effect, if the maximum of $|\hat{\lambda}^{2_1}_1 - \hat{\lambda}^{2_2}_1|$, $|\hat{\lambda}^{2_2}_1 - \hat{\lambda}^{2_3}_1|$, $|\hat{\lambda}^{2_3}_1 - \hat{\lambda}^{2_1}_1|$ is bigger than 1, and is equal to say $\hat{\lambda}^{2_2}_1 - \hat{\lambda}^{2_1}_1$, then we have

$$\begin{aligned}
&\left[\mathrm{Gr}_{\hat{G}}^{(\hat{\lambda}^0, \hat{\lambda}^1, \hat{\lambda}^{2_1}, \hat{\lambda}^{2_2}, \hat{\lambda}^{2_3})}\right] - \left[\mathrm{Gr}_{\hat{G}}^{(\hat{\lambda}^0, \hat{\lambda}^1, \hat{\lambda}^{2_1}+(1,\dots,1), \hat{\lambda}^{2_2}-(1,\dots,1), \hat{\lambda}^{2_3})}\right] \\
&= \left[\mathrm{Gr}_{\hat{G}}^{(\hat{\lambda}^0, \hat{\lambda}^1, \hat{\lambda}^{2_1}+(1,\dots,1), \hat{\lambda}^{2_2}, \hat{\lambda}^{2_3})}\right] \cdot \left(\left[\mathrm{Gr}_{\hat{G}}^{(0,0,-(1,\dots,1),0,0)}\right] - \left[\mathrm{Gr}_{\hat{G}}^{(0,0,0,-(1,\dots,1),0)}\right] \right) \\
&\in \widetilde{I}_\sigma.
\end{aligned}$$

Proceeding like this we can replace the initial $[\mathrm{Gr}^{\hat{\lambda}}_{\hat{G}}]$ with the one that is equal to it modulo $\widetilde{I}_\sigma$ but has the absolute value of differences $\hat{\lambda}^{2_i}_1 - \hat{\lambda}^{2_j}_1$ at most 1. Now if say $\hat{\lambda}^{2_2}_1 - \hat{\lambda}^{2_3}_1 = -1$ we repeat the above replacement once more to swap $\hat{\lambda}^{2_2}_1$ and $\hat{\lambda}^{2_3}_1$ and make sure $\hat{\lambda}^{2_2}_1 - \hat{\lambda}^{2_3}_1 = 1$. This way we replace the initial $[\mathrm{Gr}^{\hat{\lambda}}_{\hat{G}}]$ with the one that is equal to it modulo $\widetilde{I}_\sigma$, and has $\hat{\lambda}^{2_1}_1 \geq \hat{\lambda}^{2_2}_1 \geq \hat{\lambda}^{2_3}_1 \geq \hat{\lambda}^{2_1}_1 - 1$. To take care of the second coordinate $\hat{\lambda}^{2_i}_2$, instead of $-(1, \dots, 1)$ above we use $-(0, 1, \dots, 1)$ (that does not change the first coordinate $\hat{\lambda}^{2_i}_1$) in the above replacement procedure. Proceeding like this we arrive at the desired coweight $\hat{\mu} \in \hat{Y}'$ such that $[\mathrm{Gr}_{\hat{G}}^{\hat{\lambda}}] - [\mathrm{Gr}_{\hat{G}}^{\hat{\mu}}] \in \widetilde{I}_\sigma$. The surjectivity of $\widetilde{E} \to \operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathrm{Gr}_{\hat{G}})/\widetilde{I}_\sigma$ is proved.

Since $\operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathrm{Gr}_{\hat{G}}) = \bigotimes_i \operatorname{gr} H^{\mathrm{GL}(V_i)_\mathcal{O}}(\mathrm{Gr}_{\mathrm{GL}(V_i)})$ (the product over $i = 0, 1, 2_1, 2_2, 2_3$), and σ rotates cyclically the last three factors, we see that

$$\begin{aligned}\operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathrm{Gr}_{\hat{G}})/\widetilde{I}_\sigma \simeq{} & \operatorname{gr} H^{\mathrm{GL}(V_0)_\mathcal{O}}(\mathrm{Gr}_{\mathrm{GL}(V_0)}) \\ & \otimes \operatorname{gr} H^{\mathrm{GL}(V_1)_\mathcal{O}}(\mathrm{Gr}_{\mathrm{GL}(V_1)}) \otimes \operatorname{gr} H^{\mathrm{GL}(V_2)_\mathcal{O}}(\mathrm{Gr}_{\mathrm{GL}(V_2)}),\end{aligned}$$

and the graded dimension of the RHS coincides with the one of $\widetilde{E}$. Here the grading is by the cone of dominant coweights of G times $\mathbb{Z}$ (the homological grading). Hence the surjectivity established in the previous paragraph implies the isomorphism $\widetilde{E} \xrightarrow{\sim} \operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathrm{Gr}_{\hat{G}})/\widetilde{I}_\sigma$.

We return to the proof of isomorphism $\overline{\varphi}\colon E \xrightarrow{\sim} \mathbb{C}[\overline{\mathcal{M}}_C(\hat{G}, \mathbf{N})^\sigma]$. Consider the following commutative diagram:

$$\begin{array}{ccc} \operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathcal{R}_{\hat{G},\mathbf{N}}) & \longrightarrow & \operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathcal{R}_{\hat{G},\mathbf{N}})/\overline{I}_\sigma \\ \downarrow{\scriptstyle \mathbf{z}^*} & & \downarrow{\scriptstyle \mathbf{z}^*} \\ \operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathrm{Gr}_{\hat{G}}) & \longrightarrow & \operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathrm{Gr}_{\hat{G}})/\widetilde{I}_\sigma \end{array}$$

Here $\mathbf{z}^*$ is the restriction to the zero section (see [Part II, §5(iv)]). Thus $\mathbf{z}^*$ is injective, $\mathbf{z}^* E \subset \widetilde{E}$, and $\overline{I}_\sigma = (\mathbf{z}^*)^{-1}\widetilde{I}_\sigma$, and the right vertical arrow is injective as well. Hence the injectivity $\widetilde{E} \hookrightarrow \operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathcal{R}_{\hat{G},\mathbf{N}})/\overline{I}_\sigma$.

To prove the surjectivity, recall the setup of [Part II, §6(ix)]. We use the flavor symmetry group $\mathbb{C}^\times$, and instead of $E \subset \operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathcal{R}_{\hat{G},\mathbf{N}}) \supset \overline{I}_\sigma$ we consider the similarly defined subspace and ideal $E' \subset \operatorname{gr} H^{\mathbb{C}^\times \times \hat{G}_\mathcal{O}}(\mathcal{R}_{\hat{G},\mathbf{N}}) \supset \overline{I}'_\sigma$. It suffices to prove the surjectivity

$$\mathbb{C}(\mathbf{t}) \otimes_{\mathbb{C}[\mathbf{t}]} E' \twoheadrightarrow \mathbb{C}(\mathbf{t}) \otimes_{\mathbb{C}[\mathbf{t}]} \operatorname{gr} H^{\mathbb{C}^\times \times \hat{G}_\mathcal{O}}(\mathcal{R}_{\hat{G},\mathbf{N}})/\overline{I}'_\sigma$$

because the $\mathbf{t}$-deformation of $\operatorname{gr} H^{\hat{G}_\mathcal{O}}(\mathcal{R}_{\hat{G},\mathbf{N}})$ is trivial due to [Part II, Remark 3.24(2)]. This generic surjectivity follows from [Part II, Prop. 6.17] and the surjectivity at $\mathbf{t} = \infty$ which was already established earlier during the proof. $\square$

Remark 4.2. The proof of Proposition 4.1 works for all the twisted cases whose unfolding has no cycles (because we use [Part II, Remark 3.24(2)]. This excludes the $C_n^{(1)}$ case whose unfolding is the cyclic quiver $A_{2n-1}^{(1)}$. In this case the fixed point set of the automorphism σ of $A^{(1)}_{2n-1}$ consists of two points, and we choose a σ-invariant orientation from the first one to the second one. Then the dilatation action of $\mathbb{C}^\times$ on $\mathbf{N}$ factors through $\hat{G}$ again, and the proof of Proposition 4.1 goes through as well.[13]

Appendices

Alexander Braverman, Michael Finkelberg, Joel Kamnitzer,
Ryosuke Kodera, Hiraku Nakajima,
Ben Webster, and Alex Weekes

In the first appendix we write certain elements of quantized Coulomb branches $\mathcal{A}_\hbar$ as explicit difference operators. These elements are homology classes lived on closed $G_\mathcal{O}$-orbits, i.e., orbits Gr^λ_G for *minuscule* coweights λ, and their slight generalization corresponding to *quasi-minuscule* and *small fundamental* coweights. The first class is called minuscule monopole operators in physics literature (see [Part II, Remark 6.7]).

Examples of explicit difference operators include Macdonald operators [Mac95, Chap. VI, §3] and ones in representations of Yangian in the work of Gerasimov-Kharchev-Lebedev-Oblezin [GKLO05] and its generalization [KWWY14].

We hope that these elements, together with $H^*_G(\mathrm{pt})$, generate quantized Coulomb branches $\mathcal{A}_\hbar$ in many situations, possibly after inverting $\hbar$ (and variables for flavor symmetry groups). If this would happen, it identifies $\mathcal{A}_\hbar$ as a subalgebra in the ring of difference operators, generated by explicit elements. It gives a purely algebraic characterization of $\mathcal{A}_\hbar$. We will show that this happens for quiver gauge theories of Jordan and ADE types. In particular, we will show that the quantized Coulomb branches for quiver gauge theories of type ADE are isomorphic to truncated shifted Yangian under the dominance condition in the second appendix. (See Corollary B.28.)

Appendix A. Minuscule monopole operators as difference operators

A.1. Embedding to the ring of difference operators

Let us return back to general notation conventions in [Part II]. Let $(G, \mathbf{N})$ be a pair of a complex reductive group and its representation. Let T be

[13] We are grateful to L. Rybnikov for this observation.

a maximal torus of G and $\mathbf{N}_T$ the restriction of $\mathbf{N}$ to T as usual. Let W denote the Weyl group. Let us consider the quantized Coulomb branches for $(G,\mathbf{N})$, $(T,\mathbf{N}_T)$ and $(T,0)$. If we simply write $\mathcal{A}_\hbar$, it means the quantized Coulomb branch for $(G,\mathbf{N})$. We indicate a group and its representation for other two. We will add flavor symmetry groups in examples below, but we omit them for brevity now.

Recall we have an embedding $\mathcal{A}_\hbar \hookrightarrow \mathcal{A}_\hbar(T,0)[\hbar^{-1},(\text{root}+m\hbar)^{-1}]_{m\in\mathbb{Z}}$ in [Part II, Remark 5.23]. We thus have an algebra embedding

$$\begin{aligned}\mathbf{z}^*(\iota_*)^{-1}\colon \mathcal{A}_\hbar \hookrightarrow \tilde{\mathcal{A}}_\hbar &\stackrel{\text{def.}}{=} \mathcal{A}_\hbar(T,0)[\hbar^{-1},(\text{root}+m\hbar)^{-1}]_{m\in\mathbb{Z}} \\ &= \mathbb{C}[\hbar]\langle w_r, \mathsf{u}_r^{\pm1}, \hbar^{-1}, (\alpha+m\hbar)^{-1}\rangle \quad (\alpha\colon \text{root},\ m\in\mathbb{Z}).\end{aligned}$$

We consider $\tilde{\mathcal{A}}_\hbar$ as the localized ring of $\hbar$-difference operators on $\mathfrak{t}$: $\mathsf{u}_r^{\pm1}$ is the operator

$$(\mathsf{u}_r^{\pm1} f)(\dots,w_s,\dots) = f(\dots,w_s\pm\hbar\delta_{r,s},\dots).$$

Remark A.1. We could also consider $\mathcal{A}_\hbar(T,0)$ as the ring of differential operators on $T^\vee$: $\mathsf{u}_r^{\pm1}$ is a coordinate of $T^\vee$, and w_s is $-\hbar\mathsf{u}_s\partial/\partial\mathsf{u}_s$. But it is natural for us to consider difference operators on $\mathfrak{t}$, as we invert roots.

In general, we do not know how to characterize the image of $\mathcal{A}_\hbar$ in $\tilde{\mathcal{A}}_\hbar$ explicitly. Nevertheless, the image of a homology class associated with a closed $G_\mathcal{O}$-orbit Gr_G^λ can be explicitly written down. (See [Part II, Proposition 6.6] for $(\iota_*)^{-1}$ and [Part II, §4(vi)] for $\mathbf{z}^*$.) Note that Gr_G^λ is closed if and only if λ is minuscule. (Since $\overline{\mathrm{Gr}_G^\lambda}\supset \mathrm{Gr}^\mu$ if and only if $\lambda\ge\mu$, and the minuscule coweights are minimal in this order.)

Proposition A.2. *Let λ be a minuscule dominant coweight and W_λ its stabilizer in W. Let $f\in\mathbb{C}[\mathfrak{t}]^{W_\lambda}$. Let $\mathcal{R}_\lambda = \pi^{-1}(\mathrm{Gr}_G^\lambda)$, where $\pi\colon\mathcal{R}\to\mathrm{Gr}_G$ is the projection. Then*

$$\mathbf{z}^*(\iota_*)^{-1} f[\mathcal{R}_\lambda] = \sum_{\lambda'=w\lambda\in W\lambda} \frac{wf\times e\left(z^{\lambda'}\mathbf{N}_\mathcal{O}/z^{\lambda'}\mathbf{N}_\mathcal{O}\cap\mathbf{N}_\mathcal{O}\right)}{e(T_{\lambda'}\mathrm{Gr}_G^\lambda)}\mathsf{u}_{\lambda'},$$

where $T_{\lambda'}\mathrm{Gr}_G^\lambda$ is the tangent space of Gr_G^λ at the point $z^{\lambda'}$ and $\mathsf{u}_{\lambda'}$ is the shift operator corresponding to λ', i.e., $(\mathsf{u}_{\lambda'}f)(\bullet) = f(\bullet+\hbar\lambda')$ for $f\in\mathbb{C}[\mathfrak{t}]$.

A.2. Quiver gauge theories

Let us return back to the notational convention in this paper.

Let $(\mathrm{GL}(V), \mathbf{N})$ be a quiver gauge theory, which is not necessarily of either finite ADE or affine type. Let $T(V)$ be a maximal torus of $\mathrm{GL}(V)$, and $\mathbf{N}_{T(V)}$ is the restriction of $\mathbf{N}$ to $T(V)$. We add the flavor symmetry group $T(W)$ as in §3(v). Thus we mean $\mathcal{A}_\hbar = H_*^{(\mathrm{GL}(V)\times T(W))_\mathcal{O}\rtimes\mathbb{C}^\times}(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}})$, and $\mathcal{A}_\hbar(T(V), \mathbf{N}_{T(V)})$, $\mathcal{A}_\hbar(T(V), 0)$ are similar.

When there are several loops in the underlying graph (e.g., the Jordan quiver or an affine quiver of type A), we should also add additional flavor symmetries rescaling entries in $\mathbf{N}$ in loops. But we omit them for brevity except in §A.3.

Recall $w_{i,r}^*$ is the cocharacter of $\mathrm{GL}(V) = \prod \mathrm{GL}(V_i)$, which is equal to 0 except at the vertex i, and is $(0, \dots, 0, 1, 0, \dots, 0)$ at i. Here 1 is at the rth entry ($r = 1, \dots, a_i = \dim V_i$). We take corresponding coordinates $w_{i,r}$, $\mathsf{u}_{i,r}$ ($i \in I$, $1 \le r \le a_i$) of $\operatorname{Lie} T(V)$ and $T(V)^\vee$. The roots are $w_{i,r} - w_{i,s}$ ($r \ne s$). Furthermore, $\mathcal{A}_\hbar(T(V), 0)$ is a $\mathbb{C}[\hbar, z_1, \dots, z_N]$-algebra generated by $w_{i,r}$, $\mathsf{u}_{i,r}^{\pm 1}$ ($i \in I,\ 1 \le r \le a_i$) with relations $[\mathsf{u}_{j,s}^{\pm 1}, w_{i,r}] = \pm\delta_{i,j}\delta_{r,s}\hbar\mathsf{u}_{i,r}^{\pm 1}$. We thus have an algebra embedding

$$\begin{aligned} &\mathcal{A}_\hbar \hookrightarrow \tilde{\mathcal{A}}_\hbar \\ &\overset{\text{def.}}{=} \mathbb{C}[\hbar, z_1, \dots, z_N]\langle w_{i,r}, \mathsf{u}_{i,r}^{\pm 1}, \hbar^{-1}, (w_{i,r} - w_{i,s} + m\hbar)^{-1} (r \ne s,\ m \in \mathbb{Z})\rangle. \end{aligned}$$

We consider $\tilde{\mathcal{A}}_\hbar$ as the localized ring of $\hbar$-difference operators on $\operatorname{Lie} T(V)$ as above, and z_1, …, z_N are parameters.

Let $\varpi_{i,n}$ be the nth fundamental coweight of the factor $\mathrm{GL}(V_i)$, i.e., $(1, \dots, 1, 0, \dots, 0) = w_{i,1}^* + \dots + w_{i,n}^*$, where 1 appears n times ($1 \le n \le a_i$). Then $\mathrm{Gr}_{\mathrm{GL}(V)}^{\varpi_{i,n}}$ is closed and isomorphic to the Grassmannian $\mathrm{Gr}(V_i, n)$ of n-dimensional quotients of V_i. In fact, $\mathrm{Gr}_{\mathrm{GL}(V)}^{\varpi_{i,n}}$ is identified with the moduli space of $\mathcal{O}$-modules L such that

$$z\mathcal{O} \otimes V_i \subset L \subset \mathcal{O} \otimes V_i, \qquad \dim_\mathbb{C} \mathcal{O} \otimes V_i / L = n,$$

hence $\mathcal{O} \otimes V_i/L$ is the corresponding quotient space of $\mathcal{O} \otimes V_i / z\mathcal{O} \otimes V_i \cong V_i$.

Let $\mathcal{Q}_i$ be the vector bundle over $\mathrm{Gr}_{\mathrm{GL}(V)}^{\varpi_{i,n}}$ whose fiber at L is $\mathcal{O} \otimes V_i/L$. It is the universal rank n quotient bundle of $\mathrm{Gr}(V_i, n) \times V_i$. Its pull-back to $\mathcal{R}_{\varpi_{i,n}}$ is denoted also by $\mathcal{Q}_i$ for brevity. Let $c_p(\mathcal{Q}_i)$ denote its pth Chern class. More generally we can consider a class $f(\mathcal{Q}_i)$ for a symmetric function f in n variables so that $c_p(\mathcal{Q}_i)$ corresponds to the pth elementary symmetric polynomial.

The $T(V)$ fixed points in $\mathrm{Gr}_{\mathrm{GL}(V)}^{\varpi_{i,n}}$ are in bijection to subsets $I \subset \{1, \dots, a_i\}$ with $\#I = n$. The bijection is given by assigning a cocharacter

$\lambda_I \overset{\text{def.}}{=} \sum_{r\in I} w_{i,r}^*$ of $\mathrm{GL}(V_i)$ to I. The fixed point formula implies

$$(\iota_*)^{-1} f(\mathcal{Q}_i) \cap [\mathcal{R}_{\varpi_{i,n}}] = \sum_{\substack{I\subset\{1,\dots,a_i\}\\ \#I=n}} f(w_{i,I}) \prod_{r\in I, s\notin I} \frac{r^{\lambda_I}}{w_{i,r}-w_{i,s}},$$

where $f(w_{i,I})$ means that we substitute $(w_{i,r})_{r\in I}$ to the symmetric function f, and r^{λ_I} denote the fundamental class of the fiber of $\mathcal{R}_{T(V),\mathbf{N}_{T(V)}} \to \mathrm{Gr}_{T(V)}$ at λ_I. In view of Proposition A.2, the $T(V)$-fixed point set is the Weyl group orbit $\mathbb{W}\varpi_{i,n}$, and $\prod(w_{i,r}-w_{i,s})$ is the equivariant Euler class $e(T_{\lambda_I}\mathrm{Gr}_{\mathrm{GL}(V)}^{\varpi_{i,n}})$ of the tangent space of $\mathrm{Gr}_{\mathrm{GL}(V)}^{\varpi_{i,n}}$ at the fixed point λ_I.

Furthermore

$$e\left(z^{\lambda_I}\mathbf{N}_{\mathcal{O}}/z^{\lambda_I}\mathbf{N}_{\mathcal{O}}\cap\mathbf{N}_{\mathcal{O}}\right) = \prod_{\substack{h\in Q_1:\mathrm{o}(h)=i\\ r\in I}} \prod_{\substack{s=1\\ \mathrm{i}(h)\neq i \text{ or } s\notin I}}^{a_{\mathrm{i}(h)}} (-w_{i,r}+w_{\mathrm{i}(h),s}-\hbar/2)$$

followed by the replacement r^{λ_I} by $\prod_{r\in I}\mathsf{u}_{i,r}$. Here '$\mathrm{i}(h)\neq i$ or $s\notin I$' means that the product excludes $s\in I$ if h is an edge loop. We thus get

$$\text{(A.3)}\quad \mathbf{z}^*(\iota_*)^{-1} f(\mathcal{Q}_i)\cap[\mathcal{R}_{\varpi_{i,n}}] \\ = \sum_{\substack{I\subset\{1,\dots,a_i\}\\ \#I=n}} f(w_{i,I}) \frac{\displaystyle\prod_{\substack{h\in Q_1:\mathrm{o}(h)=i\\ r\in I}} \prod_{\substack{s=1\\ \mathrm{i}(h)\neq i \text{ or } s\notin I}}^{a_{\mathrm{i}(h)}} (-w_{i,r}+w_{\mathrm{i}(h),s}-\hbar/2)}{\displaystyle\prod_{r\in I,s\notin I}(w_{i,r}-w_{i,s})} \prod_{r\in I}\mathsf{u}_{i,r}.$$

Instead of $f(\mathcal{Q}_i)$, we can also consider the class $f(\mathcal{S}_i)$, a polynomial in Chern classes of the universal subbundle $\mathcal{S}_i$ over $\mathrm{Gr}_{\mathrm{GL}(V)}^{\varpi_{i,n}}$. Then variables $(w_{i,r})_{r\in I}$ in $f(w_{i,I})$ are replaced by $(w_{i,s})_{s\notin I}$. We will consider symmetric functions in the full variables $w_{i,r}$ $(r=1,\dots,a_i)$ later, so the difference between $\mathcal{Q}_i$ and $\mathcal{S}_i$ are not essential: the algebra generated by (A.3) and one by $f(\mathcal{S}_i)$ are the same if we add symmetric functions in the full $w_{i,r}$.

Let us recall the Δ-degree, defined in [Part II, (2.10)]. Its value for $\varpi_{i,n}$ is

$$\tag{A.4} \begin{aligned} \Delta(\varpi_{i,n}) = (\#\{i \to i \in Q_1\} - 1)n(\dim V_i - n) \\ + \frac{n}{2} \sum_{\substack{h \in Q_1 \sqcup \overline{Q_1} \\ \mathrm{o}(h)=i \\ \mathrm{i}(h) \neq i}} \dim V_{\mathrm{i}(h)} + \frac{n}{2} \dim W_i, \end{aligned}$$

where $\#\{i \to i \in Q_1\}$ is the number of edge loops at i. For a finite quiver gauge theory of type ADE and $n = 1$, this is equal to $1 + \frac{1}{2}\langle \mu, \alpha_i^\vee \rangle$. For Jordan quiver and $n = 1$, we have $\frac{1}{2} \dim W_i$.

Similarly we consider $\varpi_{i,n}^* = -w_0 \varpi_{i,n}$, where the corresponding orbit $\mathrm{Gr}_{\mathrm{GL}(V)}^{\varpi_{i,n}^*}$ is also isomorphic to the Grassmannian $\mathrm{Gr}(n, V_i)$ of n-planes in V_i. In fact, $\mathrm{Gr}_{\mathrm{GL}(V)}^{\varpi_{i,n}^*}$ is the moduli space of $\mathcal{O}$-modules L such that $\mathcal{O} \otimes V_i \subset L \subset z^{-1}\mathcal{O} \otimes V_i$ with $\dim_{\mathbb{C}} L/\mathcal{O} \otimes V_i = n$. Let $\mathcal{S}_i$ be the rank n vector bundle over $\mathrm{Gr}_{\mathrm{GL}(V)}^{\varpi_{i,n}^*}$ whose fiber over L is $L/\mathcal{O} \otimes V_i$. Its pull-back to $\mathcal{R}_{\varpi_{i,n}^*}$ is also denoted by $\mathcal{S}_i$. Then

$$\tag{A.5} \begin{aligned} &\mathbf{z}^*(\iota_*)^{-1} f(\mathcal{S}_i) \cap [\mathcal{R}_{\varpi_{i,n}^*}] \\ = &\sum_{\substack{I \subset \{1,\dots,a_i\} \\ \#I = n}} f(w_{i,I} - \hbar) \prod_{\substack{r \in I \\ k : i_k = i}} (w_{i,r} - z_k - \hbar/2) \\ &\times \frac{\displaystyle\prod_{\substack{h \in Q_1 : \mathrm{i}(h) = i \\ r \in I}} \prod_{\substack{s=1 \\ \mathrm{o}(h) \neq i \text{ or } s \notin I}}^{a_{\mathrm{o}(h)}} (w_{i,r} - w_{\mathrm{o}(h),s} - \hbar/2)}{\displaystyle\prod_{r \in I, s \notin I} (-w_{i,r} + w_{i,s})} \prod_{r \in I} \mathsf{u}_{i,r}^{-1}, \end{aligned}$$

where $f(w_{i,I} - \hbar)$ means that we substitute $(w_{i,r} - \hbar)_{r \in I}$ to f. The extra factor $w_{i,r} - z_k - \hbar/2$ came from $\mathrm{Hom}(W_i, V_i)$.

Thus elements in the right hand sides of (A.3, A.5) are in the image of $\mathbf{z}^*(\iota_*)^{-1} \colon \mathcal{A}_\hbar \hookrightarrow \tilde{\mathcal{A}}_\hbar$.

Remark A.6. We assume that $(\mathrm{GL}(V), \mathbf{N})$ is a quiver gauge theory of ADE type, so that the Coulomb branch is isomorphic to a BD slice $\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}$, $\lambda - \mu = \alpha$. Recall the function $\chi_{i,+}^\lambda$ of [BDF16, Theorem 1.6(5), Theorem 6.4]. It is a function on $\mathring{Z}^\alpha \times \mathbb{A}^N$ measuring Ext^1 of certain line bundles on $\mathbb{P}^1$ coming from the flags in $\mathring{Z}^\alpha$. More conceptually, it is the crucial part (the 4-th summand of [BDF16, (1.5)]) of the *Gaiotto-Witten superpotential*, or else the i-th summand of the *Whittaker function* (see e.g. [BDF16, 6.3]). Composing $\chi^\lambda_{i,+}$ with the projection $\overline{\mathcal{W}}^{\lambda^*}_{\mu^*} \to Z^\alpha \times \mathbb{A}^N$ we can view $\chi^\lambda_{i,+}$ as a rational function on $\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}$. Now a direct comparison of formulas (A.5) and [BDF16, (1.3)] shows that up to sign, $\chi^\lambda_{i,+}$ coincides with $\mathbf{z}^*(\iota_*)^{-1}[\mathcal{R}_{\varpi^*_{i,1}}]|_{\hbar=0}$; in particular, it is a *regular* function on $\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}$.

Recall that the logarithmic part $\log F_\alpha$ of the Gaiotto-Witten superpotential (the 3-rd summand of [BDF16, (1.5)]) is also expressed in terms of Coulomb branch, see Remark 3.5.

A.3. Jordan quiver

Consider the case of Jordan quiver. We omit the index i as we only have one vertex. For example, let $\dim V = a$, $\dim W = l$. Hence $\mathrm{GL}(V) = \mathrm{GL}(a)$. As we mentioned before, we add the dilatation on $\mathbf{N}$ as the flavor symmetry $\mathbb{C}^\times$. Let us denote the corresponding equivariant variable by $\mathbf{t}$. By Proposition 3.24, the Coulomb branch $\mathcal{A} = \mathbb{C}[\mathcal{M}_C]$ with $\hbar = \mathbf{t} = z_k = 0$ is $\mathrm{Sym}^a \mathcal{S}_l$ where $\mathcal{S}_l$ is the hypersurface $xy = w^l$ in $\mathbb{A}^3$.

Note that the equivariant variable $\mathbf{t}$ will be added each factor in the numerator of (A.3, A.5). Since it always appears with $-\hbar/2$, let us absorb $-\hbar/2$ to $\mathbf{t}$. Also we replace z_k by $z_k + \hbar + \mathbf{t}$ so that $w_r - z_k + \mathbf{t}$ becomes $w_r - \hbar - z_k$.

Then (A.3, A.5) become

(A.7)

$$E_n[f] \overset{\text{def.}}{=} \sum_{\substack{I\subset\{1,\dots,a\}\\ \#I=n}} f(w_I) \prod_{r\in I, s\notin I} \frac{w_r - w_s - \mathbf{t}}{w_r - w_s} \prod_{r\in I} \mathsf{u}_r,$$

$$F_n[f] \overset{\text{def.}}{=} \sum_{\substack{I\subset\{1,\dots,a\}\\ \#I=n}} f(w_I - \hbar) \prod_{r\in I, s\notin I} \frac{w_r - w_s + \mathbf{t}}{w_r - w_s} \prod_{r\in I} \left(\prod_{k=1}^{l} (w_r - \hbar - z_k) \cdot \mathsf{u}_r^{-1} \right),$$

where $f(w_I)$, $f(w_I - \hbar)$ are $f\left((w_r)_{r\in I}\right)$, $f\left((w_r - \hbar)_{r\in I}\right)$ respectively. We also multiply $(-1)^{n(a-n)}$ to omit the sign.

If $f \equiv 1$, $E_n[1]$ is a rational version of the nth Macdonald operator, once u_r is understood as the $\hbar$-difference operator:

$$f(w_1, \dots, w_a) \mapsto f(w_1, \dots, w_r + \hbar, \dots, w_a).$$

A little more precisely, the Macdonald operator is

$$\sum_{\substack{I\subset\{1,\dots,a\}\\ \#I=n}} \prod_{r\in I, s\notin I} \frac{tx_r - x_s}{x_r - x_s} \prod_{r\in I} T_r,$$

for $(T_r f)(x_1,\dots,x_a) = f(x_1,\dots,qx_r,\dots,x_a)$. (See e.g., [Mac95, Chap. VI, §3].) We recover $E_n[1]$ if we set $x_r = \exp(\boldsymbol{\beta} w_r)$, $q = \exp(\boldsymbol{\beta}\hbar)$, $t = \exp(-\boldsymbol{\beta}\mathbf{t})$ and take the limit $\boldsymbol{\beta}\to 0$.

Remark A.8. Let us consider the operator E_n for the K-theoretic version of the quantized Coulomb branch. The computation is the same, we just replace Euler classes by K-theoretic ones, e.g., $w_r - w_s$ by $1 - x_s x_r^{-1} = 1 - \exp(-(w_r - w_s))$ under the identification $x_r = \exp w_r$. Then $(-w_r + w_s + \mathbf{t})/(w_r - w_s)$ is replaced by

$$\frac{1 - x_r x_s^{-1}\exp(-\mathbf{t})}{1 - x_s x_r^{-1}} = -\exp(-\mathbf{t})\frac{x_r}{x_s}\frac{x_r - x_s\exp\mathbf{t}}{x_r - x_s}.$$

If we compare this with the Macdonald operator, we see the extra factor x_r/x_s. It can be regarded as the canonical bundle of $\mathrm{Gr}^{\varpi_n}_{\mathrm{GL}(a)}$, and absorbed into the symmetric function f for our purpose. However if we want to check the commutativity $[E_m, E_n] = 0$, it is true for the homology case, and need to put the extra factor for the K-theory.

By the way, we do not see *a priori* geometric reason why we have $[E_m, E_n] = 0$.

Theorem A.9. *Operators $E_n[f]$, $F_n[f]$ ($1\le n\le a$, f: a symmetric function in n variables) in* (A.7) *together with symmetric functions in w_s generate the quantized Coulomb branch $\mathcal{A}_\hbar$ over $\mathbb{C}[\hbar,\mathbf{t},z_1,\dots,z_l]$.*

This result identifies $\mathcal{A}_\hbar$ as a subalgebra in $\tilde{\mathcal{A}}_\hbar$, generated by explicit elements, as we have remarked after Proposition A.2. It is purely an algebraic problem to identify this subalgebra with the spherical part of cyclotomic rational Cherednik algebra. We will return back to this problem in [KN18, BEF16]

Let us use the vector notation $\vec{w}$ for $(w_1,\dots,w_a)$. Therefore symmetric functions f in the full w_r are denoted by $f(\vec{w})$. On the other hand, symmetric functions in less variables as still denoted like in (A.7).

Proof. At $\mathbf{t} = \hbar = z_k = 0$, $E_n[f]$, $F_n[f]$ are specialized to

$$\sum_{\substack{I\subset\{1,\dots,a\}\\ \#I=n}} f(w_I)\prod_{r\in I}\mathsf{u}_r, \qquad \sum_{\substack{I\subset\{1,\dots,a\}\\ \#I=n}} f(w_I)\prod_{r\in I} w_r^l\mathsf{u}_r^{-1}. \tag{A.10}$$

It is enough to show that these elements together with symmetric functions in $\vec{w}$ generate $\mathcal{A}$ at $\mathbf{t} = z_k = 0$ by graded Nakayama lemma.

If $a = 1$, i.e., $\dim V = 1$, we have $\mathcal{M}_C = \mathcal{S}_l = \{xy = w^l\} \subset \mathbb{A}^3$, where $w = w_1$, $x = \mathsf{u}_1$, $y = w_1^l\mathsf{u}_1^{-1}$ in the current notation. (See [Part II, Theorem 4.1].) The above elements with $f = 1$ are $x = \mathsf{u}_1$, $y = w_1^l\mathsf{u}_1^{-1}$ respectively. Therefore they together with w generate $\mathcal{A} = \mathbb{C}[\mathcal{M}_C]$.

Let us write $x_r = \mathsf{u}_r$, $y_r = w_r^l\mathsf{u}_r^{-1}$. Then we have a surjective homomorphism

$$\mathbb{C}[\mathcal{M}_C] = \operatorname{Sym}^a\mathcal{S}_l \twoheadleftarrow \mathbb{C}[\vec{x},\vec{w}]^{S_a}\otimes\mathbb{C}[\vec{y},\vec{w}]^{S_a},$$

where $\vec{x} = (x_1,\dots,x_a)$ and $\vec{y}$, $\vec{w}$ are similar. It is a classical result that the left and right elements in (A.10) and symmetric polynomials in $\vec{w}$ generate $\mathbb{C}[\vec{x},\vec{w}]^{S_a}$ and $\mathbb{C}[\vec{y},\vec{w}]^{S_a}$ respectively. (See [Wey97, §2.2].) □

A.4. Adjoint

We consider the case $\mathbf{N} = \mathfrak{g}$, the adjoint representation of a reductive group G. When $G = \mathrm{GL}(a)$, it corresponds to the case studied in the previous subsection with $W = 0$.

We add the flavor symmetry $\mathbb{C}^\times$, the dilatation on $\mathbf{N}$, and denote the corresponding equivariant variable by $\mathbf{t}$.

A.4.1. Minuscule coweights. The minuscule monopole operator in Proposition A.2 for the adjoint is given by

Proposition A.11.

$$\sum_{w\lambda\in\mathbb{W}\lambda} wf\times\prod_{\substack{\alpha^\vee\in\Delta^\vee\\ \langle\alpha^\vee,w\lambda\rangle=1}}\frac{-\alpha^\vee-\hbar/2+\mathbf{t}}{\alpha^\vee}\mathsf{u}_{w\lambda}.$$

This is a rational version of the Macdonald operator for a minuscule coweight for $f\equiv 1$. (See e.g., [Kir97].)

Proof. Let $\lambda' = w\lambda \in \mathbb{W}\lambda$. As above $(\iota_*)^{-1}$ is given by the equivariant Euler class $e(T_{\lambda'}\mathrm{Gr}_G^\lambda)$ of the tangent space at λ'. It is given by

$$e(T_{\lambda'}\mathrm{Gr}_G^\lambda) = \prod_{\substack{\alpha^\vee \in \Delta^\vee \\ \langle \alpha^\vee, \lambda' \rangle = 1}} \alpha^\vee.$$

In fact, $T_{\lambda'}\mathrm{Gr}_G^\lambda = \bigoplus_{\alpha^\vee\in\Delta^\vee} \bigoplus_{n=0}^{\max(0,\langle\alpha^\vee,\lambda'\rangle)-1} \mathfrak{g}_\alpha^\vee z^n$ as is mentioned in the proof of [Part II, Lemma 2.5]. Since λ is minuscule, $\langle \alpha^\vee, \lambda' \rangle = 0, \pm 1$. Therefore only roots with $\langle \alpha^\vee, \lambda' \rangle = 1$ contribute.

Next consider $\mathbf{z}^*$. It is the multiplication of the equivariant Euler class of $z^{\lambda'}\mathbf{N}_\mathcal{O}/z^{\lambda'}\mathbf{N}_\mathcal{O} \cap \mathbf{N}_\mathcal{O}$ by [Part II, §4(vi)] as before. We consider the decomposition $\mathbf{N} = \mathfrak{g} = \bigoplus_{\alpha^\vee\in\Delta} \mathfrak{g}_\alpha^\vee \oplus \mathfrak{t}$, and conclude that roots $\alpha^\vee$ with $\langle \alpha^\vee, \lambda' \rangle = -1$ contribute. It gives the numerator $-\alpha^\vee - \hbar/2 + \mathbf{t}$ in the formula. □

A.4.2. Quasi-minuscule coweights. We consider a generalization of Proposition A.2 to the case when λ is a *quasi-minuscule* coweight, i.e., $\lambda = \alpha_0$ where $\alpha_0^\vee$ is the highest root. Then $\langle \alpha^\vee, \lambda \rangle \le 2$ for any positive root $\alpha^\vee \in \Delta_+^\vee$, and the equality holds if and only if $\alpha^\vee = \alpha_0^\vee$. Therefore

$$\begin{aligned} &e(T_\lambda \mathrm{Gr}_G^\lambda) = (\alpha_0^\vee + \hbar) \prod_{\substack{\alpha^\vee \in \Delta^\vee \\ \langle \alpha^\vee, \lambda \rangle > 0}} \alpha^\vee, \\ &e(z^\lambda \mathbf{N}_\mathcal{O}/z^\lambda \mathbf{N}_\mathcal{O} \cap \mathbf{N}_\mathcal{O}) = \left(-\alpha_0^\vee - \frac{3\hbar}{2} + \mathsf{t}\right) \prod_{\substack{\alpha^\vee \in \Delta^\vee \\ \langle \alpha^\vee, \lambda \rangle > 0}} \left(-\alpha^\vee - \frac{\hbar}{2} + \mathsf{t}\right). \end{aligned} \tag{A.12}$$

In fact, Gr_G^λ is a line bundle L over G/P_λ. The factor $\alpha_0^\vee + \hbar$ corresponds to the tangent direction to the fiber. The space $z^\lambda \mathbf{N}_\mathcal{O}/z^\lambda \mathbf{N}_\mathcal{O} \cap \mathbf{N}_\mathcal{O}$ is the fiber of the quotient $\mathcal{T}/\mathcal{R}$ at $z^\lambda \in \mathrm{Gr}_G$. For $\mathbf{N} = \mathfrak{g}$, the quotient is the cotangent bundle of Gr_G^λ. Therefore the second formula in (A.12) is obtained from the first one by changing the sign, and then adding $-\hbar/2 + \mathsf{t}$ for each factor, which corresponds to the action on fibers.

The closure $\overline{\mathrm{Gr}}_G^\lambda = \mathrm{Gr}_G^\lambda \sqcup \mathrm{Gr}_G^0$ has a singularity at $1 = \mathrm{Gr}_G^0$ (isomorphic to the singularity of the closure of the minimal nilpotent orbit in $\mathfrak{g}$ at 0), but it has a resolution $\mathbb{P}(\mathcal{O} \oplus L)$ the projective bundle associated with L. (See [NP01, Lemma 7.3].) Also the vector bundle $\mathcal{T}/\mathcal{R}$ over Gr_G^λ extends to $\mathbb{P}(\mathcal{O} \oplus L)$ as it is the cotangent bundle. More precisely, let us denote by $p\colon \mathbb{P}(\mathcal{O} \oplus L) \to \overline{\mathrm{Gr}}_G^\lambda$ the above resolution. Then the vector subbundle $\mathcal{R}_\lambda \subset \mathcal{T}_\lambda$ over Gr_G^λ extends to a vector subbundle in $p^*\mathcal{T}$ over the whole of $\mathbb{P}(\mathcal{O} \oplus L)$, to be denoted $\widetilde{\mathcal{R}}_{\le\lambda}$, such that $p^*\mathcal{T}/\widetilde{\mathcal{R}}_{\le\lambda}$ is the cotangent bundle

$T^*\mathbb{P}(\mathcal{O}\oplus L)$. We have a proper projection $p\colon \widetilde{\mathcal{R}}_{\le\lambda}\to\mathcal{T}$ with the image lying in $\mathcal{R}_{\le\lambda}$. By base change we can compute $\mathbf{z}^*(\iota_*)^{-1}$ of a class $p_*(f[\widetilde{\mathcal{R}}_{\le\lambda}])$ over $\widetilde{\mathcal{R}}_{\le\lambda}$, where $f\in\mathbb{C}[\mathfrak{t}]^{\mathbb{W}_\lambda}$ viewed as a class in $H^*_{\mathrm{Stab}_G(\lambda)}(\mathrm{pt})\cong H^*_G(G/P_\lambda)$ pull-backed to $\widetilde{\mathcal{R}}_{\le\lambda}$.

The torus fixed points in $\widetilde{\mathcal{R}}_{\le\lambda}$ come in pairs, 0 and ∞ in $\mathbb{P}^1$ for each T-fixed point in G/P_λ, i.e., a point in the orbit $\mathbb{W}\lambda$. Let us denote them by $0_{\lambda'}$, $\infty_{\lambda'}$ for $\lambda'\in\mathbb{W}\lambda$. The points $0_{\lambda'}$ are in Gr^λ_G, hence the Euler classes are given by the formula (A.12), after applying w with $\lambda'=w\lambda$. At $\infty_{\lambda'}$, the Euler class of the tangent space $e(T_{\infty_{\lambda'}}\mathbb{P}(\mathcal{O}\oplus L))$ is almost the same as $e(T_{\lambda'}\mathrm{Gr}^\lambda_G)$, but the factor $\alpha_0+\hbar$ corresponding to the fiber of the projective bundle changes the sign. The second Euler class $e((\mathcal{T}/\mathcal{R})_{\infty_{\lambda'}})$ is obtained from $e(T_{\infty_{\lambda'}}\mathbb{P}(\mathcal{O}\oplus L))$ by the same process as before. We thus get

Theorem A.13. *Let $\lambda=\alpha_0$, the quasi-minuscule coroot. Then*

$$\begin{aligned}&\mathbf{z}^*(\iota_*)^{-1}p_*(f[\widetilde{\mathcal{R}}_{\le\lambda}])\\ &=\sum_{w\lambda\in\mathbb{W}\lambda} wf\times\Biggl(\frac{-w\alpha_0^\vee-3\hbar/2+\mathsf{t}}{w\alpha_0^\vee+\hbar}\prod_{\substack{\alpha^\vee\in\Delta\\ \langle\alpha^\vee,w\lambda\rangle>0}}\frac{-\alpha^\vee-\hbar/2+\mathsf{t}}{\alpha^\vee}\mathsf{u}_{w\lambda}\\ &\qquad\qquad+\frac{w\alpha_0^\vee+\hbar/2+\mathsf{t}}{-w\alpha_0^\vee-\hbar}\prod_{\substack{\alpha^\vee\in\Delta\\ \langle\alpha^\vee,w\lambda\rangle>0}}\frac{-\alpha^\vee-\hbar/2+\mathsf{t}}{\alpha^\vee}\Biggr).\end{aligned}$$

When $f=1$, this is a rational version of the Macdonald operator for a quasi-minuscule weight [Mac01] up to constant in $\mathbb{C}[\mathfrak{t}]^{\mathbb{W}_\lambda}$. The constant vanishes if we use the form in [vDE11].

Remark A.14. For general $\mathbf{N}$, we are not certain whether we have a resolution $\tilde{\mathcal{R}}_{\le\lambda}$ of $\mathcal{R}_{\le\lambda}$ for which we can calculate $\mathbf{z}^*(\iota_*)^{-1}$. Nevertheless it is clearly possible for $\mathbf{N}=0$: we have a resolution $\mathbb{P}(\mathcal{O}\oplus L)$ of $\overline{\mathrm{Gr}}^\lambda_G$. In this case, we get a formula as in Theorem A.13, where the numerator is replaced by 1. Its proof is contained in one in Theorem A.13.

A.4.3. Small fundamental. Recall that apart from type A, the quasi-minuscule coweight is fundamental. More generally, we consider a *small* fundamental coweight ω, i.e. $\langle\alpha^\vee,\omega\rangle\le 2$ for any $\alpha^\vee\in\Delta^\vee$ (see e.g. [vDE11]).[14]

[14] Some authors use another definition of small coweights: ω is small if in the corresponding irreducible representation V^ω of $G^\vee$ the zero weight has a nonzero multiplicity, but the weight 2α has zero multiplicity for any $\alpha\in\Delta$.

According to [vDE11, Table 1], any dominant coweight $\mu \leq \omega$ is also small fundamental, and all such coweights are totally ordered: $\omega^{(0)} < \omega^{(1)} < \cdots < \omega^{(n)} = \omega$, and $\omega^{(0)}$ is either minuscule or zero. Moreover, there is a chain of connected subdiagrams of the Dynkin diagram of G: $D^{(1)} \supset D^{(2)} \supset \cdots \supset D^{(n)}$ with the corresponding Levi subgroups $G \supset L^{(1)} \supset \cdots \supset L^{(n)} \supset T$ such that $\omega^{(i)} - \omega^{(i-1)}$ is the quasiminuscule coweight $\alpha_0^{(i)}$ of $L^{(i)}$. Note that $\alpha_0^{(n)}$ is a fundamental coweight of $L^{(n)}$, and we define $D^{(n+1)}$ as the complement in $D^{(n)}$ of the corresponding vertex; $L^{(n+1)} \subset L^{(n)}$ is the corresponding Levi subgroup. According to [MOV05, Lemma 3.1.1], there is a natural isomorphism of slices $\overline{\mathcal{W}}^{\omega^{(i)}}_{G,\omega^{(i-1)}} \simeq \overline{\mathcal{W}}^{\alpha_0^{(i)}}_{L^{(i)},0}$. Hence the above resolution $\widetilde{\mathrm{Gr}}^{\alpha_0}_G$ admits the following generalization: a resolution $\widetilde{\mathrm{Gr}}^{\omega}_G \to \overline{\mathrm{Gr}}^{\omega}_G$ constructed as an iterated blowup. We first take the blowup $\mathrm{Bl}^{(1)} := \mathrm{Bl}_{\overline{\mathrm{Gr}}^{\omega^{(0)}}_G} \overline{\mathrm{Gr}}^{\omega}_G$ at the closed $G_{\mathcal{O}}$-orbit $\overline{\mathrm{Gr}}^{\omega^{(0)}}_G$. The strict transform of $\overline{\mathrm{Gr}}^{\omega^{(1)}}_G \subset \overline{\mathrm{Gr}}^{\omega}_G$ is a resolution $\widetilde{\mathrm{Gr}}^{\omega^{(1)}}_G \subset \mathrm{Bl}^{(1)}$. The preimage of $\overline{\mathrm{Gr}}^{\omega^{(0)}}_G \subset \overline{\mathrm{Gr}}^{\omega}_G$ fibers over $\overline{\mathrm{Gr}}^{\omega^{(0)}}_G$ with fibers isomorphic to the partial flag variety $L^{(1)}/P^{(1)}_{\alpha_0^{(1)}}$ of the Levi group $L^{(1)}$. We define $\mathrm{Bl}^{(2)} := \mathrm{Bl}_{\widetilde{\mathrm{Gr}}^{\omega^{(1)}}_G} \mathrm{Bl}^{(1)}$. The strict transform of $\overline{\mathrm{Gr}}^{\omega^{(2)}}_G \subset \overline{\mathrm{Gr}}^{\omega}_G$ is a resolution $\widetilde{\mathrm{Gr}}^{\omega^{(2)}}_G \subset \mathrm{Bl}^{(2)}$. The preimage of $\widetilde{\mathrm{Gr}}^{\omega^{(1)}}_G \subset \mathrm{Bl}^{(1)}$ fibers over $\widetilde{\mathrm{Gr}}^{\omega^{(1)}}_G$ with fibers isomorphic to the partial flag variety $L^{(2)}/P^{(2)}_{\alpha_0^{(2)}}$. We continue like this till we arrive at $\widetilde{\mathrm{Gr}}^{\omega}_G := \mathrm{Bl}^{(n)} := \mathrm{Bl}_{\widetilde{\mathrm{Gr}}^{\omega^{(n-1)}}_G} \mathrm{Bl}^{(n-1)} \xrightarrow{p} \overline{\mathrm{Gr}}^{\omega}_G$. The preimage of $\widetilde{\mathrm{Gr}}^{\omega^{(n-1)}}_G \subset \mathrm{Bl}^{(n-1)}$ fibers over $\widetilde{\mathrm{Gr}}^{\omega^{(n-1)}}_G$ with fibers isomorphic to the partial flag variety $L^{(n)}/P^{(n)}_{\alpha_0^{(n)}}$. The vector subbundle $\mathcal{R}_\omega \subset \mathcal{T}_\omega$ over Gr^{ω}_G extends to a vector subbundle in $p^*\mathcal{T}$ over the whole of $\widetilde{\mathrm{Gr}}^{\omega}_G$, to be denoted $\widetilde{\mathcal{R}}_{\leq\omega}$, such that $p^*\mathcal{T}/\widetilde{\mathcal{R}}_{\leq\omega} = T^*\widetilde{\mathrm{Gr}}^{\omega}_G$. We have a proper projection $p\colon \widetilde{\mathcal{R}}_{\leq\omega} \to \mathcal{T}$ with the image lying in $\mathcal{R}_{\leq\omega}$.

Since $\overline{\mathrm{Gr}}^{\omega}_G = \bigsqcup_{0\leq i\leq n} \mathrm{Gr}^{\omega^{(i)}}_G$, the T-fixed points set $(\overline{\mathrm{Gr}}^{\omega}_G)^T$ decomposes into a disjoint union $\bigsqcup_{0\leq i\leq n} (\mathrm{Gr}^{\omega^{(i)}}_G)^T = \bigsqcup_{0\leq i\leq n} \mathbb{W}\omega^{(i)}$. From the above description of the resolutions we get

$$(\widetilde{\mathcal{R}}_{\leq\omega})^T = (\widetilde{\mathrm{Gr}}^{\omega}_G)^T = \bigsqcup_{0\leq i\leq n} \mathbb{W}\omega^{(i)} \times \mathbb{W}^{(i+1)}/\mathbb{W}^{(i+2)} \times \cdots \times \mathbb{W}^{(n-1)}/\mathbb{W}^{(n)} \times \mathbb{W}^{(n)}/\mathbb{W}^{(n+1)}.$$

We rewrite the RHS in the following form: $(\widetilde{\mathcal{R}}_{\leq\omega})^T = (\widetilde{\mathrm{Gr}}^{\omega}_G)^T = \bigsqcup_{0\leq i\leq n} \{(\nu \in \mathbb{W}\omega^{(i)},\ \eta^{(i+1)} \in w_i \mathbb{W}^{(i+1)}\omega^{(i+1)}, \ldots,\ \eta^{(n)} \in w_{n-1}\mathbb{W}^{(n)}\omega^{(n)})\}$ where w_i is an element of $\mathbb{W}$ such that $\nu = w_i\omega^{(i)}$, w_{i+1} is an element of $\mathbb{W}$ such that $\eta^{(i+1)} = w_{i+1}\omega^{(i+1)}$, and so on, and finally w_{n-1} is an element of $\mathbb{W}$ such that $\eta^{(n-1)} = w_{n-1}\omega^{(n-1)}$.

The Euler class of the tangent space $e(T_{(\nu,\eta^{(i+1)},\dots,\eta^{(n)})}\widetilde{\mathrm{Gr}}^{\omega}_G)$ equals

$$\prod_{\substack{\alpha^\vee\in\Delta^\vee\\ \langle\nu,\alpha^\vee\rangle=2}}(\alpha^\vee+\hbar)\cdot\prod_{\substack{\alpha^\vee\in\Delta^\vee\\ \langle\nu,\alpha^\vee\rangle>0}}\alpha^\vee\cdot\prod_{i+1\le j\le n}\left(\left(w_{j-1}(\alpha_0^{(j)})^\vee+\hbar\right)\prod_{\substack{\alpha^\vee\in w_{j-1}\Delta^\vee_{(j)}\\ \langle\eta^{(j)},\alpha^\vee\rangle>0}}\alpha^\vee\right).$$

Here the second product arises from the tangent bundle to the partial flag variety $(\mathrm{Gr}^{\omega^{(i)}}_G)^{\mathbb{C}^\times}$ (fixed points set of the loop rotations); the first product arises from the normal bundle $\mathcal{N}_{(\mathrm{Gr}^{\omega^{(i)}}_G)^{\mathbb{C}^\times}/\mathrm{Gr}^{\omega^{(i)}}_G}$; the last product arises from the tangent bundle to the fiber of blowup, and its prefactor arises from the normal bundle to the fiber of blowup. The Euler class of the cotangent bundle fiber at $(\nu,\eta^{(i+1)},\dots,\eta^{(n)})\in\widetilde{\mathrm{Gr}}^{\omega}_G$ is obtained from the above one by changing the sign of each factor and then adding $-\hbar+\mathsf{t}$ to each factor. The result is

$$\prod_{\substack{\alpha^\vee\in\Delta^\vee\\ \langle\nu,\alpha^\vee\rangle=2}}\left(\mathsf{t}-\alpha^\vee-\frac{3\hbar}{2}\right)\times\prod_{\substack{\alpha^\vee\in\Delta^\vee\\ \langle\nu,\alpha^\vee\rangle>0}}\left(\mathsf{t}-\alpha^\vee-\frac{\hbar}{2}\right)$$
$$\times\prod_{i+1\le j\le n}\left(\left(\mathsf{t}-w_{j-1}(\alpha_0^{(j)})^\vee-\frac{3\hbar}{2}\right)\prod_{\substack{\alpha^\vee\in w_{j-1}\Delta^\vee_{(j)}\\ \langle\eta^{(j)},\alpha^\vee\rangle>0}}\left(\mathsf{t}-\alpha^\vee-\frac{\hbar}{2}\right)\right).$$

We thus get (cf. [vDE11, Section 3])

Theorem A.15. *Let ω be a small fundamental coweight. Then*

$$\begin{aligned}&\mathbf{z}^*(\iota_*)^{-1}p_*[\widetilde{\mathcal{R}}_{\le\omega}]\\ &=\sum_{0\le i\le n}\sum_{(\nu,\eta^{(i+1)},\dots,\eta^{(n)})}\prod_{\substack{\alpha^\vee\in\Delta^\vee\\ \langle\nu,\alpha^\vee\rangle=2}}\frac{\mathsf{t}-\alpha^\vee-3\hbar/2}{\alpha^\vee+\hbar}\times\prod_{\substack{\alpha^\vee\in\Delta^\vee\\ \langle\nu,\alpha^\vee\rangle>0}}\frac{\mathsf{t}-\alpha^\vee-\hbar/2}{\alpha^\vee}\\ &\quad\times\prod_{i+1\le j\le n}\left(\frac{\mathsf{t}-w_{j-1}(\alpha_0^{(j)})^\vee-3\hbar/2}{w_{j-1}(\alpha_0^{(j)})^\vee+\hbar}\prod_{\substack{\alpha^\vee\in w_{j-1}\Delta^\vee_{(j)}\\ \langle\eta^{(j)},\alpha^\vee\rangle>0}}\frac{\mathsf{t}-\alpha^\vee-\hbar/2}{\alpha^\vee}\right)\mathsf{u}_\nu.\end{aligned}$$

Remark A.16. Note that the fundamental class $[\widetilde{\mathcal{R}}_{\le\omega}]$ does not have a coefficient $f\in\mathbb{C}[\mathfrak{t}]^{\mathbb{W}_\omega}$ as opposed to Theorem A.13, because there is no projection $[\widetilde{\mathcal{R}}_{\le\omega}]\to G/P_\omega$ for arbitrary small fundamental ω, so we do not have a way to produce natural homology classes on $\widetilde{\mathcal{R}}_{\leq\omega}$ except its fundamental class.

Question A.17. We know that $\mathbb{C}[\mathcal{M}_C] \cong \mathbb{C}[\mathfrak{t} \times T^\vee]^{\mathbb{W}}$ as a Poisson algebra. We do not know elements in Proposition A.11, Theorems A.13, A.15 with $\mathbb{C}[\mathfrak{t}]^{\mathbb{W}}$ generate $\mathbb{C}[\mathfrak{t} \times T^\vee]^{\mathbb{W}}$ as a Poisson algebra at $\hbar = \mathbf{t} = 0$, or they generate $\mathcal{A}_\hbar$ if we invert $\hbar$. Recall (see [Part II, §3(x)(b)]) that it is conjectured that $\mathcal{A}_\hbar$ is isomorphic to the spherical part of the graded Cherednik algebra. We do not know the corresponding statement for the spherical part either. These are true for type A, as we will show in a separate publication.

Appendix B. Shifted Yangians and quantization of generalized slices

In this section, we study quiver gauge theory coming from the Dynkin diagram of a simple algebraic group G. As usual we fix an orientation of the Dynkin diagram and we fix a dominant coweight λ and a coweight μ such that $\lambda - \mu = \sum a_i\alpha_i$ with $a_i \in \mathbf{N}$. We also fix a sequence of fundamental coweights $\underline{\lambda} = (\omega_{i_1}, \dots, \omega_{i_N})$ such that $\sum_{s=1}^N \omega_{i_s} = \lambda$. We will relate the quantized Coulomb branch to a generalization of the truncated shifted Yangians from [KWWY14].

B.1. Shifted Yangians

In this section, we will work with filtered algebras. We begin by recalling some basic facts about filtered algebras and the Rees construction.

Let A be a $\mathbb{C}$-algebra and let $F^\bullet A = \cdots \subseteq F^{-1}A \subseteq F^0A \subseteq F^1A \subseteq \cdots$ be a separated and exhaustive filtration, meaning that $\cap_k F^kA = 0$ and $\cup_k F^kA = A$. We assume that this filtration is compatible with the algebra structure in the sense that $F^kA \cdot F^lA \subset F^{k+l}A$ and $1 \in F^0A$.

In this case, we define the *Rees algebra* of A to be the graded $\mathbb{C}[\hbar]$–algebra $\mathrm{Rees}^F A := \oplus_k \hbar^k F^kA$, viewed as a subalgebra of $A[\hbar, \hbar^{-1}]$. We also define the associated graded of A to be the graded algebra $\mathrm{gr}^F A := \bigoplus F^kA/F^{k-1}A$. Note that we have a canonical isomorphism of graded algebras

$$\mathrm{Rees}^F A/\hbar\mathrm{Rees}^F A \cong \mathrm{gr}^F A.$$

We say that the filtered algebra A is *almost commutative* if $\mathrm{gr}^F A$ is commutative. In this case, for any $a \in F^kA$, $b \in F^lA$, we have $ab - ba \in$

$F^{k+l-1}A$. Thus in Rees^F, we can define a Poisson bracket by $\{a, b\} := \frac{1}{\hbar}(ab - ba)$.

Definition B.1. We define the "Cartan doubled Yangian" Y_∞ to be the $\mathbb{C}$-algebra with generators $E_i^{(q)}, F_i^{(q)}, H_i^{(p)}$ for $q > 0$ and $p \in \mathbb{Z}$ and $i \in Q_0$ and relations

$$\begin{aligned}
[H_i^{(p)}, H_j^{(p')}] &= 0,\\
[E_i^{(p)}, F_j^{(q)}] &= \delta_{ij} H_i^{(p+q-1)},\\
[H_i^{(p+1)}, E_j^{(q)}] - [H_i^{(p)}, E_j^{(q+1)}] &= \frac{\alpha_i \cdot \alpha_j}{2}(H_i^{(p)} E_j^{(q)} + E_j^{(q)} H_i^{(p)}),\\
[H_i^{(p+1)}, F_j^{(q)}] - [H_i^{(p)}, F_j^{(q+1)}] &= -\frac{\alpha_i \cdot \alpha_j}{2}(H_i^{(p)} F_j^{(q)} + F_j^{(q)} H_i^{(p)}),\\
[E_i^{(p+1)}, E_j^{(q)}] - [E_i^{(p)}, E_j^{(q+1)}] &= \frac{\alpha_i \cdot \alpha_j}{2}(E_i^{(p)} E_j^{(q)} + E_j^{(q)} E_i^{(p)}),\\
[F_i^{(p+1)}, F_j^{(q)}] - [F_i^{(p)}, F_j^{(q+1)}] &= -\frac{\alpha_i \cdot \alpha_j}{2}(F_i^{(p)} F_j^{(q)} + F_j^{(p)} F_i^{(q)}),\\
i \neq j, N = 1 - \alpha_i \cdot \alpha_j &\Rightarrow \mathrm{sym}[E_i^{(p_1)}, [E_i^{(p_2)}, \cdots [E_i^{(p_N)}, E_j^{(q)}] \cdots]] = 0,\\
i \neq j, N = 1 - \alpha_i \cdot \alpha_j &\Rightarrow \mathrm{sym}[F_i^{(p_1)}, [F_i^{(p_2)}, \cdots [F_i^{(p_N)}, F_j^{(q)}] \cdots]] = 0,
\end{aligned}$$

where sym denotes the symmetrization over the indices $p_1, \ldots, p_N$.

Definition B.2. The *shifted Yangian* Y_μ is the quotient of Y_∞ by the relations $H_i^{(p)} = 0$ for $p < -\langle \mu, \alpha_i^\vee \rangle$ and $H_i^{(-\langle \mu, \alpha_i^\vee \rangle)} = 1$.

Remark B.3. When $\mu = 0$, then it is easy to see that $Y = Y_0$ coincides with the Yangian, as defined in [KWWY14, Section 3.4]. On the other hand, suppose that μ is dominant. Then the map $Y_\mu \to Y$ defined by

$$H_i^{(s)} \mapsto H_i^{(s+\langle \mu, \alpha_i^\vee \rangle)}, \ E_i^{(s)} \mapsto E_i^{(s)}, \ F_i^{(s)} \mapsto F_i^{(s+\langle \mu, \alpha_i^\vee \rangle)}$$

gives an isomorphism between Y_μ and the subalgebra of Y which is also denoted Y_μ in [KWWY14].

To be a bit more precise, in [KWWY14], we worked with the corresponding graded $\mathbb{C}[\hbar]$-algebras. In fact, we made a mistake concerning these presentations of these algebras; [KWWY14, Theorem 3.5] is incorrect. We claimed to give a presentation of $(U_\hbar \mathfrak{g}[z])'$, using generators $E_\alpha^{(p)}, H_i^{(p)}, F_\alpha^{(p)}$, but we are definitely missing relations involving the $E_\alpha^{(p)}, F_\beta^{(q)}$ for α, β not simple roots. At this time, we do not know an explicit description of all the relations. In this paper, we will work with Rees algebras to avoid this problem.

Denote the generators of the shifted Yangian Y_μ by $E_i^{(r)}, H_i^{(r)}, F_i^{(r)}$, and form their respective generating series

$$E_i(z) = \sum_{r>0} E_i^{(r)} z^{-r}, \quad H_i(z) = z^{\langle \mu, \alpha_i^\vee \rangle} + \sum_{r > -\langle \mu, \alpha_i^\vee \rangle} H_i^{(r)} z^{-r},$$
$$F_i(z) = \sum_{r>0} F_i^{(r)} z^{-r}.$$

The relations for Y_μ can be written as identities of formal series. First, given a series $X(z) = \sum_{r\in\mathbb{Z}} X_i^{(r)} z^{-r}$, we write $\underline{X(z)} = \sum_{r>0} X_i^{(r)} z^{-r}$ for the principal part.

Then for all $i, j \in Q_0$ we have relations

$$\text{(B.4)} \qquad [H_i(z), H_j(y)] = 0,$$
$$\text{(B.5)} \qquad (z-y-a)H_i(z)E_j(y) = (z-y+a)E_j(y)H_i(z) - 2aE_j(z-a)H_i(z),$$
$$\text{(B.6)} \qquad (z-y-a)E_i(z)E_j(y) = (z-y+a)E_j(y)E_i(z) + [E_i^{(1)}, E_j(y)] - [E_i(z), E_j^{(1)}],$$
$$\text{(B.7)} \qquad (z-y+a)H_i(z)F_j(y) = (z-y-a)F_j(y)H_i(z) + 2aF_j(z+a)H_i(z),$$
$$\text{(B.8)} \qquad (z-y+a)F_i(z)F_j(y) = (z-y-a)F_j(y)F_i(z) + [F_i^{(1)}, F_j(y)] - [F_i(z), F_j^{(1)}],$$
$$\text{(B.9)} \qquad (z-y)[E_i(z), F_j(y)] = \delta_{i,j}\left(\underline{H_i(y)} - \underline{H_i(z)}\right),$$

where we denote $a = \frac{1}{2}\alpha_i \cdot \alpha_j$. We also have the Serre relations. First when $a_{ij} = 0$. we have

$$\text{(B.10)} \qquad [E_i(z), E_j(y)] = 0$$
$$\text{(B.11)} \qquad [F_i(z), F_j(y)] = 0$$

and for $a_{ij} = -1$ we have

$$\text{(B.12)} \qquad [E_i(z_1), [E_i(z_2), E_j(y)]] + [E_i(z_2), [E_i(z_1), E_j(y)]] = 0,$$
$$\text{(B.13)} \qquad [F_i(z_1), [F_i(z_2), F_j(y)]] + [F_i(z_2), [F_i(z_1), F_j(y)]] = 0.$$

Let μ_1, μ_2 be two coweights such that $\mu_1 + \mu_2 = \mu$. In [FKP+18], we defined filtrations $F_{\mu_1,\mu_2} Y_\mu$ of Y_μ. In this filtration, the degrees of the generators are given

$$\deg E_i^{(r)} = \langle \mu_1, \alpha_i^\vee \rangle + r,\ \deg F_i^{(r)} = \langle \mu_2, \alpha_i^\vee \rangle + r,\ \deg H_i^{(r)} = \langle \mu, \alpha_i^\vee \rangle + r$$

However, we note that these degrees do not determine the filtration because we also specify the degrees of certain PBW variables, see [FKP+18, section 5.4] for more details.

In [FKP+18], we proved that Y_μ is almost commutative with this filtration. We also proved that for any pair μ_1, μ_2 as above, the Rees algebras $\mathrm{Rees}^{F_{\mu_1,\mu_2}} Y_\mu$ are canonically isomorphic (as $\mathbb{C}[\hbar]$-algebras).

For the purposes of this paper, we will choose μ_1, μ_2 as follows

$$\langle \mu_1, \alpha_i^\vee \rangle = \langle \lambda, \alpha_i^\vee \rangle - a_i + \sum_{h:\mathrm{i}(h)=i} a_{\mathrm{o}(h)},\ \langle \mu_2, \alpha_i^\vee \rangle = -a_i + \sum_{h:\mathrm{o}(h)=i} a_{\mathrm{i}(h)}$$

where the sums are taken over all arrows h to i or from i respectively. We write $\mathbf{Y}_\mu := \mathrm{Rees}^{F_{\mu_1,\mu_2}} Y_\mu$ for this Rees algebra (with the induced grading).

B.2. A representation using difference operators

We will work with the larger algebra $Y_\mu[z_1, \ldots, z_N] = Y_\mu \otimes \mathbb{C}[z_1, \ldots, z_N]$. We extend the filtration F_{μ_1,μ_2} to $Y_\mu[z_1, \ldots, z_N]$ by placing all generators in degree 1.

Denote

$$Z_i(z) = \prod_{k:i_k=i} (z - z_k - \tfrac{1}{2}),$$

and define new "Cartan" elements $A_i^{(p)}$ for $p > 0$ by

$$H_i(z) = Z_i(z) \frac{\prod_{\substack{h \in Q_1 \sqcup \overline{Q_1} \\ \mathrm{o}(h)=i}} (z - \frac{1}{2})^{a_{\mathrm{i}(h)}}}{z^{a_i}(z-1)^{a_i}} \frac{\prod_{\substack{h \in Q_1 \sqcup \overline{Q_1} \\ \mathrm{o}(h)=i}} A_{\mathrm{i}(h)}(z - \frac{1}{2})}{A_i(z) A_i(z-1)}. \tag{B.14}$$

Consider also the $\mathbb{C}$-algebra $\tilde{\mathcal{A}} \overset{\text{def.}}{=} \mathbb{C}[z_1, \ldots, z_N]\langle w_{i,r}, \mathsf{u}_{i,r}^{\pm 1}, (w_{i,r} - w_{i,s} + m)^{-1} (r \neq s,\ m \in \mathbb{Z})\rangle$, with relations $[\mathsf{u}_{i,r}^{\pm}, w_{j,s}] = \pm \delta_{i,j} \delta_{r,s} \mathsf{u}_{i,r}^{\pm}$. Denote

$$W_i(z) = \prod_{r=1}^{a_i} (z - w_{i,r}) \quad \text{and} \quad W_{i,r}(z) = \prod_{\substack{s=1 \\ s \neq r}}^{a_i} (z - w_{i,s}).$$

We define a filtration on $\tilde{\mathcal{A}}$ by setting the degree of each $w_{i,r}$ to be 1 and the degree of $\mathsf{u}_{i,r}^{\pm}$ to be 0. The filtration degree of each $(w_{i,r} - w_{i,s} + m)^{-1}$ is also set to be -1.

Note that $\tilde{\mathcal{A}}$ is almost commutative and we have $\mathrm{Rees}\tilde{\mathcal{A}} = \tilde{\mathcal{A}}_\hbar$, the algebra defined in A.1.

The following result generalizes [KWWY14, Theorem 4.5] which was a generalization of a construction of Gerasimov-Kharchev-Lebedev-Oblezin [GKLO05].

Theorem B.15. *There is a homomorphism of filtered $\mathbb{C}$-algebras*

$$\Phi_\mu^\lambda \colon Y_\mu[z_1, \ldots, z_N] \longrightarrow \tilde{\mathcal{A}},$$

defined by

$$\begin{aligned}
A_i(z) &\mapsto z^{-a_i} W_i(z),\\
E_i(z) &\mapsto -\sum_{r=1}^{a_i} \frac{Z_i(w_{i,r}) \prod_{h \in Q_1 : \mathrm{i}(h)=i} W_{\mathrm{o}(h)}(w_{i,r} - \frac{1}{2})}{(z - w_{i,r}) W_{i,r}(w_{i,r})} \mathsf{u}_{i,r}^{-1},\\
F_i(z) &\mapsto \sum_{r=1}^{a_i} \frac{\prod_{h \in Q_1 : \mathrm{o}(h)=i} W_{\mathrm{i}(h)}(w_{i,r} + \frac{1}{2})}{(z - w_{i,r} - 1) W_{i,r}(w_{i,r})} \mathsf{u}_{i,r}.
\end{aligned}$$

Proof. The argument is basically the same as in [KWWY14, Theorem 4.5]. The proof in [KWWY14] should be considered incomplete, since we didn't have a complete presentation.

We verify the relations (B.4)–(B.13) which involve $E_i(z)$, those involving $F_i(z)$ being similar (they can also be deduced from the $E_i(z)$ cases by using certain involutions of Y_μ and $\tilde{\mathcal{A}}$).

Note that by (B.14), under Φ_μ^λ we have

$$H_i(z) \mapsto \frac{Z_i(z) \prod_{\substack{h \in Q_1 \sqcup \overline{Q_1} \\ \mathrm{o}(h)=i}} W_{\mathrm{i}(h)}(z - \frac{1}{2})}{W_i(z) W_i(z-1)},$$

and these images clearly satisfy equation (B.4).

B.3. Relation (B.5) between $H_i(z)$ and $E_j(y)$

B.3.1. The case $a_{ij} = 0$. Equation (B.5) simply says that $H_i(z)$ and $E_j(y)$ commute. It is clear that this holds true for their images under Φ_μ^λ.

B.3.2. The case $a_{ij} = -1$. Here, equation (B.5) reads

$$(z - y + \tfrac{1}{2})H_i(z)E_j(y) = (z - y - \tfrac{1}{2})E_j(y)H_i(z) + E_j(z + \tfrac{1}{2})H_i(z).$$

This is an $\mathfrak{sl}_3$ relation, and we can assume that $Q_0 = \{i, j\}$. We may also assume that Q_1 consists of a single arrow $j \to i$.

Then, the image of the left-hand side under Φ^λ_μ is

$$-(z - y + \tfrac{1}{2}) \cdot \frac{Z_i(z)W_j(z - \frac{1}{2})}{W_i(z)W_i(z-1)} \cdot \sum_{r=1}^{a_j} \frac{Z_j(w_{j,r})}{(y - w_{j,r})W_{j,r}(w_{j,r})} \mathsf{u}_{j,r}^{-1}$$

$$= -\sum_{r=1}^{a_j} \frac{(z - y + \frac{1}{2})(z - w_{j,r} - \frac{1}{2})}{y - w_{j,r}} \frac{Z_i(z)W_{j,r}(z - \frac{1}{2})}{W_i(z)W_i(z-1)} \frac{Z_j(w_{j,r})}{W_{j,r}(w_{j,r})} \mathsf{u}_{j,r}^{-1}.$$

On the other hand, the image of the right-hand side under Φ^λ_μ is

$$-\sum_{r=1}^{a_j} \left(\frac{z - y - \frac{1}{2}}{y - w_{j,r}} + \frac{1}{z - w_{j,r} + \frac{1}{2}} \right) \frac{Z_j(w_{j,r})}{W_{j,r}(w_{j,r})} \mathsf{u}_{j,r}^{-1} \cdot \frac{Z_i(z)W_j(z - \frac{1}{2})}{W_i(z)W_i(z-1)}.$$

Commuting the $\Phi^\lambda_\mu(H_i(z))$ factor to the left, this is equal to

$$\begin{aligned} -\sum_{r=1}^{a_j} &\left(\frac{z - y - \frac{1}{2}}{y - w_{j,r}} + \frac{1}{z - w_{j,r} + \frac{1}{2}} \right) \\ &\times (z - w_{j,r} + \tfrac{1}{2}) \frac{Z_i(z)W_{j,r}(z - \frac{1}{2})}{W_i(z)W_i(z-1)} \frac{Z_j(w_{j,r})}{W_{j,r}(w_{j,r})} \mathsf{u}_{j,r}^{-1}. \end{aligned}$$

So, the relation (B.5) follows in this case from an equality of rational functions:

$$\frac{(z - y + \frac{1}{2})(z - w_{j,r} - \frac{1}{2})}{y - w_{j,r}} = \left(\frac{z - y - \frac{1}{2}}{y - w_{j,r}} + \frac{1}{z - w_{j,r} + \frac{1}{2}} \right) (z - w_{j,r} + \tfrac{1}{2}).$$

B.3.3. The case $i = j$. Here, equation (B.5) says that

$$(z - y - 1)H_i(z)E_i(y) = (z - y + 1)E_i(y)H_i(z) - 2E_i(z - 1)H_i(z).$$

In this case we may assume that $\mathfrak{g} = \mathfrak{sl}_2$, and so we will temporarily drop the index i from our notation.

The image of the left-hand side under Φ_μ^λ is then

$$
\begin{aligned}
&-(z-y-1)\cdot\frac{Z(z)}{W(z)W(z-1)}\cdot\sum_{r=1}^{a}\frac{Z(w_r)}{(y-w_r)W_r(w_r)}\mathsf{u}_r^{-1}\\
&=-\sum_{r=1}^{a}\frac{(z-y-1)}{(y-w_r)(z-w_r)(z-w_r-1)}\frac{Z(z)}{W_r(z)W_r(z-1)}\frac{Z(w_r)}{W_r(w_r)}\mathsf{u}_r^{-1},
\end{aligned}
$$

while the image of the right-hand side under Φ_μ^λ is

$$
\begin{aligned}
&-\sum_{r=1}^{a}\left(\frac{z-y+1}{y-w_r}+\frac{-2}{z-w-1}\right)\frac{Z(w_r)}{W_r(w_r)}\mathsf{u}_r^{-1}\cdot\frac{Z(z)}{W(z)W(z-1)}\\
&=-\sum_{r=1}^{a}\left(\frac{z-y+1}{y-w_r}+\frac{-2}{z-w-1}\right)\\
&\quad\times\frac{1}{(z-w_r+1)(z-w_r)}\frac{Z(z)}{W_r(z)W_r(z-1)}\frac{Z(w_r)}{W_r(w_r)}\mathsf{u}_r^{-1}.
\end{aligned}
$$

So, the relation follows from the equality

$$
\begin{aligned}
&\frac{z-y-1}{(y-w_r)(z-w_r)(z-w_r-1)}\\
&=\left(\frac{z-y+1}{y-w_r}+\frac{-2}{z-w-1}\right)\frac{1}{(z-w_r+1)(z-w_r)}.
\end{aligned}
$$

B.4. Relation (B.6) between $E_i(z)$ and $E_j(y)$

We will verify that for all i, j we have

$$
\begin{aligned}
&(z-y-a)E_i(z)E_j(y)+E_i(z)E_j^{(1)}-E_i^{(1)}E_j(y)\\
&=(z-y+a)E_j(y)E_i(z)+E_j^{(1)}E_i(z)-E_j(y)E_i^{(1)},
\end{aligned}
\tag{B.16}
$$

where $a=\frac{1}{2}a_{ij}$.

B.4.1. The case $a_{ij}=0$. We need to check that

$$[\Phi_\mu^\lambda(E_i(z)),\Phi_\mu^\lambda(E_j(y))]=0,$$

which is clear.

B.4.2. The case $a_{ij} = -1$. In this case we can assume that $Q_0 = \{i, j\}$, and by the symmetry of (B.16) we may also assume that Q_1 consists of a single arrow $i \to j$. After collecting terms, the image of the left-hand side of (B.16) is

$$\begin{aligned}
&\sum_{r=1}^{a_i} \sum_{s=1}^{a_j} \left(\frac{z - y + \frac{1}{2}}{(z - w_{i,r})(y - w_{j,s})} + \frac{1}{z - w_{i,r}} - \frac{1}{y - w_{j,s}} \right) \\
&\qquad \times \frac{Z_i(w_{i,r})}{W_{i,r}(w_{i,r})} \mathsf{u}_{i,r}^{-1} \frac{Z_j(w_{j,s}) W_i(w_{j,s} - \frac{1}{2})}{W_{j,s}(w_{j,s})} \mathsf{u}_{j,s}^{-1} \\
&= \sum_{r,s} \frac{(w_{i,r} - w_{j,s} + \frac{1}{2})(w_{j,s} - w_{i,r} + \frac{1}{2})}{(z - w_{i,r})(y - w_{j,s})} \\
&\qquad \times \frac{Z_i(w_{i,r}) Z_j(w_{j,s}) W_{i,r}(w_{j,s} - \frac{1}{2})}{W_{i,r}(w_{i,r}) W_{j,s}(w_{j,s})} \mathsf{u}_{i,r}^{-1} \mathsf{u}_{j,s}^{-1}.
\end{aligned}$$

The image of the right-hand side of (B.16) reduces to the same expression, so the relation holds.

B.4.3. The case $i = j$. Here we may assume that $\mathfrak{g} = \mathfrak{sl}_2$, and we will drop the index i from our notation for this calculation. In this case, the left-hand side of (B.16) is

$$\begin{aligned}
&\sum_{r=1}^{a} \sum_{s=1}^{a} \Bigg((z - y - 1) \frac{Z(w_r)}{(z - w_r) W_r(w_r)} \mathsf{u}_r^{-1} \frac{Z(w_s)}{(y - w_s) W_s(w_s)} \mathsf{u}_s^{-1} \\
&+ \frac{Z(w_r)}{(z - w_r) W_r(w_r)} \mathsf{u}_r^{-1} \frac{Z(w_s)}{W_s(w_s)} \mathsf{u}_s^{-1} - \frac{Z(w_r)}{W_r(w_r)} \mathsf{u}_r^{-1} \frac{Z(w_s)}{(y - w_s) W_s(w_s)} \mathsf{u}_s^{-1} \Bigg).
\end{aligned}$$

Collecting terms, we express this as two sums:

$$\begin{aligned}
&\sum_{r} \left(\frac{z - y - 1}{(z - w_r)(y - w_r + 1)} + \frac{1}{z - w_r} - \frac{1}{y - w_r + 1} \right) \frac{Z(w_r) Z(w_r - 1)}{W_r(w_r) W_r(w_r - 1)} \mathsf{u}_r^{-2} \\
&+ \sum_{r \neq s} \left(\frac{z - y - 1}{(z - w_r)(y - w_s)} + \frac{1}{z - w_r} - \frac{1}{y - w_s} \right) \\
&\qquad \times \frac{Z(w_r) Z(w_s)}{W_r(w_r)(w_s - w_r + 1) W_{rs}(w_s)} \mathsf{u}_r^{-1} \mathsf{u}_s^{-1}.
\end{aligned}$$

The term in brackets in the first sum is zero, while the second sum is

$$-\sum_{r \neq s} \frac{1}{(z - w_r)(y - w_s)} \frac{Z(w_r) Z(w_s)}{W_r(w_r) W_{rs}(w_s)} \mathsf{u}_r^{-1} \mathsf{u}_s^{-1},$$

where $W_{rs}(z) = \sum_{t\neq r,s}(z - w_t)$. We get the same expression for the right-hand side of (B.16), so this relation holds.

B.5. Relation (B.9) between $E_i(z)$ and $F_j(y)$

B.5.1. The case $i \neq j$. Here, we must check that

$$[\Phi^\lambda_\mu(E_i(z)), \Phi^\lambda_\mu(F_j(y))] = 0.$$

We may assume that $Q_0 = \{i, j\}$. The only interesting case is when $a_{ij} = -1$ and $j \to i$. Then $[\Phi^\lambda_\mu(E_i(z)), \Phi^\lambda_\mu(F_j(y))]$ is equal to

$$\begin{aligned}
&-\sum_{r=1}^{a_i}\sum_{s=1}^{a_j}\left[\frac{Z_i(w_{i,r})W_j(w_{i,r}-\frac{1}{2})}{(z-w_{i,r})W_{i,r}(w_{i,r})}\mathsf{u}_{i,r}^{-1}, \frac{W_i(w_{j,s}+\frac{1}{2})}{(y-w_{j,s}-1)W_{j,s}(w_{j,s})}\mathsf{u}_{j,s}\right]\\
&= -\sum_{r=1}^{a_i}\sum_{s=1}^{a_j}\frac{Z_i(w_{i,r})}{(z-w_{i,r})W_{i,r}(w_{i,r})}\frac{1}{(y-w_{j,s}-1)W_{j,s}(w_{j,s})}\\
&\quad\times\left[W_j(w_{i,r}-\tfrac{1}{2})\mathsf{u}_{i,r}^{-1}, W_i(w_{j,s}+\tfrac{1}{2})\mathsf{u}_{j,s}\right].
\end{aligned}$$

This is indeed zero, as the commutator in each summand is zero.

B.5.2. The case $i = j$. The proof in this case is almost identical to that of [KWWY14, Theorem 4.5]. Recall that $(z-y)[\Phi^\lambda_\mu(E_i(z)), \Phi^\lambda_\mu(F_i(y))]$ is equal to

$$\begin{aligned}
(z-y)&\left[-\sum_{r=1}^{a_i}\frac{Z_i(w_{i,r})\prod_{\substack{h\in Q_1\\ \mathrm{i}(h)=i}}W_{\mathrm{o}(h)}(w_{i,r}-\frac{1}{2})}{(z-w_{i,r})W_{i,r}(w_{i,r})}\mathsf{u}_{i,r}^{-1},\right.\\
&\left.\sum_{s=1}^{a_i}\frac{\prod_{\substack{h\in Q_1\\ \mathrm{o}(h)=i}}Z_{\mathrm{i}(h)}(w_{i,s}+\frac{1}{2})}{(y-w_{i,s}-1)W_{i,s}(w_{i,s})}\mathsf{u}_{i,s}\right].
\end{aligned}$$

All terms where $r \neq s$ vanish, and what remains can be rewritten as

$$\sum_{r=1}^{a_i}\big((L_{i,r}(y) - R_{i,r}(y)) - (L_{i,r}(z) - R_{i,r}(z))\big),$$

where

$$L_{i,r}(y) = \frac{Z_i(w_{i,r}+1)\prod_{\substack{h\in Q_1\sqcup\overline{Q_1}\\ \mathrm{o}(h)=i}} W_{\mathrm{i}(h)}(w_{i,r}+\frac{1}{2})}{(y-w_{i,r}-1)W_{i,r}(w_{i,r}+1)W_{i,r}(w_{i,r})},$$
$$R_{i,r}(y) = \frac{Z_i(w_{i,r})\prod_{\substack{h\in Q_1\sqcup\overline{Q_1}\\ \mathrm{o}(h)=i}} W_{\mathrm{i}(h)}(w_{i,r}-\frac{1}{2})}{(y-w_{i,r})W_{i,r}(w_{i,r})W_{i,r}(w_{i,r}-1)}.$$

Therefore, it remains to verify that

$$\sum_{r=1}^{a_i}(L_{i,r}(y)-R_{i,r}(y)) = H_{i,+}(y).$$

As in [KWWY14], this is done by comparing coefficients at all y^{-k} for $k>0$ between the left-hand side and $H_i(y)$, using partial fractions to compute the case of $H_i(y)$.

B.6. The Serre relations

When $a_{ij}=0$, the relation is immediate, so we concentrate on the $a_{ij}=-1$ case and in particular, the version with Es, see (B.12) above. The proof of this relation is sketched out in [GKLO05]. Following their notation, let us denote

$$\chi_{i,r} = -\frac{Z_i(w_{i,r})\prod_{h\in Q_1:\mathrm{i}(h)=i} W_{\mathrm{o}(h)}(w_{i,r}-\frac{1}{2})}{W_{i,r}(w_{i,r})}\mathsf{u}_{i,r}^{-1},$$

so that $\Phi_\mu^\lambda(E_i(y)) = \sum_{r=1}^{a_i}\frac{1}{y-w_{i,r}}\chi_{i,r}$.

These elements satisfy the relations $[\chi_{i,r}, w_{i,s}] = -\delta_{r,s}\chi_{i,r}$ and

$$(w_{i,r}-w_{i,s}-1)\chi_{i,r}\chi_{i,s} = (w_{i,r}-w_{i,s}+1)\chi_{i,s}\chi_{i,r}, \qquad \text{for } r\neq s,$$
$$(w_{i,r}-w_{j,t}+\tfrac{1}{2})\chi_{i,r}\chi_{j,t} = (w_{i,r}-w_{j,t}-\tfrac{1}{2})\chi_{j,t}\chi_{i,r}.$$

Using the above relations, we find that

$$[\Phi_\mu^\lambda(E_i(y_1)),[\Phi_\mu^\lambda(E_i(y_2)),\Phi_\mu^\lambda(E_j(z))]]$$
$$= \left[\sum_{r=1}^{a_i}\frac{1}{y_1-w_{i,r}}\chi_{i,r}, \sum_{s=1}^{a_i}\sum_{t=1}^{a_j}\frac{1}{(y_2-w_{i,s})(z-w_{j,t})}\frac{-1}{w_{i,s}-w_{j,t}-\frac{1}{2}}\chi_{i,s}\chi_{j,t}\right]$$

$$
\begin{aligned}
&= \sum_r \sum_t \left(\frac{1}{(y_1 - w_{i,r})(y_2 - w_{i,r}+1)} - \frac{1}{(y_1 - w_{i,r}+1)(y_2 - w_{i,r})} \right) \\
&\qquad \times \frac{1}{z - w_{j,t}} \frac{-1}{w_{i,r} - w_{j,t} - \frac{3}{2}} \chi_{i,r}^2 \chi_{j,t} \\
&\quad + \sum_{r \neq s} \sum_t \frac{1}{(y_1 - w_{i,r})(y_2 - w_{i,s})(z - w_{j,t})} \frac{-1}{(w_{i,s} - w_{j,t} - \frac{1}{2})} \\
&\qquad \times \frac{w_{i,r} + w_{i,s} - 2w_{j,t}}{(w_{i,r} - w_{j,t} - \frac{1}{2})(w_{i,r} - w_{i,s} + 1)} \chi_{i,r}\chi_{i,s}\chi_{j,t}.
\end{aligned}
$$

The first sum is clearly skew-symmetric in y_1, y_2. The second sum is as well, which one can see by applying the above relation between $\chi_{i,r}$ and $\chi_{i,s}$. This proves the Serre relation along with the theorem.

B.7. The filtration

We are left to verify the claim that the filtrations match. To do this, it suffices to check that each PBW variable $E_\beta^{(p)}, H_i^{(q)}, F_\beta^{(p)}$ (see [FKP+18, Remark 3.4] for their definition) is sent to the correct filtered degree. When β is a simple root, this is immediate.

Now suppose that β is not a simple root. Then $E_\beta^{(p)}$ is defined by commutators. Since $\tilde{\mathcal{A}}$ is almost commutative, this immediately implies that $E_\beta^{(p)}$ is mapped into the correct filtered piece. □

Applying the Rees functor to Theorem B.15, we deduce the following result.

Corollary B.17. *There exists a unique graded* $\mathbb{C}[\hbar, z_1, \dots, z_N]$*-algebra homomorphism* $\mathbf{Y}_\mu[z_1, \dots, z_N] \to \tilde{\mathcal{A}}_\hbar$*, such that*

$$
\begin{aligned}
A_i(z) &\mapsto z^{-a_i} W_i(z), \\
E_i(z) &\mapsto -\sum_{r=1}^{a_i} \frac{Z_i(w_{i,r}) \prod_{h \in Q_1 : \mathrm{i}(h) = i} W_{\mathrm{o}(h)}(w_{i,r} - \frac{1}{2}\hbar)}{(z - w_{i,r}) W_{i,r}(w_{i,r})} \mathsf{u}_{i,r}^{-1}, \\
F_i(z) &\mapsto \sum_{r=1}^{a_i} \frac{\prod_{h \in Q_1 : \mathrm{o}(h) = i} W_{\mathrm{i}(h)}(w_{i,r} + \frac{1}{2}\hbar)}{(z - w_{i,r} - \hbar) W_{i,r}(w_{i,r})} \mathsf{u}_{i,r}.
\end{aligned}
$$

In the above corollary, we are using a slight abuse of notation. For a generator x (such as $E_i^{(p)}$ or $w_{i,r}$) of the algebra Y_μ or $\tilde{\mathcal{A}}$ which lives in filtered degree k (but not in filtered degree $k-1$) we write x for the element $\hbar^k x \in \mathrm{Rees} Y_\mu$ or $\mathrm{Rees}\tilde{\mathcal{A}}$.

B.8. Relation to quantization of Coulomb branch

Recall the setup of A.2: we have $\mathcal{A}_\hbar = H_*^{(\mathrm{GL}(V)\times T(W))_{\mathcal{O}}\rtimes\mathbb{C}^\times}(\mathcal{R}_{\mathrm{GL}(V),\mathbf{N}}) \hookrightarrow \tilde{\mathcal{A}}_\hbar$, the quantized Coulomb branch associated to the pair $(\mathrm{GL}(V),\mathbf{N})$ with flavor symmetry. This inclusion takes the homological grading on $\mathcal{A}_\hbar$ (not the Δ-grading) to the above grading on $\tilde{\mathcal{A}}_\hbar$.

Theorem B.18. *There exists a unique graded $\mathbb{C}[\hbar, z_1, \ldots, z_N]$-algebra homomorphism*

$$\overline{\Phi}_\mu^\lambda \colon \mathbf{Y}_\mu[z_1, \ldots, z_N] \to \mathcal{A}_\hbar,$$

such that

$$\begin{aligned} A_i^{(p)} &\mapsto (-1)^p e_p(\{w_{i,r}\}), \\ F_i^{(p)} &\mapsto (-1)^{\sum_{\mathrm{o}(h)=i} a_{\mathrm{i}(h)}}(c_1(\mathcal{Q}_i)+\hbar)^{p-1} \cap [\mathcal{R}_{\varpi_{i,1}}], \\ E_i^{(p)} &\mapsto (-1)^{a_i}(c_1(\mathcal{S}_i)+\hbar)^{p-1} \cap [\mathcal{R}_{\varpi_{i,1}^*}]. \end{aligned}$$

Remark B.19. This homomorphism is analogous to (and was inspired by) the action of the Yangian of $\mathfrak{gl}_n$ on the cohomology of Laumon spaces, constructed by Feigin-Finkelberg-Negut-Rybnikov [FFNR11].

Remark B.20. In the above Theorem, we use the (μ_1, μ_2)-grading on $\mathbf{Y}_\mu[z_1, \ldots, z_N]$ (where μ_1, μ_2 are defined above) and the homological grading on $\mathcal{A}_\hbar$. On the other hand, if we want to use the Δ-grading on $\mathcal{A}_\hbar$ (as defined in [Part II, Remark 2.8(2)]), then we should use the $(\mu/2, \mu/2)$-grading on $\mathbf{Y}_\mu[z_1, \ldots, z_N]$.

Recall that the $(\mu/2, \mu/2)$-grading is defined so that PBW variables $E_\beta^{(p)}$, $F_\beta^{(p)}$, $H_i^{(q)}$ have degree

$$\deg E_\beta^{(p)} = \frac{1}{2}\langle \mu, \beta\rangle + p, \quad \deg F_\beta^{(p)} = \frac{1}{2}\langle \mu, \beta\rangle + p, \quad \deg H_i^{(q)} = \langle \mu, \alpha_i\rangle + q,$$

where β is a positive root. See [FKP+18, section 5.4]. Therefore $\mathbf{Y}_\mu[z_1, \ldots, z_N]$ is $\mathbb{Z}_{\geq 0}$-graded and the degree 0 part consists only of scalars (with respect to the $(\mu/2, \mu/2)$-grading) if and only if $\langle \mu, \beta\rangle \geq -1$ for any positive root β. Note that $\mathcal{A}$ is called *good or ugly* when its Δ-grading satisfies the same property. See [Nak16, Remark 4.2]. One of the authors show that $\langle \mu, \beta\rangle \geq -1$

if $\mathcal{A}$ is good or ugly. See [Nak15, Proof of Prop. 5.9]. The converse is also true if $\overline{\Phi}_\mu^\lambda$ is surjective.

Proof. We have the graded $\mathbb{C}[\hbar, z_1, \ldots, z_N]$-algebra homomorphisms Φ_μ^λ : $\mathbf{Y}_\mu[z_1, \ldots, z_N] \to \mathcal{A}_\hbar$ and $\mathbf{z}^*(\iota_*)^{-1} : \mathcal{A}_\hbar \to \tilde{\mathcal{A}}_\hbar$, the second of which is injective. So we just need to verify that the image Φ_μ^λ is contained in the image of $\mathbf{z}^*(\iota_*)^{-1}$.

It follows immediately from equations (A.3) and (A.5) that

$$\mathbf{z}^*(\iota_*)^{-1}(\overline{\Phi}_\mu^\lambda(X_i^{(s)})) = \Phi_\mu^\lambda(X_i^{(s)})$$

for $X = A, E, F$. Now, the elements $A_i^{(s)}, E_i^{(1)}, F_i^{(1)}$ generate $\mathbf{Y}_\mu \otimes \mathbb{C}[z_1, \ldots, z_N]$ as a $\mathbb{C}[\hbar, z_1, \ldots, z_N]$ Poisson algebra (where the Poisson bracket is $\{a, b\} = \frac{1}{\hbar}(ab - ba)$). Since $\mathcal{A}_\hbar$ is almost commutative, $\mathcal{A}_\hbar$ is closed under the Poisson bracket and so the image of Φ_μ^λ is contained in $\mathbf{z}^*(\iota_*)^{-1}(\mathcal{A}_\hbar)$. $\square$

The image of $\overline{\Phi}_\mu^\lambda$ is called the truncated shifted Yangian and is denoted $\mathbf{Y}_\mu^\lambda$.

Remark B.21. It is easy to see that the elements $A_i^{(p)}$ for $p > a_i$ are sent to 0 under $\overline{\Phi}_\mu^\lambda$. We conjecture that these elements generate the kernel of $\overline{\Phi}_\mu^\lambda$ and thus we get a presentation of Y_μ^λ (the $\hbar = 1$ specialization of $\mathbf{Y}_\mu^\lambda$).

B.9. Specialization to the dominant case

Now, let us assume that μ is dominant.

B.9.1. The scheme $\mathcal{G}_\mu$. Consider the scheme $\mathcal{W}_\mu$ defined as the locus $G_1[[z^{-1}]]z^\mu \subset G((z^{-1}))/G[z]$. It is the moduli space of pairs $(\mathcal{P}, \sigma)$ where $\mathcal{P}$ is a G-bundle on $\mathbb{P}^1$ of isomorphism type μ and σ is a trivialization in the formal neighbourhood of ∞, such that $\mathcal{P}$ has isomorphism type μ and such that the Harder-Narasimhan flag of $\mathcal{P}$ at ∞ is compatible with $B_- \subset G$ (under σ). For any $\underline{\lambda}$, we have a morphism $\overline{\mathcal{W}}_\mu^{\underline{\lambda}} \to \mathcal{W}_\mu$ and a closed embedding $\overline{\mathcal{W}}_\mu^{\underline{\lambda}} \hookrightarrow \mathcal{W}_\mu \times \mathbb{A}^N$.

For any $\underline{\lambda}$ and any point $\underline{z} \in \mathbb{A}^N$, let $\overline{\mathcal{W}}_\mu^{\underline{\lambda},\underline{z}}$ be the fibre of $\overline{\mathcal{W}}_\mu^{\underline{\lambda}} \to \mathbb{A}^N$ over the point $\underline{z}$. The open locus $\mathcal{W}_\mu^{\underline{\lambda},\underline{z}}$ embeds into $\mathcal{W}_\mu$ as the intersection $\mathcal{W}_\mu \cap G[z]z^{\underline{\lambda},\underline{z}} \subset G((z^{-1}))/G[z]$, where $z^{\underline{\lambda},\underline{z}} = \prod_{s=1}^N (z - z_s)^{\omega_{i_s}}$.

In [KWWY14], we constructed a Poisson structure on $\mathcal{W}_\mu$. Now [KWWY14, Theorem 2.5] generalizes immediately to show that $\mathcal{W}_\mu^{\underline{\lambda},\underline{z}}$ is a symplectic leaf of $\mathcal{W}_\mu$. (In the case when $G = SL_n$, this is closely related to [Sha16, Theorem 2.2]).

Now, consider the subgroup of $G_1[[z^{-1}]]$ defined as

$$\mathcal{G}_\mu = \{g \in G_1[[z^{-1}]] \mid z^{-\mu} g z^\mu \in G_1[[z^{-1}]]\}.$$

The natural map $g \mapsto gz^\mu$ provides an isomorphism $\mathcal{G}_\mu \cong \mathcal{W}_\mu$.

The following result is [KWWY14, Theorem 3.12].

Theorem B.22. *There is an isomorphism of Poisson algebras*

$$\Psi \colon \mathbf{Y}_\mu / \hbar \mathbf{Y}_\mu \to \mathbb{C}[\mathcal{G}_{\mu^*}]$$

given by

$$\begin{aligned} H_i(z) &\mapsto z^{\langle \mu, \alpha_i \rangle} \prod_{\substack{h \in Q_1 \sqcup \overline{Q_1} \\ \mathrm{o}(h) = i}} \Delta_{w_0 \omega^\vee_{\mathrm{i}(h)}, w_0 \omega^\vee_{\mathrm{i}(h)}}(z) \Delta_{w_0 \omega_i^\vee, w_0 \omega_i^\vee}(z)^{-2}, \\ F_i(z) &\mapsto \Delta_{w_0 s_i \omega_i^\vee, w_0 \omega_i^\vee}(z) \Delta_{w_0 \omega_i^\vee, w_0 \omega_i^\vee}(z)^{-1}, \\ E_i(z) &\mapsto z^{\langle \mu, \alpha_i \rangle} \Delta_{w_0 \omega_i^\vee, w_0 s_i \omega_i^\vee}(z) \Delta_{w_0 \omega_i^\vee, w_0 \omega_i^\vee}(z)^{-1}. \end{aligned}$$

Here $\Delta_{w_0 \omega_i^\vee, w_0 \omega_i^\vee}$, etc. are generalized minors (see [KWWY14, Section 2] for more explanation) and we define $\Delta_{w_0 \omega_i^\vee, w_0 \omega_i^\vee}(z) \in \mathbb{C}[\mathcal{G}_{\mu^*}]((z^{-1}))$ by

$$\Delta_{w_0 \omega_i^\vee, w_0 \omega_i^\vee}(z)(g) = \Delta_{w_0 \omega_i^\vee, w_0 \omega_i^\vee}(g).$$

B.9.2. Involutions. Let $G \to G$, $g \mapsto g^t$ denote the transpose involution (it is an antiautomorphism which corresponds to the Lie algebra antiautomorphism given $E_i \mapsto F_i, F_i \mapsto E_i, H_i \mapsto H_i$). Also, let $\varkappa_{-1} \colon G_1[[z^{-1}]] \to G_1[[z^{-1}]]$ be the involution given by $z \mapsto -z$.

If $g \in \mathcal{G}_{\mu *}$, then $z^{-\mu^*} g z^{\mu^*} \in G_1[[z^{-1}]]$ and so $(z^{-\mu^*} g z^{\mu^*})^t = z^{\mu^*} g^t z^{-\mu^*} \in \mathcal{G}_{\mu^*}$.

We define an involution $\mathbf{i} \colon \mathcal{G}_{\mu^*} \to \mathcal{G}_{\mu^*}$ by $\mathbf{i}(g) = z^{-\mu^*} \varkappa_{-1}(g^t) z^{\mu^*}$. We can extend $\mathbf{i}$ to $\mathcal{G}_{\mu^*} \times \mathbb{A}^N$ by acting by multiplication by -1 on the second factor.

Following Remark 3.16, we consider an involution $\mathbf{i} \colon \overline{\mathcal{W}}^{\lambda^*}_{\mu^*} \to \overline{\mathcal{W}}^{\lambda^*}_{\mu^*}$ as the composition of ι^λ_μ and $\varkappa_{-1}$ and the action of $\beta(-1)$, where β is the coweight defined by

$$\beta = \sum_i \left(a_i - \sum_{h \in Q_1 : \mathrm{o}(h) = i} a_{\mathrm{i}(h)} \right) \omega_i.$$

Let us write $\mho \colon \overline{\mathcal{W}}^{\lambda^*}_{\mu^*} \to \mathcal{W}_{\mu^*} \times \mathbb{A}^N \cong \mathcal{G}_{\mu^*} \times \mathbb{A}^N$ for the natural composition. The following result is immediate.

Lemma B.23. *Up to $\beta(-1)$, the involutions are compatible with $\mho$. More precisely,*

$$\mho \circ a(\beta(-1)) \circ \mathbf{i} = \mathbf{i} \circ \mho.$$

We also can define an involution on $\mathbf{i}\colon \mathbf{Y}_\mu \to \mathbf{Y}_\mu$ by

$$E_i^{(p)} \mapsto (-1)^p F_i^{(p)},\ H_i^{(p)} \mapsto (-1)^{p+\langle\mu,\alpha_i\rangle} H_i^{(p)},\ F_i^{(p)} \mapsto (-1)^{p+\langle\mu,\alpha_i\rangle} E_i^{(p)}.$$

Above we defined the map $\Psi\colon \mathbf{Y}_\mu/\hbar\mathbf{Y}_\mu \to \mathbb{C}[\mathcal{G}_{\mu^*}]$. A simple computation shows the following result.

Lemma B.24. *The involutions are compatible with Ψ. More precisely,*

$$\Psi \circ \mathbf{i} = \mathbf{i} \circ \Psi.$$

Finally, we also have the involution $\mathfrak{i}^\lambda_{\mu*}\colon \mathcal{A}_0 \to \mathcal{A}_0$ (where $\mathcal{A}_0 = \mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar$) defined as in Remark 3.16, which comes from the isomorphism of varieties $\mathfrak{i}^\lambda_\mu\colon \mathcal{R}_{\mathrm{GL}(V),\mathbf{N}^\lambda_\mu} \xrightarrow{\sim} \mathcal{R}_{\mathrm{GL}(V^*),\mathbf{N}^\lambda_\mu}$. Note that the $\mathbf{N}^\lambda_\mu$ on the right hand side is computed with respect to the opposite orientation.

In Theorem B.18, we defined a homomorphism $\overline{\Phi}\colon \mathbf{Y}_\mu[z_1,\dots,z_N] \to \mathcal{A}_\hbar$ and thus a homomorphism $\mathbf{Y}_\mu[z_1,\dots,z_N]/\hbar\mathbf{Y}_\mu[z_1,\dots,z_N] \to \mathcal{A}_0$. We extend the involution $\mathbf{i}$ from $\mathbf{Y}_\mu$ to $\mathbf{Y}_\mu[z_1,\dots,z_N]$ by setting $\mathbf{i}(z_k) = -z_k$ for all k.

Lemma B.25. *Up to $\beta(-1)$, the involutions are compatible with $\overline{\Phi}$. More precisely,*

$$\overline{\Phi} \circ \mathbf{i} = a(\beta(-1)) \circ \mathfrak{i}^\lambda_{\mu*} \circ \overline{\Phi}$$

as maps $\mathbf{Y}_\mu[z_1,\dots,z_N]/\hbar\mathbf{Y}_\mu[z_1,\dots,z_N] \to \mathcal{A}_0$.

Proof. Note that we have $\mathfrak{i}^\lambda_{\mu*}([\mathcal{R}_{\varpi_{i,1}}]) = [\mathcal{R}_{\varpi^*_{i,1}}]$ and $\mathfrak{i}^\lambda_{\mu*}([\mathcal{R}_{\varpi^*_{i,1}}]) = [\mathcal{R}_{\varpi_{i,1}}]$. Also $\mathfrak{i}^\lambda_{\mu*}(c_1(\mathcal{S}_i)) = -c_1(\mathcal{Q}_i)$ since under the isomorphism $\mathfrak{i}^\lambda_\mu\colon \mathcal{R}_{\mathrm{GL}(V),\mathbf{N}^\lambda_\mu} \xrightarrow{\sim} \mathcal{R}_{\mathrm{GL}(V^*),\mathbf{N}^\lambda_\mu}$ from Remark 3.16, we have that $\mathfrak{i}^\lambda_{\mu*}(\mathcal{S}_i) = \mathcal{Q}_i^*$. Finally, we have that $\mathfrak{i}^\lambda_{\mu*}(w_{i,r}) = -w_{i,r}$.

Hence examining the formulas for $\overline{\Phi}$ given in Theorem B.18, the result follows. $\square$

Remark B.26. The involution $\mathfrak{i}^\lambda_{\mu*}\colon \mathcal{A}_0 \to \mathcal{A}_0$ extends to an involution $\mathfrak{i}^\lambda_{\mu*}\colon \mathcal{A}_\hbar \to \mathcal{A}_\hbar$. However, it is easy to see that the map $\overline{\Phi}\colon \mathbf{Y}_\mu[z_1,\dots,z_N] \to \mathcal{A}_\hbar$ is not compatible with this involution (not even up to sign). It is possible to modify the involution of $\mathbf{Y}_\mu$ to make it compatible up to sign, but it will be given by a bit more complicated formulae (for example $E_i(z) \mapsto -F_i(-z+\hbar)$). However, we will not need compatibility at the non-commutative level in this paper.

B.9.3. Commutativity. We have a surjection

$$\mho \circ \Psi \colon \mathbf{Y}_\mu[z_1,\dots,z_N]/\hbar\mathbf{Y}_\mu[z_1,\dots,z_N] \xrightarrow{\sim} \mathbb{C}[\mathcal{G}_{\mu^*} \times \mathbb{A}^N] \to \mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}].$$

Recall that in the previous section, we constructed a map

$$\overline{\Phi}\colon \mathbf{Y}_\mu[z_1,\dots,z_N] \to \mathcal{A}_\hbar.$$

On the other hand, in Theorem 3.20, we have constructed an isomorphism $\Xi\colon \mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}] \xrightarrow{\sim} \mathcal{A}_\hbar/\hbar\mathcal{A}_\hbar$.

Lemma B.27. *The composition* $\Xi^{-1}\circ\overline{\Phi}$ *equals* $\mho\circ\Psi$ *as morphisms*

$$\mathbf{Y}_\mu[z_1,\dots,z_N]/\hbar\mathbf{Y}_\mu[z_1,\dots,z_N] \to \mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}].$$

Proof. Since all the morphisms involved are Poisson $\mathbb{C}[z_1,\dots,z_N]$-algebra morphisms, it suffices to check the statement on the generators $E_i^{(s)}, A_i^{(s)}, F_i^{(s)}$ of $\mathbf{Y}_\mu$.

Recall that from §2(x) we have the morphism $s^{\lambda^*}_{\mu^*}\colon \overline{\mathcal{W}}^{\lambda^*}_{\mu^*} \to Z^\alpha \times \mathbb{A}^N$, where $\alpha = \lambda-\mu$. Given a point $([gz^{\mu^*}],(z_1,\dots,z_N)) \in \overline{\mathcal{W}}^{\lambda^*}_{\mu^*}$, the corresponding principal G-bundle $\mathcal{P}$ has associated vector bundle $\mathcal{V}^{\lambda^\vee}_{\mathcal{P}} = gz^{\mu^*}(V^{\lambda^\vee}\otimes\mathcal{O}_{\mathbb{P}^1})$ and invertible subsheaf $\mathcal{L}_{\lambda^\vee} = gz^{\mu^*}(V^{\lambda^\vee}_{w_0\lambda^\vee}\otimes\mathcal{O}_{\mathbb{P}^1})$. Thus the image of $([gz^{\mu^*}],(z_1,\dots,z_N))$ under $s^{\lambda^*}_{\mu^*}$ gives the collection of invertible subsheaves

$$gz^{\mu^*}(V^{\lambda^\vee}_{w_0\lambda^\vee}\otimes\mathcal{O}_{\mathbb{P}^1})\left(\sum_{s=1}^N \langle w_0\omega_{i_s^*},\lambda^\vee\rangle\cdot z_s\right) \subset V^{\lambda^\vee}\otimes\mathcal{O}_{\mathbb{P}^1}.$$

Now, we specialize to $\lambda^\vee = \omega_i^\vee$. Then the invertible subsheaf is generated over $\mathcal{O}_{\mathbb{P}^1}$ by

$$\left(\prod_{s=1}^N (z-z_s)^{-\langle w_0\omega_{i_s^*},\omega_i^\vee\rangle}\right) gz^{\mu^*}(v_{w_0\omega_i}) = Q_i(z)v_{w_0\omega_i^\vee} + P_i(z)v_{w_0 s_i\omega_i^\vee} + \cdots$$

where

$$\begin{aligned} Q_i(z) &= \Delta_{w_0\omega_i^\vee, w_0\omega_i^\vee}\left(\prod_{s=1}^N (z-z_s)^{-\langle w_0\omega_{i_s^*}, \omega_i^\vee\rangle} g z^{\mu^*}\right) \\ &= z^{-\langle \mu, \omega_i^\vee\rangle} \prod_{s=1}^N (z-z_s)^{\langle \omega_{i_s}, \omega_i^\vee\rangle} \Delta_{w_0\omega_i^\vee, w_0\omega_i^\vee}(g), \end{aligned}$$

and

$$\begin{aligned} P_i(z) &= \Delta_{w_0 s_i\omega_i^\vee, w_0\omega_i^\vee}\left(\prod_{s=1}^N (z-z_s)^{-\langle w_0\omega_{i_s^*}, \omega_i^\vee\rangle} g z^{\mu^*}\right) \\ &= z^{-\langle \mu, \omega_i^\vee\rangle} \prod_{s=1}^N (z-z_s)^{\langle \omega_{i_s}, \omega_i^\vee\rangle} \Delta_{w_0 s_i\omega_i^\vee, w_0\omega_i^\vee}(g). \end{aligned}$$

By definition (see [BDF16, 2.2]), $Q_i(z), P_i(z)$ are related to the coordinates $(w_{i,r}, y_{i,r})$ by $Q_i(w_{i,r}) = 0, P_i(w_{i,r}) = y_{i,r}$.

Now using the definition of $\Psi(H_i(z))$ given in Theorem B.22 and the definition of $A_i(z)$ given in (B.14), we deduce that

$$\Psi(A_i(z)) = z^{-a_i} \prod_{s=1}^N (z-z_s)^{\langle \omega_{i_s}, \omega_i^\vee\rangle} z^{-\langle \mu, \omega_i^\vee\rangle} \Delta_{w_0\omega_i^\vee, w_0\omega_i^\vee}(z),$$

and so $\Psi(A_i(z)) = z^{-a_i} Q_i(z)$ which agrees with $\overline{\Phi}$.

Next, we consider $F_i(z)$. First, we have that

$$\Psi(F_i(z)) = \Delta_{w_0 s_i\omega_i^\vee, w_0\omega_i^\vee}(z) \Delta_{w_0\omega_i^\vee, w_0\omega_i^\vee}(z)^{-1} = \frac{P_i(z)}{Q_i(z)}$$

by the above analysis.

We also have that

$$\Phi_\mu^\lambda(F_i(z)) = \sum_{r=1}^{a_i} \frac{\displaystyle\prod_{h\in Q_1: \mathrm{o}(h)=i} \prod_{s=1}^{a_j} (w_{i,r} - w_{\mathrm{i}(h),s})}{(z - w_{i,r}) \displaystyle\prod_{s\neq r} (w_{i,r} - w_{i,s})} \mathsf{u}_{i,r}.$$

On the other hand, we have

$$\Xi^{-1}\left(\prod_{h\in Q_1:\mathrm{o}(h)=i}\prod_{s=1}^{a_j}(-w_{i,r}+w_{\mathrm{i}(h),s})\mathsf{u}_{i,r}\right)=y_{i,r}$$

(see §3(i), §3(iii)) and thus

$$\Xi^{-1}(\overline{\Phi}(F_i(z))=\sum_{r=1}^{a_i}\frac{y_{i,r}}{(z-w_{i,r})\prod_{s\neq r}(w_{i,r}-w_{i,s})}=\frac{P_i(z)}{Q_i(z)},$$

where the last equality is obtained by Lagrange interpolation.

Thus, the statement holds for $F_i^{(p)}$.

Finally, we wish to show that $\Xi^{-1}\circ\overline{\Phi}(E_i^{(p)})=\mho\circ\Psi(E_i^{(p)})$. It suffices to prove that this equation holds after applying $\mathbf{i}$.

Applying Remark 3.16 and Lemma B.25, we deduce that

$$\mathbf{i}(\Xi^{-1}\circ\overline{\Phi}(E_i^{(p)}))=(-1)^{b_i}\Xi^{-1}\circ\overline{\Phi}(\mathbf{i}(E_i^{(p)})),$$

where $b_i=a_i-\sum_{i\to j}a_j$. However $\mathbf{i}(E_i^{(p)})=(-1)^pF_i^{(p)}$ and above we proved that $\Xi^{-1}\circ\overline{\Phi}(F_i^{(p)})=\mho\circ\Psi(F_i^{(p)})$. Thus we conclude that

$$\mathbf{i}(\Xi^{-1}\circ\overline{\Phi}(E_i^{(p)}))=(-1)^{b_i}\mho\circ\Psi(\mathbf{i}(E_i^{(p)})).$$

Now, applying Lemmas B.23 and B.24, we deduce that

$$\mho\circ\Psi(\mathbf{i}(E_i^{(p)}))=(-1)^{b_i}\mathbf{i}(\mho\circ\Psi(E_i^{(p)})).$$

Thus, we conclude that

$$\mathbf{i}(\Xi^{-1}\circ\overline{\Phi}(E_i^{(p)}))=\mathbf{i}(\mho\circ\Psi(E_i^{(p)})),$$

and hence $\Xi^{-1}\circ\overline{\Phi}(E_i^{(p)})=\mho\circ\Psi(E_i^{(p)})$ as desired. □

Corollary B.28. *We have an equality* $\mathbf{Y}_\mu^\lambda=\mathcal{A}_\hbar$ *and in particular, we have an isomorphism* $\mathbf{Y}_\mu^\lambda/\hbar\mathbf{Y}_\mu^\lambda\cong\mathbb{C}[\overline{\mathcal{W}}_{\mu^*}^{\lambda^*}]$.

Proof. The above theorem shows that the inclusion $\mathbf{Y}_\mu^\lambda\hookrightarrow\mathcal{A}_\hbar$ gives as isomorphism $\mathbf{Y}_\mu^\lambda/\hbar\mathbf{Y}_\mu^\lambda\cong\mathcal{A}_\hbar/h\mathcal{A}_\hbar$.

Thus each element of $\mathcal{A}_\hbar$ admits a lift modulo $\hbar$ to $\mathbf{Y}_\mu^\lambda$. Since $\mathbf{Y}_\mu^\lambda,\mathcal{A}_\hbar$ are graded and the grading is bounded below, this proves the equality. □

Remark B.29. The isomorphism $\mathbf{Y}^\lambda_\mu/\hbar\mathbf{Y}^\lambda_\mu \cong \mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}]$ was conjectured in [KWWY14]. More precisely, in [KWWY14], we proved that the map Ψ descended to a surjection $\mathbf{Y}^\lambda_\mu/(\hbar, z_1, \dots, z_N) \to \mathbb{C}[\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}]$ which was an isomorphism modulo nilpotents. The above Corollary shows that this map is an isomorphism. For other points $\underline{z} \in \mathbb{A}^N$, this also proves that the corresponding quotient in $\mathbf{Y}^\lambda_\mu$ is isomorphic to the corresponding fibre of $\overline{\mathcal{W}}^{\lambda^*}_{\mu^*}$. In [KWWY14], we made a mistake on this point (we stated that this would always quantize the central fibre).

Remark B.30. If we take μ^* not dominant, then some of the results of this section continue to hold. In this case, we defined a version of $\mathcal{W}_{\mu^*}$ and we directly constructed the isomorphism $\mathbf{Y}/\hbar\mathbf{Y} \cong \mathbb{C}[\mathcal{W}_{\mu^*}]$ in [FKP$^+$18, Theorem 5.15] . However, we do not know how to see that $\mathcal{W}_{\mu^*}$ has an intrinsic Poisson structure, nor have we proven the surjectivity of $\mathbf{Y}_\mu \to \mathcal{A}_\hbar$ in this situation.

References

[Part II] A. Braverman, M. Finkelberg, and H. Nakajima, *Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N}=4$ gauge theories, II*, Adv. Theor. Math. Phys. **22** (2018), no. 5, 1071–1147.

[Affine] A. Braverman, M. Finkelberg, and H. Nakajima, *Ring objects in the equivariant derived Satake category arising from Coulomb branches*, arXiv e-prints (2017), `arXiv:1706.02112 [math.RT]`.

[AH88] M. Atiyah and N. Hitchin, The Geometry and Dynamics of Magnetic Monopoles, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 1988.

[BC11] C. D. A. Blair and S. A. Cherkis, *Singular monopoles from Cheshire bows*, Nuclear Phys. B **845** (2011), no. 1, 140–164.

[BDF16] A. Braverman, G. Dobrovolska, and M. Finkelberg, *Gaiotto-Witten superpotential and Whittaker D-modules on monopoles*, Adv. Math. **300** (2016), 451–472.

[BDG17] M. Bullimore, T. Dimofte, and D. Gaiotto, *The Coulomb Branch of 3d $\mathcal{N}=4$ Theories*, Commun. Math. Phys. **354** (2017), no. 2, 671–751.

[BEF16] A. Braverman, P. Etingof, and M. Finkelberg, *Cyclotomic double affine Hecke algebras (with an appendix by Hiraku Nakajima and Daisuke Yamakawa)*, arXiv e-prints (2016), `arXiv: 1611.10216 [math.RT]`.

[BF10] A. Braverman and M. Finkelberg, *Pursuing the double affine Grassmannian I: transversal slices via instantons on A_k-singularities*, Duke Math. J. **152** (2010), no. 2, 175–206.

[BF12] A. Braverman and M. Finkelberg, *Pursuing the double affine Grassmannian II: Convolution*, Adv. Math. **230** (2012), no. 1, 414–432.

[BF13] A. Braverman and M. Finkelberg, *Pursuing the double affine Grassmannian III: Convolution with affine zastava*, Mosc. Math. J. **13** (2013), no. 2, 233–265, 363.

[BF14] A. Braverman and M. Finkelberg, *Semi-infinite Schubert varieties and quantum K-theory of flag manifolds*, J. Amer. Math. Soc. **27** (2014), no. 4, 1147–1168.

[BF17] A. Braverman and M. Finkelberg, *Twisted zastava and q-Whittaker functions*, J. Lond. Math. Soc. (2) **96** (2017), no. 2, 309–325.

[BFG06] A. Braverman, M. Finkelberg, and D. Gaitsgory, *Uhlenbeck spaces via affine Lie algebras*, The Unity of Mathematics, Progr. Math., Vol. 244, pp. 17–135, Birkhäuser Boston, Boston, MA, 2006, see `arXiv:math/0301176` for erratum.

[BFGM02] A. Braverman, M. Finkelberg, D. Gaitsgory, and I. Mirković, *Intersection cohomology of Drinfeld's compactifications*, Selecta Math. (N.S.) **8** (2002), no. 3, 381–418, see `arXiv:math/0012129v3` or Selecta Math. (N.S.) **10** (2004), 429–430, for erratum.

[BFM05] R. Bezrukavnikov, M. Finkelberg, and I. Mirković, *Equivariant homology and K-theory of affine Grassmannians and Toda lattices*, Compos. Math. **141** (2005), no. 3, 746–768.

[BFN16] A. Braverman, M. Finkelberg, and H. Nakajima, *Instanton moduli spaces and $\mathcal{W}$-algebras*, Astérisque (2016), no. 385, vii+128.

[BG01] A. Braverman and D. Gaitsgory, *Crystals via the affine Grassmannian*, Duke Math. J. **107** (2001), 561–575.

[BG08] P. Baumann and S. Gaussent, *On Mirković-Vilonen cycles and crystal combinatorics*, Represent. Theory **12** (2008), 83–130.

[BLPW16] T. Braden, A. Licata, N. Proudfoot, and B. Webster, *Quantizations of conical symplectic resolutions II: category $\mathcal{O}$ and symplectic duality*, Astérisque (2016), no. 384, 75–179, with an appendix by I. Losev.

[BM19] T. Braden and C. Mautner, *Ringel duality for perverse sheaves on hypertoric varieties*, Adv. Math. **344** (2019), 35–98.

[Bra03] T. Braden, *Hyperbolic localization of intersection cohomology*, Transform. Groups **8** (2003), no. 3, 209–216.

[CB01] W. Crawley-Boevey, *Geometry of the moment map for representations of quivers*, Compositio Math. **126** (2001), no. 3, 257–293.

[CFHM14] S. Cremonesi, G. Ferlito, A. Hanany, and N. Mekareeya, *Coulomb branch and the moduli space of instantons*, JHEP **1412** (2014), 103.

[CH08] B. Charbonneau and J. Hurtubise, *Calorons, Nahm's equations on S^1 and bundles over $\mathbb{P}^1 \times \mathbb{P}^1$*, Comm. Math. Phys. **280** (2008), no. 2, 315–349.

[CH10] B. Charbonneau and J. Hurtubise, *The Nahm transform for calorons*, The Many Facets of Geometry, Oxford Univ. Press, Oxford, 2010, pp. 34–70.

[Che09] S. A. Cherkis, *Moduli spaces of instantons on the Taub-NUT space*, Comm. Math. Phys. **290** (2009), no. 2, 719–736.

[Che10] S. A. Cherkis, *Instantons on the Taub-NUT space*, Adv. Theor. Math. Phys. **14** (2010), no. 2, 609–641.

[CK98] S. A. Cherkis and A. Kapustin, *Singular monopoles and supersymmetric gauge theories in three dimensions*, Nuclear Phys. B **525** (1998), no. 1-2, 215–234.

[dBHO+97] J. de Boer, K. Hori, H. Ooguri, Y. Oz, and Z. Yin, *Mirror symmetry in three-dimensional gauge theories,* $\mathrm{SL}(2,\mathbf{Z})$ *and D-brane moduli spaces*, Nuclear Phys. B **493** (1997), no. 1-2, 148–176.

[dBHOO97] J. de Boer, K. Hori, H. Ooguri, and Y. Oz, *Mirror symmetry in three-dimensional gauge theories, quivers and D-branes*, Nuclear Phys. B **493** (1997), no. 1-2, 101–147.

[Del87] P. Deligne, *Le déterminant de la cohomologie*, Contemp. Math. **67** (1987), 93–177.

[Don84] S. K. Donaldson, *Nahm's equations and the classification of monopoles*, Comm. Math. Phys. **96** (1984), no. 3, 387–407.

[Fal03] G. Faltings, *Algebraic loop groups and moduli spaces of bundles*, J. European Math. Soc. **5** (2003), no. 1, 41–68.

[FFNR11] B. Feigin, M. Finkelberg, A. Negut, and L. Rybnikov, *Yangians and cohomology rings of Laumon spaces*, Selecta Math. (N.S.) **17** (2011), no. 3, 573–607.

[Fin18] M. Finkelberg, *Double affine Grassmannians and Coulomb branches of 3d $\mathcal{N}=4$ quiver gauge theories*, Proceedings of the International Congress of Mathematicians (2018), Rio de Janeiro, Vol. 2, 1283–1302.

[FKMM99] M. Finkelberg, A. Kuznetsov, N. Markarian, and I. Mirković, *A note on the symplectic structure on the space of g-monopoles*, Comm. Math. Phys. **201** (1999), no. 2, 411–421, see `arXiv:math/9803124v6` or Comm. Math. Phys. **334** (2015), no. 2, 1153–1155, for erratum.

[FKP+18] M. Finkelberg, J. Kamnitzer, K. Pham, L. Rybnikov, and A. Weekes, *Comultiplication for shifted Yangians and quantum open Toda lattice*, Adv. Math. **327** (2018), 349–389.

[FKR18] M. Finkelberg, A. Kuznetsov, and L. Rybnikov, *Towards a cluster structure on trigonometric zastava (with appendix by Galyna Dobrovolska)*, Selecta Math. (N.S.) **24** (2018), no. 1, 187–225.

[FL06] G. Fourier and P. Littelmann, *Tensor product structure of affine Demazure modules and limit constructions*, Nagoya Math. J. **182** (2006), 171–198.

[FM99] M. Finkelberg and I. Mirković, *Semi-infinite flags. I. Case of global curve* $\mathbb{P}^1$, Amer. Math. Soc. Transl. Ser. 2 **194** (1999), 81–112.

[FR14] M. Finkelberg and L. Rybnikov, *Quantization of Drinfeld zastava in type A*, J. Eur. Math. Soc. **16** (2014), no. 2, 235–271.

[Gai08] D. Gaitsgory, *Twisted Whittaker model and factorizable sheaves*, Selecta Math. (N.S.) **13** (2008), no. 4, 617–659.

[Gin95] V. Ginzburg, *Perverse sheaves on a Loop group and Langlands' duality*, arXiv e-prints (1995), `arXiv:alg-geom/9511007 [alg-geom]`.

[GKLO05] A. Gerasimov, S. Kharchev, D. Lebedev, and S. Oblezin, *On a class of representations of the Yangian and moduli space of monopoles*, Comm. Math. Phys. **260** (2005), no. 3, 511–525.

[Hit83] N. J. Hitchin, *On the construction of monopoles*, Comm. Math. Phys. **89** (1983), no. 2, 145–190.

[HM89] J. Hurtubise and M. K. Murray, *On the construction of monopoles for the classical groups*, Comm. Math. Phys. **122** (1989), no. 1, 35–89.

[Hur85] J. Hurtubise, *Monopoles and rational maps: a note on a theorem of Donaldson*, Comm. Math. Phys. **100** (1985), no. 2, 191–196.

[Hur89] J. Hurtubise, *The classification of monopoles for the classical groups*, Comm. Math. Phys. **120** (1989), no. 4, 613–641.

[HW97] A. Hanany and E. Witten, *Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics*, Nuclear Phys. B **492** (1997), no. 1-2, 152–190.

[Ill71] L. Illusie, Complexe Cotangent et Déformations. I, Lecture Notes in Mathematics **239**, Springer-Verlag, Berlin – New York, 1971.

[Jar98] S. Jarvis, *Euclidean monopoles and rational maps*, Proc. London Math. Soc. (3) **77** (1998), no. 1, 170–192.

[Kas95] M. Kashiwara, *On crystal bases*, CMS Conf. Proc. **16** (1995), 155–197.

[Kir97] A. A. Kirillov, Jr., *Lectures on affine Hecke algebras and Macdonald's conjectures*, Bull. Amer. Math. Soc. (N.S.) **34** (1997), no. 3, 251–292.

[KN18] R. Kodera and H. Nakajima, *Quantized Coulomb branches of Jordan quiver gauge theories and cyclotomic rational Cherednik algebras*, String-Math 2016, Proc. Symops. Pure Math. **98**, Amer. Math. Soc., Providence, RI, 2018, 49–78.

[Kro85] P. B. Kronheimer, *Monopoles and Taub-NUT metrics*, Master's thesis, Oxford, 1985.

[Kry18] V. Krylov, *Integrable crystals and restriction to Levi via generalized slices in the affine Grassmannian*, Funct. Anal. Appl. **52** (2018), no. 2, 113–133.

[KS90] M. Kashiwara and P. Schapira, Sheaves on Manifolds, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 292, Springer-Verlag, Berlin, 1990, With a chapter in French by Christian Houzel.

[KW07] A. Kapustin and E. Witten, *Electric-magnetic duality and the geometric Langlands program*, Commun. Number Theory Phys. **1** (2007), no. 1, 1–236.

[KWWY14] J. Kamnitzer, B. Webster, A. Weekes, and O. Yacobi, *Yangians and quantizations of slices in the affine Grassmannian*, Algebra and Number Theory **8** (2014), no. 4, 857–893.

[Lus83] G. Lusztig, *Singularities, character formulas, and a q-analog of weight multiplicities*, Astérisque **101-102** (1983), 208–229.

[Lus93] G. Lusztig, *Introduction to quantum groups*, Progress in Mathematics, vol. 110, Birkhäuser Boston Inc., Boston, MA, 1993.

[Mac95] I. G. Macdonald, Symmetric Functions and Hall Polynomials, second ed., Oxford Mathematical Monographs, The Clarendon Press Oxford University Press, New York, 1995, With contributions by A. Zelevinsky, Oxford Science Publications.

[Mac01] I. G. Macdonald, *Orthogonal polynomials associated with root systems*, Sém. Lothar. Combin. **45** (2000/01), Art. B45a, 40.

[Mat86] H. Matsumura, Commutative Ring Theory, Cambridge Studies in Advanced Mathematics **8**, Cambridge University Press, Cambridge, 1986.

[Mek15] N. Mekareeya, *The moduli space of instantons on an ALE space from 3d $\mathcal{N}=4$ field theories*, Journal of High Energy Physics **2015** (2015), no. 12, 1–30.

[Mir14] I. Mirković, *Loop Grassmannians in the framework of local spaces over a curve*, Contemp. Math. **623** (2014), 215–226.

[MO19] D. Maulik and A. Okounkov, *Quantum groups and quantum cohomology*, Astérisque (2019), no. 408, ix+209.

[MOV05] A. Malkin, V. Ostrik, and M. Vybornov, *The minimal degeneration singularities in the affine Grassmannians*, Duke Math. J. **126** (2005), no. 2, 233–249.

[MV07] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143.

[Nak93] H. Nakajima, *Monopoles and Nahm's equations*, Einstein Metrics and Yang-Mills Connections (Sanda, 1990), Lecture Notes in Pure and Appl. Math., Vol. 145, Dekker, New York, 1993, pp. 193–211.

[Nak94] H. Nakajima, *Instantons on ALE spaces, quiver varieties, and Kac-Moody algebras*, Duke Math. J. **76** (1994), no. 2, 365–416.

[Nak98] H. Nakajima, *Quiver varieties and Kac-Moody algebras*, Duke Math. J. **91** (1998), no. 3, 515–560.

[Nak01] H. Nakajima, *Quiver varieties and tensor products*, Invent. Math. **146** (2001), no. 2, 399–449.

[Nak15] H. Nakajima, *Questions on provisional Coulomb branches of 3-dimensional $\mathcal{N}=4$ gauge theories*, Sūrikaisekikenkyūsho Kōkyūroku (2015), no. 1977, 57–76.

[Nak16] H. Nakajima, *Towards a mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N}=4$ gauge theories, I*, Adv. Theor. Math. Phys. **20** (2016), no. 3, 595–669.

[Nak17] H. Nakajima, *Lectures on perverse sheaves on instanton moduli spaces*, Geometry of moduli spaces and representation theory, 381–436, IAS/Park City Math. Ser. 24, Amer. Math. Soc., Providence, RI, 2017.

[NP01] B. C. Ngô and P. Polo, *Résolutions de Demazure affines et formule de Casselman-Shalika géométrique*, J. Algebraic Geom. **10** (2001), no. 3, 515–547.

[NP12] N. Nekrasov and V. Pestun, *Seiberg-Witten geometry of four dimensional $N = 2$ quiver gauge theories*, arXiv e-prints (2012), `arXiv:1211.2240 [hep-th]`.

[NS00] T. M. W. Nye and M. A. Singer, *An L^2-index theorem for Dirac operators on $S^1 \times \mathbf{R}^3$*, J. Funct. Anal. **177** (2000), no. 1, 203–218.

[NT17] H. Nakajima and Y. Takayama, *Cherkis bow varieties and Coulomb branches of quiver gauge theories of affine type A*, Selecta Mathematica **23** (2017), no. 4, 2553–2633.

[Nye01] T. M. W. Nye, *The geometry of calorons*, Ph.D. thesis, U. of Edinburgh, 2001.

[Sch16] S. Schieder, *Geometric Bernstein asymptotics and the Drinfeld-Lafforgue-Vinberg degeneration for arbitrary reductive groups*, arXiv e-prints (2016), `arXiv:1607.00586 [math.AG]`.

[Sha16] A. Shapiro, *Poisson geometry of monic matrix polynomials*, Int. Math. Res. Not. IMRN (2016), no. 17, 5427–5453.

[Slo12] W. Slofstra, *A Brylinski filtration for affine Kac-Moody algebras*, Adv. Math. **229** (2012), no. 2, 968–983.

[Tak16] Y. Takayama, *Nahm's equations, quiver varieties and parabolic sheaves*, Publ. Res. Inst. Math. Sci. **52** (2016), no. 1, 1–41.

[Ton99] D. Tong, *Three-dimensional gauge theories and ADE monopoles*, Phys. Lett. B **448** (1999), no. 1-2, 33–36.

[vDE11] J. F. van Diejen and E. Emsiz, *A generalized Macdonald operator*, Int. Math. Res. Not. IMRN (2011), no. 15, 3560–3574.

[Wey97] H. Weyl, The Classical Groups, Their Invariants and Representations, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997, Fifteenth printing, Princeton Paperbacks.

Department of Mathematics, University of Toronto
and Perimeter Institute of Theoretical Physics
Waterloo, Ontario, N2L 2Y5, Canada
E-mail address: `braval@math.toronto.edu`

Department of Mathematics
National Research University Higher School of Economics
6 Usacheva st, Moscow 119048, Russian Federation;
Skolkovo Institute of Science and Technology
Moscow 121205, Russia
and Institute for Information Transmission Problems
Moscow 127051, Russia
E-mail address: `fnklberg@gmail.com`

Research Institute for Mathematical Sciences
Kyoto University, Kyoto 606-8502, Japan
E-mail address: `nakajima@kurims.kyoto-u.ac.jp`
Current address:
Kavli Institute for the Physics and Mathematics of the Universe (WPI), The University of Tokyo
5-1-5 Kashiwanoha, Kashiwa, Chiba, 277-8583, Japan
E-mail address: `hiraku.nakajima@ipmu.jp`