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COULOMBIC-HADRONIC INTERFERENCE IN AN EIKONAL MODEL

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A B S T R A C T

The problem of extracting the phase of the hadronic proton-proton scattering amplitude from its interference with the Coulombic amplitude near the forward direction is re-examined using an eikonal model. The results are in accord with the Feynman diagrammatic calculation of West and Yennie, with some small corrections. An especially compact form is derived for the electromagnetic shift in the hadronic amplitude's phase, which includes the effect of the electromagnetic form factor. The largest modification of the previous results comes from the effect of the form factor on the phase of the electromagnetic amplitude itself.

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1. - INTRODUCTION

Continuing advances in accelerator technology are providing the opportunity to study pp and $p\bar{p}$ elastic scattering at ever higher centre-of-mass energies. Of special interest are the total cross-sections, which are related by the optical theorem to the imaginary parts of the forward scattering amplitudes, and the ratios of the real to imaginary parts of the amplitudes, since there are bounds on combinations of these quantities which can be derived from basic principles[1].

The traditional technique for determining the forward amplitude utilizes interference with the Coulomb amplitude. If we normalize the scattering amplitudes so that

$$\frac{d\sigma}{dt} = \frac{\pi}{s\, p_{cm}^2} \, |F|^2 , \tag{1}$$

then the Coulomb amplitude is, for charges of like sign, essentially

$$F^c = -\frac{\alpha s}{\vec{q}^{\,2}} , \tag{2}$$

where $\vec{q}$ is the centre-of-mass momentum transfer. We have assumed that the momentum transfer is small. Throughout, we shall assume we are considering pp or $p\bar{p}$ scattering (with its equal mass kinematics) at very high energies: $s \gg M_p^2$. We shall always consider particle-particle scattering first. The antiparticle-particle result will be obtained by a judicious reversal of appropriate signs.

The hadronic elastic amplitude may be conveniently parametrized in the low momentum transfer domain as

$$F^N = A\, e^{-B\vec{q}^{\,2}/2} , \tag{3}$$

where $B \approx 10 - 15\ \mathrm{GeV}^{-2}$. The optical theorem states that

$$\sigma_{TOT} = \frac{4\pi}{p_{cm}\sqrt{s}}\, \mathrm{Im}\, F^N(\vec{q}=0) . \tag{4}$$

For very high energies the total pp cross-section is roughly 40 mb so that Im A ≈ (4 GeV^{-2})s. The Coulomb and hadronic amplitudes for elastic scattering are comparable then for a momentum transfer $q^2 \approx \alpha/4\ \text{GeV}^2$. The hadronic elastic amplitude is mostly, but not entirely, imaginary. By observing the interference between the known Coulomb amplitude and the hadronic amplitude, it is possible to determine the phase between the two.

This simple description, while suggestive, is really not adequate for several reasons. The Coulomb amplitude, Eq. (2), is really just the Born amplitude. The actual Coulomb amplitude has a characteristic Coulomb phase and, in fact, a pathology associated with the long-range nature of the Coulomb force. Moreover, it is not really possible to describe the scattering amplitude as a sum of a Coulombic and a hadronic amplitude. There is a single amplitude, dominated by the Coulomb force at very low momentum transfer and by the hadronic force at higher momentum transfer. Even in the hadronic region, the amplitude must have the well-known problems associated with the exchange of soft virtual photons. There is inevitably an electromagnetic influence on the hadronic amplitude.

The seminal treatment of this problem is that of Bethe[2]. His analysis used a WKB approach in potential theory and was directed at the study of proton-nucleus scattering. The result was that the full amplitude including the electromagnetic and hadronic forces could be expressed as

$$F^{TOT} = F^N + e^{i\alpha\phi} F^c , \tag{5}$$

where ϕ depended on the momentum transfer, q, and a parameter, a, characterizing the size of the nucleus:

$$\phi_{\text{Bethe}} \cong 2 \ln (1.06/qa) . \tag{6}$$

Coulomb-nuclear interference has been re-examined outside potential theory using Feynman diagrams most definitively by West and Yennie[3]. They succeeded in finding a general formula for ϕ in terms of the elastic hadronic amplitude:

$$\phi_{W-Y} = -\ln \frac{q^2}{s} - \int_0^s \frac{dq'^2}{|q'^2 - q^2|} \left[1 - \frac{F^N(q'^2)}{F^N(q^2)} \right] . \tag{7}$$

For a hadronic amplitude with the conventional parametrization, $F^N \propto \exp(-Bq^2/2)$, this yields

$$\phi_{W-Y} = -\left[\ln (Bq^2/2) + \gamma + O(Bq^2) \right] . \tag{8}$$

West and Yennie treated several other aspects of the electromagnetic corrections to hadronic amplitudes. They considered the emission of real photons, which is important for πp scattering but which we shall ignore for the case of pp scattering. The diagrams they analyzed were those with potential infra-red divergences and they stressed the importance of realizing that the diagrams ignored would contribute to ϕ as well. A further point which has perhaps not been heeded is the necessity of including vacuum polarization in the Coulomb force[4]. This has the effect of making the electromagnetic coupling q^2 dependent:

$$\alpha(q^2) = \alpha\left(1 + \frac{\alpha}{3\pi} \ln \frac{q^2}{4m_e^2}\right) . \tag{9}$$

This results in an increase of about one-half a per cent in α for the q^2 range of interest.

Below, we reconsider the issues involved in determining ϕ within the context of an eikonal model. This provides a quite physical picture for the process and gives a slightly different perspective from the West-Yennie calculation. The results are in complete agreement with their work. In particular, a formula essentially equivalent to Eq. (7) is obtained. The effect of electromagnetic form factors is considered in some detail and a general expression obtained for ϕ. This formula generalizes Eq. (7) and puts it in a rather appealing form. Finally, the effect of the form factor on the Coulomb amplitude itself is calculated using a Born series expansion of the eikonalized amplitude This contributes a term to the relative phase ϕ of order $(qR)^2 \log (qR)^2$.

2. - THE EIKONAL MODEL AND INTERFERENCE WITH A PURE COULOMB FIELD

The eikonal model[5] is useful in describing scattering at small angles since it is based on an approximation in which the particles are treated as if they travelled on a straight path. In traversing this path, they accumulate a phase δ which is a function of the impact parameter b as well as the centre-of-mass energy. At very high energies, δ is given by

$$\delta(b) = \frac{1}{2\pi s} \int d^2q \; e^{-i\vec{q}\cdot\vec{b}} F_{Born}(q^2) , \tag{10}$$

where we have suppressed the s dependence of the Born amplitude. The eikonal amplitude is, again at large s,

$$F_{eik}(q^2) = \frac{s}{4\pi i} \int d^2b \; e^{i\vec{q}\cdot\vec{b}} \left[e^{2i\delta(b)} - 1 \right] . \tag{11}$$

We shall suppose there are two eikonal phases, δ^C and δ^N, which describe Coulomb and nuclear scattering so that the full amplitude is

$$F^{N+C}(q^2) = \frac{s}{4\pi i} \int d^2b \; e^{i\vec{q}\cdot\vec{b}} \left[e^{2i\left(\delta^C(b)+\delta^N(b)\right)} - 1 \right] . \tag{12}$$

In the absence of nuclear forces, the Coulomb amplitude would be

$$F^C(q^2) = \frac{s}{4\pi i} \int d^2b \; e^{i\vec{q}\cdot\vec{b}} \left[e^{2i\delta^C(b)} - 1 \right] , \tag{13}$$

while in the absence of Coulombic forces we would have the hadronic amplitude

$$F^N(q^2) = \frac{s}{4\pi i} \int d^2b \; e^{i\vec{q}\cdot\vec{b}} \left[e^{2i\delta^N(b)} - 1 \right] . \tag{14}$$

The combined effect can be written conveniently as

$$\begin{aligned} F^{N+C}(q^2) &= F^C(q^2) + F^N(q^2) + \frac{s}{4\pi i} \int d^2b \, e^{i\vec{q}\cdot\vec{b}} \left[e^{2i\delta^C(b)} - 1 \right]\left[e^{2i\delta^N(b)} - 1 \right] \\ &= F^C(q^2) + F^N(q^2) + \frac{i}{\pi s} \int d^2q' \, F^C(q'^2) \, F^N([\vec{q}-\vec{q}']^2) . \end{aligned} \tag{15}$$

Equation (15) summarizes the way the eikonal model interweaves two interactions. To apply it to the problem at hand we need the eikonal approximation to the Coulomb amplitude. This is obtained from Eqs. (2) and (10), with a fictitious photon mass, λ, inserted to tame the infra-red problem

$$\begin{aligned}\delta^c(b) &= \frac{1}{2\pi}\int d^2b\; e^{-i\vec{q}\cdot\vec{b}}\left(\frac{-\alpha}{q^2+\lambda^2}\right)\\ &= -\alpha K_0(b\lambda)\\ &= \alpha\left[\ln\left(\tfrac{1}{2}b\lambda\right)+\gamma+O(b\lambda)\right].\end{aligned} \tag{16}$$

The terms $O(b\lambda)$ can be ignored since the photon mass will finally be taken to zero. Thus

$$F^c(q^2) = \frac{s}{2i}\int_0^\infty db\; b\, J_0(qb)\left[\left(\frac{\lambda e^\gamma}{2q}\right)^{2i\alpha}(bq)^{2i\alpha}-1\right]. \tag{17}$$

Using (and slightly abusing) the formula[6)]

$$\int_0^\infty dx\; x^\mu J_0(x) = 2^\mu\,\frac{\Gamma\left(\frac{1+\mu}{2}\right)}{\Gamma\left(\frac{1-\mu}{2}\right)}\quad, \tag{18}$$

we have to lowest order in α,

$$\begin{aligned}F^c(q^2) &= \frac{s}{2iq^2}\left(\frac{\lambda e^\gamma}{2q}\right)^{2i\alpha}2^{1+2i\alpha}\,\frac{\Gamma(1+i\alpha)}{\Gamma(-i\alpha)}\\ &= -\frac{s\alpha}{q^2}\,e^{i\alpha\eta^c_{eik}(q^2)}\end{aligned} \tag{19}$$

where

$$\eta^c_{eik}(q^2) = \ln\left(\frac{\lambda^2}{q^2}\right). \tag{20}$$

Because of the singularity in $F^C(q^2)$ at $q^2 = 0$, it is convenient to write Eq. (15) as

$$F^{N+C}(q^2) = F^C(q^2) + F^N(q^2)\left\{1 + \frac{i}{\pi s}\int d^2q'\, F^C(q'^2) + \frac{i}{\pi s}\int d^2q'\, F^C(q'^2)\left[\frac{F^N([\vec{q}-\vec{q}']^2)}{F^N(q^2)} - 1\right]\right\}. \tag{21}$$

In the last integral we may safely take $\lambda = 0$ in the denominator. In fact, we can write

$$F^{N+C}(q^2)\, e^{-i\alpha\eta^C_{eik}(q^2)} = -\frac{\alpha s}{q^2} + F^N(q^2)\left\{e^{-i\alpha\eta^C_{eik}(q^2)} + \frac{i}{\pi s}\int^Q d^2q'\left(\frac{-\alpha s}{q'^2+\lambda^2}\right)\left(\frac{q^2}{q'^2}\right)^{i\alpha} + \frac{i}{\pi s}\int^Q d^2q'\left(\frac{-\alpha s}{[\vec{q}-\vec{q}']^2}\right)\left[\frac{F^N(q'^2)}{F^N(q^2)} - 1\right]\right\}. \tag{22}$$

We have indicated an upper limit of integration, Q, as a reminder that the two integrals are separately divergent at large q^2. The sum of the two is finite as $Q^2 \to \infty$. The sum of the first two terms in Eq. (22) is

$$\left(\frac{\lambda^2}{q^2}\right)^{-i\alpha} + \left(\frac{Q^2}{q^2}\right)^{-i\alpha} - \left(\frac{\lambda^2}{q^2}\right)^{-i\alpha} \cong 1 - i\alpha \ln Q^2/q^2 . \tag{23}$$

Altogether, then

$$F^{N+C}(q^2)\, e^{-i\alpha\eta^C_{eik}(q^2)} = -\frac{\alpha s}{q^2} + F^N(q^2)\left\{1 - i\alpha \ln Q^2/q^2 - i\alpha\int_0^{Q^2} dq'^2\,\frac{1}{|q^2-q'^2|}\left[\frac{F^N(q'^2)}{F^N(q^2)} - 1\right]\right\}, \tag{24}$$

where we have used

$$\int_0^{2\pi} d\phi \, \frac{1}{q^2 + 2qq'\cos\phi + q'^2} = \frac{2\pi}{|q^2 - q'^2|} \tag{25}$$

The integrand in Eq. (24) is not singular at $q = q'$. Comparing Eq. (24) with Eq. (5), we find the eikonal model value for the phase

$$\phi_{eik} = \lim_{Q^2 \to \infty} \left[-\ln q^2/Q^2 + \int_0^{Q^2} dq'^2 \frac{1}{|q^2 - q'^2|} \left[\frac{F^N(q'^2)}{F^N(q^2)} - 1 \right] \right], \tag{26}$$

which agrees with the result of West and Yennie obtained from Feynman diagrams.

3. - THE INCLUSION OF ELECTROMAGNETIC FORM FACTORS

A realistic treatment of the Coulomb interference problem must include the electromagnetic form factor of the proton. The form factor will modify the Coulomb amplitude first by changing the Born term by a factor $f^2(q^2)$ and second by modifying the phase η^c_{eik}. In this section we consider only the former and thus take

$$F^c(q^2) = \left(\frac{-\alpha s}{q^2 + \lambda^2} \right) \left(\frac{\lambda^2}{q^2} \right)^{i\alpha} f^2(q^2) \tag{27}$$

In the next section we shall consider the modification of the phase.

Using this modified Coulomb amplitude in Eq. (21) we have

$$\begin{aligned} F^{N+C}(q^2) \left(\frac{\lambda^2}{q^2} \right)^{-i\alpha} = & - \frac{\alpha s}{q^2} f^2(q^2) + F^N(q^2) \left\{ \left(\frac{\lambda^2}{q^2} \right)^{-i\alpha} \right. \\ & + \frac{i}{\pi s} \int d^2 q' \left(\frac{-\alpha s}{q'^2 + \lambda^2} \right) \left(\frac{q^2}{q'^2} \right)^{i\alpha} f^2(q'^2) \\ & \left. + \frac{i}{\pi s} \int d^2 q' \left(\frac{-\alpha s}{q'^2} \right) f^2(q'^2) \left[\frac{F^N([\vec{q} - \vec{q}\,']^2)}{F^N(q^2)} - 1 \right] \right\}. \end{aligned} \tag{28}$$

where terms of higher order in α have been dropped from the last term. The next-to-the-last integral we evaluate as

$$\frac{i}{\pi s}\int d^2q'\left(\frac{-\alpha s}{q'^2+\lambda^2}\right)\left(\frac{q^2}{q'^2}\right)^{i\alpha} f^2(q'^2) \approx -i\alpha\int dq'^2\,\frac{(q^2)^{i\alpha}}{(q'^2+\lambda^2)^{1+i\alpha}}\,f^2(q'^2)$$
$$= -\left(\frac{q^2}{\lambda^2}\right)^{i\alpha} - \int_0^\infty dq'^2\left(\frac{q'^2+\lambda^2}{q^2}\right)^{-i\alpha}\left[f^2(q'^2)\right]'$$
$$= -\left(\frac{q^2}{\lambda^2}\right)^{i\alpha} - \int_0^\infty dq'^2\left(1-i\alpha\ln\frac{q'^2}{q^2}\right)\left[f^2(q'^2)\right]' \tag{29}$$
$$= 1-\left(q^2/\lambda^2\right)^{i\alpha} + i\alpha\int_0^\infty dq'^2\ln\frac{q'^2}{q^2}\left[f^2(q'^2)\right]'.$$

Here, the prime indicates differentiation with respect to q'^2. Including this in Eq. (28) we have

$$F^{N+C}(q^2)\left(\frac{\lambda^2}{q^2}\right)^{-i\alpha} = -\frac{\alpha s}{q^2}f^2(q^2) + F^N(q^2)\Bigg\{1 +$$
$$+ i\alpha\int_0^\infty dq'^2\ln\frac{q'^2}{q^2}\left[f^2(q'^2)\right]' - \frac{i\alpha}{\pi}\int d^2q'\,\frac{f^2(q'^2)}{q'^2}\left[\frac{F^N([\vec q-\vec q\,']^2)}{F^N(q^2)}-1\right]\Bigg\}. \tag{30}$$

Now it is convenient to define an angularly averaged hadronic amplitude

$$\bar F^N(q'^2,q^2) = \frac{1}{2\pi}\int d\phi\ \ F^N([q^2+2qq'\cos\phi+q'^2])/F^N(q^2)\ , \tag{31}$$

which thus has the property

$$\bar F^N(0,q^2) = 1\ . \tag{32}$$

It follows that

$$\int d^2q'\,\frac{f^2(q'^2)}{q'^2}\left[\frac{F^N([\vec q-\vec q\,']^2)}{F^N(q^2)}-1\right] = \pi\int_0^\infty dq'^2\,\frac{f^2(q'^2)}{q'^2}\left[\bar F^N(q'^2,q^2)-1\right]$$
$$= \pi\int_0^\infty dq'^2\ln q'^2\left[\bar F^N(q'^2,q^2)f^2(q'^2) - f^2(q'^2)\right]'. \tag{33}$$

Using this, Eq. (30) becomes

$$F^{N+C}(q^2)\left(\frac{\lambda^2}{q^2}\right)^{-i\alpha} = \frac{-\alpha s}{q^2} + F^N(q^2)\Big\{1 + i\alpha\int_0^\infty dq'^2 \ln\frac{q'^2}{q^2}\left[f^2(q'^2)\,\bar{F}^N(q'^2,q^2)\right]'\Big\} . \tag{34}$$

Referring back to Eq. (5), we see that

$$\phi = -\int_0^\infty dq'^2 \ln\frac{q'^2}{q^2}\left[f^2(q'^2)\;\bar{F}^N(q'^2,q^2)\right]' \tag{35}$$

In fact, since q^2 is small compared to the scale set by the inverse size of the proton, we can generally approximate

$$\bar{F}^N(q'^2,q^2) \cong F^N(q'^2)/F^N(0) \tag{36}$$

to produce our final general expression for ϕ:

$$\phi = -\int_0^\infty dq'^2 \ln\frac{q'^2}{q^2}\;\frac{d}{dq'^2}\left[f^2(q'^2)F^N(q'^2)/F^N(0)\right] . \tag{37}$$

The standard parametrization of the hadronic amplitude at low to moderate q^2 is given in Eq. (3). The conventional form factor parametrization is[7)]

$$f(q^2) = \left(\frac{\Lambda^2}{q^2+\Lambda^2}\right)^2 , \quad \Lambda^2 = 0.71 \text{ GeV}^2 . \tag{38}$$

An alternative, approximate form with the same low q^2 dependence is

$$f(q^2) = e^{-2q^2/\Lambda^2} . \tag{39}$$

Using the latter, we find ($\gamma = 0.577...$ is Euler's constant)

$$\phi = -\ln\left(\frac{Bq^2}{2}\right) - \gamma - \ln\left(1+\frac{8}{B\Lambda^2}\right) . \tag{40}$$

The dipole form factor, on the other hand, gives

$$\phi = -\ln\left(\frac{Bq^2}{2}\right) - \gamma - g\left(\frac{B\Lambda^2}{2}\right) \tag{41}$$

where

$$g(z) = \left(1 - z + \frac{z^2}{2} - \frac{z^3}{6}\right)e^z E_1(z) + \frac{11}{6} - \frac{2z}{3} + \frac{z^2}{6} \tag{42}$$

and where $E_1(z)$ is the exponential integral[8)]

$$E_1(z) = \int_1^\infty \frac{dt}{t} e^{-zt} \tag{43}$$

These two results can be compared for a specific choice of B, the slope parameter. For $B = 13\ \mathrm{GeV}^{-2}$ the exponential form factor gives $\phi = -\ln(Bq^2/2) - \gamma - 0.62$, while the dipole form is $\phi = -\ln(Bq^2/2) - \gamma - 0.60$. Clearly, the difference is unimportant.

As a further check of the insensitivity of the result to various parametrizations, consider the Wu-Yang[9)] model in which the elastic pp differential cross-section is proportional to the fourth power of the electromagnetic form factor. In this model, B and Λ^2 are related by $B\Lambda^2 = 8$. For the exponential parametrization we find $\phi = -\ln(Bq^2/2) - \gamma - \ln 2$. If, instead, the dipole parametrization is used (both for the form factor and the hadronic amplitude), the result is $\phi = -\ln(Bq^2/2) - \gamma + (\gamma + \ln 4 - 363/140)$. The final term in parentheses is -0.63 which is quite close to $-\ln 2 = -0.69$. Of course, the Wu-Yang model cannot be considered accurate, even for our purposes, since B is s dependent unlike Λ^2.

Since the difference between Eqs. (40) and (41) is slight, the former, which is simpler, provides a convenient and reliable representation of the contributions to ϕ calculated in this section.

4. - THE PHASE OF THE MODIFIED COULOMB AMPLITUDE

The inclusion of a form factor modifies the Born amplitude for Coulomb scattering by $f^2(q^2)$. In addition, it modifies the phase of the amplitude. No longer do we expect iterations to modify the Born term simply by a phase as is the case for the pure Coulomb amplitude. That is, the corrections to the expression in Eq. (27) will not simply be a phase. Nevertheless, the first corrections in an expansion in α can be so regarded, as we show below.

To determine the corrections to Eq. (27), we return to the eikonal model, Eqs. (10) and (11), and expand in an eikonal-Born series:

$$F_{eik}(q^2) = \frac{s}{4\pi i}\int d^2b\ e^{i\vec{q}\cdot\vec{b}}\left[e^{2i\delta(b)}-1\right] \tag{44}$$

$$= \frac{s}{4\pi i}\int d^2b\ e^{i\vec{q}\cdot\vec{b}}\left[2i\delta(b)-2\delta^2(b)+\ldots\right] \tag{45}$$

$$= F_{Born}(q^2) + \frac{i}{2\pi s}\int d^2q'\, F_{Born}(q'^2)\, F_{Born}([\vec{q}-\vec{q}']^2) \tag{46}$$

$$+ \ldots$$

As an example of the use of this formula, consider the pure Coulomb situation:

$$F_{Born}(q^2) = \frac{-\alpha s}{q^2+\lambda^2} \tag{47}$$

The second Borm term is

$$\frac{i}{2\pi s}(\alpha s)^2\int d^2q'\ \frac{1}{q'^2+\lambda^2}\ \frac{1}{(\vec{q}-\vec{q}')^2+\lambda^2}$$

$$= \frac{i}{2\pi s}\int_0^1 dx\int d^2q'\ \frac{(\alpha s)^2}{[q'^2-2\vec{q}\cdot\vec{q}'x+xq^2+\lambda^2]^2} \tag{48}$$

$$\simeq -\left(-\frac{\alpha s}{q^2}\right)(i\alpha)\ln q^2/\lambda^2 .$$

Thus to this order, we have for the eikonal amplitude,

$$F_{eik}(q^2) = -\frac{\alpha s}{q^2}\left(1+i\alpha\ln\lambda^2/q^2\right)$$

$$\cong -\frac{\alpha s}{q^2}\, e^{i\alpha\ln\lambda^2/q^2} \tag{49}$$

in agreement with Eq. (20).

Next we consider the Coulomb field with a form factor. For simplicity we shall take

$$F_{Born}(q^2) = \frac{-\alpha s}{q^2+\lambda^2}\left(\frac{L^2}{L^2+q^2}\right) , \tag{50}$$

which will give the correct low q^2 behaviour if we choose $L^2 = \Lambda^2/4$. The second Born term in the eikonal model can be evaluated using the standard Feynamn trick, albeit in two space dimensions:

$$F^{(2)}_{eik}(q^2) = \frac{i}{2\pi s}(\alpha s L^2)^2 \int d^2q' \frac{1}{[q'^2+\lambda^2][q'^2+L^2][(\vec{q}-\vec{q}\,')^2+\lambda^2][(\vec{q}-\vec{q}\,')^2+L^2]}$$

$$= \frac{i}{2\pi s}(\alpha s L^2)^2 \int_0^1 dz \int_0^{1-z} dy \int_0^{1-y-z} dx \int d^2q'\, 6\Big\{(1-x-y-z)(q'^2+\lambda^2) \tag{51}$$

$$+ x(q'^2+L^2) + y[(\vec{q}-\vec{q}\,')^2+\lambda^2] + z[(\vec{q}-\vec{q}\,')^2+L^2]\Big\}^{-4} .$$

Taking $\vec{p} = \vec{q}\,' - (y+z)\vec{q}$, completing the square and doing the $\vec{p}$ integration, we find

$$F^{(2)}_{eik}(q^2) = \frac{i}{2\pi s}(\alpha s L^2)^2 \pi \int_0^1 dz \int_0^{1-z} dy \int_0^{1-y-z} dx$$

$$\times 2\Big\{q^2[(y+z)-(y+z)^2] + \lambda^2(1-x-z) + \Lambda^2(x+z)\Big\}^{-3} . \tag{52}$$

The x integral is straightforward:

$$F^{(2)}_{eik}(q^2) = \frac{-i}{2s}(\alpha s L^2)^2 \int_0^1 dz \int_0^{1-z} dy \frac{1}{L^2-\lambda^2}\Big\{\Big[q^2([y+z]-[y+z]^2)+\lambda^2 y + L^2(1-y)\Big]^{-2} \tag{53}$$

$$- \Big[q^2([y+z]-[y+z]^2) + \lambda^2(1-z) + L^2 z\Big]^{-2}\Big\} .$$

Thus if we define

$$I(q^2,\lambda^2,L^2) = \frac{1}{L^2-\lambda^2}\int_0^1 dz\int_0^{1-z} dy\left\{q^2\left[(y+z)-(y+z)^2\right] + \lambda^2 y + L^2(1-y)\right\}^{-2}, \tag{54}$$

then

$$F^{(2)}_{eik}(q^2) = \frac{-i}{2s}(\alpha s L^2)^2\left[I(q^2,\lambda^2,L^2) + I(q^2,L^2,\lambda^2)\right]. \tag{55}$$

In Eq. (54), we set $t = y + z$. Then

$$\begin{aligned} I(q^2,\lambda^2,L^2) &= \frac{1}{L^2-\lambda^2}\int_0^1 dt\int_0^t dy\left\{q^2\left[t-t^2\right] + \lambda^2 y + L^2(1-y)\right\}^{-2} \\ &= \frac{-1}{L^2-\lambda^2}\int_0^1 dt\left\{\left[q^2(t-t^2)+L^2\right]^{-1}\right. \\ &\qquad \left. -\left[q^2(t-t^2)+\lambda^2 t+L^2(1-t)\right]^{-1}\right\}. \end{aligned} \tag{56}$$

Defining

$$J(q^2,\lambda^2,L^2) = \int_0^1 dt\left[q^2(t-t^2)+\lambda^2 t+L^2(1-t)\right]^{-1} \tag{57}$$

we have the relations

$$I(q^2,\lambda^2,L^2) = \frac{1}{(L^2-\lambda^2)^2}\left[J(q^2,\lambda^2,L^2) - J(q^2,L^2,L^2)\right] \tag{58}$$

and

$$\begin{aligned} F^{(2)}_{eik}(q^2) &= \frac{-i}{2s}(\alpha s L^2)^2\frac{1}{(L^2-\lambda^2)^2} \\ &\quad \times\left[J(q^2,\lambda^2,L^2) + J(q^2,L^2,\lambda^2) - J(q^2,\lambda^2,\lambda^2) - J(q^2,L^2,L^2)\right]. \end{aligned} \tag{59}$$

It is easy to evaluate $J(q^2,\lambda^2,L^2)$. The result is

$$J(q^2,\lambda^2,L^2) = \frac{1}{S(q^2,\lambda^2,L^2)} \ln \frac{[S(q^2,\lambda^2,L^2)+q^2]^2-(L^2-\lambda^2)^2}{[S(q^2,\lambda^2,L^2)-q^2]^2-(L^2-\lambda^2)^2} \tag{60}$$

where

$$\begin{aligned} S(q^2,\lambda^2,L^2) &= \sqrt{(q^2-L^2+\lambda^2)^2+4L^2q^2} \\ &= S(q^2,L^2,\lambda^2) \end{aligned} \tag{61}$$

In particular, $J(q^2,\lambda^2,L^2) = J(q^2,L^2,\lambda^2)$. Always $\lambda << L$, so

$$S(q^2,\lambda^2,L^2) \cong q^2+L^2 + \lambda^2\left(\frac{q^2-L^2}{q^2+L^2}\right) , \tag{62}$$

$$J(q^2,\lambda^2,L^2) \cong \frac{1}{q^2+L^2} \ln \frac{(q^2+L^2)^2}{L^2\lambda^2} . \tag{63}$$

On the other hand,

$$J(q^2,L^2,L^2) = \frac{2}{\sqrt{q^4+4q^2L^2}} \ln \frac{\left(\sqrt{q^4+4q^2L^2}+q^2\right)^2}{4L^2q^2} , \tag{64}$$

$$J(q^2,\lambda^2,\lambda^2) \cong \frac{2}{q^2} \ln \frac{q^2}{\lambda^2} . \tag{65}$$

We shall be interested in the phase of the modified Coulomb amplitude only for very small q^2 compared to L^2. Thus we have the approximations

$$J(q^2, \lambda^2, L^2) \cong \frac{1}{q^2+L^2} \ln q^2/\lambda^2 + \frac{1}{L^2} \ln \frac{L^2}{q^2} , \tag{66}$$

$$J(q^2, L^2, L^2) \cong \frac{1}{L^2} , \tag{67}$$

$$J(q^2, \lambda^2, \lambda^2) = \frac{2}{q^2} \ln \frac{q^2}{\lambda^2} . \tag{68}$$

We find

$$2J(q^2, \lambda^2, L^2) - J(q^2, \lambda^2, \lambda^2) - J(q^2, L^2, L^2)$$
$$\cong \frac{-2L^2}{q^2(L^2+q^2)} \left[\ln q^2/\lambda^2 - \frac{q^2}{L^2} \ln \frac{L^2}{q^2} - \frac{q^2}{2L^2} \right] . \tag{69}$$

Using Eq. (59) and adding the Born term we have the eikonal approximation through second order in α:

$$F_{eik}^{(1)+(2)}(q^2) = \left(-\frac{\alpha s}{q^2}\right)\left(\frac{L^2}{L^2+q^2}\right)\left[1 + i\alpha\left(\ln \lambda^2/q^2 + \frac{q^2}{L^2} \ln \frac{L^2}{q^2} - \frac{q^2}{2L^2}\right)\right] . \tag{70}$$

In addition to the usual Coulomb phase of Eq. (20), there appears a phase, ν, which vanishes in the limit of a point charge:

$$\nu = \frac{q^2}{L^2} \ln \frac{L^2}{q^2} - \frac{q^2}{2L^2} . \tag{71}$$

5. - DISCUSSION

The full result for the phase is

$$\phi_{TOT} = -\left(\gamma + \ln \frac{Bq^2}{2} + \ln\left(1+\frac{8}{B\Lambda^2}\right)\right)$$
$$+ \frac{4q^2}{\Lambda^2} \ln \frac{\Lambda^2}{4q^2} - \frac{2q^2}{\Lambda^2} \tag{72}$$

obtained from Eqs. (40) and (71), using the correspondence $L^2 = \Lambda^2/4$. The phase as a function of q^2 is shown in Fig. 1. The most important shift from the West-Yennie result, Eqs. (40) and (41), is the shift of the Coulomb amplitude itself due to the form factor's influence on the phase, Eq. (71). The contribution of ν to the phase is shown in Fig. 2.

From a rather different perspective we have rederived the basic result of West and Yennie, Eqs. (7) and (26). While the approaches are technically quite different, it is clear they are similar in spirit and ignore some of the same contributions which would be included in a more complete treatment. In particular, it would be more satisfactory if some diffractive dissociation of the nucleon were considered, rather than just elastic scattering. As a result, we cannot disagree with the conclusion of West and Yennie that there are significant theoretical uncertainties in the result. It is reassuring, however, to see that some of these, like the phase of the Coulomb amplitude with a form factor, are not large.

The treatment of the electromagnetic form factor presented here is more convincing than that of West and Yennie. It led also to an especially concise result, Eq. (35). Nevertheless, the modification of the previous result is very small.

The eikonal model is a very physical and very convenient basis for analyzing the Coulomb-nuclear interference. It might well provide an appropriate starting point for analyzing those effects which have not been included in the present treatment or in the treatment of West and Yennie.

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APPENDIX

The evaluation of the phase ϕ for a dipole form factor, Eq. (38) and a Gaussian hadronic amplitude, proceeds as follows. In accordance with Eq. (36), we first consider the approximation

$$\bar{F}^N(q'^2, q^2) = e^{-Bq'^2/2} . \tag{A.1}$$

Using Eq. (35) and defining the dimensionless variable $\alpha = q^2/\Lambda^2$, $\beta = B\Lambda^2/2$, and $z = q'^2/\Lambda^2$, we can write

$$\phi = -\int_0^\infty dz \, \ln z/\alpha \, \frac{d}{dz}\left[e^{-\beta z}(1+z)^{-4}\right] . \tag{A.2}$$

If we differentiate with respect to β, the result can be integrated by parts to give

$$\frac{\partial\phi}{\partial\beta} = -e^{\beta} E_4(\beta) , \tag{A.3}$$

where[10)]

$$E_n(z) = \int_1^\infty dt \, \frac{e^{-zt}}{t^n} . \tag{A.4}$$

Now as $\beta \to \infty$

$$\begin{aligned} \phi &\to -\int_0^\infty dz \, \ln z/\alpha \, \frac{d}{dz} e^{-\beta z} \\ &= -\gamma - \ln \alpha\beta \end{aligned} \tag{A.5}$$

where γ is the Euler constant, $\gamma = 0.577...$ Now ϕ is determined by finding the indefinite integral

$$-\int^{\beta} dz \, e^{z} E_4(z) \tag{A.6}$$

and adjusting the β independent constant to agree with Eq. (A.5). From the definition Eq. (A.4) it is easy to derive the relations[11]

$$\frac{dE_n}{dz} = -E_{n-1}(z) , \qquad n>1 \tag{A.7}$$

$$\frac{d[E_1(z)e^z]}{dz} = E_1(z)e^z - \frac{1}{z} , \tag{A.8}$$

$$E_n(z) = \frac{1}{n-1}\left[e^{-z} - z E_{n-1}(z)\right] , \qquad n>1 . \tag{A.9}$$

Using Eqs. (A.7) and (A.8), we find

$$-\int^{\beta} dz\, e^z E_4(z) = -e^{\beta}\left[E_4(\beta) + E_3(\beta) + E_2(\beta) + E_1(\beta)\right] - \ln\beta + C(\alpha). \tag{A.10}$$

Repeated application of Eq. (A.9) yields

$$\begin{aligned} -\int^{\beta} dz\, e^z E_4(z) = &- \left[1-\beta + \beta^2/2 - \beta^3/6\right] e^{\beta} E_1(\beta) \\ &- \frac{11}{6} + \frac{2\beta}{3} - \frac{\beta^2}{6} - \ln\beta + C(\alpha) . \end{aligned} \tag{A.11}$$

Using the large argument asymptotic expansion[12]

$$E_1(z) \rightarrow \frac{e^{-z}}{z}\left[1 - \frac{1}{z} + \frac{2!}{z^2} - \frac{3!}{z^3} + \cdots\right] , \tag{A.12}$$

the right-hand side of Eq. (A.11) becomes

$$-\ln\beta + C(\alpha) - \frac{4}{\beta} + \frac{10}{\beta^2} - \frac{40}{\beta^3} + \cdots \tag{A.13}$$

Thus in accordance with Eq. (A.5),

$$C(\alpha) = -\gamma - \ln\alpha ,$$

so altogether

$$\phi = -\gamma - \ln \alpha\beta - \left\{ \left[1 - \beta + \frac{\beta^2}{2} - \frac{\beta^3}{6}\right] e^{\beta} E_1(\beta) + \frac{11}{6} - \frac{2\beta}{3} + \frac{\beta^2}{6} \right\}. \tag{A.14}$$

This result can be compared to that obtained with an exponential parametrization of the form factor, Eq. (40), by expanding. The dipole form gives

$$\phi \xrightarrow{\Lambda\to\infty} -\left(\gamma + \ln \frac{Bq^2}{2} + \frac{8}{B\Lambda^2} - \frac{40}{B^2\Lambda^4} + \cdots\right) \tag{A.15}$$

while the exponential yields

$$\phi \xrightarrow{\Lambda\to\infty} -\left(\gamma + \ln \frac{Bq^2}{2} + \frac{8}{B\Lambda^2} - \frac{32}{B^2\Lambda^4} + \cdots\right) . \tag{A.16}$$

In fact, these expansions are not very good for the values of interest since $B\Lambda^2 \approx 10$.

The next term in the expansion of $\bar{F}^N(q'^2,q^2)$, Eqs. (36) and (37), gives a contribution which is easier to evaluate:

$$\delta\phi = -\int_0^\infty dz \ \ln z/\alpha \ \frac{d}{dz}\left[\beta\alpha z e^{-\beta z}(1+z)^{-4}\right] \tag{A.17}$$

$$= \int_0^\infty dz \ \beta\alpha \ e^{-\beta z}(1+z)^{-4}$$

$$= \beta\alpha \ e^{\beta} E_4(\beta) . \tag{A.18}$$

Using the standard values for B and Λ^2, this becomes

$$\delta\phi \approx \frac{1}{2} Bq^2 \times (0.12) , \tag{A.19}$$

a negligible correction.

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FIGURE CAPTIONS

Fig. 1 : The phase ϕ_{TOT}, in radians, calculated from Eq. (72) as a function of q^2 in GeV^2. The values $B = 13\ GeV^{-2}$ and $\Lambda^2 = 0.71\ GeV^2$ are used. The solid line represents the full result. The dashed line represents the result without the effect of the form factor on the phase of the Coulombic amplitude, that is, without the contribution of Eq. (71).

Fig. 2 : The contribution $\nu(g^2)$, in radians, to the total phase $\phi_{TOT} = \phi + \nu$. The function $\nu(g^2)$ arises from the influence of the form factor on the phase of electromagnetic scattering.

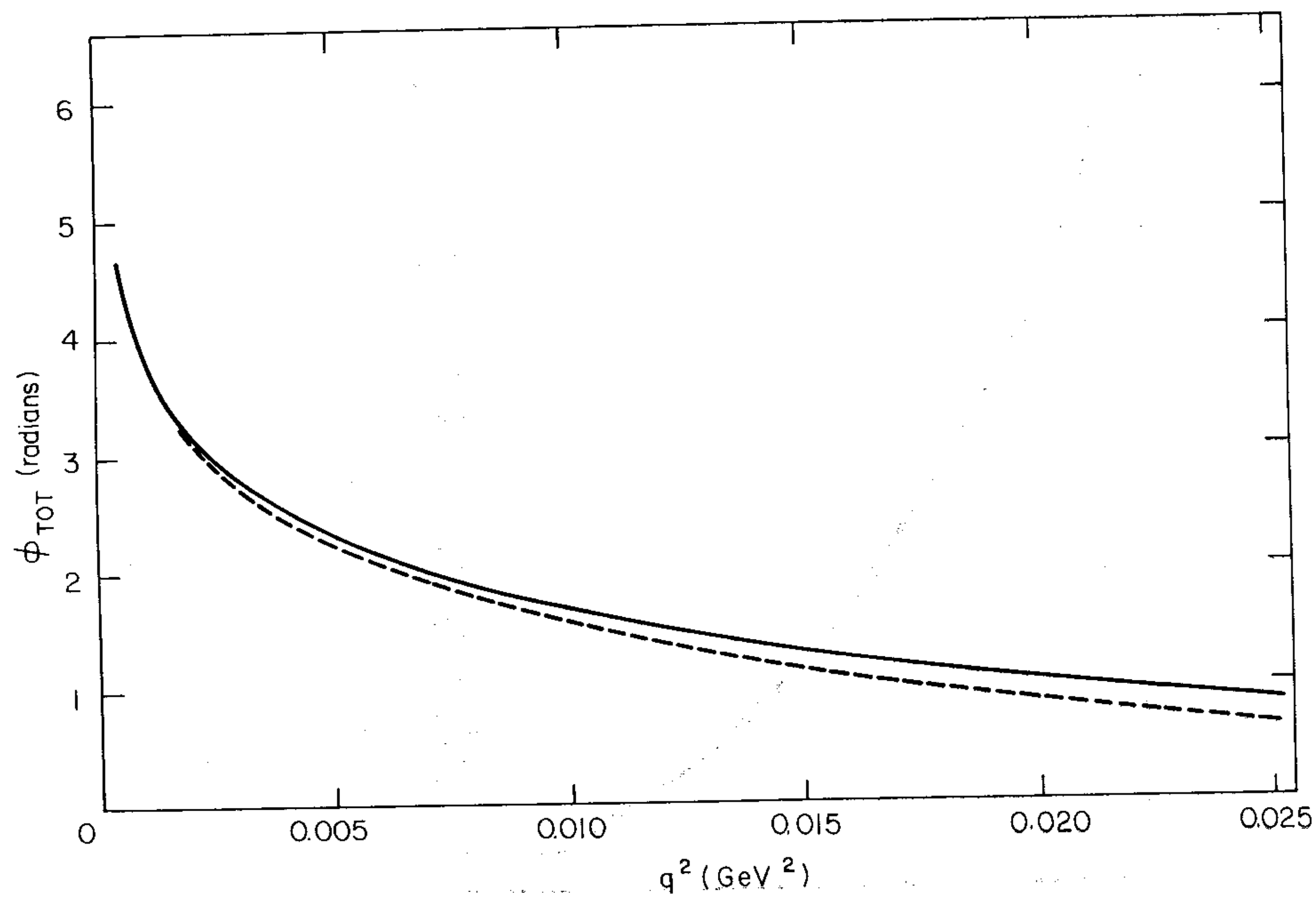

Fig. 1

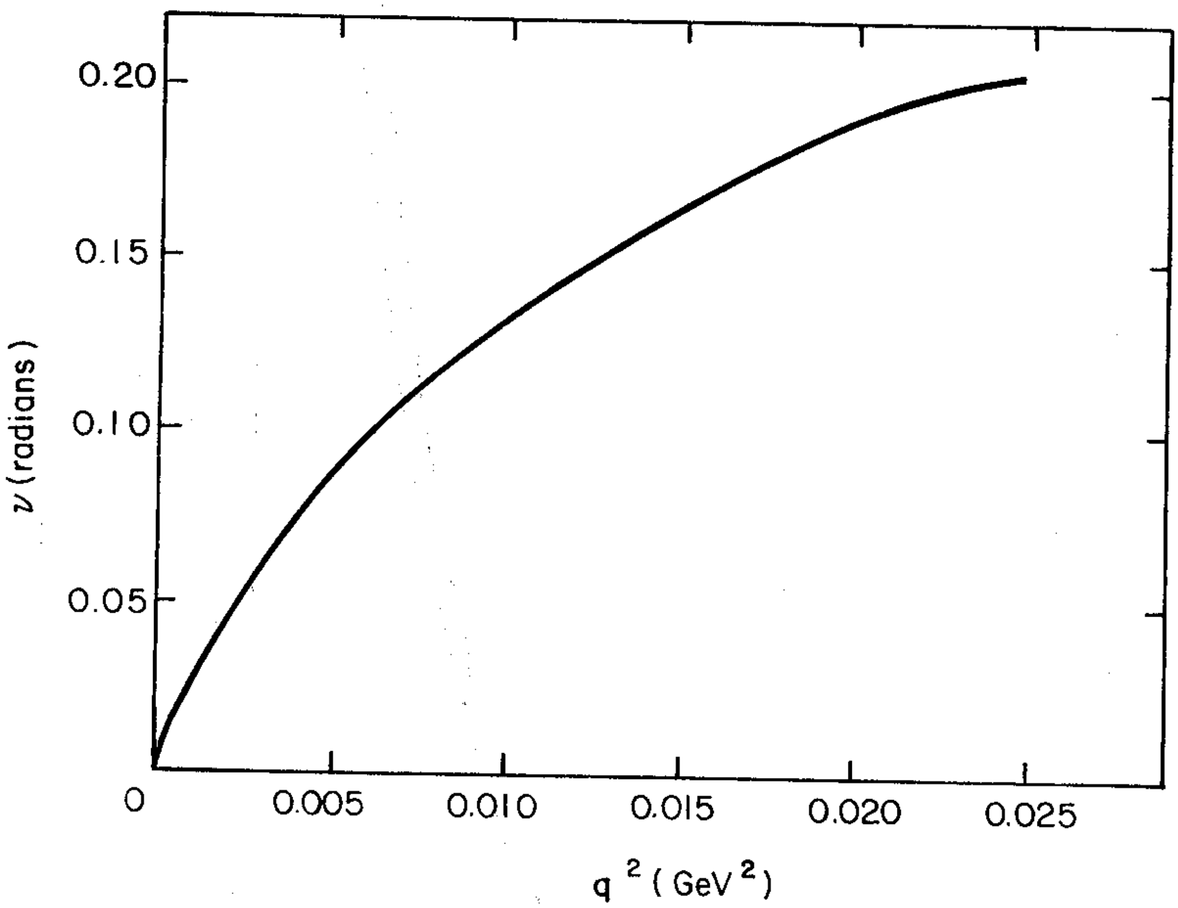

Fig. 2