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COUNTABLE DIMENSIONAL UNIVERSAL SETS

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ABSTRACT. The main results of this paper are a construction of a countable union of zero dimensional sets in the Hilbert cube whose complement does not contain any subset of finite dimension $n \geqslant 1$ (Theorem 2.1, Corollary 2.3) and a construction of universal sets for the transfinite extension of the Menger-Urysohn inductive dimension (Theorem 2.2, Corollary 2.4).

1. Terminology and notation. All spaces considered in this paper are metrizable and separable. Our terminology follows Kuratowski [**Ku**] and Nagata [**Na 2**].

1.1. *Notation.* We denote by I the interval $[-1,1]$, I^∞ is the countable product of I, i.e. the Hilbert cube, p_i: $I^\infty \to I$ is the projection onto the ith coordinate, P is the set of the irrationals from I and ω is the set of natural numbers. Given a point $t \in I$, we let $Q_t = \{(x_1, x_2, \dots) \in I^\infty \colon x_1 = t\}$.

1.2. *Partitions.* A partition in a space X between a pair of disjoint sets A and B is a closed set L such that $X \setminus L = U \cup V$, where U and V are disjoint open sets with $A \subset U$ and $B \subset V$.

1.3. *Countable dimensional spaces and the transfinite inductive dimension* ind. A space X is countable dimensional if it is a countable union $X = \bigcup_{i=1}^{\infty} X_i$ of zero dimensional sets X_i [**Hu**].

The transfinite dimension ind is the extension by transfinite induction of the classical Menger-Urysohn inductive dimension: $\operatorname{ind} X = -1$ means $X = \varnothing$, $\operatorname{ind} X \leqslant \alpha$ if and only if each point x in X can be separated in X from any closed set not containing x by a partition L with $\operatorname{ind} L < \alpha$, α being an ordinal, we let $\operatorname{ind} X$ be the smallest ordinal α with $\operatorname{ind} X \leqslant \alpha$ if such an ordinal exists, and we put $\operatorname{ind} X = \infty$ otherwise. If $\operatorname{ind} X \neq \infty$, then $\operatorname{ind} X$ is a countable ordinal, X having a countable base.

The transfinite dimension ind was first discussed by Hurewicz [**Hu**, §5], [**H-W**, p. 50] (although the idea goes back to Urysohn's memoir [**Ur**, p. 66]). A comprehensive survey of the topic is given by Engelking [**En** 2].

Hurewicz [**Hu, En 2**, 4.1, 4.15] proved that *for a complete space X,* $\operatorname{ind} X \neq \infty$ *if and only if X is countable dimensional and that each space X with* $\operatorname{ind} X \neq \infty$ *has a countable dimensional compactification.*

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1.4. *Hereditarily infinite dimensional spaces.* We say that an infinite dimensional space X is hereditarily infinite dimensional (hereditarily uncountable dimensional) if each nonempty subset of X is either zero dimensional or infinite dimensional (uncountable dimensional), cf. [**G-S**, **Wa**, **Tu**].

2. Introduction. The following two theorems are main results of this paper.

2.1. THEOREM. *There exists a countable dimensional set C in the Hilbert cube I^∞ such that for each countable dimensional subset A of I^∞ the difference $A \setminus C$ is at most zero dimensional.*

2.2. THEOREM. *For each countable ordinal α there exists a G_δ-set E_α in I^∞ with transfinite inductive dimension* $\operatorname{ind} E_\alpha = \alpha$ *such that for every G_δ-set G in I^∞ with* $\operatorname{ind} G \leqslant \alpha$ *there is an irrational $t \in I$ for which $G \cap Q_t = E_\alpha \cap Q_t$, where $Q_t = \{(x_1, x_2, \ldots) \in I^\infty \colon x_1 = t\}$.*

Since each separable metrizable space embeds in I^∞, Theorem 2.1 yields the following corollary.

2.3. COROLLARY. *Each uncountable dimensional separable metrizable space contains a countable dimensional subset with hereditarily uncountable dimensional complement.*

Each subset of I^∞ can be enlarged to a G_δ-set in I^∞ with the same transfinite dimension ind [**En 2**, 5.5] and hence the sets E_α in Theorem 2.2 have the following property:

2.4. COROLLARY. *For each countable ordinal α, every separable metrizable space X with* $\operatorname{ind} X \leqslant \alpha$ *can be embedded homeomorphically into the space E_α.*

Therefore, E_α is a universal space in the class of separable metrizable spaces with transfinite dimension $\operatorname{ind} \leqslant \alpha$. The question about the existence of such universal spaces for $\alpha \geqslant \omega_0$ was asked by Engelking [**En 2**, Problem 5.11], cf. also Luxemburg [**Lu 2**, Problem 8.4].

Using the zero dimensional set E_0 described in Theorem 2.2 one obtains the following fact (cf. §4.2):

2.5. COROLLARY. *There exists a countable dimensional $G_{\delta\sigma}$-set E_∞ in I^∞ such that for every countable dimensional $G_{\delta\sigma}$-set G in I^∞ there is an irrational $t \in I$ with $G \cap Q_t = E_\infty \cap Q_t$.*

The results of this paper are based implicitly on a notion of "universal functions" for a given collection of sets and some related diagonal arguments; these ideas go back to the origins of descriptive set theory, cf. Moschovakis [**Mo**, Remark 15 on p. 63], Kuratowski [**Ku**, §30, XIII]. Certain "universal functions" for the collections of compacta in I^∞ with transfinite dimension $\operatorname{ind} \leqslant \alpha$ have been considered in [**Po 2**, §4] and a significant part of the present paper is a modification and an extension of the methods from [**Po 2**, §4].

Theorem 2.1 can be proved by making use of the zero dimensional universal set E_0 described in Theorem 2.2 (cf. Remark 4.2.1). We give, however, an independent direct proof of this theorem (though based on the same ideas as the construction of the sets E_α in Theorem 2.2). This proof also provides a simple construction of sets with properties only slightly weaker than those of $E_0, E_1, \ldots$ (for finite α) and the set E_∞ in Corollary 2.5.

The paper is organized as follows.

In §3.1 we construct a "universal sequence" of sets for zero dimensional sets in I^∞ and in §3.2 we intensify certain singular properties of these sets, following an idea from Walsh [**Wa**], to obtain the countable dimensional set C described in Theorem 2.1.

In §4.1 we construct "universal functions" $M_\alpha \subset P \times I^\infty$, in the product of the irrationals and the Hilbert cube, for the collection of G_δ-sets in I^∞ with transfinite dimension ind $\leqslant \alpha$ and in §4.2 we apply a standard diagonal construction to get from these M_α's the sets E_α described in Theorem 2.2.

§5 is a slight departure from the main subject of this paper (and it is formally independent of the other sections). We define here, by a method similar to that in §3.1, a "universal sequence" of partitions between the opposite faces in I^∞, and we use these partitions along a path outlined by Walsh [**Wa**, §§3 and 7] to obtain a rather unexpectedly simple construction of hereditarily infinite dimensional compacta.

In §6 we collect some comments related to the subject of this paper.

I would like to thank Henryk Toruńczyk for pointing out a direct argument used in the proofs of property (I) in §3.1 and Lemma 4.1.3(ii), which simplified my original proofs.

3. A countable union of 0-dimensional sets in I^∞ whose complement has no subsets of dimension $n \geqslant 1$. In this section we give a proof of Theorem 2.1.

3.1. *A universal sequence $N_1, N_2, \ldots$ for 0-dimensional sets in I^∞.* Let T be an arbitrary set in I homeomorphic to the irrationals P, let Γ be the space of all homeomorphic embeddings $h\colon I^\infty \to I^\infty$ of the Hilbert cube into itself endowed with the topology of uniform convergence and let

$$u = (u_1, u_2, \ldots)\colon T \to \Gamma \times \Gamma \times \cdots$$

be a continuous map of the set T onto the countable product of the completely metrizable separable space Γ, cf. [**Ku**, §36, II].

For each $i = 1, 2, \ldots$ we let

(1) $N_i = \{(x_1, x_2, \ldots) \in I^\infty\colon x_1 \in T \text{ and } u_i(x_1)(x_1, x_2, \ldots) \in P \times P \times \cdots\}$.

We shall verify that the sets N_i have the following two properties, where for each $t \in I$,

$$Q_t = \{(x_1, x_2, \ldots) \in I^\infty\colon x_1 = t\}$$

(cf. §4.2(I) and (II)):

(I) *The sets N_i are zero dimensional.*

(II) *Given an arbitrary sequence $G_1, G_2, \ldots$ of zero dimensional sets in I^∞ there exists $t \in T$ such that $G_i \cap Q_t \subset N_i \cap Q_t$ for each $i = 1, 2, \ldots$.*

Proof of (I). Let $Q_T = \{(x_1, x_2, \ldots) \in I^\infty\colon x_1 \in T\}$ and let, for each $i \in \omega$, a continuous map $f_i\colon Q_T \to T \times I^\infty$ be defined by

$$f_i(x_1, x_2, \ldots) = (x_1, u_i(x_1)(x_1, x_2, \ldots)).$$

The map f_i is closed (the projection $Q_T \to T$ being parallel to a compact factor, see [**Bo**, Chapter I, §§10, 1 and 2]) and injective (the maps $u_i(t)$ being embeddings), and hence f_i embeds Q_T homeomorphically into the product $T \times I^\infty$. Property (I) follows now from the fact that f_i embeds N_i into the zero dimensional space $T \times (P \times P \times \cdots)$.

PROOF OF (II). We shall use the following universal property of the product of the irrationals $P \times P \times \cdots$ established by Nagata [**Na 1**, **Na 2**, VI.2.A] (a simple proof is given in §6.4):

3.1.1. LEMMA (NAGATA). *For each zero dimensional set G in a metrizable separable space X there exists a homeomorphic embedding h: $X \to I^\infty$ such that $h(G) \subset P \times P \times \cdots$.*

Let $G_1, G_2, \ldots$ be an arbitrary sequence of zero dimensional sets in I^∞ and, for each $i \in \omega$, let h_i: $I^\infty \to I^\infty$ be an embedding such that

$$h_i(G_i) \subset P \times P \times \cdots. \tag{2}$$

Let us choose a $t \in T$ such that

$$u(t) = (u_1(t), u_2(t), \ldots) = (h_1, h_2, \ldots). \tag{3}$$

Then, for each $i \in \omega$, we have (see (2), (3), (1))

$$\begin{aligned} G_i \cap Q_t &\subset \{(t, x_2, x_3, \ldots): h_i(t, x_2, x_3, \ldots) \in P \times P \times \cdots\} \\ &= \{(t, x_2, x_3, \ldots): u_i(t)(t, x_2, x_3, \ldots) \in P \times P \times \cdots\} = N_i \cap Q_t, \end{aligned}$$

which proves property (II).

We close this section with an observation that property (II) yields, by a simple diagonal argument, the following property of the union

$$N_\infty = \bigcup_{i=1}^{\infty} N_i. \tag{4}$$

(III) *If A is a subset of I^∞ disjoint from N_∞ whose projection onto the first coordinate contains the set T, then A is uncountable dimensional.*

Assume on the contrary that $A = \bigcup_{i=1}^\infty G_i$, where the sets G_i are zero dimensional. By property (II) there exists a $t \in T$ such that $G_i \cap Q_t \subset N_i \cap Q_t$ for all $i \in \omega$ and hence $\varnothing \neq A \cap Q_t = \bigcup_{i=1}^\infty G_i \cap Q_t \subset N_\infty \cap Q_t$, contradicting the fact that A was disjoint from N_∞.

3.2. PROOF OF THEOREM 2.1. Reasoning in this section follows some ideas of Walsh [**Wa**].

Let $T_1, T_2, \ldots$ be a sequence of topological copies of the irrationals in I, with pairwise disjoint closures in I, such that each nondegenerate interval in I contains some T_i.

For each $i \in \omega$, the construction described in §3.1 with $T = T_i$ yields a countable dimensional set $N_\infty^{(i)}$ in I^∞ such that (see property (III)) each subset in I^∞ disjoint from $N_\infty^{(i)}$ whose projection onto the first coordinate contains T_i is uncountable dimensional. Therefore, the countable dimensional set

$$C' = N_\infty^{(1)} \cup N_\infty^{(2)} \cup \cdots$$

has the following property:

(IV) *If A is a countable dimensional set in I^∞ disjoint from C' then the projection of A onto the first coordinate does not contain any nondegenerate interval in I, i.e. the projection is zero dimensional.*

Let π_i: $I^\infty \to I^\infty$ be the permutation of the coordinates interchanging the first coordinate with the ith one and let $C_i = \pi_i(C')$ for $i \in \omega$. Each countable dimensional set C_i has then the property analogous to that of C' stated in (IV), where the first coordinate is replaced by the ith one. The countable dimensional set $C = C_1 \cup C_2 \cup \cdots$ satisfies the assertion of Theorem 2.1: if A is a nonempty countable dimensional set in I^∞ disjoint from C then, A being disjoint from each C_i, for every $i \in \omega$ the projection $p_i(A)$ of A onto the ith coordinate is zero dimensional and so is the set $A \subset \prod_{i=1}^\infty p_i(A)$, cf. Walsh [**Wa**, §3].

4. Universal sets for the transfinite extension of the inductive Menger-Urysohn dimension. In this section we construct the sets E_α described in Theorem 2.2 and we prove Corollary 2.5.

4.1. *Universal functions M_α.* Given a set S in the product $P \times I^\infty$ of the irrationals P and the Hilbert cube I^∞, for each $t \in P$, we let

$$(1) \qquad S(t) = \{x \in I^\infty \colon (t, x) \in S\}.$$

4.1.1. Proposition. *For each countable ordinal α there exists a G_δ-set M_α in $P \times I^\infty$ such that*

(i) $\operatorname{ind} M_\alpha = \alpha$,

(ii) *for each G_δ-set G in I^∞ with* $\operatorname{ind} G \leqslant \alpha$ *there is a $t \in P$ with $M_\alpha(t) = G$.*

Proof. Let p_i: $I^\infty \to I$ be the projection onto the ith coordinate and let

$$(2) \qquad C_i = p_i^{-1}(-1), \quad D_i = p_i^{-1}(1), \quad H_i = p_i^{-1}(0),$$

i.e. H_i is a partition between the pair C_i and D_i of the ith opposite faces in I^∞. Let

$$(3) \qquad Z = I^\infty \setminus \bigcup\{C_{2i-1} \cup D_{2i-1} \colon i = 1, 2, \ldots\}.$$

We shall construct the sets M_α by transfinite induction. Let $M_{-1} = \varnothing$ and let us assume that for some ordinal α the sets M_β with $\beta < \alpha$ have been already constructed. We shall define the set M_α.

(I) Let us split the set of even natural numbers into disjoint infinite sets Σ_{-1}, Σ_0, $\Sigma_1, \ldots, \Sigma_\beta, \ldots, \beta < \alpha$, and let Γ be the space of all homeomorphic embeddings h: $I^\infty \to I^\infty$ satisfying the following two conditions

($*$) *if $x \in h^{-1}(Z)$ and F is a closed set in I^∞ not containing x, then $h(x) \in C_{2i}$ and $h(F) \subset D_{2i}$ for some $i \in \omega$;*

($**$) *if $j \in \Sigma_\beta$ then* $\operatorname{ind} h^{-1}(Z \cap H_j) \leqslant \beta$.

We shall consider Γ with the topology of uniform convergence.

4.1.2. LEMMA. *For each G_δ-set G in I^∞ with* $\operatorname{ind} G \leqslant \alpha$ *there exists an embedding* $h \in \Gamma$ *such that* $G = h^{-1}(Z)$.

PROOF. Let $\mathscr{B}$ be a countable base in I^∞. Let us consider the collection of all pairs (A, B) of disjoint closed sets in I^∞, each of which being a finite sum of the closures of the elements of $\mathscr{B}$ such that there exists a partition L in I^∞ between A and B with $\operatorname{ind}(L \cap G) = \gamma < \alpha$, let $\gamma(A, B)$ be the minimal such ordinal γ for the pair (A, B), and finally, let us arrange this collection of pairs into a sequence $(A_1, B_1), (A_2, B_2), \ldots$ letting $\gamma(i) = \gamma(A_i, B_i)$. Choose an injection $\tau: \omega \to \omega$ such that $\tau(i) \in \Sigma_{\gamma(i)}$ and let $f_i: I^\infty \to I$ be continuous maps with

$$(4) \qquad f_i^{-1}(-1) = A_i, \qquad f_i^{-1}(1) = B_i,$$

$$(5) \qquad \operatorname{ind}\big(G \cap f_i^{-1}(0)\big) = \gamma(i).$$

Let $I^\infty \setminus G = X_1 \cup X_2 \cup \cdots$, where X_i are compact sets and let $g_i: I^\infty \to [0,1]$ be continuous maps with $g_i^{-1}(1) = X_i$. Let us finally split the odd natural numbers into two disjoint infinite sets Σ' and Σ'' and let us choose bijections $\nu: \Sigma' \to \omega$ and $\mu: \Sigma'' \to \omega$.

An embedding $h: I^\infty \to I^\infty$ with required properties can be defined now by $h(x_1, x_2, \ldots) = (y_1, y_2, \ldots)$ where

$$(6) \qquad y_j = \begin{cases} g_{\nu(j)}(x_1, x_2, \ldots), & \text{if } j \in \Sigma', \\ \frac{1}{2} x_{\mu(j)}, & \text{if } j \in \Sigma'', \\ f_i(x_1, x_2, \ldots), & \text{if } j = \tau(i), \\ 1 & \text{if } j \notin \Sigma' \cup \Sigma'' \cup \tau(\omega). \end{cases}$$

The first two formulas in (6) guarantee that h is an embedding and $G = h^{-1}(Z)$. Given an $x \in G$ and a closed set F in I^∞ not containing x, the assumption $\operatorname{ind} G \leqslant \alpha$ yields the existence of a pair (A_i, B_i) such that $x \in A_i$ and $F \subset B_i$ (cf. [**En 1**, Lemma 1.2.9]). Then (see (2), (4), (5) and (6)), $x \in h^{-1}(C_{\tau(i)})$, $F \subset h^{-1}(D_{\tau(i)})$ and $\operatorname{ind} h^{-1}(Z \cap H_{\tau(i)}) = \operatorname{ind}(G \cap f_i^{-1}(0)) = \gamma(i)$. Therefore h satisfies condition (*) and condition (**) holds for all $j \in \tau(\omega)$ (recall that $\tau(i) \in \Sigma_{\gamma(i)}$). But if j is an even number not belonging to $\tau(\omega)$, the last formula in (6) shows that $h(I^\infty) \cap H_j = \varnothing$ and so (**) is satisfied also in that case.

(II) For each even number $2i$ let $\beta(i)$ be the ordinal such that $2i \in \Sigma_{\beta(i)}$ (see (I)). Let us consider an embedding $h \in \Gamma$, where Γ is the space defined in (I). For every even number $2i$ property (**) and universality of the set $M_{\beta(i)}$ yield the existence of an irrational $t_i \in P$ such that

$$(7) \qquad h^{-1}(Z \cap H_{2i}) = M_{\beta(i)}(t_i).$$

Let $\Lambda \subset \Gamma \times P \times P \times \ldots$ be the space of all sequences $(h, t_1, t_2, \ldots)$ such that for each $i \in \omega$ the pair (h, t_i) satisfies condition (7). The space Λ being metrizable and separable, there exists a subset S of the irrationals in I and a continuous map $u = (u_0, u_1, u_2, \ldots): S \to \Lambda$ onto Λ (cf. [**Ku**, §36, III]). Let us define a continuous map $k: S \times I^\infty \to I^\infty$ by $k(s, x) = u_0(s)(x)$ and let

$$(8) \qquad M = \{(s, x) \in S \times I^\infty: u_0(s)(x) \in Z\} = k^{-1}(Z),$$

$$(9) \qquad L_i = \{(s, x) \in S \times I^\infty: u_0(s)(x) \in H_{2i}\} = k^{-1}(H_{2i}).$$

4.1.3. LEMMA. *The set M and the sets L_i have the following properties*:

(i) *for each G_δ-set G in I^∞ with* $\operatorname{ind} G \leqslant \alpha$ *there exists an $s \in S$ such that* $M(s) = G$,

(ii) $\operatorname{ind}(L_i \cap M) \leqslant \beta(i)$, $i \in \omega$,

(iii) $\operatorname{ind} M = \alpha$.

PROOF. (i) Let G be a G_δ-set in I^∞ with $\operatorname{ind} G \leqslant \alpha$. By Lemma 4.1.2 there exists an embedding $h \in \Gamma$ such that $G = h^{-1}(Z)$ and since $u_0(S) = \Gamma$, $h = u_0(s)$ for some $s \in S$. Then (see (8)) $M(s) = u_0(s)^{-1}(Z) = G$.

(ii) By (7), for each $s \in S$ and $i \in \omega$, we have

$$u_0(s)^{-1}(Z \cap H_{2i}) = M_{\beta(i)}(u_i(s)),$$

i.e. (see (8) and (9))

$$(M \cap L_i)(s) = M_{\beta(i)}(u_i(s)). \tag{10}$$

For each $i \in \omega$ define a continuous map g_i: $S \times I^\infty \to S \times P \times I^\infty$ by

$$g_i(s, x) = (s, u_i(s), x).$$

The map g_i homeomorphically embeds the set $M \cap L_i$ into the product $S \times M_{\beta(i)}$ (see (10)) and therefore $\operatorname{ind}(M \cap L_i) \leqslant \operatorname{ind} M_{\beta(i)} = \beta(i)$, the space S being zero dimensional.

(iii) Let $(s, x) \in M$ and let F be a closed set in $S \times I^\infty$ not containing (s, x). Since $u_0(s) \in \Gamma$, property $(*)$ in (I) and (8) yield the existence of an $i \in \omega$ such that $u_0(s)(x) \in C_{2i}$ and $u_0(s)(F(s)) \subset D_{2i}$. Therefore the set L_i separates in $S \times I^\infty$ the point (s, x) from the closed set $F \cap (\{s\} \times I^\infty)$ (see (9)). The projection $S \times I^\infty \to S$ parallel to the compact factor being closed, there exists an open and closed neighborhood W of s in S such that $L_i \cap (W \times I^\infty)$ is a partition in $S \times I^\infty$ between the point (s, x) and the set F. Since, by (ii), $\operatorname{ind}(M \cap L_i) < \alpha$, this shows that $\operatorname{ind} M \leqslant \alpha$ and it completes the proof, as $\operatorname{ind} M \geqslant \alpha$, by (i).

(III) It remains to modify slightly the set M constructed in (II) to obtain a G_δ-set in $P \times I^\infty$ satisfying conditions (i) and (ii) in Lemma 4.1.3.

By (8), M is a G_δ-set in $S \times I^\infty \subset P \times I^\infty$ and therefore there exists a G_δ-set M^* in $P \times I^\infty$ such that $\operatorname{ind} M^* = \operatorname{ind} M$ and $M^* \cap (S \times I^\infty) = M$, cf. [**En 2**, 5.5]. Let S^* be the projection of the set M^* onto the P-coordinate, let w: $P \to S^*$ be a continuous map onto S^*, cf. [**Ku**, §37, I], and let

$$M_\alpha = \{(t, x) \in P \times I^\infty : (w(t), x) \in M^*\}.$$

The set M_α is a G_δ-set in $P \times I^\infty$ (being the preimage of the set M^* under the map $(t, x) \to (w(t), x)$) and for each G_δ-set G in I^∞ with $\operatorname{ind} G \leqslant \alpha$, there exists an irrational $t \in P$ with $M_\alpha(t) = G$ (as $G = M(s)$ for some $s \in S$ and $s = w(t)$ for some $t \in P$). Finally, $\operatorname{ind} M_\alpha \leqslant \alpha$, since the map $(t, x) \to (t, w(t), x)$ embeds homeomorphically the set M_α into the product $P \times M^*$ (cf. the proof of Lemma 4.1.3(ii)).

This completes the inductive proof of Proposition 4.1.1.

4.2. *Diagonal constructions related to the universal functions M_α.* For each countable ordinal α, let $M_\alpha \subset P \times I^\infty$ be the universal function constructed in §4.1 and let

$$(11) \qquad E_\alpha = \{(x_1, x_2, \dots) \in I^\infty : (x_1, (x_1, x_2, \dots)) \in M_\alpha\}.$$

Given a point $t \in I$, we put

$$Q_t = \{(x_1, x_2, \dots) \in I^\infty : x_1 = t\}.$$

Let G be an arbitrary G_δ-set in I^∞ with $\operatorname{ind} G \leqslant \alpha$. By Proposition 4.1.1, there exists an irrational $t \in P$ such that $M_\alpha(t) = G$ and hence

$$\begin{aligned} E_\alpha \cap Q_t &= \{(x_1, x_2, \dots) : x_1 = t \text{ and } (x_1, (x_1, x_2, \dots)) \in M_\alpha\} \\ &= \{(x_1, x_2, \dots) : x_1 = t \text{ and } (x_1, x_2, \dots) \in M_\alpha(t)\} \\ &= \{(x_1, x_2, \dots) : x_1 = t \text{ and } (x_1, x_2, \dots) \in G\} \\ &= G \cap Q_t. \end{aligned}$$

Moreover, Proposition 4.1.1 shows also that E_α is a G_δ-set in I^∞ with $\operatorname{ind} E_\alpha = \alpha$. Therefore the sets E_α satisfy the assertions of Theorem 2.2.

Let us construct now the set E_∞ described in Corollary 2.5.

Let $u = (u_1, u_2, \dots) : P \to P \times P \times \cdots$ be a continuous map of the irrationals onto its countable product and let

$$(12) \quad N_i^* = \{(x_1, x_2, \dots) \in I^\infty : x_1 \in P \text{ and } (u_i(x_1), (x_1, x_2, \dots)) \in M_0\},$$

where $M_0 \subset P \times I^\infty$ is the universal function for zero dimensional sets in I^∞. Let us verify that the sets N_i^* have the following two properties (cf. §3.1(I) and (II)):

(I) *Each N_i^* is a zero dimensional G_δ-set in I^∞.*

(II) *Given an arbitrary sequence G_1, $G_2, \dots$ of zero dimensional G_δ-sets in I^∞ there exists an irrational $t \in P$ such that $G_i \cap Q_t = N_i^* \cap Q_t$ for each $i = 1, 2, \dots$.*

Property (I) follows from the fact that the map $(x_1, x_2, \dots) \to (x_1, u_i(x_1), (x_1, x_2, \dots))$ homeomorphically embeds the set N_i^* into the product $P \times M_0$.

Let $G_1, G_2, \dots$ be a sequence of zero dimensional G_δ-sets in I^∞, let $t_1, t_2, \dots$ be irrationals such that $M_0(t_i) = G_i$, for $i = 1, 2, \dots$ and let t be an irrational such that $u(t) = (u_1(t), u_2(t), \dots) = (t_1, t_2, \dots)$. Then, for each $i \in \omega$, we obtain (see 12)

$$\begin{aligned} G_i \cap Q_t &= \{(x_1, x_2, \dots) : x_1 = t \text{ and } (t_i, (x_1, x_2, \dots)) \in M_0\} \\ &= \{(x_1, x_2, \dots) : x_1 = t \text{ and } (u_i(t), (x_1, x_2, \dots)) \in M_0\} \\ &= N_i^* \cap Q_t. \end{aligned}$$

Let us put now (cf. §3.1(4))

$$(13) \qquad E_\infty = \bigcup_{i=1}^{\infty} N_i^*$$

and let us verify that the countable dimensional $G_{\delta\sigma}$-set E_∞ satisfies the assertions of Corollary 2.5. Let G be an arbitrary countable dimensional $G_{\delta\sigma}$-set in I^∞; one can find zero dimensional G_δ sets G_1, $G_2, \dots$ in I^∞ such that $G = G_1 \cup G_2 \cup \cdots$. By property (II) there exists an irrational $t \in P$ such that $G_i \cap Q_t = N_i^* \cap Q_t$ for each $i \in \omega$, and therefore $G \cap Q_t = E_\infty \cap Q_t$.

4.2.1. REMARK. The sequence N_1^*, $N_2^*, \dots$ has properties slighty stronger than those of the sequence $N_1, N_2, \dots$ described in §3.1 and therefore one can obtain the results in §3 using the sets N_i^* instead of N_i. Let us notice that in the case of the universal set M_0, used to define the sets N_i^*, the construction given in §4.1 simplifies essentially: the sets Σ_β in (I) do not appear and "ind $< \alpha$" means just "empty". Still, however, the proof of Theorem 2.1 given in §3 seems more direct than an alternative one based on the construction of the space M_0. Let us observe finally that if N is a zero dimensional G_δ-set in I^∞ containing the set N_1 defined in §3.1, $G \subset P \times I^\infty$ is a G_δ-set universal for G_δ-sets in I^∞ [**Ku**, §30, XIII], and $w = (w_1, w_2)$ maps continuously P onto $P \times P$, then the set

$$M = \{(t, x_1, x_2, \dots) \in P \times I^\infty: (w_1(t), w_2(t), x_1, x_2, \dots) \in (P \times N) \cap G\}$$

is a G_δ-set in $S \times I^\infty$, S being the projection of M onto P-coordinate, and each zero dimensional G_δ-set in I^∞ is of the form $M(t)$ for some $t \in S$, cf. Jayne and Rogers [**J-R**, proof of Theorem 9.1], and therefore the space M_0 can be easily obtained from M by the method described in §4.1(III).

5. A universal sequence of partitions between the opposite faces in I^∞. In this section we show that the method of parametrizing function spaces applied in §3.1 (cf. also the proof of Proposition 4.1.1(ii)) can be used to define a sequence of partitions between the opposite faces in I^∞ with some "universal" properties. This, combined with ideas of Walsh [**Wa**] and Rubin [**Ru 1**] provides a quite simple construction of hereditarily strongly infinite dimensional compacta.

Recall that $p_i: I^\infty \to I$ is the projection onto the ith coordinate and let

$$C_i = p_i^{-1}(-1), \quad D_i = p_i^{-1}(1), \quad H_i = p_i^{-1}(0),$$

i.e. C_i and D_i is the pair of the ith opposite faces in I^∞ and H_i is a partition between them.

Let $0 < a_i < 1$ and let

$$C_i^* = p_i^{-1}[-1, -a_i], \quad D_i^* = p_i^{-1}[a_i, 1].$$

Finally, let T be any subset of I homeomorphic to the irrationals and, for each $t \in I$, let

$$Q_t = \{(x_1, x_2, \dots) \in I^\infty: x_1 = t\}.$$

A space X is strongly infinite dimensional if there exists an infinite sequence (A_1, B_1), $(A_2, B_2), \dots$ of pairs of disjoint closed sets in X such that if S_i is a partition between A_i and B_i in X ($i \in \omega$), then $\bigcap_{i=1}^\infty S_i = \emptyset$; strongly infinite dimensional spaces are uncountable dimensional, cf. [**Na 2**, Chapter VI].

5.1. PROPOSITION. *There exist partitions L_i between the ith opposite faces C_i and D_i in I^∞ ($i \in \omega$), such that for every sequence of partitions S_i between the enlarged opposite faces C_i^* and D_i^* in I^∞ ($i \in \omega$), there exists a $t \in T$ such that $L_i \cap Q_t = S_i \cap Q_t$ for each $i = 1, 2, \dots$.*

PROOF. Let Λ be the space of all continuous maps $f\colon I^\infty \to I^\infty$ such that

(1) $$f(C_i^*) \subset C_i \quad \text{and} \quad f(D_i^*) \subset D_i,$$

Λ being endowed with the topology of uniform convergence, and let $u\colon T \to \Lambda$ be a continuous map onto the completely metrizable separable space Λ, cf. [**Ku**, §36, II]. Let $Q_T = \{(x_1, x_2, \ldots) \in I^\infty\colon x_1 \in T\}$ and let $F\colon Q_T \to I^\infty$ be a continuous map defined by (cf. §3.1(1))

(2) $$F(x_1, x_2, \ldots) = u(x_1)(x_1, x_2, \ldots).$$

By (1), the set $F^{-1}(H_i)$ is a partition in Q_T between $C_i^* \cap Q_T$ and $D_i^* \cap Q_T$ and since C_i and D_i are in the interior of C_i^* and D_i^* respectively, there exists a partition L_i in I^∞ between C_i and D_i extending $F^{-1}(H_i)$, cf. Engelking [**En 2**, Lemma 1.2.9], i.e.

(3) $$L_i \cap Q_T = F^{-1}(H_i), \qquad i = 1, 2, \ldots.$$

We shall verify that the sequence $L_1, L_2, \ldots$ has the required property. Given partitions S_i in I^∞ between C_i^* and D_i^* $(i \in \omega)$, let $f_i\colon I^\infty \to I$ be continuous functions such that $C_i^* = f_i^{-1}(-1)$, $D_i^* = f_i^{-1}(1)$ and $S_i = f_i^{-1}(0)$. The diagonal map $f = (f_1, f_2, \ldots)\colon I^\infty \to I^\infty$ belongs to Λ and therefore $f = u(t)$ for some $t \in T$. Since $S_i = f^{-1}(H_i)$, for $i \in \omega$, we obtain (see (3) and (2))

$$L_i \cap Q_t = F^{-1}(H_i) \cap Q_t = u(t)^{-1}(H_i) \cap Q_t = f^{-1}(H_i) \cap Q_t = S_i \cap Q_t.$$

5.2. COROLLARY. *Let $L_1, L_2, \ldots$ be the sequence of partitions between the opposite faces in I^∞ described in Proposition* 5.1. *For each $\sigma \subset \omega \setminus \{1\}$ and for each set $M \subset \bigcap\{L_i\colon i \in \sigma\}$ whose projection onto the first coordinate contains T, we have*

(i) *if σ is a k-element set, then M is at least k-dimensional;*

(ii) *if σ is infinite, then M is strongly infinite dimensional.*

PROOF. The reasoning in both cases is the same. Assume that the assertion is not true. Then (using again a simple lemma on extension of partitions [**En 2**, Lemma 1.2.9]) one can find partitions S_i between C_i and D_i in I^∞, where $i \in \sigma$, such that $M \cap \bigcap\{S_i\colon i \in \sigma\} = \varnothing$, cf. [**Ku**, §27, II; **R-S-W**, §3]. But, on the other hand, there exists a $t \in T$ such that $S_i \cap Q_t = L_i \cap Q_t$ for each i, and this would yield a contradiction, $\varnothing \neq M \cap Q_t \subset M \cap \bigcap\{S_i\colon i \in \sigma\} = \varnothing$.

5.3. *Hereditarily strongly infinite dimensional compacta.* We shall repeat in this section some arguments due to Walsh [**Wa**, §§3, 7] and Rubin [**Ru 1**, §6] to derive from Corollary 5.2 a construction of hereditarily strongly infinite dimensional spaces.

Choose in I a collection $T_1, T_2, \ldots$ of homeomorphic copies of the irrationals such that each nondegenerate interval in I contains some T_i and let $\sigma_{ik} \subset \omega \setminus \{1, i\}$, where $i, k = 1, 2, \ldots$, be pairwise disjoint infinite sets.

Let us fix a pair of natural numbers i, k. Changing the ith coordinate with the first one and letting $T = T_k$, we obtain from Corollary 5.2 partitions L_j between the jth opposite faces in I^∞, where $j \in \sigma_{ik}$, such that each subset of the intersection $L_{ik} = \bigcap\{L_j\colon i \in \sigma_{ik}\}$ whose projection onto the ith coordinate contains T_k is strongly infinite dimensional. Since $1 \notin \bigcup_{i,k}\sigma_{ik}$, each partition in I^∞ between the opposite faces C_1 and D_1 hits the intersection $L = \bigcap\{L_j\colon j \in \bigcup_{i,k}\sigma_{ik}\} = \bigcap_{i,k}L_{ik}$, cf. [**Ku**, §28, IV], and therefore L is a compactum of positive dimension. If M is a nonempty set in L, then either for some i and k, $T_k \subset p_i(M)$ and then M is strongly infinite dimensional, or else no projection $p_i(M)$ contains a nondegenerate interval, and then M is zero dimensional, being a subset of the product $\prod_{i=1}^{\infty} p_i(M)$ of zero dimensional sets.

5.4. Remark. For other constructions of hereditarily infinite dimensional compacta we refer the reader to the papers by Walsh [**Wa**], Rubin [**Ru 1**, **Ru 3**] and Krasinkiewicz [**Kr**]; an illuminating account of the topic is given by Garity and Schori [**G-S**, §2], cf. also Nagata [**Na 2**, p. 125].

Separators of certain special type between the opposite faces in I^∞ with properties similar to (i) and (ii) in Corollary 5.2 were constructed by Walsh [**Wa**, §4], cf. also [**R-S-W** and **S-W**]. A sequence of partitions between the opposite faces in I^∞ satisfying condition (ii) in Corollary 5.2 was constructed by Rubin [**Ru 1**] (a simplified, but still rather involved, exposition of this construction was given in [**Ru 3**]). Rubin [**Ru 2**] has shown that the existence of such partitions yields a result that *each strongly infinite dimensional space contains a closed hereditrarily strongly infinite dimensional subspace*.

An important element in the constructions in [**R-S-W**, **Wa**, **Ru 1** and **Kr**] is a continuous parametrization of some collections of compacta and forming a "diagonal compactum" for that collection, and this element is also hidden in the proof of Proposition 5.1. This idea can be traced back to Mazurkiewicz [**Ma**] and Knaster [**Kn**], cf. Lelek [**Le**, Example, p. 80].

6. Comments.

6.1. *Tumarkin's property.* The following property of separable metrizable spaces X was considered by Tumarkin [**Tu**, **Na**, p. 125]:

(T) *each infinite dimensional subspace of X contains subsets of arbitrarily large finite dimension.*

Corollary 2.5. and the fact that the spaces X with $\operatorname{ind} X \neq \infty$ have property (T) yield the following fact.

6.1.1. Proposition. *Property* (T) *implies countable dimensionality, and in the class of completely metrizable separable spaces countable dimensionality is equivalent to property* (T).

It is an open problem whether there exists a countable dimensional space which fails property (T), cf. [**Tu**, **Wa**, §7, **En 2**, 4.14]. In connection with this problem, let us make the following remark. One can repeat the construction in §3, starting with a continuous mapping $u = (u_1, u_2)\colon T \to \Gamma \times \Gamma$ onto the square of Γ instead of its countable product. This yields zero dimensional sets N_1, N_2 (see 3.1(1)), a one dimensional set $E = N_1 \cup N_2$ (cf. 3.1(4)), and finally, it provides a one dimensional set D' in I^∞ defined analogously to the set C' described in §3.2. The set D' has the property that each one dimensional set $S \subset I^\infty \setminus D'$ has zero dimensional projection onto the first coordinate (cf. §3.2 (IV)). Therefore, if we let $D = \pi_1(D') \cup \pi_2(D') \cup \cdots$, where $\pi_i\colon I^\infty \to I^\infty$ is the permutation of the coordinates changing the ith one with the first one, we obtain a countable dimensional set D intersecting each one dimensional set in I^∞ (see the reasoning at the end of §3.2) and hence, the complement $I^\infty \setminus D$ does not contain any subset of positive finite dimension (as any such set contains a one-dimensional subset). It is still conceivable that there exists an infinite dimensional countable dimensional set S in $I^\infty \setminus D$ (any such S would provide a solution to the problem we have formulated), however, the nature of the construction of D makes it difficult to clarify what are exactly the properties of this set.

Let us also mention that *there exist uncountable dimensional compacta all of whose closed infinite dimensional subspaces contain closed subsets of arbitrarily large finite dimension*—such a compactum is defined in [**Po 1**].

6.2. *Totally disconnected complete spaces D_α with* $\operatorname{ind} D_\alpha = \alpha$. We shall use the spaces E_α defined in Theorem 2.2 to obtain spaces D_α described in the title of this section. Various constructions of totally disconnected complete spaces of arbitrarily large finite dimension can be found in [**Ma**, **Le**, p. 80; **R-S-W**, **Kr**].

Let $\alpha = \beta + 1$ be a non-limit-countable ordinal and let E_β be the universal set described in Theorem 2.2. Let $P \times I^\infty$ be the product of the irrationals in I and the Hilbert cube and let

$$G_\beta = \{(t,(x_1,x_2,\ldots)) \in P \times I^\infty\colon (t,x_1,x_2,\ldots) \in E_\beta\}.$$

Let $p\colon P \times I^\infty \to P$ be the projection. The universal properties of E_β yield immediately that G_β intersects each set S in $P \times I^\infty$ with $\operatorname{ind} S \leqslant \beta$ and $p(S) = P$. Let $K_\alpha \subset I^\infty$ be a compactum with $\operatorname{ind} K_\alpha = \alpha$, cf. [**En 2**, 2.2; **Na 2**, p. 148]. The set $P \times K_\alpha \setminus G_\beta = F_\alpha$ is an F_σ-set in $P \times K_\alpha$ and, since $\operatorname{ind} G_\beta = \beta < \operatorname{ind} K_\alpha = \alpha$, each vertical section $F_\alpha \cap p^{-1}(t)$ is nonempty. Therefore, there exists a function $f\colon P \to K_\alpha$ of the first Baire class whose graph D_α is contained in F_α, see [**Ku**, §43, IX]. The set D_α is a totally disconnected G_δ-set in $P \times K_\alpha$. Moreover, D_α is disjoint from G_β and $p(D_\alpha) = P$, so $\beta < \operatorname{ind} D_\alpha \leqslant \operatorname{ind}(P \times K_\alpha) = \beta + 1 = \alpha$.

If α is a limit ordinal it is enough to let D_α be the free union of the sets $D_{\beta+1}$ with $\beta < \alpha$.

6.3. *Compactifications of spaces with* $\operatorname{ind} = \alpha$. Let M_α be the universal space described in Proposition 4.1. By the theorem of Hurewicz quoted at the end of §1.3, there exists a countable dimensional compact extension M_α^* of M_α and let $\phi(\alpha)$ be the minimal transfinite dimension of such extensions. The function ϕ augmented by $\phi(\infty) = \infty$ has the following property:

For each separable metrizable space X there exists a compactification X^* of X such that $\operatorname{ind} X \leqslant \operatorname{ind} X^* \leqslant \phi(\operatorname{ind} X)$.

Luxemburg [**Lu 1**] constructed examples which show that for any limit ordinal α, $\phi(\alpha) > \alpha$. The exact nature of the function ϕ is, however, unknown, see [**Lu 1**, Conjecture on p. 443, **En 2**, 5.10].

6.4. *A proof of Nagata's Lemma* 3.1.1. We shall give a simple proof of Lemma 3.1.1. By a standard diagonal embedding argument it is enough to show that given a zero dimensional set G in X and a pair of disjoint closed sets A and B in X, there is a continuous map $f\colon X \to J$ into the interval $J = [-1/\sqrt{2}, 1/\sqrt{2}]$ such that $A \subset f^{-1}(-1/\sqrt{2})$, $B \subset f^{-1}(1/\sqrt{2})$ and $f(G) \subset P$, P being the irrationals.

Nagata [**Na 1**, **Na 2**, VI.2.A] constructed such a map f by a modification of a standard proof of Urysohn's Lemma. We shall construct this map by a simple approximation procedure.

Arrange all rational numbers from T into a sequence $r_1, r_2, \ldots$ (without repetitions) and let $\delta_i = \min\{|r_j - r_k|, |r_j \pm 1/\sqrt{2}|\colon k < j \leqslant i\}$, $\varepsilon_i = 2^{-(i+3)} \cdot \delta_i$, $a_i = r_i - \varepsilon_i$, $b_i = r_i + \varepsilon_i$, $A_i = [-1/\sqrt{2}, a_i]$, $B_i = [b_i, 1/\sqrt{2}]$, $J_i = (a_i, b_i)$. Let us define continuous maps $f_i\colon X \to J$, $i = 0, 1, 2, \ldots$, inductively as follows: let f_0 be such that $A \subset f_0^{-1}(-1/\sqrt{2})$, $B \subset f_0^{-1}(1/\sqrt{2})$, assume that the map f_i has been already defined and put $C = f_i^{-1}(A_{i+1})$, $D = f_i^{-1}(B_{i+1})$. Choose an open set U in X such that $C \subset U$, $\overline{U} \cap D = \varnothing$ and $(\overline{U} \setminus U) \cap G = \varnothing$ and let $f_{i+1}\colon X \to J$ be a continuous map such that $f_{i+1}^{-1}(A_{i+1} \cup B_{i+1}) = C \cup D$, f_{i+1} coincides with f_i on $C \cup D$ and $f_{i+1}^{-1}(r_{i+1}) = \overline{U} \setminus U$. Since $|f_{i+1}(x) - f_i(x)| < 2 \cdot \varepsilon_{i+1} = 2^{-(i+3)} \cdot \delta_{i+1}$, the sequence $\{f_i\}_{i=0}^{\infty}$ converges uniformly to a continuous map $f\colon X \to J$. Since all f_i coincide with f_0 on $A \cup B$, $A \subset f^{-1}(-1/\sqrt{2})$ and $B \subset f^{-1}(1/\sqrt{2})$ and it is routine to check that for every rational number r_i from J, $f^{-1}(r_i) \cap G = \varnothing$, i.e. $f(G) \subset P$.

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