PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 133, Number 3, Pages 937–943
S 0002-9939(04)07684-1
Article electronically published on September 29, 2004

COUNTABLY COMPACT GROUPS FROM A SELECTIVE ULTRAFILTER

S. GARCIA-FERREIRA, A. H. TOMITA, AND S. WATSON

(Communicated by Alan Dow)

ABSTRACT. We prove that the existence of a selective ultrafilter on ω implies the existence of a countably compact group without non-trivial convergent sequences all of whose powers are countably compact. Hence, by using a selective ultrafilter on ω, it is possible to construct two countably compact groups without non-trivial convergent sequences whose product is not countably compact.

1. INTRODUCTION

The first example of a countably compact group without non-trivial convergent sequences was constructed, assuming CH, by A. Hajnal and I. Juhász [7]. A second example was discovered by E. K. van Douwen [3] under the assumption of MA, and one of the most recent examples lies in [10]. All known examples of such a topological group use some form of MA. A similar situation holds in the problem of the existence, in ZFC, of two countably compact groups whose product is not countably compact (see, for instance, [3], [8], [9] and [10]). In this paper, we will construct two countably compact groups without non-trivial convergent sequences whose product is not countably compact from a selective ultrafilter. We also construct a countably compact group without non-trivial convergent sequences all of whose powers are countably compact from a selective ultrafilter on ω.

We shall use standard notation. If $\{x_\xi : \xi < \mathfrak{c}\} \subseteq \{0,1\}^{\mathfrak{c}}$ and $F \in [\mathfrak{c}]^{<\omega}$, then $x_F = \sum_{\xi \in F} x_\xi$. The *type* of a point $p \in \beta(\omega) \setminus \omega = \omega^*$ is denoted by $T(p) = \{q \in \omega^* : \exists \text{ a bijection } f : \omega \to \omega(\overline{f}(p) = q)\}$, where $\overline{f} : \beta(\omega) \to \beta(\omega)$ denotes the Stone-Čech extension of f. An ultrafilter $p \in \omega^*$ is called *selective* if for every $f : \omega \to \omega$ there is $A \in p$ such that $f|_A$ is either constant or one-to-one (the reader may find other combinatorial statements equivalent to selectivity in the book [2]).

The following concept has been very useful in the construction of countably compact spaces with certain properties.

Received by the editors March 3, 2003 and, in revised form, November 20, 2003.

2000 *Mathematics Subject Classification.* Primary 54G20, 54D80, 22A99; Secondary 54H11.

Key words and phrases. p-limit, p-compact, selective ultrafilter, countably compact group, topological group.

This research was supported by CONACYT grant no. 40057-F and DGAPA grant no. IN104601.

Definition 1.1 (A. R. Bernstein [1]). Let $p \in \omega^*$, and let $(x_n)_{n<\omega}$ be a sequence in a space X. We say that x is a p-limit point of $(x_n)_{n<\omega}$, we write $x = p\text{-}\lim_{n\to\omega} x_n$, if for every neighborhood V of x, $\{n < \omega : x_n \in V\} \in p$.

It is not difficult to prove that a space X is countably compact iff every sequence of points in X has a p-limit point in X, for some $p \in \omega^*$. The following class of spaces was introduced by A. R. Bernstein [1].

Definition 1.2. Let $p \in \omega^*$. A space X is said to be p-compact if for every sequence $(x_n)_{n<\omega}$ of points of X there is $x \in X$ such that $x = p\text{-}\lim_{n\to\omega} x_n$.

We know that p-compactness is preserved under arbitrary products, for each $p \in \omega^*$. Hence, we can find countably compact spaces that are not p-compact for any $p \in \omega^*$ (see [5]). J. Ginsburg and V. Saks [6] showed that all powers of a space X are countably compact iff there is $p \in \omega^*$ such that X is p-compact.

For $p \in \omega^*$, we shall use the properties of the ultrapower $([\mathfrak{c}]^{<\omega})^\omega/p$ considered as a vector space over the field $\{0, 1\}$ with the symmetric difference $A\Delta B = (A \setminus B) \cup (B \setminus A)$ as addition. For $p \in \omega^*$, an element of the ultrapower $([\mathfrak{c}]^{<\omega})^\omega/p$ will be denoted by $[f]_p$, where $f : \omega \to [\mathfrak{c}]^{<\omega}$ is a function. For $F \in [\mathfrak{c}]^{<\omega}$, the constant function whose domain is ω and takes only the value F will be denoted by $\vec{F}$. If $\alpha < \mathfrak{c}$ is an ordinal, then $\vec{\{\alpha\}}$ will be denoted by $\vec{\alpha}$.

2. The examples

Our group G will be generated by a linearly independent subset of $\{0, 1\}^{\mathfrak{c}}$.

For every selective ultrafilter $p \in \omega^*$, it is evident that

$$([\mathfrak{c}]^{<\omega})^\omega/p = \{[f]_p : f \in ([\mathfrak{c}]^{<\omega})^\omega \text{ is one-to-one }\} \cup \{[\vec{F}]_{\mathrm{p}} : \mathrm{F} \in [\mathfrak{c}]^{<\omega}\}.$$

Lemma 2.1. *Let $p \in \omega^*$ be selective. Then, there exists a family of one-to-one functions $\{f_\xi : \xi < \mathfrak{c}\} \subseteq ([\mathfrak{c}]^{<\omega})^\omega$ such that:*

1) $\bigcup_{n<\omega} f_\xi(n) \subseteq max\{\omega, \xi\}$, *for every $\xi < \mathfrak{c}$.*
2) $\{[f_\xi]_p : \xi < \mathfrak{c}\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ *is a base for* $([\mathfrak{c}]^{<\omega})^\omega/p$.
3) *For every one-to-one function $g \in ([\mathfrak{c}]^{<\omega})^\omega$, there are distinct $\zeta_0, \zeta_1 < \mathfrak{c}$ and two increasing sequences of positive integers $(n^0_k)_{k<\omega}$ and $(n^1_k)_{k<\omega}$ such that $f_{\zeta_i}(k) = g(n^i_k)$, for every $k < \omega$ and $i \in \{0, 1\}$.*

Proof. Let $\{g_\xi : \xi < \mathfrak{c}\}$ be an enumeration of all one-to-one functions of $([\mathfrak{c}]^{<\omega})^\omega$ in such a way that each element is listed two times, and $\bigcup_{n<\omega} g_\xi(n) \subseteq max\{\omega, \xi\}$, for every $\xi < \mathfrak{c}$. We proceed by transfinite induction. Let $\alpha < \mathfrak{c}$ and suppose that, for each $\xi < \alpha$, we have defined a one-to-one function $f_\xi : \omega \to [\mathfrak{c}]^{<\omega}$ such that:

i) For every $m < \omega$ there is $n < \omega$ such that $f_\xi(m) = g_\xi(n)$, for every $\xi < \alpha$.
ii) $\{[f_\zeta]_p : \zeta < \xi\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is linearly independent, for every $\xi < \alpha$.
iii) If $\{[f_\zeta]_p : \zeta < \xi\} \cup \{[g_\xi]_p\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is linearly independent, then $f_\xi = g_\xi$, for every $\xi < \alpha$.

If $\{[f_\xi]_p : \xi < \alpha\} \cup \{[g_\alpha]_p\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is linearly independent, then we define $f_\alpha = g_\alpha$. Let us assume that $\{[f_\xi]_p : \xi < \alpha\} \cup \{[g_\alpha]_p\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is not linearly independent. Now, let $\{A_\mu : \mu < \mathfrak{c}\}$ be an almost disjoint family of infinite subsets of ω. For each $\mu < \mathfrak{c}$, let $h_\mu : \omega \to A_\mu$ be a bijection. Then, we define $h_{\alpha,\mu} : \omega \to [\mathfrak{c}]^{<\omega}$ by $h_{\alpha,\mu}(n) = g_\alpha(h_\mu(n))$, for each $n < \omega$. It is evident that $\{n < \omega : h_{\alpha,\mu}(n) = h_{\alpha,\nu}(n)\}$ is finite for $\mu < \nu < \mathfrak{c}$. Hence, $\{[h_{\alpha,\mu}]_p : \mu < \mathfrak{c}\}$

are pairwise distinct. So, we can find $\mu_\alpha < \mathfrak{c}$ such that $[h_{\alpha,\mu_\alpha}]_p \notin \langle \{[f_\zeta]_p : \zeta < \xi\} \cup \{[\vec{\beta}]_p : \beta < max\{\omega, \alpha\}\}\rangle$. Put $f_\alpha = h_{\alpha,\mu_\alpha}$. Clearly, conditions $i)$ and $iii)$ are satisfied. We know that $\{[f_\xi]_p : \xi \leq \alpha\} \cup \{[\vec{\beta}]_p : \beta < max\{\omega, \alpha\}\}$ is linearly independent. Since $\bigcup_{n<\omega} f_\xi(n) \subseteq max\{\omega, \alpha\}$, for every $\xi \leq \alpha$, we also have that $\{[f_\xi]_p : \xi \leq \alpha\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is linearly independent. This shows that condition $ii)$ holds. We claim that the family $\{f_\xi : \xi < \mathfrak{c}\}$ satisfies all the conditions. Indeed, by the construction, $\{[f_\xi]_p : \xi < \mathfrak{c}\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is a base for $([\mathfrak{c}]^{<\omega})^\omega/p$. Let us prove that condition 3) is satisfied. For this, take $\zeta_0, \zeta_1 < \mathfrak{c}$ so that $\xi_0 < \xi_1$ and $g = g_{\zeta_0} = g_{\zeta_1}$. From condition $i)$ we can find two increasing sequences of positive integers $(n^0_k)_{k<\omega}$ and $(n^1_k)_{k<\omega}$ such that $f_{\zeta_i}(k) = g(n^i_k)$, for every $k < \omega$ and $i \in \{0, 1\}$. $\square$

In what follows, we fix a family $\{f_\xi : \xi < \mathfrak{c}\} \subseteq ([\mathfrak{c}]^{<\omega})^\omega$ satisfying the three properties stated in Lemma 2.1, and enumerate $[\mathfrak{c}]^{<\omega} \setminus \{\emptyset\}$ as $\{F_\alpha : \alpha < \mathfrak{c}\}$.

Lemma 2.2. *Let $p \in \omega^*$ be selective. Suppose that for every $\alpha < \mathfrak{c}$ we have a non-trivial homomorphism $\Phi_\alpha : [\mathfrak{c}]^{<\omega} \to \{0, 1\}$ such that*

i) $\Phi_\alpha(\{\xi\}) = p\text{-}\lim_{n\to\omega} \Phi_\alpha(f_\xi(n))$, *for every $\xi < \mathfrak{c}$; and*
ii) $\Phi_\alpha(F_\alpha) = 1$.

For $\xi < \mathfrak{c}$, we define $x_\xi \in \{0, 1\}^\mathfrak{c}$ by $x_\xi(\alpha) = \Phi_\alpha(\{\xi\})$, for every $\alpha < \mathfrak{c}$. Then, the set $X = \{x_\xi : \xi < \mathfrak{c}\}$ is linearly independent in $\{0, 1\}^\mathfrak{c}$ and $G = \langle X\rangle$ is a p-compact group without non-trivial convergent sequences.

Proof. Let $\{\xi_0, ..., \xi_k\} \in [\mathfrak{c}]^{<\omega}$. Choose $\alpha < \mathfrak{c}$ such that $F_\alpha = \{\xi_0, ..., \xi_k\}$. Then, by $ii)$,

$$(x_{\xi_0} + ... + x_{\xi_k})(\alpha) = \Phi_\alpha(\{\xi_0\}) + ... + \Phi_\alpha(\{\xi_k\}) = \Phi_\alpha(F_\alpha) = 1.$$

This shows that $\{x_\xi : \xi < \mathfrak{c}\}$ is linearly independent in $\{0, 1\}^\mathfrak{c}$. Now we will show that G is p-compact. Before proving this, notice from clause $i)$ that

$$x_\xi = p\text{-}\lim_{n\to\omega} \sum_{\mu\in f_\xi(n)} x_\mu = p\text{-}\lim_{n\to\omega} x_{f_\xi(n)},$$

for every $\xi < \mathfrak{c}$. Let $(a_n)_{n<\omega}$ be a sequence in G. Choose $g \in ([\mathfrak{c}]^{<\omega})^\omega$ such that $a_n = x_{g(n)}$, for every $n < \omega$. Since p is selective, there is $A \in p$ such that $g|_A$ is either constant or one-to-one. If $g|_A$ is constant, then there is $F \in [\mathfrak{c}]^{<\omega}$ such that $\{n < \omega : x_{g(n)} = x_F\} \in p$ and so $x_F = p\text{-}\lim_{n\to\omega} x_{g(n)}$. Let us assume that there is a one-to-one function $h \in ([\mathfrak{c}]^{<\omega})^\omega$ such that $h|_A = g|_A$. Since $\{[f_\xi]_p : \xi < \mathfrak{c}\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is a base for $([\mathfrak{c}]^{<\omega})^\omega/p$, there are $\xi_0, ..., \xi_k < \mathfrak{c}$ and $E \in [\mathfrak{c}]^{<\omega}$ such that $[h]_p = (\Delta_{i\leq k}[f_{\xi_i}]_p)\Delta(\Delta_{\mu\in E}[\vec{\mu}]_p)$. Hence, we can find $B \in p$ such that $B \subseteq A$ and $h(n) = (\Delta_{i\leq k} f_{\xi_i}(n))\Delta E$, for every $n \in B$. It then follows that $x_{h(n)} = \sum_{i\leq k} x_{f_{\xi_i}(n)} + x_E$, for all $n \in B$. So,

$$\sum_{i\leq k} x_{\xi_i} + x_E = p\text{-}\lim_{n\to\omega} x_{h(n)}.$$

This shows that G is p-compact. Let $(y_n)_{n<\omega}$ be a non-trivial sequence in G, and assume that there is a one-to-one function $g \in ([\mathfrak{c}]^{<\omega})^\omega$ such that $y_n = x_{g(n)}$. By clause 3) of Lemma 2.1, there are distinct $\zeta_0, \zeta_1 < \mathfrak{c}$ and two increasing sequences of positive integers $(n^0_k)_{k<\omega}$ and $(n^1_k)_{k<\omega}$ such that $f_{\zeta_i}(k) = g(n^i_k)$, for every $k < \omega$ and $i \in \{0, 1\}$. Since $x_{\zeta_i} = p\text{-}\lim_{k\to\omega} x_{f_{\zeta_i}(k)} = p\text{-}\lim_{k\to\omega} x_{g(n^i_k)} = p\text{-}\lim_{k\to\omega} y_{n^i_k}$, for each $i \in \{0,1\}$, we must have that x_{ζ_0} and x_{ζ_1} are both cluster points of $\{y_n : n < \omega\}$. □

Lemma 2.3. *Let $p \in \omega^*$ be selective. For every $E_0 \in [\mathfrak{c}]^{<\omega} \setminus \{\emptyset\}$, there are $\{b_i : i < \omega\} \in p$ and $\{E_i : 0 < i < \omega\} \subseteq [\mathfrak{c}]^{<\omega}$ such that*

1) *$\omega \subseteq \bigcup_{i<\omega} E_i$;*
2) *$E_i \cup [\bigcup_{\xi \in E_i} f_\xi(b_i)] \subseteq E_{i+1}$, for every $i < \omega$; and*
3) *$\{f_\xi(b_i) : \xi \in E_i\} \cup \{\{\mu\} : \mu \in E_i\}$ is linearly independent, for every $i < \omega$.*

Proof. Put $F_0 = E_0$, and let $F_{n+1} = n \cup F_n \cup [\bigcup_{\xi \in F_n} \bigcup_{m \leq n} f_\xi(m)]$, for every $1 \leq n < \omega$. Since $\{[f_\xi]_p : \xi \in S\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is linearly independent, we have that

$$A_n = \{k < \omega : \{f_\xi(k) : \xi \in F_n\} \cup \{\{\mu\} : \mu \in F_n\} \\ \text{is linearly independent}\} \in p,$$

for every $n < \omega$. By the selectivity of p, we can find $A = \{a_n : n < \omega\} \in p$ such that $m < a_m < a_n$ and $a_n \in A_n$, for every $m < n < \omega$. Let us define a coloring P_0 and P_1 on $[\omega]^2$ as $\{a,b\} \in P_0$ iff $a < b$, $a, b \in A$, $a = a_m$, $b = a_n$ and $a_m < n$, and $\{a,b\} \in P_1$ otherwise. Since p is selective, there is $B \in p$ such that $B \subseteq A$ and either $[B]^2 \subseteq P_0$ or $[B]^2 \subseteq P_1$. Choose $I \in [\omega]^\omega$ such that $B = \{a_n : n \in I\}$. Let $\{i_k : k < \omega\}$ be the enumeration of I in increasing order. Suppose that $[B]^2 \subseteq P_1$. Since $\{a_{i_0}, a_{i_k}\} \in P_1$, then $a_{i_0} \geq i_k$, for every $1 \leq k < \omega$, but this is a contradiction. Therefore, $[B]^2 \subseteq P_0$. Hence, we have that $i_k < a_{i_k} < i_{k+1}$, for every $k < \omega$. By using this, we obtain that

$$F_{i_k} \cup [\bigcup_{\xi \in F_{i_k}} f_\xi(a_{i_k})] \subseteq F_{i_{k+1}} \cup [\bigcup_{\xi \in F_{i_k}} \bigcup_{m < i_{k+1}} f_\xi(m)] \subseteq F_{i_{k+1}},$$

for all $k < \omega$. Notice that, for every $k < \omega$,

$$\{f_\xi(a_{i_k}) : \xi \in F_{i_k}\} \cup \{\{\mu\} : \mu \in F_{i_k}\}$$

is linearly independent, since $a_{i_k} \in A_{i_k}$. Then, for every $1 \leq k < \omega$, we define $E_k = F_{i_k}$ and, for every $k < \omega$, we put $b_k = a_{i_k}$. It is evident that 2) and 3) are satisfied. We remark that $E_0 \subseteq F_{i_0}$ and

$$\omega \subseteq \bigcup_{n<\omega} F_n = \bigcup_{k<\omega} F_{i_k} = \bigcup_{k<\omega} E_k,$$

and $B = \{b_k : k < \omega\} \in p$. □

Example 2.4. If $p \in \omega^*$ is selective, then there is a p-compact subgroup of size $\mathfrak{c}$ without non-trivial convergent sequences.

Proof. According to Lemma 2.2, it suffices to construct, for each $\alpha < \mathfrak{c}$, a non-trivial homomorphism $\Phi_\alpha : [\mathfrak{c}]^{<\omega} \to \{0,1\}$ such that

i) $\Phi_\alpha(\{\xi\}) = p\text{-}\lim_{n\to\omega} \Phi_\alpha(f_\xi(n))$, for every $\xi < \mathfrak{c}$; and
ii) $\Phi_\alpha(F_\alpha) = 1$.

Fix $\alpha < \mathfrak{c}$. By applying Lemma 2.3 to $E_0 = F_\alpha$, we get $\{b_i : i < \omega\} \in p$ and $\{E_i : 0 < i < \omega\} \subseteq [\mathfrak{c}]^{<\omega}$ such that

1) $\omega \subseteq \bigcup_{i<\omega} E_i =: E$;
2) $E_i \cup [\bigcup_{\xi \in E_i} f_\xi(b_i)] \subseteq E_{i+1}$, for every $i < \omega$; and
3) $\{f_\xi(b_i) : \xi \in E_i\} \cup \{\{\mu\} : \mu \in E_i\}$ is linearly independent, for every $i < \omega$.

Now, suppose that for $i < \omega$, we have defined Φ_α on $[E_i]^{<\omega}$ so that $\Phi_\alpha(F_\alpha) = 1$ and $\Phi_\alpha(f_\xi(b_i)) = \Phi_\alpha(\{\mu\})$, for every $\xi, \mu \in E_i$. Since $\{f_\xi(b_{i+1}) : \xi \in E_{i+1}\} \cup \{\{\mu\} : \mu \in E_{i+1}\}$ is linearly independent, and $E_i \cup \bigcup_{\xi \in E_i} f_\xi(b_i) \subseteq E_{i+1}$, we may extend $\Phi_\alpha : [E_i]^{<\omega} \to \{0,1\}$ to a homomorphism from $[E_{i+1}]^{<\omega}$ to $\{0,1\}$ in such a way that $\Phi_\alpha(f_\xi(b_{i+1})) = \Phi_\alpha(\{\xi\})$, for every $\xi \in E_{i+1}$. Thus, we have defined Φ_α on $[E]^{<\omega}$. Observe that $\Phi_\alpha(f_\xi(b_i)) = \Phi_\alpha(\{\xi\})$, for every $\xi \in E_i$ and $i < \omega$. Hence, $\{n < \omega : \Phi_\alpha(f_\xi(n)) = \Phi_\alpha(\{\xi\})\} \in p$, for every $\xi \in E$. Our next task is to extend Φ_α to $[\mathfrak{c}]^{<\omega}$. We will do this by transfinite induction on $\mathfrak{c} \setminus E$. Let $\gamma \in \mathfrak{c} \setminus E$ and suppose that Φ_α has been defined on $[E \cup \gamma]^{<\omega}$. Since $f_\gamma(n) \subseteq \gamma$, for every $n < \omega$, $\Phi_\alpha(\{\mu\})$ has been defined for each $\mu < \gamma$ and $\{\{\gamma\}\} \cup \{\{\mu\} : \mu < \gamma\}$ is linearly independent, Φ_α can be extended to $[E \cup (\gamma + 1)]^{<\omega}$ in such a way that

$$\Phi_\alpha(\{\gamma\}) = p\text{-}\lim_{n\to\omega} \Phi_\alpha(f_\gamma(n)).$$

It is evident that Φ_α satisfies the required properties. □

The following example follows from E. K. van Douwen's construction [3, 6.1] applied to Example 2.4.

Example 2.5. If there is a selective ultrafilter on ω, then there are two countably compact groups without non-trivial convergent sequences whose product is not countably compact.

3. One more example

In this section, we will improve a little bit Example 2.5.

For $p \in \omega^*$, we say that a space is *almost p-compact* if for every sequence $(x_n)_{n<\omega}$ in X there is a function $\sigma : \omega \to \omega$ such that $\overline{\sigma}(p) \in \omega^*$ and $\overline{\sigma}(p)\text{-}\lim_{n\to\omega} x_n \in X$ (this concept was introduced in [4]). It is evident that every p-compact space is almost p-compact, and every almost p-compact space is countably compact. All these notions are different from each other.

The following lemma is a generalization of Lemma 2.1.

Lemma 3.1. *Let $p \in \omega^*$ be a selective ultrafilter. Then, there exists a family of one-to-one functions $\{f_\xi : \omega \le \xi < \mathfrak{c}\} \subseteq ([\mathfrak{c}]^{<\omega})^\omega$ and pairwise disjoint sets $I_0, I_1, I_2, I_3 \in [\mathfrak{c} \setminus \omega]^{\mathfrak{c}}$ such that:*

- a) $\bigcup_{n<\omega} f_\xi(n) \subseteq \xi$ *for every* $\omega \le \xi < \mathfrak{c}$.
- b) b.0) $\bigcup_{n<\omega} f_\xi(n) \subseteq \omega$, *for every* $\xi \in I_0$.
  - b.1) $\bigcup_{n<\omega} f_\xi(n) \subseteq \omega$, *for every* $\xi \in I_1$.
  - b.2) $\bigcup_{n<\omega} f_\xi(n) \subseteq I_0 \cup I_2$, *for every* $\xi \in I_2$.
  - b.3) $\bigcup_{n<\omega} f_\xi(n) \subseteq I_1 \cup I_3$, *for every* $\xi \in I_3$.
- c) $\{[f_\xi]_p : \omega \le \xi < \mathfrak{c}\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ *is linearly independent.*
- d) *For every $j \in \{0,1\}$ and for every one-to-one function $g \in ([\omega]^{<\omega})^\omega$, there exists a bijection $\sigma : \omega \to \omega$ and $\xi \in I_j$ such that $[g \circ \sigma]_p = [f_\xi]_p$.*
- e) *For every $j \in \{0,1\}$, $\{[f_\xi]_p : \xi \in I_{j+2}\} \cup \{[\vec{\beta}]_p : \beta \in I_j \cup I_{j+2}\}$ is a base for $([I_j \cup I_{j+2}]^{<\omega})^\omega/p$.*

Proof. Let I_0, I_1, I_2 and I_3 be a partition of $\mathfrak{c} \setminus \omega$ in subsets of size $\mathfrak{c}$, and let $\{g_\xi : \omega \le \xi < \mathfrak{c}\}$ be such that:

- i) For each $j \in \{0,1\}$, we have that $\{g_\xi : \xi \in I_j\}$ is an enumeration of all one-to-one functions in $([\omega]^{<\omega})^\omega$.

ii) For each $j \in \{0,1\}$, we have that $\{g_\xi : \xi \in I_{j+2}\}$ is an enumeration of all one-to-one functions in $([I_j \cup I_{j+2}]^{<\omega})^\omega$ in such a way that $\bigcup_{n<\omega} g_\xi(n) \subseteq \xi$, for every $\xi \in I_{j+2}$.

By applying the proof of Lemma 2.1 to $\{g_\xi : \xi \in I_j\}$, for $j \in 2, 3$, we get a set of one-to-one functions $\{f_\xi : \xi \in I_j\}$ satisfying a), b.2), b.3), and e). On the other hand, we apply the proof of Lemma 2.1 to $\{g_\xi : \xi \in I_0 \cup I_1\}$ to obtain a family of one-to-one functions $\{f_\xi : \xi \in I_j\}$ satisfying b.0) and b.1). Furthermore, $\{[f_\xi]_p : \xi \in I_0 \cup I_1\} \cup \{[\vec{n}]_p : n < \omega\}$ is linearly independent. Thus, condition c) also holds. Let us see how we get condition d). Following the notation of the proof of Lemma 2.1, at stage $\alpha < \mathfrak{c}$, we choose $\mu_\alpha < \mathfrak{c}$ such that $\{[h_{\alpha,\mu_\alpha}]_p : \alpha < \mathfrak{c}\} \cup \{[f_\xi]_p : \xi < \alpha\} \cup \{[\vec{\beta}]_p : \beta < \mathfrak{c}\}$ is linearly independent. We know that $h_{\mu_\alpha} : \omega \to A_{\mu_\alpha}$ is a bijection. Now, pick $B \subseteq A_{\mu_\alpha}$ such that $|B| = |A_{\mu_\alpha} \setminus B| = \omega$ and $h_{\mu_\alpha}^{-1}(B) \in p$. Choose a bijection $\sigma : \omega \to \omega$ for which $\sigma(n) = h_{\mu_\alpha}(n)$ for every $n \in h_{\mu_\alpha}^{-1}(B)$ and $\sigma[\omega \setminus h_{\mu_\alpha}^{-1}(B)] = \omega \setminus B$. So, we define $f_\alpha(n) = g_\alpha(\sigma(n))$, for every $n < \omega$. Then, we have that $f_\alpha = g_\alpha(\sigma(n)) = g_\alpha(h_{\mu_\alpha}(n)) = h_{\alpha,\mu_\alpha}(n)$, for every $n \in h_{\mu_\alpha}^{-1}(B)$. Therefore, $[f_\alpha]_p = [g_\alpha \circ \sigma]_p = [h_{\alpha,\mu_\alpha}]_p$. This shows condition d). $\square$

In the next example, we fix a family $\{f_\xi : \omega \le \xi < \mathfrak{c}\} \subseteq ([\mathfrak{c}]^{<\omega})^\omega$ and sets $I_0, I_1, I_2, I_3 \in [\mathfrak{c} \setminus \omega]^{\mathfrak{c}}$ satisfying all the properties of Lemma 3.1.

Example 3.2. If there is a selective ultrafilter on ω, then there are two almost p-compact groups whose product is not countably compact.

Proof. By using clause c), Lemma 2.3 and the proof of Example 2.4, we can define, for every $\alpha < \mathfrak{c}$, a non-trivial homomorphism $\Phi_\alpha : [\mathfrak{c}]^{<\omega} \to \{0,1\}$ so that

i) $\Phi_\alpha(\{\xi\}) = p\text{-}\lim_{n\to\omega} \Phi_\alpha(f_\xi(n))$, for every $\omega \le \xi < \mathfrak{c}$; and
ii) $\Phi_\alpha(F_\alpha) = 1$.

Hence, for each $\xi < \mathfrak{c}$ we define $x_\xi(\alpha) = \Phi_\alpha(\{\xi\})$, for every $\alpha < \mathfrak{c}$. Then, we have that $\{x_\xi : \xi < \mathfrak{c}\}$ is a linearly independent subset of $\{0,1\}^{\mathfrak{c}}$ and $x_\xi = p\text{-}\lim_{n\to\omega} x_{f_\xi(n)}$, for every $\omega \le \xi < \mathfrak{c}$. We put

$E = \langle\{x_n : n < \omega\}\rangle$,
$H_0 = \langle\{x_\xi : \xi \in I_0 \cup I_2\}\rangle$,
$H_1 = \langle\{x_\xi : \xi \in I_1 \cup I_3\}\rangle$,
$G_0 = E + H_0 = \langle\{x_\xi : \xi \in \omega \cup I_0 \cup I_2\}\rangle$ and
$G_1 = E + H_1 = \langle\{x_\xi : \xi \in \omega \cup I_1 \cup I_3\}\rangle$.

It is evident that $G_0 \cap G_1 = E$. Hence, we deduce that $G_0 \times G_1$ is not countably compact. As in Lemma 2.2, both H_0 and H_1 are p-compact groups. We shall show that G_j is almost p-compact, for $j \in \{0,1\}$. For this, fix a sequence $(a_n)_{n<\omega}$ in G_j. Choose two sequences $(e_n)_{n<\omega}$ in E and $(h_n)_{n<\omega}$ in H_j so that $a_n = e_n + h_n$, for every $n < \omega$. By the selectivity of p, there is $A \in p$ such that either $e_n = e$, for all $n \in A$, for some $e \in E$, or the function $n \to e_n$, for $n \in A$, is one-to-one. In the former case, $e + h = p\text{-}\lim_{n\to\omega}(e_n + h_n) \in E + H_j = G_j$, where $h = p\text{-}\lim_{n\to\omega} h_n$. In the latter case, we can find a one-to-one function $g \in ([\omega]^{<\omega})^\omega$ such that $e_n = x_{g(n)}$, for every $n \in A$. According to clause e) of Lemma 3.1, there are a bijection $\sigma : \omega \to \omega$ and $\xi \in I_j$ such that $[g \circ \sigma]_p = [f_\xi]_p$. Pick $B \in p$ so that $B \subseteq A$ and $g(\sigma(n)) = f_\xi(n)$, for every $n \in B$. Hence, $e_{\sigma(n)} = x_{g(\sigma(n))} = x_{f_\xi(n)}$, for every $n \in B$. This implies that

$$p\text{-}\lim_{n\to\omega} e_{\sigma(n)} = p\text{-}\lim_{n\to\omega} x_{f_\xi(n)} = x_\xi \in H_j.$$

So, $x_\xi = \overline{\sigma}(p)\text{-}\lim_{n\to\omega} e_n$ and $q = \overline{\sigma}(p) \in T(p)$. Since H_j is p-compact, it is also q-compact. Thus, $q\text{-}\lim_{n\to\omega} h_n = h \in H_j$. Hence, $q\text{-}\lim_{n\to\omega} a_n = x_\xi + h \in H_j + H_j \subseteq G_j$. Therefore, G_j is almost p-compact. □

Finally, we list some open problems that the authors were unable to solve.

Question 3.3. For an arbitrary $p \in \omega^*$, is there a p-compact group without non-trivial convergent sequences?

Question 3.4. Does the existence of a P-point in ω^* imply the existence of two countably compact groups whose product is not countably compact?

Question 3.5. Does the existence of a selective ultrafilter on ω imply the existence of a countably compact group whose square is not countably compact?

Acknowledgment

We are grateful to the referee for his helpful remarks and suggestions.

References

1. A. R. Bernstein, *A new kind of compactness for topological spaces*, Fund. Math. **66** (1970), 185-193. MR0251697 (40:4924)
2. W. Comfort and S. Negrepontis, *The Theory of Ultrafilters*, Springer-Verlag, Berlin, 1974. MR0396267 (53:135)
3. E. K. van Douwen, *The product of two countably compact topological groups*, Trans. Amer. Math. Soc. **262** (1980), 417 - 427. MR0586725 (82b:22002)
4. S. Garcia-Ferreira, *Quasi M-compact spaces*, Czechoslovak Math. J. **46** (1996), 161 - 177. MR1371698 (97b:54033)
5. L. Gillman and M. Jerison, *Rings of continuous functions*, Lectures Notes in Mathematics No. **27**, Springer-Verlag, 1976. MR0407579 (53:11352)
6. J. Ginsburg and V. Saks, *Some applications of ultrafilters in topology*, Pacific J. Math. **57** (1975), 403-418. MR0380736 (52:1633)
7. A. Hajnal and I. Juhász, *A separable normal topological group need not be Lindelöf*, Gen. Topology Appl. **6** (1976), 199-205. MR0431086 (55:4088)
8. K. P. Hart and J. van Mill, *A countably compact topological group H such that $H \times H$ is not countably compact*, Trans. Amer. Math. Soc. **323** (1991), 811- 821. MR0982236 (91e:54025)
9. A. H. Tomita, *A group under $MA_{countable}$ whose square is countably compact but whose cube is not*, Topology Appl. **91** (1999), 91-104. MR1664516 (2000d:54039)
10. A. H. Tomita, *Countable compactness and finite powers of topological groups without convergent sequences*, submitted.

Instituto de Matemáticas (UNAM), Apartado Postal 61-3, Xangari, 58089, Morelia, Michoacán, México

E-mail address: sgarcia@matmor.unam.mx

Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, CEP 05315-970, São Paulo, Brasil

E-mail address: tomita@ime.usp.br

Department of Mathematics, York University, 474700 Keele Street, M3P 1P3, Toronto, Ontario, Canada

E-mail address: watson@@mathstat.yorku.ca