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COUNTING FACES AND CHAINS IN POLYTOPES AND POSETS

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COUNTING FACES AND CHAINS IN POLYTOPES AND POSETS

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ABSTRACT. The purpose of this paper is to provide an overview of a class of problems concerning the enumeration of faces in convex polytopes or general triangulated spheres, and of chains in certain related posets. The degree to which we refine the objects being counted may vary; for example, we may count the number of chains consisting of k different faces of a given polytope, or we may instead ask for the number of chains of faces having dimensions given by a prespecified k-set of integers. Throughout, the unifying problem will be to determine all the affine linear relations satisfied by the numbers in question. The best known of all such relations is the Euler equation, $f_0-f_1+f_2 = 2$, which relates the number of vertices, edges and 2-faces of any 3-polytope.

While the paper is mostly expository, sections 4, 6 and 7 discuss new results. A listing of the section headings follows.

1. f-vectors of convex polytopes
2. Dehn-Sommerville equations
3. Spanning the Euler hyperplane and the Dehn-Sommerville space
4. Labeled simplicial complexes
5. Some other proofs of $h_S = h_{\tilde{S}}$
6. Affine span for completely balanced spheres
7. Eulerian poset complexes
8. Concluding remarks

1. f-VECTORS OF CONVEX POLYTOPES. By a convex polytope P we mean the convex hull of a finite point set in a real Euclidean space. Equivalently, P can be defined as the bounded intersection of finitely many closed half-spaces. By a face F of P we mean the intersection of P with a hyperplane having the property that P is contained in one of its closed half-spaces. Thus, the empty set is always a face of P, and we call P a face of P (whether or not it arises in the above manner). All other faces will be called proper faces, and they are finite in number. Each face of a polytope P is again a polytope.

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We define the dimension of a polytope P, $\dim P$, to be the dimension of $\mathrm{aff}(P)$, its affine hull, and say that P is a d-polytope if $\dim P = d$. In this case each face of P, except P itself, has dimension less than d. For each $i = -1,0,1,\ldots,d-1$, let $f_i(P)$ denote the number of i-dimensional faces of P. In particular, $f_{-1}(P) = 1$ counts the empty face, $f_0(P)$ is the number of vertices, $f_1(P)$ is the number of edges and $f_{d-1}(P)$ is the number of facets of P. We denote by $f(P)$ the vector $(f_{-1}(P),f_0(P),\ldots,f_{d-1}(P))$, called the f-vector of P. For a comprehensive treatment of the theory of convex polytopes and, in particular, of f-vectors see [15], [23] or [34]. For a survey of the latter topic which includes a discussion of the more recent results, see [29].

Let $f(P^d)$ denote the set of all f-vectors of d-polytopes. There is considerable interest in describing the set $f(P^d)$ exactly, but this remains unsettled in general. However, we can describe $\mathrm{aff}(f(P^d))$ and certain inequalities satisfied by each $f \in f(P^d)$. First, each $f \in f(P^d)$ satisfies the Euler Equation

$$f_0 - f_1 + f_2 - \ldots \pm f_{d-1} = 1 - (-1)^d,$$

and, further, this equation specifies the affine hull, namely

$$\mathrm{aff}(f(P^d)) = \{(f_{-1},f_0,\ldots,f_{d-1}): f_{-1} = 1,\ f_0 - f_1 + \ldots = 1 - (-1)^d\}.$$

(We will give a proof of the latter assertion in Section 3.)

The inequalities are somewhat harder to describe. To this end, consider the moment curve in R^d given by $x(t) = (t,t^2,t^3,\ldots,t^d)$ and choose real numbers $t_1 < t_2 < \ldots < t_n$, with $n > d$. Define $C(n,d)$ to be the convex hull of $V = \{x(t_1),x(t_2),\ldots,x(t_n)\}$. While the actual polytope obtained by this procedure depends on the choices of the t_i's, it is known that its combinatorial structure, in particular, its f-vector, is independent of the t_i's. We use the symbol $C(n,d)$ to refer to this combinatorial type. (In general, when we refer to a polytope, we will be concerned only with its combinatorial type, that is, its face lattice.) It is easily seen that $C(n,d)$ is a simplicial d-polytope, that is, each facet (and thus, each proper face) is a simplex (an $(r-1)$-polytope having just r vertices).

One of the most remarkable properties of the polytope $C(n,d)$ is that it is neighborly, that is, each pair of vertices forms an edge of $C(n,d)$. In fact, for $k = 1,\ldots,[d/2]$, the convex hull of any k-subset of V is a face of $C(n,d)$. (Here $[x]$ denotes the largest integer less than or equal to x.) Thus among all d-polytopes with n vertices, $C(n,d)$ clearly has the maximum number of i-faces for $i = 0,1,\ldots,[d/2]-1$. That $C(n,d)$ has the maximum number of i-faces, among all d-polytopes with n vertices, for all i is the content of the Upper Bound Theorem, first formulated by Motzkin [36] and proved by McMullen [31]. Thus we have that for all d-polytopes P with n vertices

$$f(P) \leq f(C(n,d)).$$

Since the number of i-faces of $C(n,d)$ is known for each i as a function of n and d, this gives upper bounds for each $f_i(P)$ in terms of $n = f_0(P)$ (and d), and thus inequalities which must be satisfied for each $f \in f(P^d)$. By the above discussion we have

$$f_i(C(n,d)) = \binom{n}{i+1}$$

for $i = 0,1,\ldots,[d/2]-1$. See [23] or [34] for a general expression for the remaining coordinates of $f(C(n,d))$ and its derivation.

A complete description of $f(P^d)$ has remained elusive for general d. There do not seem to be even reasonable conjectures as to a final set of conditions. However, if one restricts to the case of simplicial polytopes, then the situation is considerably better understood. In fact, the set $f(P_s^d)$ of all f-vectors of simplicial d-polytopes is completely known, being specified entirely by a list of linear equations, linear inequalities and nonlinear inequalities. We will describe these in turn. First note that by the usual polyhedral polarity [23], to each d-polytope P there corresponds another d-polytope P^* which has the property that $f_i(P) = f_{d-1-i}(P^*)$. In particular, when P is simplicial, then P^* is simple, that is, each vertex is on precisely d facets. Thus, describing $f(P_s^d)$ is equivalent to describing the f-vectors of all simple d-polytopes. Further, one proves the Upper Bound

Theorem by first showing that the maximum number of faces must occur in simplicial polytopes, and then proving the Upper Bound Theorem for simplicial polytopes.

First, we note that each $f \in f(P_s^d)$ satisfies the <u>Dehn-Sommerville Equations</u>

$$E_k^d: \sum_{j=k}^{d-1} (-1)^j \binom{j+1}{k+1} f_j = (-1)^{d-1} f_k$$

for $k = -1,0,1,\dots,d-1$. The equation E_{-1}^d is just the Euler equation. It is known that $[(d+1)/2]$ of these equations are independent, and they completely determine $\mathrm{aff}(f(P_s^d))$. Thus the dimension of $\mathrm{aff}(f(P_s^d))$ is $[d/2]$. We treat these questions in detail in sections 2 and 3.

To be able to describe the remaining conditions, we must apply a change of variables to the space of f-vectors, first used by Sommerville [40], which recasts the Dehn-Sommerville equations in a particularly simple form. If $f = f(P)$ for a d-polytope P, then we define the <u>h-vector</u> of P to be the vector $h(P) = (h_0,h_1,\dots,h_d)$, where for each i

$$h_i = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}.$$

Note that $h_0(P) = 1$ since $f_{-1}(P) = 1$. (Also, note the dependence of h on $d = \dim P$.) These relations can be inverted to give

$$f_j = \sum_{i=0}^{j+1} \binom{d-i}{d-j-i} h_i.$$

Thus f_j is a <u>non-negative</u> linear combination of $h_0,\dots,h_{j+1}$, and so an inequality of the form $h(P) \leq h(P')$ implies the corresponding inequality $f(P) \leq f(P')$. In fact, the proof of the Upper Bound Theorem proceeds by showing $h(P) \leq h(C(n,d))$ for the simplicial d-polytopes with n vertices. In terms of the h-vector of P, the Dehn-Sommerville equations become $h_i = h_{d-i}$, for $i = 0,1,\dots,[d/2]$. (See [34] or [35] where our h_{k+1} corresponds to their $g_k^d(P)$.)

Let $h(P_s^d)$ denote the set of h-vectors of simplicial d-polytopes. By the above discussion, knowing $h(P_s^d)$ is equivalent to knowing $f(P_s^d)$. We can now describe the set of linear inequalities satisfied by all $f \in f(P_s^d)$. They were first proposed by McMullen and Walkup in the form of a Generalized Lower Bound Conjecture, which stated that $h_{i+1} \geq h_i$, for $i = 0,1,\ldots,[d/2]-1$. Thus, in light of the Dehn-Sommerville equations, these inequalities imply that the h-vector must be unimodal. In terms of the f_i's, $h_{i+1} \geq h_i$ implies a lower bound on f_i as a linear function of the f_j's for $j < i < [d/2]$; these lower bounds imply the lower bounds given in the so-called Lower Bound Theorem proved by Barnette [6] (see [35]).

To complete the description, we must establish a last bit of notation. For positive integers h and i, we note that h can always be written uniquely in the form

$$h = \binom{n_i}{i} + \binom{n_{i-1}}{i-1} + \ldots + \binom{n_j}{j}.$$

where $n_i > n_{i-1} > \ldots > n_j \geq j \geq 1$. (Choose n_i to be the largest integer with $h \geq \binom{n_i}{i}$, etc.) Define the ith pseudopower of h to be

$$h^{\langle i\rangle} = \binom{n_i+1}{i+1} + \binom{n_{i-1}+1}{i} + \ldots + \binom{n_j+1}{j+1}.$$

Put $0^{\langle i\rangle} = 0$ for all i.

We state the nonlinear inequalities on the components of the h-vector (and thus the f-vector) together with the earlier conditions in the form of a characterization of $h(P_s^d)$.

THEOREM (MCMULLEN'S CONDITIONS). An integer vector $h = (h_0,h_1,\ldots,h_d)$ is the h-vector of a simplicial convex d-polytope if and only if the following three conditions hold:

(i) $h_i = h_{d-i}$, $i = 0,1,\ldots,[d/2]$,

(ii) $h_{i+1} \geq h_i$, $i = 0,1,\ldots,[d/2]-1$, and

(iii) $h_0 = 1$ and $h_{i+1}-h_i \leq (h_i-h_{i-1})^{\langle i\rangle}$, $i = 1,\ldots,[d/2]-1$.

This characterization was conjectured in 1971 by McMullen [33], [34], and proved by him for $d \leq 5$ and for the case of d-polytopes having n vertices where $d < n < d+3$. The sufficiency of these conditions was proved by Billera and Lee [11], [12]; the proof of necessity was given by Stanley [47]. The proof of sufficiency depends heavily on insights provided by earlier work of Stanley [43] in which the Upper Bound Theorem was extended to general triangulations of spheres by means of techniques of commutative algebra. The proof of necessity extends this earlier work, introducing powerful new techniques from algebraic geometry. See [10] and [50] for overviews of these developments.

Throughout the paper we will use the following notational conventions. N will denote the set of natural numbers. A point $z \in N^{r+1}$ will have coordinates $(z_0, z_1, \ldots, z_r)$. If $z, w \in N^{r+1}$, the inequality $z \leq w$ means $z_i \leq w_i$ for $0 \leq i \leq r$. If further for some i, $z_i < w_i$ we will write $z < w$. For $r \in N$ write $\langle r\rangle = \{0,1,\ldots,r-1\}$. For $S \subseteq \langle r\rangle$, we sometimes denote $\langle r\rangle \setminus S$ by $\tilde{S}$. It is convenient to assign the following values to binomial coefficients: $\binom{n}{0} = 0$ if $n < 0$, $\binom{n}{-1} = 0$ if $n \neq -1$, $\binom{-1}{-1} = 1$ and $\binom{n}{m} = 0$ if $0 \leq n < m$.

2. DEHN-SOMMERVILLE EQUATIONS. In 1905 Dehn conjectured the existence of $[d/2]$ linear relations on the f-vectors of simplicial d-polytopes. These equations, which form part of the McMullen conditions, were discovered and proved by Sommerville in 1927 [40]. They were largely forgotten until Klee reproved them in 1963 in a more general context [26]. They are referred to as the Dehn-Sommerville equations. We give here Sommerville's original proof. First we need the definition of an interval in the face lattice of a polytope.

Let F_1 and F_2 be i- and k-faces, respectively, of a d-polytope P, and suppose $F_1 \subseteq F_2$. (Here we allow $F_1 = \emptyset$ or $F_2 = P$.) Define the <u>interval</u> $[F_1, F_2]$ to be the set of faces G of P such that $F_1 \subseteq G \subseteq F_2$. $[F_1, F_2]$ is ordered by inclusion, and is isomorphic to the face lattice of a $(k-i-1)$-polytope. We will assume the Euler relation holds for any polytope. (See [23] for a proof.)

THEOREM 2.1. Dehn-Sommerville Equations. If $f(P) = (f_{-1}, f_0, \ldots, f_{d-1})$ is the f-vector of a simplicial d-polytope P, then for $-1 \leq k \leq d-2$,

$$f_k = \sum_{j=k}^{d-1} (-1)^{d-1-j} \binom{j+1}{k+1} f_j.$$

PROOF. Let F be a k-face of P, and write P_F for the $(d-k-1)$-polytope with face lattice $[F,P]$. Applying Euler's formula to P_F we get

$$1 = \sum_{i=-1}^{d-k-2} (-1)^{d-k-i} f_i(P_F).$$

If we sum this equation over all k-faces of P we get

$$f_k(P) = \sum_{i=-1}^{d-k-2} (-1)^{d-k-i} \sum_{\dim F=k} f_i(P_F).$$

The i-faces of P_F correspond to $(k+i+1)$-faces of P containing F. So

$$\sum_{\dim F=k} f_i(P_F) = \text{the number of pairs } F^k \subseteq F^{k+i+1}$$

$$= \sum_{\dim G=k+i+1} f_k(G),$$

where F^k is a k-face, and F^{k+i+1} is a $(k+i+1)$-face of P. Now, since P is simplicial, G is a $(k+i+1)$-simplex and $f_k(G) = \binom{k+i+2}{k+1}$. So

$$f_k(P) = \sum_{i=-1}^{d-k-2} (-1)^{d-k-i} \sum_{\dim G=k+i+1} \binom{k+i+2}{k+1}$$

$$= \sum_{i=-1}^{d-k-2} (-1)^{d-k-i} \binom{k+i+2}{k+1} f_{k+i+1}(P)$$

$$= \sum_{j=k}^{d-1} (-1)^{d-1-j} \binom{j+1}{k+1} f_j(P). \quad \square$$

Note that Euler's formula is the Dehn-Sommerville equation with $k = -1$. The equation for $k = d-2$ is $2f_{d-2} = df_{d-1}$. This simply says that each $(d-2)$-face is on exactly two facets, and each facet has exactly d $(d-2)$-faces. Sommerville noted that in terms of the h-vector of a polytope the Dehn-Sommerville equations have a very nice form.

COROLLARY 2.2. If $h(P) = (h_0, h_1, \ldots, h_d)$ is the h-vector of a simplicial d-polytope P, then for $0 \leq r \leq d$, $h_r = h_{d-r}$.

PROOF. Let E_k^d be the Dehn-Sommerville equation:

$$f_k = \sum_{j=k}^{d-1} (-1)^{d-1-j} \binom{j+1}{k+1} f_j.$$

For $0 \leq r \leq d$ we take the following linear combination of the equations: $\sum_{i=0}^{r} (-1)^i \binom{d-i}{d-r} E_{i-1}^d$. On the left-hand side we get

$$\sum_{i=0}^{r} (-1)^i \binom{d-i}{d-r} f_{i-1} = (-1)^r h_r.$$

On the right-hand side we get

$$\sum_{i=0}^{r} (-1)^i \binom{d-i}{d-r} \sum_{j=i-1}^{d-1} (-1)^{d-1-j} \binom{j+1}{i} f_j = \sum_{i=0}^{r} (-1)^i \binom{d-i}{d-r} \sum_{j=i}^{d} (-1)^{d-j} \binom{j}{i} f_{j-1}$$

$$= \sum_{j=0}^{d} (-1)^{d-j} f_{j-1} \sum_{i=0}^{j} (-1)^i \binom{d-i}{d-r} \binom{j}{i}.$$

We use the identity $\sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{i+m}{t} = (-1)^n \binom{m}{t-n}$ to simplify the right-hand sum:

$$\sum_{i=0}^{j} (-1)^i \binom{d-i}{d-r} \binom{j}{i} = \sum_{s=0}^{j} (-1)^{s-i} \binom{j}{s} \binom{d-j+s}{r} = \binom{d-j}{d-r-j}.$$

So the right-hand side of $\sum_{i=0}^{r} (-1)^i \binom{d-i}{d-r} E_{i-1}^d$ is

$$\sum_{j=0}^{d} (-1)^{d-j} f_{j-1} \binom{d-j}{r} = \sum_{j=0}^{d-r} (-1)^{d-j} \binom{d-j}{r} f_{j-1} = (-1)^r h_{d-r}.$$

Thus the combination of E_k^d gives $h_r = h_{d-r}$. □

Since $h_0 = 1$, we see immediately that at most $[d/2]$ of the h_i, and thus of the f_i, can be independent. It is easy to show that the equations $h_r = h_{d-r}$ are actually equivalent to the equations E_k^d. In the next section we will do this by showing that the affine span of the f-vectors of simplicial d-polytopes is determined by the equations $h_r = h_{d-r}$. Although Sommerville observed this form of the equations, he did not realize the significance of the h-vector, which was extremely important in the discovery and proof of the McMullen conditions (see [34], [45], [47] and [12]). Since the links of simplices in homology spheres are again homology spheres, Sommerville's proof of the Dehn-Sommerville equations extends to this more general case as well.

3. SPANNING THE EULER HYPERPLANE AND THE DEHN-SOMMERVILLE SPACE. In this section we describe some operations on convex polytopes and their effect on the f- and h-vectors. We use these to give simple direct proofs that the Euler equation and the Dehn-Sommerville equations are the only affine linear equations satisfied by the f-vectors of convex polytopes and simplicial convex polytopes, respectively. Note first that for any d-polytope Q, the Euler equation is equivalent to the relation $h_d(Q) = 1$.

If Q is a d-polytope, the pyramid on Q, $P(Q)$, is the convex $(d+1)$-polytope formed by taking the convex hull of Q with a point not in the affine span of Q. In terms of f-vectors we have

$$f_i(P(Q)) = f_i(Q) + f_{i-1}(Q) \quad \text{for} \quad i \leq d-1$$

and

$$f_d(P(Q)) = 1 + f_{d-1}(Q),$$

with the convention that $f_j(Q) = 0$ if $j < -1$. (See [23, §4.2].) The following is due to Sommerville [40].

PROPOSITION 3.1. For any d-polytope Q,

$$h(P(Q)) = (h(Q),1).$$

PROOF. Since the Euler equation for $P(Q)$ gives $h_{d+1}(P(Q)) = 1$, we need show $h_i(P(Q)) = h_i(Q)$ for $i \leq d$. Since $P(Q)$ is a $(d+1)$-polytope, if $i \leq d$,

$$\begin{aligned} h_i(P(Q)) &= \sum_{j=0}^{i} (-1)^{i-j}\binom{d+1-j}{d+1-i} f_{j-1}(P(Q)) \\ &= \sum_{j=0}^{i} (-1)^{i-j}\binom{d+1-j}{d+1-i}[f_{j-1}(Q) + f_{j-2}(Q)] \\ &= \sum_{j=0}^{i} (-1)^{i-j}\binom{d+1-j}{d+1-i} f_{j-1}(Q) + \sum_{j=0}^{i-1} (-1)^{i-j-1}\binom{d-j}{d+1-i} f_{j-1}(Q) \\ &= \sum_{j=0}^{i} (-1)^{i-j}\binom{d-j}{d-i} f_{j-1}(Q) = h_i(Q). \quad \square \end{aligned}$$

Note that $P(Q)$ is not simplicial unless Q is itself a simplex since Q is a facet of $P(Q)$. In this case $P(Q)$ is again a simplex, and we get by induction

COROLLARY 3.2. If Q is a simplex then

$$h(Q) = (1,1,\ldots,1). \quad \square$$

For a d-polytope Q, the <u>bipyramid</u> over Q, $B(Q)$, is defined to be the convex $(d+1)$-polytope formed by taking the convex hull of Q with a line segment which meets Q in a relative interior point of each. For example, the bipyramid over an interval is a square and the bipyramid over a square is an octahedron. For the f-vectors, we have [23, §4.3]

$$f_i(B(Q)) = f_i(Q) + 2f_{i-1}(Q) \quad \text{for} \quad i \leq d-1$$

and

$$f_d(B(Q)) = 2f_{d-1}(Q).$$

In terms of h-vectors, we have the following

PROPOSITION 3.3. For any d-polytope Q,

$$h(B(Q)) = (h(Q),0) + (0,h(Q)).$$

PROOF. Again, $h_{d+1}(B(Q)) = 1 = h_d(Q)$, so we must show $h_i(B(Q)) = h_i(Q) + h_{i-1}(Q)$ for $i \leq d$. By the proof of Proposition 3.1,

$$\begin{aligned} h_i(B(Q)) &= \sum_{j=0}^{i} (-1)^{i-j}\binom{d+1-j}{d+1-i}[f_{j-1}(Q) + 2f_{j-2}(Q)] \\ &= h_i(Q) + \sum_{j=0}^{i} (-1)^{i-j}\binom{d-(j-1)}{d-(i-1)}f_{j-2}(Q) \\ &= h_i(Q) + h_{i-1}(Q). \quad \square \end{aligned}$$

Note that if Q is simplicial, then so is $B(Q)$. For example, since the h-vector of the interval is $(1,1)$, that of the square is $(1,2,1) = (1,1,0) + (0,1,1)$, and that of the octahedron is $(1,3,3,1) = (1,2,1,0) + (0,1,2,1)$. While we do not in this section make use of the bipyramid operation, it will prove useful later on.

To define the final operation, let Q be a simplicial d-polytope and F a proper face of Q. If H is a ($(d-1)$-dimensional) hyperplane containing Q in one of its closed half spaces, then a point $x \notin H$ is said to be <u>beneath</u> H if it is on the same side of H as Q, and <u>beyond</u> H otherwise. Now let $F_1,\ldots,F_k$ be all the facets ($(d-1)$-dimensional faces) of Q which contain F. Let x be a point which is beyond the hyperplanes generated by these F_i's and beneath the hyperplanes generated by any other facets. (A point $x \notin Q$ sufficiently close to the centroid of F will do.) Define the stellar subdivision of the face F in Q, st(F,Q), to be the (simplicial) d-polytope which is the convex hull of $Q \cup \{x\}$. (See [18] where this operation is described for nonsimplicial Q as well.)

To describe st(F,Q) combinatorially, let Δ be the boundary complex of Q, and let σ be the set of vertices of F. Then the boundary complex of st(F,Q) is the complex $st(\sigma,\Delta)$, the stellar subdivision of simplex σ in Δ, where

$$st(\sigma,\Delta) = (\Delta\backslash\sigma) \cup \overline{x}\cdot\partial\sigma\cdot \ell k_\Delta \sigma .$$

Here $\Delta\backslash\sigma = \{\tau \in \Delta \mid \tau \not\supseteq \sigma\}$, $\ell k_\Delta \sigma$ is the link of σ in Δ, defined by

$$\ell k_\Delta \sigma = \{\tau \in \Delta \mid \tau \cap \sigma = \emptyset,\quad \tau \cup \sigma \in \Delta\},$$

$\partial\sigma$ is the complex of proper subsets of σ, $\overline{x}$ denotes the complex consisting of $\{x\}$ and $\emptyset$ and $\cdot$ denotes the join of simplicial complexes. (See [18].)

The complex $\ell k_\Delta \sigma$ is also the boundary complex of a polytope, the one whose face lattice is isomorphic to the interval [F,Q] in the face lattice of Q. This polytope will be of dimension k-1 if dim F = d-k; let h([F,Q]) denote its h-vector. For vectors $a = (a_0,a_1,\ldots,a_k)$, $b = (b_0,b_1,\ldots,b_\ell)$, let $a*b = (c_0,c_1,\ldots,c_{k+\ell})$ denote their convolution, where $c_i = \sum_{j=0}^{i} a_j b_{i-j}$. The following is proved in [28; Proposition 2.10.1].

LEMMA 3.4. If Q is a simplicial d-polytope and F is a (d-k)-face of Q, then

$$h(st(F,Q)) = h(Q) + \underbrace{(0,1,1,\ldots,1,0)}_{d-k+2} * h([F,Q]).$$

We wish to apply this when Q is a simplex. In this case, [F,Q] is the face lattice of a (k-1)-simplex, and so

$$h([F,Q]) = \underbrace{(1,1,\ldots,1)}_{k}.$$

So when Q is a simplex and F is a (d-k)-face of Q, we have

$$h(st(F,Q)) = h(Q) + \underbrace{(0,1,\ldots,1,0)}_{d-k+2} * \underbrace{(1,\ldots,1)}_{k}.$$

We denote the polytopes $st(F,Q)$ in this case by T^d_k; they are all the simplicial d-polytopes with $d+2$ vertices (see [23, §6.1]). For convenience, let T^d_0 denote the d-simplex.

PROPOSITION 3.5. For $0 \leq k \leq [d/2]$,

$$h(T^d_k) = (1,2,3,\ldots,k,k+1,k+1,\ldots,k+1,k,\ldots,3,2,1)$$

PROOF. It is enough to note in the expression above for $h(st(F,Q))$ that

$$h(Q) = (1,1,\ldots,1)$$

and $(0,1,\ldots,1,0)*(1,\ldots,1)$ is just the vector of (reverse) diagonal sums of the matrix

$$k \left(\overbrace{\begin{matrix} 0 & 1 & \ldots & 1 & 0 \\ 0 & 1 & \ldots & 1 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ 0 & 1 & \ldots & 1 & 0 \end{matrix}}^{d-k+2} \right)$$

starting in the upper left corner, i.e. 0, 0+1, 0+1+1, etc. □

We can now prove

THEOREM 3.6. The f-vectors of the simplicial d-polytopes T^d_k, $0 \leq k \leq [d/2]$ span the Dehn-Sommerville subspace, that is, the set

$$\{f(T^d_k) \mid 0 \leq k \leq [d/2]\}$$

is affinely independent.

PROOF. Since $f_{-1}(T^d_k) = 1$ for each k, it is enough to show the matrix of these f-vectors has full row rank. But since the transformation from f to h is invertible, it is enough to consider the matrix of h-vectors. By Proposition 3.5, this matrix is

$$\begin{pmatrix} 1 & 1 & 1 & \cdots\cdots & 1 & 1 & 1 \\ 1 & 2 & 2 & \cdots\cdots & 2 & 2 & 1 \\ 1 & 2 & 3 & \cdots\cdots & 3 & 2 & 1 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 1 & 2 & 3 & \cdot\,\cdot\,[d/2]+1\,\cdot\,\cdot & 3 & 2 & 1 \end{pmatrix},$$

and it is easily seen to have independent rows. □

To demonstrate a basis for the Euler hyperplane, we introduce another class of polytopes. Define $T_k^{d,r}$ to be the r-fold pyramid over the (d-r)-polytope T_k^{d-r}, where $0 \leq r \leq d-2$ and $1 \leq k \leq [\frac{1}{2}(d-r)]$ (that is, the result of performing the pyramid operation r times, beginnning with T_k^{d-r}). These polytopes constitute all the d-polytopes with d+2 vertices (see [23; §6.1]). It is straightforward, using Propositions 3.1 and 3.5 to write down the vectors $h(T_k^{d,r})$. However, for our purposes, we need consider only the case k = 1.

THEOREM 3.7. The f-vectors of the d-polytopes $T_1^{d,r}$, $0 \leq r \leq d-2$, together with that of the d-simplex T_0^d, span the Euler hyperplane.

PROOF. Again, it is enough to show that the matrix of h-vectors has full row rank. By Propositions 3.1 and 3.5, this matrix (with $h(T_0^d)$ first) is easily seen to be

$$\begin{pmatrix} 1 & 1 & \cdots\cdots & & & 1 & 1 \\ 1 & 2 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 2 & 2 & 1 & \cdots & 1 & 1 \\ \vdots & \vdots & \vdots & & & \vdots & \vdots \\ 1 & 2 & 2 & \cdots\cdots & & 2 & 1 \end{pmatrix}. \quad \square$$

We conclude that the Euler relation is the only linear relation holding for f-vectors of all polytopes. For similar results for general Euler-like relations, see [39].

4. LABELED SIMPLICIAL COMPLEXES.

DEFINITIONS. We start with some definitions pertaining to simplicial complexes.

Let Δ be a simplicial complex with n vertices, $v_1, v_2, \ldots, v_n$. Associated with Δ is its Stanley-Reisner ring A_Δ. Take the polynomial ring in n indeterminates corresponding to the vertices: $K[x_1, x_2, \ldots, x_n]$. (We will assume throughout that K is the field of rational numbers, although much of what we do works for more general fields.) Define the support of a monomial $m = \Pi_{i=1}^{n} x_i^{r_i}$ to be $\mathrm{supp}(m) = \{v_i : r_i > 0\}$. Let I_Δ be the ideal generated by those monomials m whose supports are not faces of Δ. Then the Stanley-Reisner ring of Δ is $A_\Delta = K[x_1, \ldots, x_n]/I_\Delta$. Note that as a K-vector space, A_Δ is generated by those monomials m whose supports are faces of Δ.

Let A be a K-algebra. An N-grading of A is a decomposition of A as a direct sum of K-vector spaces $A = \sum_{i \in N} A_i$, such that $\forall\, i,j \in N$, $A_i A_j \subseteq A_{i+j}$. (Throughout this paper, $\sum$ will denote direct sum.) More generally, if M is a commutative monoid with operation $+$, A is M-graded if $A = \sum_{m \in M} A_m$ with $A_m A_n \subseteq A_{m+n}$ $\forall\, m,n \in M$. We will assume our graded rings A are finitely generated K-algebras, so that each graded piece A_m is a finite-dimensional K-vector space, and that $A_0 = K$. In this case, $A_0 A_m = K A_m = A_m$ for all $m \in M$. So A is an M-graded K-vector space. We can then define the Hilbert function of A to be $H(A,m) = \dim_K A_m$ for $m \in M$. In the case where $M = N^{r+1}$ we define a generating function for the Hilbert function as follows. Let $t_0, t_1, t_2, \ldots, t_r$ be indeterminates and for $z = (z_0, z_1, z_2, \ldots, z_r) \in N^{r+1}$ write $t^z = t_0^{z_0} t_1^{z_1} t_2^{z_2} \ldots t_r^{z_r}$. Then the Hilbert (or Poincaré) series for A is $P(A,t) = \sum_{z \in N^{r+1}} H(A,z) t^z$. The Hilbert series of A_Δ contains important combinatorial information about Δ.

A labeling of a simplicial complex Δ is a partition of the vertex set of Δ into subsets: $V(\Delta) = V_0 \cup \ldots \cup V_r$. The vertices in V_i are said to be labeled i. If the labeling is such that each maximal face of Δ contains exactly one vertex from such V_i, then Δ is said to be completely balanced [46]. In this case Δ is a pure simplicial r-complex, that is, a simplicial complex in which every maximal face has $r+1$ vertices. Maximal faces are then called facets.

A combinatorially interesting class of simplicial complexes comes from partially ordered sets (posets). We assume all posets have least and greatest elements, $\hat{0}$ and $\hat{1}$. For P a finite poset, we form a simplicial complex $\Delta(P)$: $\Delta(P)$ has as vertices the elements of P (except $\hat{0}$ and $\hat{1}$), and a set of elements $\{x_1,x_2,\ldots,x_k\}$ is a face of $\Delta(P)$ if and only if $\hat{0} < x_{\sigma(1)} < x_{\sigma(2)} < \ldots < x_{\sigma(k)} < \hat{1}$ for some permutation σ of $\{1,2,\ldots,k\}$. We call simplicial complexes arising in this way poset complexes. If we wish to stress the poset giving rise to the complex, we call $\Delta(P)$ the order complex of P.

We give here a characterization of poset complexes due to Stanley [46]. A cycle of length k of a complex Δ is a sequence of vertices $v_1,v_2,\ldots,v_k,v_{k+1} = v_1$, allowing repetitions, where $\{v_i,v_{i+1}\}$ is an edge of Δ and no pair of vertices occurs twice in the same order. Such a cycle has a triangular chord if for some i, $\{v_i,v_{i+2}\}$ (subscripts modulo k) is an edge of Δ. A simplicial complex Δ is the order complex of a finite poset if and only if (i) any minimal set of vertices not forming a face of Δ has two elements; and (ii) every odd cycle of Δ has a triangular chord (that is, the 1-skeleton of Δ is a comparability graph [20]).

A poset P is called ranked if for every $x \in P$ all maximal chains up to x, $\hat{0} < x_0 < \ldots < x_k = x$, have the same length $k+1$. We then call k the rank of x, written $r(x)$. We make the convention $r(\hat{0}) = -1$ and define $r(P)$ to be $r(\hat{1})$. (Note that this rank function corresponds to the usual rank function shifted down by 1; it corresponds to the usual rank in $P\setminus\{\hat{0}\}$.) A labeling of the vertices of $\Delta(P)$ with the ranks of the corresponding elements in P makes $\Delta(P)$ completely balanced, since every maximal chain contains exactly one element of each rank. Conversely, if P is any finite poset whose order complex is completely balanced then every maximal chain in P has the same length, so P is ranked.

If Δ is a simplicial complex labeled by $0,1,2,\ldots,r$, define, for each $z = (z_0,\ldots,z_r) \in N^{r+1}$, $f_z(\Delta)$ to be the cardinality of the set $\Delta_z = \{\sigma \in \Delta: |\sigma \cap V_i| = z_i$ for each i, $0 \leq i \leq r\}$. Then the total number of j-faces of

Δ is $f_j(\Delta) = \sum_{\substack{z \\ \sum z_i = j+1}} f_z(\Delta)$. In the case where the labeling makes Δ completely balanced, we adopt a notation that will prove useful later. In this case, $f_z(\Delta) = 0$ unless $z_i \leq 1$ for all i. Write $Z = \text{supp } z = \{i\colon z_i = 1\}$ and define $\Delta_Z = \Delta_z$ and $f_Z(\Delta) = f_z(\Delta)$; then $f_j(\Delta) = \sum_{\substack{T \subseteq \langle r+1 \rangle \\ |T| = j+1}} f_T(\Delta)$. If Δ is completely balanced, we define $h_S(\Delta)$ in analogy to h_i:

$$h_S(\Delta) = \sum_{T \subseteq S} (-1)^{|S|-|T|} f_T(\Delta).$$

Letting $d = r+1$, it is easy to check that

$$\sum_{\substack{S \subseteq \langle d \rangle \\ |S| = i}} h_S(\Delta) = \sum_{j=0}^{i} (-1)^{i-j} \binom{d-j}{d-i} f_{j-1}(\Delta) = h_i(\Delta).$$

Because of the relationships between the numbers f_S and f_i, h_S and h_i, we will refer to the vectors $(f_S)_{S \subseteq \langle d \rangle}$ and $(h_S)_{S \subseteq \langle d \rangle}$ as the <u>extended</u> f-vector and h-vector.

If P is a poset of rank d (i.e., $r(\hat{1}) = d$), then $\Delta(P)$ is completely balanced with labels $\{0,1,\ldots,d-1\}$. For $T = \{i_1,\ldots,i_k\} \subseteq \{0,1,\ldots,d-1\}$, $f_T(\Delta(P))$ is the number of chains of P of the form $\hat{0} < x_{i_1} < \ldots < x_{i_k} < \hat{1}$, where $r(x_j) = j$. In this case we will often write $f_T(P) = f_T(\Delta(P))$, and $h_S(P) = h_S(\Delta(P))$.

An important special case of ranked poset complexes occurs when P is the lattice of faces of a convex d-polytope Q. (Here our choice of rank function leads to $r(P) = \dim Q$.) In this case $\Delta(P)$ is the complete barycentric subdivision of the polytope Q and is itself a convex polytope [18]. Each vertex of $\Delta(P)$ corresponds to a face of Q, its label being the dimension of that face. Here, $f_T(P)$ is the number of chains of faces of Q having precisely the dimensions in T. In particular $f_i(Q) = f_{\{i\}}(P)$, so information on the extended f-vector of P will yield information on $f(Q)$. In the remainder of this paper, especially in section 7, we will extend the methods and results of sections 2 and 3 to the study of extended f-vectors.

Another labeled complex associated with Q is the minimal subdivision $\sigma(Q)$ discussed in [7; section 2.5]. It is the result of performing stellar subdivisions on all the non-simplex faces of Q in order of decreasing dimension and so by [18] is a simplicial polytope. Vertices of $\sigma(Q)$ are labeled 0 if they are vertices of Q; otherwise they are labeled with the dimension of the face that they subdivide. This is not a balanced labeling (e.g. each original edge has two vertices with 0 labels and there are no vertices with label 1), but the results of this section will apply here.

Another way to describe the complex $\sigma(Q)$ is as follows. If V_Q is the set of vertices of Q, then the vertex set V of $\sigma(Q)$ is

$$V_Q \cup \{v_F : F \text{ a nonsimplex face of } Q\}$$

where the v_F's are new symbols. Each $v \in V$ is labeled with the appropriate dimension. Let

$$\phi = F_0 \subset F_1 \subset F_2 \subset \dots \subset F_k$$

be a chain of faces of Q; a simplex in $\sigma(Q)$ can be defined as follows. Let F_i, $0 \le i \le k$ be the largest face in the chain which is a simplex. Then

$$F_i \cup \{v_{F_{i+1}}, \dots, v_{F_k}\}$$

is a simplex in $\sigma(Q)$; by considering all chains we obtain all simplices in $\sigma(Q)$. For example, if Q is the pyramid over a square base, $\sigma(Q)$ is an octahedron with one vertex labeled 2, the rest labeled 0.

We now describe an N^{r+1}-grading on the ring A_Δ associated with a complex Δ labeled by $0,1,\dots,r$. If the generator x in A_Δ corresponds to a vertex v of Δ with label i, let the degree of x ($\deg x$) be e_i, the ith unit vector in N^{r+1}. Multiplication of monomials in A_Δ results in addition of degrees in N^{r+1}. So if we write $(A_\Delta)_z$ for the subspace of A_Δ spanned by monomials of degree $z \in N^{r+1}$, then $(A_\Delta)_z(A_\Delta)_w \subseteq (A_\Delta)_{z+w}$. Thus $A_\Delta = \sum_{z \in N^{r+1}} (A_\Delta)_z$ defines a grading on A_Δ. Stanley [46] derives the Hilbert function of A_Δ with respect to the N^{r+1}-grading. (Although he states the proposition for balanced complexes the proof uses only the fact that the complex is labeled.)

PROPOSITION 4.1. (Stanley). If Δ is a simplicial complex labeled by $0,1,\ldots,r$, then for all $w \in N^{r+1}$,

$$H(A_\Delta, w) = \sum_{z \in N^{r+1}} f_z(\Delta) \prod_{i=0}^{r} \binom{w_i - 1}{z_i - 1}. \quad \square$$

Recall that we use the conventions $\binom{n}{0} = 0$ if $n < 0$, $\binom{n}{-1} = 0$ if $n \neq -1$, and $\binom{-1}{-1} = 1$.

Let us apply Proposition 4.1 in the case where Δ is completely balanced. In this case $f_z(\Delta) = 0$ if $z_i > 1$ for some i. On the other hand the product $\Pi_{i=0}^{r} \binom{w_i - 1}{z_i - 1}$ is nonzero if and only if $z \leq w$ and, for all i, $z_i = 0$ if and only if $w_i = 0$. Together these mean that the term $f_z(\Delta)\ \Pi_{i=0}^{r} \binom{w_i - 1}{z_i - 1}$ is nonzero if and only if z is given by

$$z_i = \begin{cases} 1 & \text{if } w_i \geq 1 \\ 0 & \text{if } w_i = 0. \end{cases}$$

For this z, $\Pi_{i=0}^{r} \binom{w_i - 1}{z_i - 1} = 1$, so $H(A_\Delta, w) = f_z(\Delta) = f_{\text{supp } w}(\Delta)$.

AN EXACT SEQUENCE FOR SPHERES. This subsection proves that a certain sequence of graded vector spaces associated with a homology sphere is exact. The exact sequence is given without proof by Danilov [17], who referred to a related result of Kouchnirenko [27]. The proof here is based on that of Kouchnirenko. The exact sequence enables us to prove generalizations of the Dehn-Sommerville equations for labeled simplicial spheres.

Any set L of faces of a simplicial complex determines a complex $\overline{L}$ consisting of all faces of elements of L. Recall that for σ a face of a complex Δ the <u>link</u> of σ in Δ is $\ell k_\Delta \sigma = \{\tau \in \Delta : \tau \cap \sigma = \emptyset, \tau \cup \sigma \in \Delta\}$; define the <u>star</u> of σ in K to be $\text{star}_\Delta \sigma = \{\tau \in \Delta : \sigma \subseteq \tau\}$. Then $\overline{\text{star}_\Delta \sigma}$ is the join $\ell k_\Delta \sigma \cdot \overline{\{\sigma\}} = \{\tau \cup \rho : \tau \in \ell k_\Delta \sigma, \rho \subseteq \sigma\}$. Also define the <u>boundary</u> of $\text{star}_\Delta E$ to be $(\text{star}_\Delta \sigma)^\bullet = \overline{\text{star}_\Delta \sigma} \setminus \text{star}_\Delta \sigma = \{\tau \in \overline{\text{star}_\Delta} \sigma : \sigma \not\subseteq \tau\}$. A $(d-1)$-dimensional complex Δ is said to be a <u>homology</u> $(d-1)$-<u>sphere</u> if for each k-face $\sigma \in \Delta$, $\ell k_\Delta \sigma$ is a $(d-k-2)$-dimensional complex having the rational homology of a $(d-k-2)$-sphere, $-1 \leq k \leq d-1$.

Let Δ be a homology (d-1)-sphere, and let A_Δ be its Stanley-Reisner ring. As a vector space basis for A_Δ we choose the set of monomials whose supports are faces of Δ. Let M be the semigroup of all monomials in $\{x_1, x_2, \ldots, x_n\}$ $(n = f_0(\Delta))$, including the "empty" monomial 1, with standard monomial multiplication. Then A_Δ is an M-graded algebra, and if supp m is a face of Δ, we write $A_\Delta(m)$ for the subspace of A_Δ generated by m.

Let Δ_j be the set of j-faces of Δ $(-1 \leq j \leq d-1)$, and for $\sigma \in \Delta_j$, $\sigma = \{v_{i_0}, v_{i_1}, \ldots, v_{i_j}\}$, let $A_\sigma = K[x_{i_0}, x_{i_1}, \ldots, x_{i_j}]$. Define $C_j = \sum_{\sigma\in\Delta_j} A_\sigma$. C_j with componentwise multiplication is then a K-algebra. A typical element of C_j is written $(a_\sigma)_{\sigma\in\Delta_j}$. For $m \in M$ the <u>homogeneous elements of degree</u> m in C_j are the elements $(q_\sigma m)_{\sigma\in\Delta_j}$, where $\forall_\sigma\ q_\sigma \in K$ and $q_\sigma = 0$ unless supp $m \subseteq \sigma$. If we write $C_j(m)$ for the set of homogeneous elements of degree m in C_j, then $C_j = \sum_{m\in M} C_j(m)$, and for $m,n \in M$, $C_j(m)\cdot C_j(n) \subseteq C_j(mn)$, so this defines an M-grading of C_j. Note that $C_j(m) = 0$ for $j < \dim(\text{supp } m)$.

If A and B are M-graded vector spaces and $g: A \to B$ is a linear transformation, we say g is <u>homogeneous with respect to M</u> (or M-homogeneous or a homomorphism of M-graded vector spaces) if $\forall_{m\in M}\ g(A_m) \subseteq B_m$. In the context defined above, a linear transformations $g: C_j \to C_{j-1}$ is M-homogeneous if $g(C_j(m)) \subseteq C_{j-1}(m)$.

THEOREM 4.2. For Δ a homology (d-1)-sphere, A_Δ, C_j and M defined as above, there exist M-homogeneous linear transformations ∂_i such that the sequence

$$0 \longrightarrow A_p \xrightarrow{\partial_d} C_{d-1} \xrightarrow{\partial_{d-1}} C_{d-2} \xrightarrow{\partial_{d-2}} \cdots$$

$$\xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \longrightarrow 0$$

is exact.

PROOF. First we define ∂_j, $0 \leq j < d-1$. To simplify notation write $m_\sigma = (0,\ldots,0,m,0,\ldots,0)$, the element of C_j with monomial m in the σ component and zeros elsewhere. As a K-vector space, C_j has basis $\{m_\sigma : \sigma \in \Delta_j,\ \text{supp } m \subseteq \sigma\}$. We define the σ' component $(\sigma' \in \Delta_{j-1})$ of $\partial_j(m_\sigma)$ by analogy to the differential on the ordered chain complex of Δ.

Assume the vertices of Δ are ordered once and for all $v_1, v_2, \ldots, v_n$. Write $\sigma = \{v_{i_0}, v_{i_1}, \ldots, v_{i_j}\}$ with $i_0 < i_1 < \ldots < i_j$. If $\sigma' \subseteq \sigma$ and $\dim \sigma' = j-1$ then $\sigma' = \{v_{i_0}, v_{i_1}, \ldots, v_{i_j}\} \setminus \{v_{i_q}\}$. In this case, define $s(\sigma,\sigma') = (-1)^q$. If $\sigma' \not\subseteq \sigma$ let $s(\sigma,\sigma') = 0$. Now define the σ' component of $\partial_j(m_\sigma)$ to be

$$(\partial_j(m_\sigma))_{\sigma'} = \begin{cases} s(\sigma,\sigma')m, & \text{if } \operatorname{supp} m \subseteq \sigma' \\ 0, & \text{otherwise.} \end{cases}$$

This map is clearly M-homogeneous.

Next we show the exactness of the sequence $0 \to A_\Delta \to C_{d-1} \to \ldots \to C_0 \to C_{-1} \to 0$ (by showing that the sequence "restricted to each $m \in M$" is exact. Let $m \in M$, $m \neq 1$; let $\sigma = \operatorname{supp} m$, $k = \dim \sigma$. At m we get the sequence $C_{d-1}(m) \to C_{d-2}(m) \to \ldots \to C_{k+1}(m) \to C_k(m) \to 0$ or

$$(4.3) \qquad \sum_{\substack{\tau \in \Delta_{d-1} \\ \sigma \subseteq \tau}} Km \to \sum_{\substack{\tau \in \Delta_{d-2} \\ \sigma \subseteq \tau}} Km \to \ldots \to \sum_{\substack{\tau \in \Delta_{k+1} \\ \sigma \subseteq \tau}} Km \to Km \to 0.$$

The proof that this part of the sequence is exact will also produce the map $\partial_d\colon A_\Delta(m) = Km \to C_{d-1}(m)$.

We claim that the restricted sequence (4.3) is a chain complex isomorphic to the relative chain complex $C(\overline{\operatorname{star}_\Delta \sigma})/C((\operatorname{star}_\Delta \sigma)^\cdot)$ over K. $C_j(m)$ is generated as a K-vector space by elements m_τ, where $\tau \in \Delta_j$, $\sigma \subseteq \tau$. Such τ are precisely the j-dimensional elements of $\overline{\operatorname{star}_\Delta \sigma} \setminus (\operatorname{star}_\Delta \sigma)^\cdot$; thus they form a basis for $C_j(\overline{\operatorname{star}_\Delta \sigma})/C_j((\operatorname{star}_\Delta \sigma)^\cdot)$.

We use this correspondence to define the linear map $g_j\colon C_j(m) \to C_j(\overline{\operatorname{star}_\Delta \sigma})/C_j((\operatorname{star}_\Delta \sigma)^\cdot)$ such that $g_j(m_\tau) = \tau$; this map is clearly invertible. We want the following square to commute:

$$\begin{array}{ccc} C_j(m) & \xrightarrow{\partial_j} & C_{j-1}(m) \\ \Big\downarrow g_j & & \Big\downarrow g_{j-1} \\ C_j(\overline{\operatorname{star}_\Delta \sigma})/C_j((\operatorname{star}_\Delta \sigma)^\cdot) & \xrightarrow{d_j} & C_{j-1}(\overline{\operatorname{star}_\Delta \sigma})/C_{j-1}((\operatorname{star}_\Delta \sigma)^\cdot). \end{array}$$

Here ∂_j means $\partial_j | C_j(m)$, and d_j is the differential of the relative chain complex. But the ∂_j were defined precisely to make this diagram commute. We have

$$g_{j-1}(\partial_j(m_\tau)) = g_{j-1}\Big(\sum_{\substack{\tau' \in \Delta_{j-1} \\ \text{supp } m \subseteq \tau' \subseteq \tau}} s(\tau,\tau')m_{\tau'}\Big)$$

$$= \sum_{\substack{\tau' \in \Delta_{j-1} \\ \text{supp } m \subseteq \tau' \subseteq \tau}} s(\tau,\tau')\tau'.$$

On the other hand,

$$d_j(g_j(m_\tau)) = d_j(\tau) = \sum s(\tau,\tau')\tau',$$

the summation being over $\tau' \in \overline{\text{star}_\Delta \sigma} \setminus (\text{star}_\Delta \sigma)^\bullet$ with $\dim \tau' = j-1$. By the preceding paragraph this shows $g_{j-1} \circ \partial_j = d_j \circ g_j$.

So g_j is an isomorphism between chain complexes. By [24; Corollary 2.10.12],

$$H_j(\overline{\text{star}_\Delta \sigma}, (\text{star}_\Delta \sigma)^\bullet) = \tilde{H}_{j-k-1}(\ell k_\Delta \sigma) = \begin{cases} K & \text{if } j = d-1 \\ 0 & \text{else.} \end{cases}$$

Thus for $j \leq d-2$, $\text{Im}(\partial_{j+1} | C_{j+1}(m)) = \text{Ker}(\partial_j | C_j(m))$. Now we are ready to define ∂_d. Let z be a generator of $H_{d-1}(\overline{\text{star}_\Delta \sigma}, (\text{star}_\Delta \sigma)^\bullet)$, so $g_{d-1}^{-1}(z)$ generates $\text{Ker}(\partial_{d-1} | C_{d-1}(m))$. Define $\partial_d(qm) = qg_{d-1}^{-1}(z)$, $q \in K$. Then $\partial_d | A_\Delta(m)$ is an injection and $\text{Im}(\partial_d | A_\Delta(m)) = \text{Ker}(\partial_{d-1} | C_{d-1}(m))$. So for $m \in M$, $m \neq 1$, and $k = \dim \text{supp } m$, the sequence

$$0 \to A_\Delta(m) \to C_{d-1}(m) \to \dots \to C_{k+1}(m) \to C_k(m) \to 0$$

is exact.

Now we deal with the case $m = 1$, for which $\text{supp } m = \emptyset \subseteq \tau$ for all faces τ of Δ. We want exactness of the sequence

$$0 \to A_\Delta(1) \to C_{d-1}(1) \to C_{d-2}(1) \to \dots \to C_0(1) \to C_{-1}(1) \to 0$$

or

$$0 \to K \to \sum_{\tau \in \Delta_{d-1}} K \to \sum_{\tau \in \Delta_{d-2}} K \to \dots \to \sum_{\tau \in \Delta_0} K \to K \to 0.$$

Just as before we get an isomorhism between $C(1)$ and a simplicial chain complex. Let $C(\Delta)$ be the simplicial chain complex associated with Δ, and augment $C(\Delta)$ with $C_0(\Delta) \overset{\varepsilon}{\to} C_{-1}(\Delta) = K \to 0$ (here $\varepsilon(v) = 1$ for all vertices v). $C_j(1)$ is generated by $\{1_\tau : \tau \in \Delta_j\}$; a basis for $C_j(\Delta)$ is just Δ_j. So if we define $g_j(1_\tau) = \tau$ for $0 \leq j \leq d-1$, then the same calculation as in the previous case shows that the square

$$\begin{array}{ccc} C_j(1) & \xrightarrow{\partial_j} & C_{j-1}(1) \\ {\scriptstyle g_j}\downarrow & & \downarrow{\scriptstyle g_{j-1}} \\ C_j(\Delta) & \xrightarrow{d_j} & C_{j-1}(\Delta) \end{array}$$

commutes. Now, defining $g_{-1}(q) = q \in C_{-1}(\Delta)$ we check that the square

$$\begin{array}{ccc} C_0(1) & \xrightarrow{\partial_0} & C_{-1}(1) \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle g_{-1}} \\ C_0(\Delta) & \xrightarrow{\varepsilon} & C_{-1}(\Delta) \end{array}$$

commutes. We have $\partial_0(1_\tau) = 1$ for all $\tau \in \Delta_0$, so

$$g_{-1}(\partial_0(1_\tau)) = g_{-1}(1) = 1 = \varepsilon(\tau) = \varepsilon(g_0(1_\tau)).$$

Now $\tilde{H}_i(\Delta) = 0$ for $i \neq d-1$ so the sequence

$$C_{d-1}(1) \to C_{d-2}(1) \to \dots \to C_0(1) \to C_{-1}(1) \to 0$$

is exact. Finally we define $\partial_d(1)$. We know $\mathrm{Ker}(\partial_{d-1}|C_{d-1}(1)) \cong \tilde{H}_{d-1}(\Delta) \cong K$. Let z be a generator of $\tilde{H}_{d-1}(\Delta)$, and for $q \in K$ define $\partial_d(q) = qg_{d-1}^{-1}(z)$. As in the case $m \neq 1$ we get that the sequence

$$0 \to A_\Delta(1) \to C_{d-1}(1) \to \dots \to C_0(1) \to C_{-1}(1) \to 0$$

is exact.

Since the maps ∂_i are homogeneous,

$$\mathrm{Im}\ \partial_j = \textstyle\sum_{m\in M} \mathrm{Im}(\partial_j|C_j(m)) \quad \text{and} \quad \mathrm{Ker}\ \partial_j = \sum_{m\in M} \mathrm{Ker}(\partial_j|C_j(m)),$$

so the exactness of the restricted sequences implies the exactness of the sequence

$$0 \longrightarrow A_\Delta \xrightarrow{\partial_d} C_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \longrightarrow 0. \square$$

REMARK. Consider the exact sequence restricted to m, where $\sigma = \text{supp } m$ is a k-face of Δ. Then by the additivity of vector space dimension $\dim_K A_\Delta(m) = \sum_{j=k}^{d-1} (-1)^{d-1-j} \dim_K C_j(m)$. Now $\dim_K A_\Delta(m) = 1$ and $\dim_K C_j(m) = \left|\{\tau \in \Delta_j : \sigma \subseteq \tau\}\right| = f_{j-k-1}(\ell k_\Delta \sigma)$. Recall that $\ell k_\Delta \sigma$ has the homology of a (d-k-2)-sphere. The equation on dimensions just says

$$1 = \sum_{j=k}^{d-1} (-1)^{d-1-j} f_{j-k-1}(\ell k_\Delta \sigma) = \sum_{i=-1}^{d-k-2} (-1)^{d-k-i-2} f_i(\ell k_\Delta \sigma) \quad \text{or}$$

$$1 - (-1)^{d-k-1} = \sum_{i=0}^{d-k-2} (-1)^{d-k-i} f_i(\ell k_\Delta \sigma).$$

This is equivalent to Euler's formula for $\ell k_\Delta \sigma$: multiplying it by $(-1)^{d-k}$ we get

$$1 - (-1)^{d-k-1} = \sum_{i=0}^{d-k-2} (-1)^i f_i(\ell k_\Delta \sigma).$$

GRADINGS ON RINGS. We show that by assigning certain gradings to the rings A_Δ and C_j the exact sequence of Theorem 4.2 will give interesting numerical criteria for spheres. We continue to denote by M the semigroup of monomials in $x_1, x_2, \ldots, x_n$. Let M' be a commutative monoid, and let $\phi: M \to M'$ be a monoid homomorphism. Assume further that $\phi^{-1}(x)$ is finite for every $x \in M'$. Then we can decompose the rings associated with a simplicial complex Δ as follows:

$$A_\Delta = \sum_{x \in M'} A_\Delta(x) = \sum_{x \in M'} \Big(\sum_{m \in \phi^{-1}(x)} Km \Big),$$

$$C_j = \sum_{x \in M'} C_j(x) = \sum_{x \in M'} \Big(\sum_{m \in \phi^{-1}(x)} C_j(m) \Big).$$

The finiteness condition above allows us to define the Hilbert functions with respect to M'; for $x \in M'$

$$H(A_\Delta, x) = \dim_K A_\Delta(x), \quad H(C_j, x) = \dim_K C_j(x).$$

The maps ∂_i of Theorem 4.2 are homogeneous with respect to this grading by

M', so the exact sequence gives

$$H(A_\Delta,x) = \sum_{j=-1}^{d-1} (-1)^{d-1-j} H(C_j,x).$$

We apply this in the case where Δ is a labeled homology (d-1)-sphere, $M' = N^{r+1}$, and ϕ is the degree map defined prior to Proposition 4.1 (ϕ is induced by the labeling of the vertices of the sphere). In this case it is convenient to write the relation in terms of Hilbert series (in $r+1$ variables $(t_0,\ldots,t_r) = t$):

$$P(A_\Delta,t) = \sum_{j=-1}^{d-1} (-1)^{d-1-j} P(C_j,t),$$

where, by Proposition 4.1, the left hand side can be written

$$P(A_\Delta,t) = \sum_{w\in N^{r+1}} \sum_{z\leq w} f_z(P) \prod_{i=0}^{r} \binom{w_i-1}{z_i-1} t^w.$$

Now we wish to compute $P(C_j,t)$. Recall that $C_j = \sum A_\sigma$, the summation being over all j-faces σ of Δ. A_σ inherits the N^{r+1} grading in the obvious way, and $P(C_j,t) = \sum P(A_\sigma,t)$. Now A_σ is just the polynomial ring in variables corresponding to the vertices of σ. For $0 \leq i \leq r$, let $z_i = |\sigma \cap V_i|$; since $\dim \sigma = j$, σ has $j+1$ vertices, so $\sum_{i=0}^{r} z_i = j+1$. The coefficient of t^w in the series $1/\prod_{i=0}^{r} (1-t_i)^{z_i}$ is the number of monomials of degree w in A_σ. So

$$P(C_j,t) = \sum_{\substack{\sigma \\ \dim \sigma=j}} P(A_\Delta,t) = \sum_{\substack{z \\ \sum z_i=j+1}} \sum_{\substack{\sigma \\ \sigma\in\Delta_z}} P(A_\sigma,t)$$

$$= \sum_{\substack{z \\ \sum z_i=j+1}} f_z(\Delta)/\prod_{i=0}^{r} (1-t_i)^{z_i}$$

$$= \sum_{\substack{z \\ \sum z_i=j+1}} f_z(\Delta) \prod_{i=0}^{r} \left(\sum_{k=0}^{\infty} \binom{k+z_i-1}{z_i-1} t_i^k\right).$$

So the relation on Hilbert series becomes

$$\sum_{w\in N^{r+1}} \sum_{z\leq w} f_z(\Delta) \prod_{i=0}^{r} \binom{w_i-1}{z_i-1} t^w$$

$$= \sum_{j=0}^{d} (-1)^{d-j} \sum_{\substack{z\in N^{r+1} \\ \sum z_i=j}} f_z(\Delta) \prod_{i=0}^{r} \left(\sum_{k=0}^{\infty} \binom{k+z_i-1}{z_i-1} t_i^k\right).$$

By equating coefficients we get

THEOREM 4.4. Let Δ be a homology (d-1)-sphere with vertices labeled by $\{0,1,\ldots,r\}$. Then for all $w \in N^{r+1}$

$$\sum_{z\leq w} f_z(\Delta) \prod_{i=0}^{r} \binom{w_i-1}{z_i-1} = \sum_{z\in N^{r+1}} (-1)^{d-\sum z_i} f_z(\Delta) \prod_{i=0}^{r} \binom{w_i+z_i-1}{z_i-1}. \quad \square$$

An alternate proof of Theorem 4.4 has been suggested (privately) by Stanley. It is based on a result of his [48; Theorem 7.1] and avoids the use of the exact sequence of Theorem 4.2. We feel that this sequence is of independent interest, apart from the current application.

COROLLARY 4.5. The f-vector of a homology (d-1)-sphere satisfies the Dehn-Sommerville equations:

$$f_k = \sum_{j=k}^{d-1} (-1)^{d-j-1} \binom{j+1}{k+1} f_j, \quad -1 \leq k \leq d-2.$$

PROOF. We apply Theorem 4.4 to the case $r = 0$, but we must be careful with notation. Conventionally the f-vector of a sphere Δ is $f(\Delta) = (f_0, f_1, \ldots, f_{d-1})$, where f_i is the number of faces of dimension i. In the vector-subscripted notation $f_{(i)}(\Delta)$ is the number of faces of cardinality i, so $f_{(i)}(\Delta) = f_{i-1}$. With this translation, Theorem 4.4 with $w = (q+1)$ and $q \geq 0$ gives

$$\sum_{k=0}^{q} \binom{q}{k} f_k = \sum_{j=0}^{d-1} (-1)^{d-j-1} \binom{q+1+j}{j} f_j. \tag{4.6}$$

For $q = -1$ the theorem gives the Euler equation (which is the Dehn-Sommerville equation for $k = -1$): $1 = \sum_{j=-1}^{d-1} (-1)^{d-j-1} f_j$. We now use (4.6) and induction to prove the Corollary. Assume the kth Dehn-Sommerville equation holds for $-1 \leq k \leq q-1$ $(q \geq 0)$. By (4.6)

$$f_q = \sum_{j=0}^{d-1} (-1)^{d-j-1} \binom{q+1+j}{j} f_j - \sum_{k=0}^{q-1} \binom{q}{k} f_k$$

$$= \sum_{j=0}^{d-1} (-1)^{d-j-1} \binom{q+1+j}{j} f_j - \sum_{k=0}^{q-1} \binom{q}{k} \sum_{j=k}^{d-1} (-1)^{d-j-1} \binom{j+1}{k+1} f_j$$

(by the induction hypothesis)

$$= \sum_{j=0}^{d-1} (-1)^{d-j-1} \sum_{k=0}^{q} \binom{j+1}{k+1}\binom{q}{k} f_j - \sum_{j=0}^{d-1} (-1)^{d-j-1} \sum_{k=0}^{q-1} \binom{j+1}{k+1}\binom{q}{k} f_j$$

$$= \sum_{j=0}^{d-1} (-1)^{d-j-1} \binom{j+1}{q+1} f_j = \sum_{j=q}^{d-1} (-1)^{d-j-1} \binom{j+1}{q+1} f_j.$$

The next-to-last equality follows from the combinatorial identity $\binom{m+1+j}{j} = \sum_{k=0}^{m} \binom{j+1}{k+1}\binom{m}{k}$ which, in turn, follows easily by considering an $(m+1+j)$-set partitioned into a $(j+1)$-set and an m-set. □

REMARK. Conversely, the equations (4.6) are implied by the Dehn-Sommerville equations as follows:

$$\sum_{k=0}^{q} \binom{q}{k} f_k = \sum_{k=0}^{q} \binom{q}{k} \sum_{j=k}^{d-1} (-1)^{d-j-1} \binom{j+1}{k+1} f_j$$

$$= \sum_{j=0}^{d-1} (-1)^{d-j-1} \sum_{k=0}^{q} \binom{q}{k}\binom{j+1}{k+1} f_j$$

$$= \sum_{j=0}^{d-1} (-1)^{d-j-1} \binom{q+1+j}{j} f_j.$$

The following corollary has been proved for certain classes of poset complexes in [14] and [49]. Further results for these complexes are derived in Section 7.

COROLLARY 4.7. If Δ is a completely balanced homology $(d-1)$-sphere, then for all $S \subseteq \{0,1,\ldots,d-1\}$

$$f_S(\Delta) = \sum_{\substack{T \\ S \subseteq T \subseteq \langle d \rangle}} (-1)^{d-|T|} f_T(\Delta)$$

or, equivalently, for all S,

$$h_S(\Delta) = h_{\tilde{S}}(\Delta).$$

PROOF. Apply Theorem 4.4 to Δ with $w \in N^d$ such that $w_i \leq 1$ for all i.

$$(4.8) \qquad \sum_{z \leq w} f_z(\Delta) \prod_{i=0}^{d-1} \binom{w_i - 1}{z_i - 1} = \sum_{z \in N^d} (-1)^{d - \Sigma z_i} f_z(\Delta) \prod_{i=0}^{d-1} \binom{w_i + z_i - 1}{z_i - 1}.$$

Now $\binom{w_i - 1}{z_i - 1} = \binom{w_i + z_i - 1}{z_i - 1} = 0$ if $w_i = 1$ and $z_i = 0$. On the left-hand side, then, the only nonzero term is f_w. For the right-hand side note that $f_z(\Delta) = 0$ if $z_i > 1$ for some i. So the vectors z making nonzero contribution on the right hand side are those for which $w \leq z \leq (1,1,\ldots,1)$. For such z, $\Pi_{i=0}^{d-1}\binom{w_i + z_i - 1}{z_i - 1}$ is the product of terms $\binom{w_i}{0} = 1$ (when $z_i = 1$) and terms $\binom{-1}{-1} = 1$ (when $z_i = w_i = 0$). So (4.8) says for all $w \leq (1,1,\ldots,1)$,

$$f_w(\Delta) = \sum_{w \leq z \leq (1,1,\ldots,1)} (-1)^{d - \Sigma z_i} f_z(\Delta).$$

In the notation introduced earlier for completely balanced complexes this says that for all $S \subseteq \{0,1,\ldots,d-1\}$

$$f_S(\Delta) = \sum_{\substack{T \\ S \subseteq T \subseteq \langle d \rangle}} (-1)^{d - |T|} f_T(\Delta).$$

Now substituting the resulting expression for $f_S(\Delta)$ into the definition of $h_S(\Delta)$ we get:

$$\begin{aligned} h_S(\Delta) &= \sum_{T \subseteq S} (-1)^{|S| - |T|} f_T(\Delta) \\ &= \sum_{T \subseteq S} (-1)^{|S| - |T|} \sum_{U \supseteq T} (-1)^{d - |U|} f_U(\Delta) \\ &= \sum_{U \subseteq \langle d \rangle} (-1)^{d - |S| - |U|} f_U(\Delta) \sum_{T \subseteq U \cap S} (-1)^{|T|}. \end{aligned}$$

If $U \cap S \neq \emptyset$, $\sum_{T \subseteq U \cap S} (-1)^{|T|} = (1-1)^{|U \cap S|} = 0$, so

$$h_S(\Delta) = \sum_{U \subseteq \tilde{S}} (-1)^{d - |S| - |U|} f_U(\Delta) = h_{\tilde{S}}(\Delta).$$

Similarly, the equations $h_S(\Delta) = h_{\tilde{S}}(\Delta)$ imply the equations $f_S(\Delta) = \sum_{T \supseteq S} (-1)^{d - |T|} f_T(\Delta)$. Assume $h_S(\Delta) = h_{\tilde{S}}(\Delta)$. Then

$$\sum_{T\supseteq S}(-1)^{d-|T|}f_T(\Delta) = \sum_{T\supseteq S}(-1)^{d-|T|}\sum_{U\subseteq T}h_U(\Delta)$$

$$= \sum_{U\subseteq \langle d\rangle} h_U(\Delta)\Big(\sum_{T\supseteq S\cup U}(-1)^{d-|T|}\Big).$$

If $S \cup U \neq \{0,1,\ldots,d-1\}$ then $\sum_{T\supseteq S\cup U}(-1)^{d-|T|} = \sum_{V\subseteq \widetilde{(S\cup U)}}(-1)^{|V|} = 0$. So

$$\sum_{T\supseteq S}(-1)^{d-|T|}f_T(\Delta) = \sum_{U\supseteq \tilde{S}}h_U(\Delta) = \sum_{U\supseteq \tilde{S}}h_{\tilde{U}}(\Delta) = \sum_{V\subseteq S}h_V(\Delta) = f_S(\Delta). \quad \Box$$

We note here that the Dehn-Sommerville equations in the form $h_i = h_{d-i}$ follow immediately from the equations $h_S = h_{\tilde{S}}$ and the fact that $h_i = \sum_{|S|=i} h_S$.

A Gorenstein complex is the join of a homology sphere with a simplex [14], [44]. If Δ is a completely balanced Gorenstein complex, then the homology sphere is itself a completely balanced complex (as is the simplex -- on a disjoint set of labels). If S is the label set for the homology sphere and $T \subseteq \{0,\ldots,d-1\}$ then it follows from Corollary 4.7 that

$$h_T(\Delta) = \begin{cases} 0 & \text{if } T \not\subseteq S \\ h_{S\setminus T}(\Delta) & \text{if } T \subseteq S. \end{cases}$$

Corollary 4.7 can also be obtained by ring theoretic techniques as in [51]. Alternatively, in the case of certain poset complexes, Stanley has given an elementary proof using the Möbius function [49]. We describe these proofs in the next section.

Finally, we state for completeness the result of applying Theorem 4.4 when $\Delta = \sigma(Q)$, the minimal subdivision of the d-polytope Q. The proof is fairly direct and can be found in [7]. Recall that the label set is $\{0,1,\ldots,d-1\}$, although the label 1 never appears.

COROLLARY 4.9. Let Q be a d-polytope and $\Delta = \sigma(Q)$. Let $w = (w_0,\ldots,w_{d-1}) \in N^d$ be such that $w_1 = 0$ and $w_i \leq 1$ for $i \geq 2$. Then

$$f_w(\Delta) = \sum_{z\geq w}(-1)^{d-\sum z_i}\binom{z_1}{w_1}f_z(\Delta). \quad \Box$$

5. SOME OTHER PROOFS OF $h_S = h_{\tilde{S}}$

In this section we describe other approaches to the proof of Corollary 4.7 in general or for special cases. We attempt here to give the flavor of the arguments and refer to the literature for important details.

Suppose first that Δ is a completely balanced homology (d-1)-sphere. We sketch a ring theoretic proof of Corollary 4.7 based on a proof of $h_i = h_{d-i}$ for Gorenstein complexes given by Stanley in [51]. We assume here the terminology of [46].

By [46; Corollary 4.2], A_Δ has a system of parameters $\theta_0,\ldots,\theta_{d-1}$ which is homogeneous in the N^d grading on A_Δ $(d = r+1)$; in fact, $\deg \theta_i = e_i$, the i^{th} unit vector in N^d. Thus the ring $B = A_\Delta/(\theta_0,\ldots,\theta_{d-1})$ inherits an N^d grading; by [46; Proposition 3.2 and proof of Theorem 4.4] we have that $B = \sum_{S\subseteq\langle d\rangle} B_S$ (direct sum) where $\dim_K B_S = h_S$, $B_S\cdot B_T \subseteq B_{S\cup T}$ if $S \cap T = \emptyset$ and $B_S\cdot B_T = 0$ otherwise. Let $B_+ = \sum_{\substack{S\subseteq\langle d\rangle \\ S\neq\emptyset}} B_S$; then since A_Δ is Gorenstein (e.g. [44], [51]), it follows that the subspace

$$C = \{x \in B: xB_+ = 0\}$$

has K-dimension 1 [51].

SECOND PROOF OF COROLLARY 4.7. First we recall that $h_{\langle d\rangle}(\Delta) = h_d(\Delta) = 1$ by the Euler relation for Δ. Thus $B_{\langle d\rangle} \neq 0$. But $B_{\langle d\rangle}\cdot B_S = 0$ for all $S \neq \emptyset$, so $B_{\langle d\rangle} \subseteq C$; since $\dim_K C = 1$, $B_{\langle d\rangle} = C$.

We next claim that if $T \subseteq \langle d\rangle$ and $x_T \in B_T$, $x_T \neq 0$, then $x_T\cdot B_{\langle d\rangle\setminus T} \neq 0$. Since $B_\phi = K$, if the claim fails for some T, we must have $T \neq \langle d\rangle$. Suppose T is a maximal set for which $x_T\cdot B_{\langle d\rangle\setminus T} = 0$ for some $x_T \in B_T$, $x_T \neq 0$. For $R \subseteq \langle d\rangle\setminus T$, $R \neq \emptyset$, and $x_R \in B_R$, $x_Tx_R \in B_{T\cup R}$ so if $x_Tx_R \neq 0$, then by the maximality of T we have $x_Tx_RB_{\langle d\rangle\setminus(T\cup R)} \neq 0$. But $x_RB_{\langle d\rangle\setminus(T\cup R)} \subseteq B_{\langle d\rangle\setminus T}$ contradicting $x_T\cdot B_{\langle d\rangle\setminus T} = 0$. Thus we conclude $x_TB_R = 0$ for each $R \subseteq \langle d\rangle\setminus T$. On the other hand, if $R \cap T \neq \emptyset$, then $x_TB_R = 0$ as well. So $x_TB_+ = 0$, i.e., $x_T \in C = B_{\langle d\rangle}$. But $x_T \in B_T$, $T \neq \langle d\rangle$, and so $x_T \in B_T \cap B_{\langle d\rangle} = 0$; this contradiction proves the claim.

Now, for each $S \subseteq \langle d\rangle$ we have a pairing of K-vector spaces

$$B_S \times B_{\langle d\rangle\setminus S} \to B_{\langle d\rangle} = K$$

defined by ring multiplication, which gives us a linear map

$$B_S \to \mathrm{Hom}_K(B_{\langle d\rangle\setminus S}, K).$$

The claim allows us to conclude that this map is an injection, and so for all S,

$$\dim_K B_S \leq \dim_K B_{\langle d\rangle\setminus S},$$

completing the proof. □

Suppose now that Δ is a shellable completely balanced homology (d-1)-sphere, i.e. there is an ordering $\sigma_1, \sigma_2, \ldots, \sigma_k$ of all the (d-1)-simplices of Δ so that for $2 \leq i \leq k$, $(\sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_{i-1}) \cap \sigma_i$ is a nonempty union of (d-2)-faces of σ_i. The following interpretation of the $h_S(\Delta)$ for shellable completely balanced Δ is a special case of [46; Proposition 3.3], which extends the interpretation of h_i in [31].

PROPOSITION 5.1. Let Δ be a completely balanced (d-1)-complex, and suppose $\sigma_1, \sigma_2, \ldots, \sigma_k$ is a shelling of Δ. For each i, $1 \leq i \leq k$, define τ_i to be the unique minimal face of σ_i not contained in $\sigma_1 \cup \sigma_2 \cup \ldots \cup \sigma_{i-1}$. Let S_i be the subset of $\{0,1,\ldots,d-1\}$ which labels τ_i. Then for any $S \subseteq \{0,1,\ldots,d-1\}$, $h_S(\Delta)$ is the number of i for which $S_i = S$. □

To prove Corollary 4.7 in the special case of shellable completely balanced homology spheres (these are always spheres; for example, barycentric subdivisions of polytopes [18], [16]) we need the fact that if $\sigma_1, \ldots, \sigma_k$ is a shelling of a homology sphere Δ, then so is $\sigma_k, \ldots, \sigma_1$ [28; Proposition 3.3.11]. Then if τ_i is the unique minimal face of σ_i not contained in $\sigma_1 \cup \ldots \cup \sigma_{i-1}$, then $\sigma_i \setminus \tau_i$ has the same property for $\sigma_k \cup \ldots \cup \sigma_{i+1}$. If τ_i has label set S_i then $\sigma_i \setminus \tau_i$ has label set $\langle d\rangle \setminus S_i$ and so

$$\begin{aligned} h_S &= \left|\{i : \tilde{S}_i = S\}\right| \\ &= \left|\{i : S_i = \tilde{S}\}\right| = h_{\tilde{S}}, \end{aligned}$$

the first equality coming from the shelling $\sigma_k,\ldots,\sigma_1$, the last from the shelling $\sigma_1,\ldots,\sigma_k$.

Finally, we wish to prove the equations $h_S = h_{\tilde{S}}$ for an interesting class of poset complexes. Recall that the Möbius function of a poset P is an integer-valued function μ on $P\times P$ defined by $\mu(x,y) = 0$ if $x \not\leq y$, $\mu(x,x) = 1$ for all x and $\sum_{x\leq y\leq z} \mu(x,y) = 0$ if $x < z$ [21]. A ranked poset P is said to be Eulerian if its Möbius function satisfies $\mu(x,y) = (-1)^{r(y)-r(x)}$ for all $x \leq y$ in P. (Recall $\hat{0},\hat{1} \in P$.) We will write μ_P when confusion may arise as to the appropriate poset.

For any ranked poset P, we have $\mu(P) \equiv \mu_P(\hat{0},\hat{1}) = \chi(\Delta) - 1$, where $\chi(\Delta)$ is the Euler characteristic of the complex $\Delta = \Delta(P)$, and it follows that

$$h_{\langle d\rangle}(\Delta) = \sum_{T\subseteq\langle d\rangle} (-1)^{d-|T|} f_T(\Delta) = (-1)^{d-1}\mu(P),$$

where $d = r(P) = r(\hat{1})$. (See, for example, [21] or [46].) By considering the "rank-selected" sub-poset

$$P_S = \{\hat{0},\hat{1}\} \cup \{x \in P\colon r(x) \in S\}$$

for $S \subseteq \langle d\rangle$, we obtain

$$h_S(\Delta) = (-1)^{|S|-1}\mu(P_S),$$

where $\mu(P_S) = \mu_{P_S}(\hat{0},\hat{1})$. By [4; Lemma 4.6] we can write

$$\mu(P_S) = \sum_{\substack{x_1<\cdots<x_k \\ r(x_i)\notin S}} (-1)^k \mu(\hat{0},x_1)\mu(x_1,x_2)\cdots\mu(x_k,\hat{1})$$

where the sum ranges over chains $x_0 = \hat{0} < x_1 < \ldots < x_k < 1$ in $P_{\tilde{S}}$ $(k \geq 0)$ and $\mu = \mu_P$ on the right hand side.

Now suppose P is Eulerian; we show $h_S(\Delta(P)) = h_{\tilde{S}}(\Delta(P))$. The following proof is due to Stanley [49; Proposition 2.2]. We have, with $\Delta = \Delta(P)$

$$h_S(\Delta) = (-1)^{|S|-1}\mu(P_S)$$

$$= (-1)^{|S|-1} \sum_{\substack{x_1<\cdots<x_k \\ r(x_i)\notin S}} (-1)^k \mu(\hat{0},x_1)\cdots\mu(x_k,\hat{1})$$

$$= (-1)^{|S|-1} \sum_{\substack{x_1<\cdots<x_k \\ r(x_i)\notin S}} (-1)^{k+r(\hat{1})-r(\hat{0})}$$

$$= (-1)^{|S|+d} \sum_{T\subset\tilde{S}} (-1)^{|T|} f_T(\Delta)$$

$$= h_{\tilde{S}}(\Delta)$$

(recall the convention that $r(\hat{0}) = 1$).

We will treat the case of Eulerian posets in detail in section 7 where we will find all the linear relations holding among the numbers $f_S(\Delta)$. The face lattice of a convex polytope is an Eulerian poset [30], [38]. We will, in fact, get all the linear relations holding for the chain numbers of an arbitrary convex polytope.

6. AFFINE SPAN FOR COMPLETELY BALANCED SPHERES

We consider in this section the question of whether the equations given by Corollary 4.7 are the only linear relations on the extended f-vector which hold for all competely balanced homology spheres. We give an affirmative answer by exhibiting a set of completely balanced simplicial polytopes whose extended h-vectors span the space determined by the relations $h_S = h_{\tilde{S}}$ along with the trivial relation $h_\phi = 1$.

We will define an operation that subdivides a face of a completely balanced complex, resulting in another such complex. This notion of a completely balanced stellar subdivision is then used to create the desired basis in the same way that we produced a basis for the Dehn-Sommerville space in Section 3 by performing stellar subdivisions on a simplex. To this end, we define a class of polytopes to play the role of simplices.

Suppose in R^d we have d mutually orthogonal segments, $[v_i, w_i]$ $(0 \leq i \leq d-1)$, that intersect at a single point interior to each of the segments. The convex hull of these segments is a simplicial polytope, called the d-crosspolytope Q^d or just Q (see [23; §4.3]). Alternatively, Q is the polar to the d-cube, and its boundary complex can be viewed as the order complex of the rank d poset having 2 elements of each rank, any two elements of different rank being comparable.

For $0 \leq k \leq d-1$, the k-faces of Q are determined by sets F of $k+1$ of the points $\{v_i\} \cup \{w_i\}$ where no pair $\{v_i, w_i\}$ is in F. In particular, the facets consist of exactly one element from each pair $\{v_i, w_i\}$, $0 \leq i \leq d-1$. Thus, the sets $V_i = \{v_i, w_i\}$ $(0 \leq i \leq d-1)$ partition the vertices of Q, making Q a completely balanced simplicial complex. For $S \subseteq \{0,1,\ldots,d-1\}$, the S-labeled faces of Q are the sets $\{y_i : i \in S\}$ where $y_i \in V_i = \{v_i, w_i\}$. There are clearly $2^{|S|}$ such sets, i.e., $f_S(Q) = 2^{|S|}$. Also,

$$h_S(Q) = \sum_{T \subseteq S} (-1)^{|S|-|T|} f_T(Q)$$

$$= \sum_{T \subseteq S} (-1)^{|S|-|T|} 2^{|T|} = (2-1)^{|S|} = 1$$

for all $S \subseteq \{0,1,\ldots,d-1\}$. This makes the crosspolytope the completely balanced analog of the simplex, which, as we have seen has $h_i = 1$ for all i, $0 \leq i \leq d$.

Suppose F is a j-face of Q labeled by some subset $X \subseteq \{0,1,\ldots,d-1\}$ $(|X| = j+1)$. Then $\ell k_Q F$ is a complex whose facets consist of exactly one element from each V_i, for $i \in \{0,1,\ldots,d-1\} \setminus X$. So $\ell k_Q F$ is itself a $(d-j-1)$-crosspolytope with vertices labeled by $\{0,1,\ldots,d-1\} \setminus X$.

For a completely balanced complex Δ, we define the completely balanced subdivision of a face $\sigma \in \Delta$ as follows. Suppose Δ is labeled with $0,1,\ldots,r$ and suppose for ease of exposition that σ has label set $\{0,1,\ldots,k\}$, $k \leq r$. Define $\Delta_0 = st(\sigma, \Delta)$, the usual stellar subdivision of complexes discussed in Section 3. Assign the new vertex introduced in Δ_0 the label 0. If $k > 0$, Δ_0 is not completely balanced; there will be a unique edge having two vertices with label 0. Denote this edge by σ_0 and define $\Delta_1 = st(\sigma_0,\Delta_0)$. Label the new vertex in Δ with a 1; if $k > 1$, then there is a unique edge with two vertices labeled 1. Continuing this process, the final complex in the sequence, Δ_k, will be completely balanced, and we define this complex to be the desired subdivision of σ in Δ. A proof that this procedure is well-defined is included in the proof of Theorem 3.1 in [9]. It is clear from this definition that if Δ is the boundary complex of a simplicial polytope, then so is Δ_k.

To view this subdivision another way, consider first the case $k = r$. Let $\hat{\Delta}$ be the (boundary complex of the) (r+1)-crosspolytope and choose a maximal simplex $\hat{\sigma} \in \hat{\Delta}$; $\hat{\sigma}$ is completely labeled with $0,1,\ldots,r$. Then Δ_r is the complex $(\Delta\backslash\sigma) \cup (\hat{\Delta}\backslash\hat{\sigma})$, where we identify the corresponding faces of σ and $\hat{\sigma}$. In the case $k < r$, take $\hat{\Delta}$ to be a (k+1)-crosspolytope, $\hat{\sigma} \in \hat{\Delta}$ a k-simplex (with labels $0,1,\ldots,k$). The balanced subdivision Δ_k is the union of $\Delta\backslash\sigma$ and $(\hat{\Delta}\backslash\hat{\sigma})\cdot \ell k_{\Delta}\sigma = (\hat{\Delta}\cdot \ell k_{\Delta}\sigma)\backslash\hat{\sigma}$, again identifying the corresponding faces of σ and $\hat{\sigma}$. See [9] for details.

Now let Q be the d-crosspolytope. (For the remainder of this section, when we refer to a (simplicial) polytope Q we will mean its boundary complex; faces of Q will be considered as sets of vertices.) For each subset $X \subseteq \langle d\rangle$, $X \neq \emptyset$, let P^X denote the completely balanced simplicial polytope which results from a completely balanced subdivision of a face of Q having label set X. Note that if $|X| = 1$ then $P^X = Q$.

THEOREM 6.1. For each $S \subseteq \langle d\rangle$,

$$h_S(P^X) = \begin{cases} 1 & \text{if } S \cap X = \emptyset \text{ or } X \subseteq S \\ 2 & \text{otherwise.} \end{cases}$$

PROOF. Suppose $|X| = k+1$ and let F be the face of Q which is subdivided. Let $\hat{Q}$ be the (k+1)-crosspolytope; F' a facet of $\hat{Q}$ labeled by X; and $Q' = \hat{Q}\cdot \ell k_Q F$. By the above discussion (with $\hat{\Delta} = \hat{Q}$ and $\hat{\sigma} = F'$) the faces of P^X are either faces of Q not containing F or faces of Q' not containing F'; faces of the latter type get vertices with labels in X

from $\hat{Q}$, the others from $\ell k_Q F$.

Thus for any $T \subseteq \{0,1,\ldots,d-1\}$,

$$f_T(P^X) = f_T(Q\backslash F) + f_{T\backslash X}(\ell k_Q F)(f_{T\cap X}(\hat{Q}) - f_{T\cap X}(F')).$$

(Here $f_{T\cap X}(F') = 1$ denotes the number of faces of the simplex F' having label set $T \cap X$.) If $X \not\subseteq T$ then $f_T(Q\backslash F) = f_T(Q)$; if $X \subseteq T$ then

$$\begin{aligned} f_T(Q\backslash F) &= f_T(Q) - f_{T\backslash X}(\ell k_Q F) f_X(F) \\ &= f_T(Q) - f_{T\backslash X}(\ell k_Q F). \end{aligned}$$

So if we write $\chi(X \subseteq T) = \begin{cases} 1 & \text{if } X \subseteq T \\ 0 & \text{if } X \not\subseteq T \end{cases}$, then we get

$$\begin{aligned} f_T(P^X) &= f_T(Q) + f_{T\backslash X}(\ell k_Q F)(f_{T\cap X}(\hat{Q}) - f_{T\cap X}(F') - \chi(X \subseteq T)) \\ &= 2^{|T|} + 2^{|T\backslash X|}(2^{|T\cap X|} - 1 - \chi(X \subseteq T)). \end{aligned}$$

So

$$\begin{aligned} h_S(P^X) &= \sum_{T\subseteq S} (-1)^{|S|-|T|} f_T(P^X) \\ &= \sum_{T\subseteq S} (-1)^{|S|-|T|}(2^{|T|}+2^{|T|}-2^{|T\backslash X|}-2^{|T\backslash X|}\chi(X \subseteq T)). \end{aligned}$$

Now $\sum_{T\subseteq S} (-1)^{|S|-|T|} 2^{|T|} = 1$;

$$\begin{aligned} \sum_{T\subseteq S} (-1)^{|S|-|T|} 2^{|T\backslash X|} &= \sum_{U\subseteq S\cap X} \sum_{V\subseteq S\backslash X} (-1)^{|S|-|U|-|V|} 2^{|V|} \\ &= \sum_{U\subseteq S\cap X} (-1)^{|S\cap X|-|U|} = (1-1)^{|S\cap X|} = \chi(S \cap X = \emptyset); \end{aligned}$$

and

$$\begin{aligned} &\sum_{T\subseteq S} (-1)^{|S|-|T|} 2^{|T\backslash X|} \chi(X \subseteq T) \\ &\quad = \chi(X \subseteq S) \sum_{V\subseteq S\backslash X} (-1)^{|S|-|X|-|V|} 2^{|V|} = \chi(X \subseteq S). \end{aligned}$$

So

$$\begin{aligned} h_S(P^X) &= 2 - \chi(S \cap X = \emptyset) - \chi(X \subseteq S) \\ &= \begin{cases} 1 & \text{if } S \cap X = \emptyset \text{ or } X \subseteq S \\ 2 & \text{else.} \end{cases} \end{aligned}$$

(Note that when $|X| > 0$ we cannot have $S \cap X = \emptyset$ and $X \subseteq S$ simultaneously.) □

Let C^d be the set of all completely balanced homology (d-1)-spheres and $(h_S(C^d))$ the set of the vectors $(h_S(\Delta))_{S \subseteq \langle d \rangle} \in N^{2^d}$ for $\Delta \in C^d$.

THEOREM 6.2. $\dim \operatorname{aff}(h_S(C^d)) = 2^{d-1}-1$.

PROOF. For all $\Delta \in C^d$ we have $h_\emptyset = 1$ and (by Corollary 2.4) $h_S(\Delta) = h_{\tilde{S}}(\Delta)$ for $S \subseteq \{0,1,\ldots,d-1\}$. These are clearly independent linear equations, and there are $2^{d-1}+1$ of them, so $\dim \operatorname{aff} h_S(C^d) \leq 2^d - (2^{d-1}+1) = 2^{d-1}-1$. The rest of the proof will consist in showing the other inequality by demonstrating 2^{d-1} affinely independent vectors in $h_S(C^d)$.

We first define a lexicographic order on the subsets of $\{0,1,\ldots,d-1\}$ as follows. If $S = \{s_1,\ldots,s_k\}$, $s_1 < s_2 < \ldots < s_k$, and $T = \{t_1,\ldots,t_{k'}\}$, $t_1 < t_2 < \ldots < t_{k'}$, then $S < T$ if $k < k'$ or if $k = k'$ and for some $j \leq k$, $s_j < t_j$, while for $i < j$, $s_i = t_i$. In this ordering $S < T$ if and only if $\tilde{S} > \tilde{T}$, so the complement of the nth subset is the (2^d-n+1)st subset.

Now define a $(2^{d-1} \times 2^{d-1})$ matrix A with columns indexed by the first 2^{d-1} subsets of $\{0,1,\ldots,d-1\}$ in increasing order, and the rows indexed by the last 2^{d-1} subsets of $\{0,1,\ldots,d-1\}$ arranged in decreasing order. If S is one of the first 2^{d-1} subsets and X is one of the last 2^{d-1} subsets, then the (X,S) entry in A is

$$a_{X,S} = 2 - h_S(P^X) = \begin{cases} 1 & \text{if } S \cap X = \emptyset \text{ or } X \subseteq S \\ 0 & \text{else,} \end{cases}$$

where P^X is given by Theorem 6.1. Note that for X and S within the range defined, $S < X$, so $X \not\subseteq S$.

The matrix A is lower triangular with ones along the diagonal. To see this, let $1 \leq q < n \leq 2^{d-1}$, and let X be the (2^d-q+1)st set (the set indexing row q) and S the nth set. Then $\tilde{X}$ is the qth set, so $\tilde{X} < S$. This implies $S \not\subseteq \tilde{X}$, i.e., $S \cap X \neq \emptyset$. So $a_{X,S} = 0$. The diagonal elements of A are $a_{\tilde{S},S} = 1$. Thus, rank $A = 2^{d-1}$. This says that the polytopes P^X, as X ranges over the last 2^{d-1} subsets of $\{0,1,\ldots,d-1\}$, have affinely independent vectors $(h_S) \in N^{2^{d-1}}$ (S ranges over the first 2^{d-1} subsets of $\{0,1,\ldots,d-1\}$). But then their complete h-vectors $(h_S(P^X))_{S\subseteq\langle d\rangle}$ must be affinely independent. Thus $\dim \operatorname{aff} h(C^d) \geq 2^{d-1}-1$; combined with the other inequality, this gives the desired result. □

Since the basis constructed for the proof of Theorem 6.2 actually consists of polytopes, we have the following.

COROLLARY 6.3. The dimension of the affine span of the extended h-vectors of completely balanced simplicial d-polytopes is $2^{d-1}-1$.

We note, finally, that various nonlinear conditions are known to hold for the extended h-vectors of the more general balanced Cohen-Macaulay complexes [46]. In particular, it is known that if Δ is such a complex, then $h_S(\Delta) = f_S(\Lambda)$, where Λ is another labeled complex having the property that each simplex has _at most_ one vertex of each label. We call a complex having this property _colored_, and note that it need not be a pure complex, i.e., maximal simplices may be of different sizes.

Conversely, if Λ is any colored complex, then $f_S(\Lambda) = h_S(\Delta)$ for some completely balanced simplicial Cohen-Macaulay complex Δ, in fact for a shellable completely balanced complex. We sketch a proof of this originally suggested by Björner. Order the faces of Λ by cardinality, starting with ϕ and continuing with the vertices (in any order), and so on. Suppose $\dim \Lambda = r$. Define $r+1$ _new_ vertices, labeled $0,1,\ldots,r$, and define a complex Δ on these new vertices plus those of Λ. The maximal faces of Δ will be $(r+1)$-sets consisting of a face λ of Λ augmented by those new vertices whose labels do not appear on the vertices of λ. Call this facet λ^*. The order on Λ defines an order on the maximal faces of Δ which is easily seen to be a shelling order. In fact, the minimal face of λ^* not in the union of those facets preceding λ^* is λ, and so we get $h_S(\Delta) = f_S(\Lambda)$ by a direct application of Proposition 5.1.

It seems to be a difficult problem to characterize numerically the extended f-vectors of colored complexes. It seems necessary to have such a characterization before one could hope to extend McMullen's conditions to a characterization of the extended h-vectors of completely balanced polyhedral complexes.

7. EULERIAN POSET COMPLEXES. In this section we take a further look at a special class of completely balanced complexes: the order complexes of Eulerian posets. Recall that a poset P is called Eulerian if for every $x < y$ in P, $\mu(x,y) = (-1)^{r(y)-r(x)}$. We shall see that this means that Euler's formula holds for intervals in the poset, and so, in particular, the order complexes of these posets are Eulerian manifolds [26].

THEOREM 7.1. Let P be an Eulerian poset of rank d, and $S \subseteq \{0,1,\ldots,d-1\}$. If $\{i,k\} \subseteq S \cup \{-1,d\}$ and S contains no j such that $i < j < k$, then

$$\sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S\cup j}(P) = f_S(P)(1 - (-1)^{k-i-1})$$

PROOF. Let C be a chain in P with rank set S. Let x be the element of C with rank i ($\hat{0}$ if $i = -1$), and y the element with rank k ($\hat{1}$ if $i = d$). For $i \le j \le k$ write $f_j(x,y)$ for the number of rank j elements of P between x and y. Since P is Eulerian, $\mu(x,y) = (-1)^{k-i}$, and we use the fact that $\sum_{x\le z\le y} \mu(x,z) = 0$ to get

$$\begin{aligned}(-1)^{k-i} = \mu(x,y) &= -\sum_{x\le z<y} \mu(x,z)\\ &= -\sum_{x\le z<y} (-1)^{r(z)-i}\\ &= -\sum_{j=i}^{k-1} \sum_{\substack{x\le z<y\\ r(z)=j}} (-1)^{j-i}\\ &= -\sum_{j=i}^{k-1} (-1)^{j-i} f_j(x,y).\end{aligned}$$

Then, since $f_i(x,y) = 1$ we get a form of Euler's equation:

$$1 - (-1)^{k-i-1} = \sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_j(x,y).$$

Summing over all S-chains C we get (here x and y depend on C)

$$f_S(P)(1 - (-1)^{k-i-1}) = \sum_{C \text{ an } S\text{-chain}} \sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_j(x,y)$$

$$= \sum_{j=i+1}^{k-1} (-1)^{j-i-1} \sum_{C \text{ an } S\text{-chain}} f_j(x,y)$$

$$= \sum_{j=i+1}^{k-1} (-1)^{j-i-1} f_{S\cup j}(P). \quad \square$$

The motivating example of an Eulerian poset is the face lattice of a polytope [30], [38]. For arbitrary polytopes, the equations of this theorem are the analogs of the Dehn-Sommerville equations, which hold for the f-vectors of simplicial polytopes. Note that taking $S = \emptyset$, $i = -1$, $k = d$, the equation given by Theorem 7.1 is Euler's formula.

We now analyze the dependencies among the variables f_S given by the equations of Theorem 7.1.

PROPOSITION 7.2. For $d \geq 1$, let Ψ^d be the set of subsets $S \subseteq \{0,1,\ldots,d-2\}$ such that S contains no two consecutive integers. Then for all $T \subseteq \{0,1,\ldots,d-1\}$ such that $T \notin \Psi^d$, there is a nontrivial linear relation expressing $f_T(P)$ in terms of $f_S(P)$, $S \in \Psi^d$, which holds for all Eulerian posets P of rank d. The cardinality of Ψ^d is c_d, the dth Fibonacci number ($c_d = c_{d-1} + c_{d-2}$, $c_1 = 1$, $c_2 = 2$).

PROOF. Order the subsets of $\{0,1,\ldots,d-1\}$ by increasing cardinality, and within cardinality lexicographically. Thus, if $S = \{s_1,s_2,\ldots,s_k\}$ and $V = \{v_1,v_2,\ldots,v_{k'}\}$, then $S < V$ if and only if either $k < k'$ or $k = k'$ and for some j, $1 \leq j \leq k$, $s_j < v_j$ while for $1 \leq i < j$, $s_i = v_i$. If $T \notin \Psi^d$ then for some k, $1 \leq k \leq d$, $\{k-1,k\} \subseteq T \cup \{d\}$. Let $S = T\backslash\{k-1\}$, and $i = \max\{j \in T \cup \{-1\}: j < k-1\}$. Then Theorem 7.1 for these S, i and k says

$$f_T(P) = \sum_{j=i+1}^{k-2} (-1)^{k-j} f_{S\cup j}(P) + f_S(P)(1 - (-1)^{k-i-1}).$$

All the subscripts appearing on the right-hand side of this equation are less than T in the lexicographic order. Repeating the process for any subscript not in Ψ^d we eventually get the desired linear relation.

To compute $|\Psi^d|$, note that any element S of Ψ^d is one of two types: either $d-2 \notin S$ or $d-2 \in S$. In the first case $S \in \Psi^{d-1}$; in the second case $d-3 \notin \Psi$, so $S\setminus\{d-2\} \in \Psi^{d-2}$. Thus $|\Psi^d| = |\Psi^{d-1}| + |\Psi^{d-2}|$; it is easy to see $|\Psi^1| = 1$, $|\Psi^2| = 2$, so the proposition is proved. □

Adding the relation $f_\phi = 1$ we get that the dimension of the affine span of the extended f-vectors of Eulerian posets is at most c_d-1. In fact, this upper bound is the actual dimension, and its value gives us a hint as to the proof. We need to exhibit c_d affinely independent extended f-vectors; it turns out we can do this within the class of d-polytopes. Since $c_d = c_{d-1}+c_{d-2}$ we try to use bases for aff $f_S(P^{d-1})$ and aff $f_S(P^{d-2})$ to create a basis for aff $f_S(P^d)$. To do this we use the operations of taking the pyramid and bipyramid over a polytope. We will use the convention that the symbol P alone stands for the "0-dimensional" polytope, i.e., a single point. An ordered string or word made up of the symbols B and P, and ending in P, stands for the polytope obtained by taking successive pyramids and bipyramids over P in the order indicated by the word. (Alternatively, one could consider the first P to denote taking the pyramid over the empty polytope.) Thus $P^2 = PP$ is an interval, P^3 is a triangle, BP^2 is a square and B^2P^2 is an octahedron. We choose to write words ending in P^2 to avoid redundancy, since $BP = P^2$. Clearly the dimension of the polytope is one less than the length of the word.

PROPOSITION 7.3. For $d \geq 1$, let Ω^d be the set of d-polytopes named by words of length d+1 in B and P that end in P^2 and contain no two adjacent B's. The extended f-vectors of elements of Ω^d are affinely independent, and Ω^d contains c_d elements.

SKETCH OF PROOF. First we count the elements of Ω^d. Consider the two types of words in Ω^d: those beginning with P and those beginning with B. Words of the first type are of the form PQ', where Q' is any word in Ω^{d-1}.

Words of the second type must start with BP (since B^2 is not allowed) and thus are of the form BPQ", where Q" is any word in Ω^{d-2}. So $|\Omega^d| = |\Omega^{d-1}| + |\Omega^{d-2}|$; i.e., the cardinality of Ω^d satisfies the Fibonacci recursion. Since $|\Omega^1| = |\{P^2\}| = 1$, $|\Omega^2| = |\{P^3, BP^2\}| = 2$, we get $|\Omega^d| = c_d$.

The proof that the extended f-vectors of elements in Ω^d are independent is difficult because the effect on the extended f-vector of taking a pyramid or bipyramid is not easily described. It is relatively easy, however, to describe the faces of a pyramid or bipyramid in terms of the faces of the original polytope. In particular, all the faces of a polytope in Ω^d are in $(\cup_{i=1}^{d-1} \Omega^i) \cup \{\phi, P\}$, which we will call M^{d+1}. (Recall P is a single point.) The idea of the proof is to work with the $c_d \times c_d$ matrix A^d, whose typical entry is a_{QM} where for $Q \in \Omega^d$, $M \in M^d$, a_{QM} is the number of faces of Q of combinatorial type M. One can find an invertible transformation that takes A^d to the matrix whose rows are the vectors $(f_S(Q))_{S \in \Psi^d}$ for $Q \in \Omega^d$. Given this it suffices to show A^d is non-singular. The proof of this latter fact makes crucial use of the recursive construction of Ω^d. It consists in exhibiting row and column operations that have the following effect.

$$\left(\begin{array}{c|c} A^{d-1} & * \\ \hline 0 & I \end{array}\right) \to \left(\begin{array}{c|c} A^{d-1} & * \\ \hline A^{d-2} & \text{shifted} \end{array}\right) \to A^d.$$

Then rank A^d = rank A^{d-1} + rank $A^{d-2} = |\Omega^{d-1}| + |\Omega^{d-2}| = |\Omega^d| = c_d$. So the extended f-vectors of the elements of Ω^d are affinely independent. For complete details, see [9]. □

Together Propositions 7.2 and 7.3 give us the following result.

THEOREM 7.4. For $d \geq 1$

$$\dim \operatorname{aff}\{(f_S(P))_{S \subseteq \langle d \rangle} : P \text{ is an Eulerian poset of rank } d\}$$
$$= \dim \operatorname{aff}\{(f_S(P))_{S \subseteq \langle d \rangle} : P \text{ is a d-polytope}\} = c_d - 1$$

where c_d is the dth Fibonacci number. □

In particular, the extended f-vectors of Eulerian posets (or polytopes) are contained in a proper subspace of the affine span of the extended f-vectors of completely balanced homology spheres. In other words the equations $h_S = h_{\tilde{S}}$ for Eulerian posets are dependent on the equations given by Theorem 7.1.

Already at dimension 4, the f-vectors of polytopes have not been characterized. The results of this section show that the extended f-vectors of 4-polytopes are determined linearly by the values of f_0, f_1, f_2 and $f_{\{0,2\}}$ (here we have dropped the set brackets on $f_{\{i\}}$, because it coincides with f_i in the original f-vector).

It is interesting to note here that for simplicial polytopes (and hence for simple polytopes) the extended f-vector is linearly determined by the usual f-vector, and so the affine hull of these remains $[d/2]$-dimensional. To see this, suppose that $S = \{i_1, \ldots, i_s\}$, $s \geq 2$, $i_1 < i_2 < \ldots < i_s < d$. Then for a simplicial d-polytope P,

$$f_S(P) = f_{i_S}(P) \cdot f_{S \setminus \{i_s\}}(T^{i_s}), \tag{7.5}$$

where T^{i_s} denotes the i_s-dimensional simplex. Thus the difference between $c_d - 1$ and $[d/2]$ can be thought of as a crude indication of how special simplicial (or simple) polytopes are.

Further, by an extension of the usual argument of "pulling vertices" [23; p. 80], one can show that, for a fixed number of vertices, the number of chains (for any dimension set) will be maximized by a simplicial polytope. Thus by (7.5) and the Upper Bound Theorem, we get that, for d-polytopes with n vertices, the numbers $f_S(P)$ are simultaneously maximized (for all S) by the cyclic polytope $C(n,d)$.

8. CONCLUDING REMARKS. The results on f-vectors and extended f-vectors surveyed in this paper serve to make combinatorial distinctions among different classes of complexes. The f-vectors of simplicial d-polytopes span a $[d/2]$-dimensional affine subspace of the Euler hyperplane (the (d-1)-space spanned by the f-vectors of all d-polytopes). The extended f-vectors of the (complete

barycentric subdivisions of) simplicial d-polytopes still have dimension $[d/2]$. They are properly contained in the (c_d-1)-dimensional space spanned by the extended f-vectors of rank d Eulerian posets (alternatively, of complete barycentric subdivisions of arbitrary d-polytopes). This in turn is a subspace of the (2^d-1)-dimensional space determined by the extended f-vectors of completely balanced homology $(d-1)$-spheres.

Determining affine dimensions is the first step towards the goal of characterizing the f-vectors of these classes of complexes. The characterization of the f-vectors of simplicial polytopes was motivated by the particular form $h_i = h_{d-i}$ of the Dehn-Sommerville equations. It is natural,therefore, to look at the generalizations of the Dehn-Sommerville equations surveyed in this paper for clues towards nonlinear conditions on the numbers of chains. Of primary interest is characterizing the extended f-vectors of arbitrary d-polytopes. Perhaps what is needed is a change of variables from the f_S, which will put the equations of Theorem 7.1 into a simple form (the transformation to the extended h-vector does not do this).

Part of the motivation for considering extended f-vectors is to derive information on the original f-vectors themselves. The linear equations obtained do not help with this problem. We mention here a conjecture which attempts to generalize the Dehn-Sommerville equations in another way.

CONJECTURE. If $f(P) = (f_0, f_1, \ldots, f_{d-1})$ is the f-vector of a d-polytope P, then for $0 \leq k \leq d-2$

$$f_k \geq \sum_{j=k}^{d-1} (-1)^{d-1-j} \binom{j+1}{k+1} f_j .$$

The inequality holds for $k = d-2$ and, equivalently, for $k = d-3$. The conjecture is true for polytopes of dimension ≤ 4; for simple polytopes (and, of course, simplicial polytopes, for which the relations are equalities); and for prisms on simplicial polytopes. If the inequalities hold for some polytope, then they hold for the pyramid and bipyramid over that polytope. (For details see [8].) Note that Theorem 3.7 says that the f-vectors of all d-polytopes are spanned by f-vectors of (r-fold) pyramids over simplicial (d-r)-polytopes and simplicial d-polytopes. All such polytopes satisfy the inequalities of the conjecture. Thus, the conjecture is related to the following question: can the f-vector of any polytope be written as a linear combination of the f-vectors of simplicial polytopes and the f-vectors of pyramids over simplicial polytopes with nonnegative coefficients on the latter?

A resolution of the conjecture would be relevant to the question of characterizing the f-vectors of d-polytopes. We note that the f-vectors of 3-polytopes are characterized by the inequality of the conjecture and its "polar" (obtained by interchanging f_i with f_{d-1-i}), the inequalities $f_i \geq \binom{d+1}{i+1}$, and the Euler equation [23; p. 190]. That is

$$f(P^3) = \{(f_0,f_1,f_2): f_0-f_1+f_2 = 2, \ f_0 \geq 4, \ f_2 \geq 4, \ 3f_2 \leq 2f_1, \ 3f_0 \leq 2f_1\}.$$

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