# Counting Independent Sets up to the Tree Threshold

Dror Weitz
Tel Aviv

AISP, Santa Fe
May 2007

# What is this work about?

Novel exact tree representation for the marginal probability at a vertex in any binary spin system.

➡ The regular tree is the worst-case graph for an appropriate notion of spatial decay of correlations (Strong Spatial Mixing).

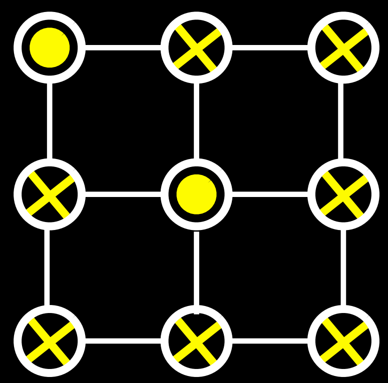
➡ New efficient algorithm for approximating marginals (and hence the partition function) in the regime where the regular tree exhibits SSM.

- Strong application: hard-core model (independent sets).

# The Hard-Core Model (Independent Sets)

- Count/sample weighted independent sets of a graph G.
- Weights are determined by an activity parameter $\lambda$:

$$w(I) = \lambda^{|I|}$$



- Occupied vertex

- Unoccupied vertex

# The Hard-Core Model (Independent Sets)

- Count/sample weighted independent sets of a graph G.
- Weights are determined by an activity parameter $\lambda$:

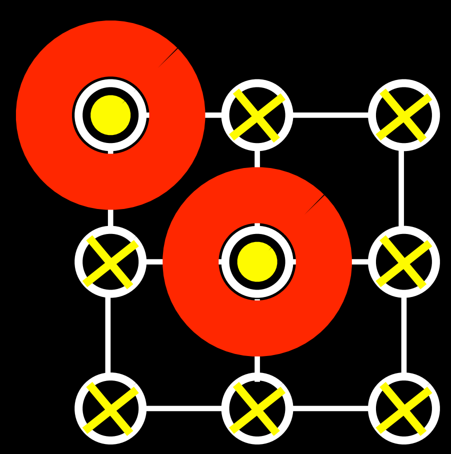
$$w(I) = \lambda^{|I|}$$


- Model for lattice gas, communication networks, ...

# The Hard-Core Model (Independent Sets)

- Count/sample weighted independent sets of a graph G.
- Weights are determined by an activity parameter $\lambda$:

$$w(I) = \lambda^{|I|}$$



- Occupied vertex
- Unoccupied vertex

- Model for lattice gas, communication networks, ...

# Computational Problem

- Aim: $(1+\epsilon)$-approximation of the partition function –

$$Z \equiv Z_G^\lambda = \sum_I \lambda^{|I|}$$

Equivalently: approximately sample independent sets where $\Pr(I) = \lambda^{|I|}/Z$.

# Computational Problem

- Aim: $(1+\epsilon)$-approximation of the partition function –

$$Z \equiv Z_G^{\lambda} = \sum_I \lambda^{|I|}$$

Equivalently: approximately sample independent sets where $\Pr(I) = \lambda^{|I|}/Z$.

- Intuitively, the problem becomes harder as $\lambda$ grows.

(Sampling with large $\lambda$ will output a maximum ind. set.)

# Known bounds

- NP-hard to approximate $Z$ within a polynomial factor for: max degree $\Delta$ and $\lambda \geq c/\Delta$, where $c$ is a (large enough) constant. [Luby-Vigoda]

# Known bounds

- NP-hard to approximate $Z$ within a polynomial factor for: max degree $\Delta$ and $\lambda \geq c/\Delta$, where $c$ is a (large enough) constant. [Luby-Vigoda]

- FPRAS exists for (based on the Glauber dynamics) -

  easy: $\lambda \leq \frac{1}{\Delta - 1}$ (Dobrushin's uniqueness condition)

  best: $\lambda \leq \frac{2}{\Delta - 2}$ [Dyer-Greenhill, Vigoda]

# Known bounds

- NP-hard to approximate $Z$ within a polynomial factor for: max degree $\Delta$ and $\lambda \geq c/\Delta$, where $c$ is a (large enough) constant. [Luby-Vigoda]

- FPRAS exists for (based on the Glauber dynamics) -

  easy: $\lambda \leq \frac{1}{\Delta-1}$ (Dobrushin's uniqueness condition)

  best: $\lambda \leq \frac{2}{\Delta-2}$ [Dyer-Greenhill, Vigoda]

- Finding out exact constants is important -

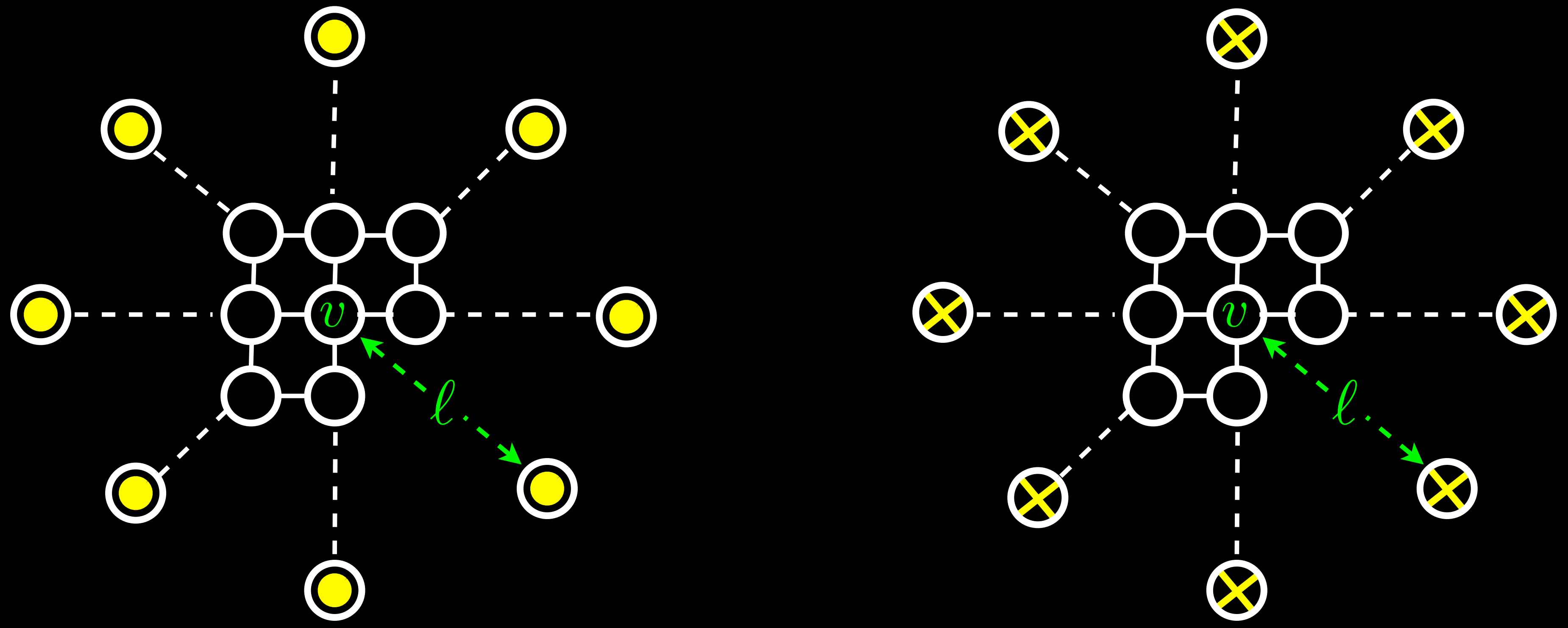
  most interesting graphs are low dimensional lattices.

# Combinatorial Problem

For what values of $\lambda$ is the 'Gibbs' measure unique?

uniqueness of Gibbs measure:

$$|\Pr(v \text{ is occupied} \mid \sigma_\ell) - \Pr(v \text{ is occupied} \mid \tau_\ell)| \underset{\ell \to \infty}{\longrightarrow} 0$$



# Uniqueness for General Graphs

For what values of $\lambda$ is there a decaying rate $\delta(\ell) \underset{\ell \to \infty}{\longrightarrow} 0$

such that for every graph $G$ of maximum degree $\Delta$

and every $v \in G$,

$$|\Pr(v \text{ is occupied} \,|\, \sigma_\ell) - \Pr(v \text{ is occupied} \,|\, \tau_\ell)| \;\leq\; \delta(\ell)$$

?

# Known Bounds

- Gibbs measure is unique on all graphs of maximum degree $\Delta$ for $\lambda < \frac{2}{\Delta - 2}$. [Vigoda]

Same bound as the algorithmic one; uses essentially the same argument. (Part of a general correspondence between computational complexity and decay of correlations in the Gibbs distribution.)

# Known Bounds

- Gibbs measure is unique on all graphs of maximum degree $\Delta$ for $\lambda < \frac{2}{\Delta-2}$. [Vigoda]

- On the $\Delta$-regular tree, Gibbs measure is unique if and only if $\lambda \leq \lambda_c = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}} \ \left(\geq \frac{e}{\Delta-2}\right)$.

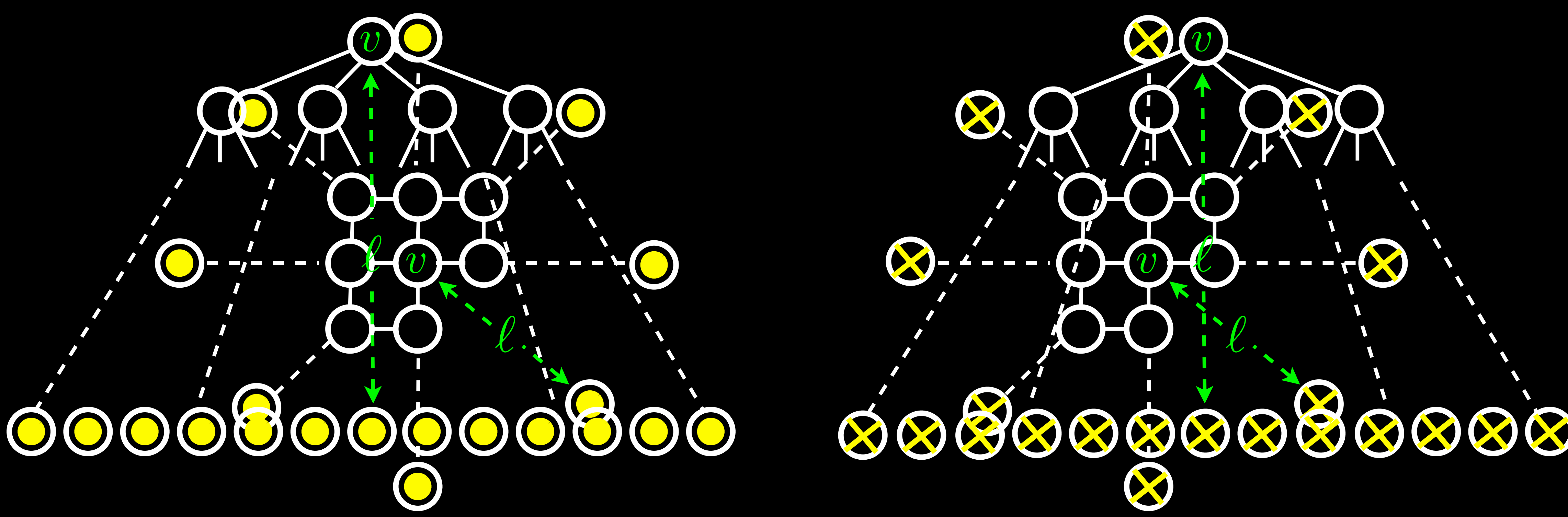
Algorithmic implications: although it is easy to count independent sets of the tree for arbitrary $\lambda$, arguments that imply uniqueness are bound to fail above $\lambda_c$ .

# Known Bounds

- Gibbs measure is unique on all graphs of maximum degree $\Delta$ for $\lambda < \frac{2}{\Delta-2}$. [Vigoda]
- On the $\Delta$-regular tree, Gibbs measure is unique if and only if $\lambda \leq \lambda_c = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}} \left(\geq \frac{e}{\Delta-2}\right)$.
- Conjecture [Sokal]: the tree is the worst case - uniqueness on all graphs for $\lambda \leq \lambda_c$.

# Main Result

**Theorem:** Fix $\Delta$ and $\lambda$. For a general graph $G$ of maximum degree $\Delta$, consider the influence of placing conditions at any given distance. This influence is maximized by taking $G$ to be the regular tree.



# Main Result

**Theorem:** Fix $\Delta$ and $\lambda$. For a general graph $G$ of maximum degree $\Delta$, consider the influence of placing conditions at any given distance. This influence is maximized by taking $G$ to be the regular tree.

**Corollary:** The Gibbs measure is unique for all graphs of maximum degree $\Delta$ and $\lambda \leq \lambda_c = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^{\Delta}}$.

# Algorithmic Implications

**New algorithm**: fix $\Delta$ and $\lambda < \lambda_c$; deterministic $(1+\epsilon)$-approximation for any graph of max degree $\Delta$ in time $\mathrm{poly}(n, 1/\epsilon)$.
(degree of poly depends on $\Delta$ and $\lambda$.)

**Corollary:** For all graphs of *'sub-exponential growth'* and $\lambda < \lambda_c$ the Glauber dynamics is rapidly mixing.
$(\Rightarrow \mathbf{FPRAS})$

# Interesting Specific Cases

- Uniformly weighted independent sets ($\lambda = 1$):
  - New: efficient approximation scheme for $\Delta \leq 5$.
  - Previous bound is $\Delta \leq 4$.
  - Believed to be hard for $\Delta \geq 6$.
  - First deterministic approx scheme for #P-complete problem.

# Interesting Specific Cases

- Uniformly weighted independent sets $(\lambda = 1)$:
  - New: efficient approximation scheme for $\Delta \leq 5$.
  - Previous bound is $\Delta \leq 4$.
  - Believed to be hard for $\Delta \geq 6$.
  - First deterministic approx scheme for #P-complete problem.

- The sqaure lattice $\mathbb{Z}^2$:
  - Believed to have a critical activity at $\sim 3.79$.
  - Previously best known lower bound: $1.25$ $\{1.45\}$ (site-perc.)
  - New bound: $1.6875$.

# Proof of Main Theorem

**Theorem:** Fix $\Delta$ and $\lambda$. For a general graph $G$ of maximum degree $\Delta$, consider the influence of placing conditions at any given distance. This influence is maximized by taking $G$ to be the regular tree.

**Part 1:** prove the theorem when $G$ is a general (irregular) tree.

In other words: on the regular tree SSM holds all the way up to the uniqueness threshold.

# Tree Representation for General Graphs

**Theorem:** For every graph $G$ and vertex $v \in G$ there exists a tree $T(G,v)$ of the same maximum degree such that

$$\Pr_G(v \text{ is occupied}) = \Pr_{T(G,v)}(\text{root is occupied}).$$

# Tree Representation for General Graphs

**Theorem:** For every graph $G$ and vertex $v \in G$ there exists a tree $T(G,v)$ of the same maximum degree such that

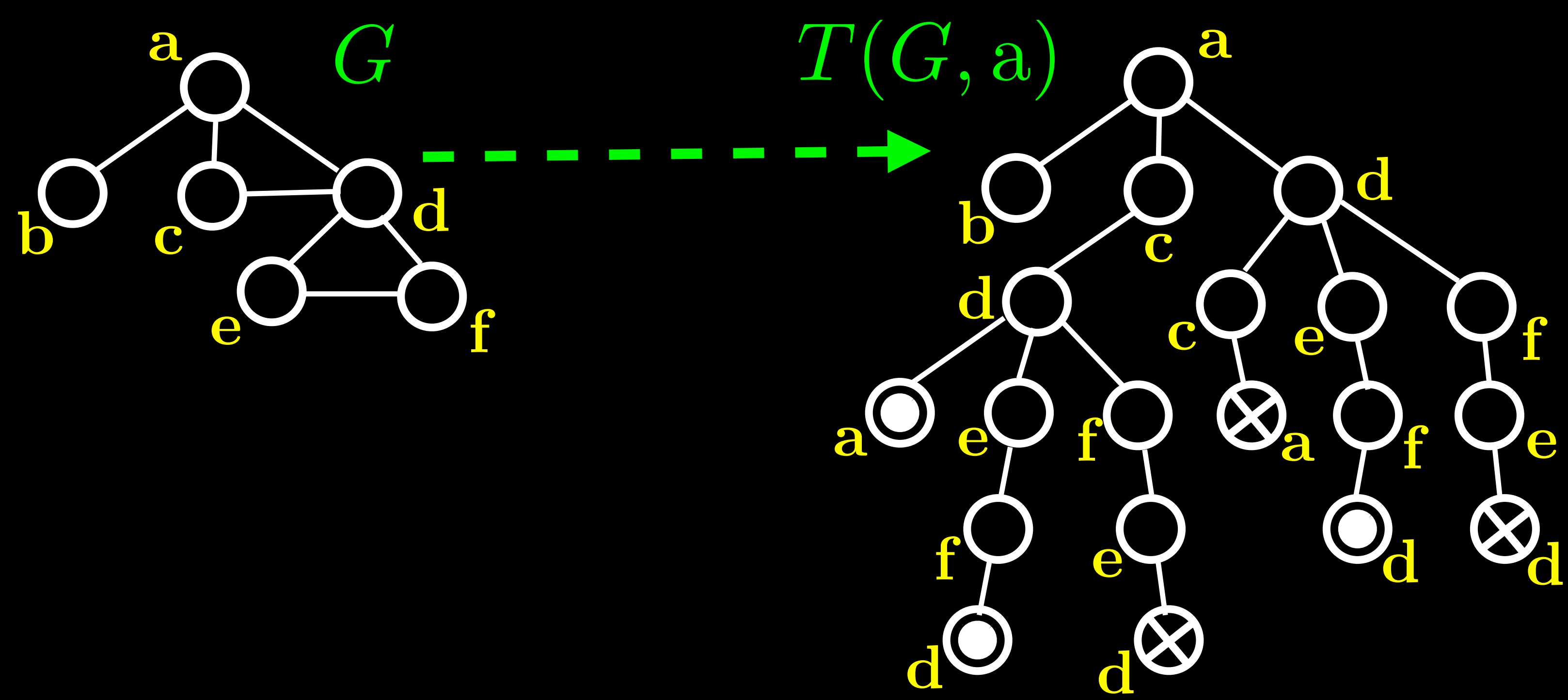
$$\Pr_G(v \text{ is occupied} \mid \sigma_\ell) \;=\; \Pr_{T(G,v)}(\text{root is occupied} \mid \widehat{\sigma_\ell}).$$

Furthermore, the correspondence (with the same tree) continues to hold when placing a condition on $G$ (and a corresponding condition on $T(G,v)$).

# Construction of $T(G, v)$

Similar to the tree of self-avoiding walks originating at $v$:

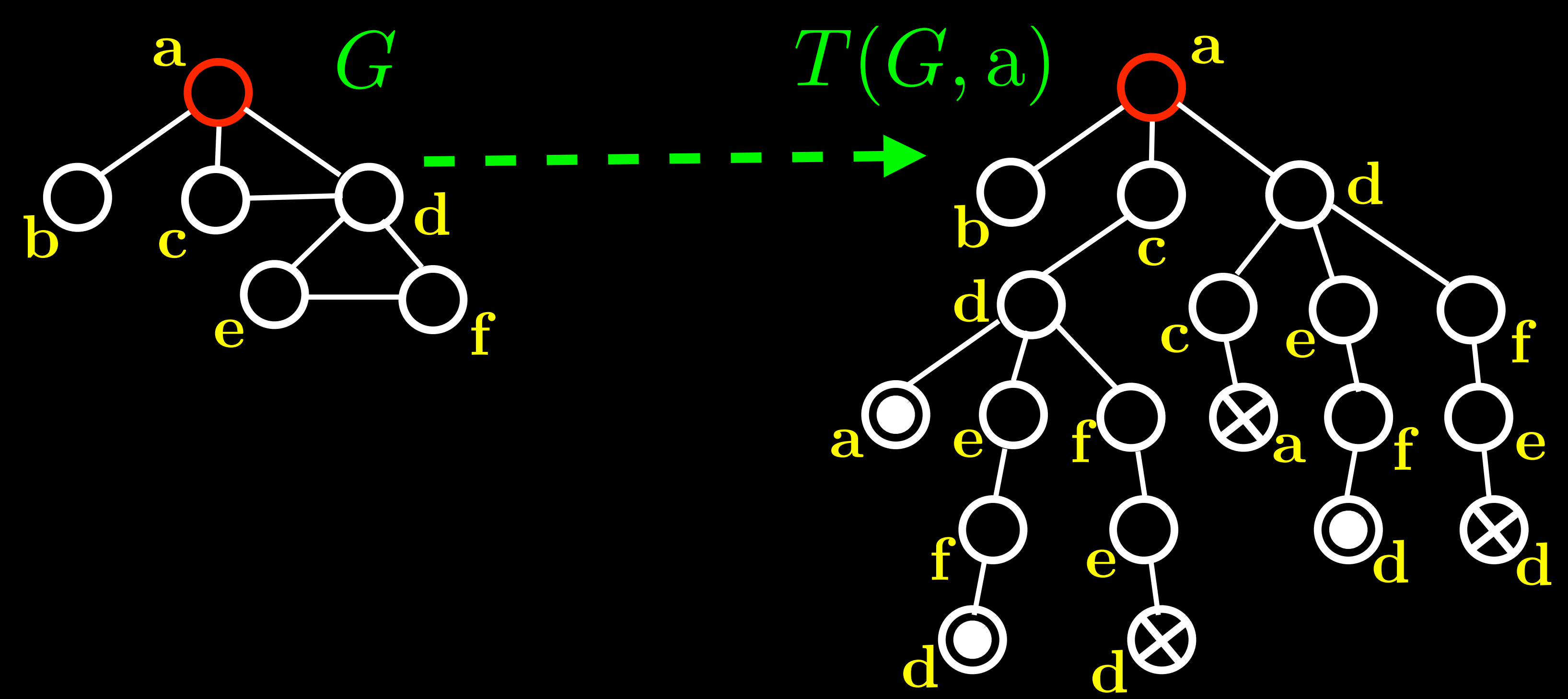
- order the neighbors of each vertex;
- construct the tree of paths originating at $v$;
- vertices that close cycles are fixed to be occupied or unoccupied (determined by the above ordering).



# Construction of $T(G, v)$

Similar to the tree of self-avoiding walks originating at $v$:

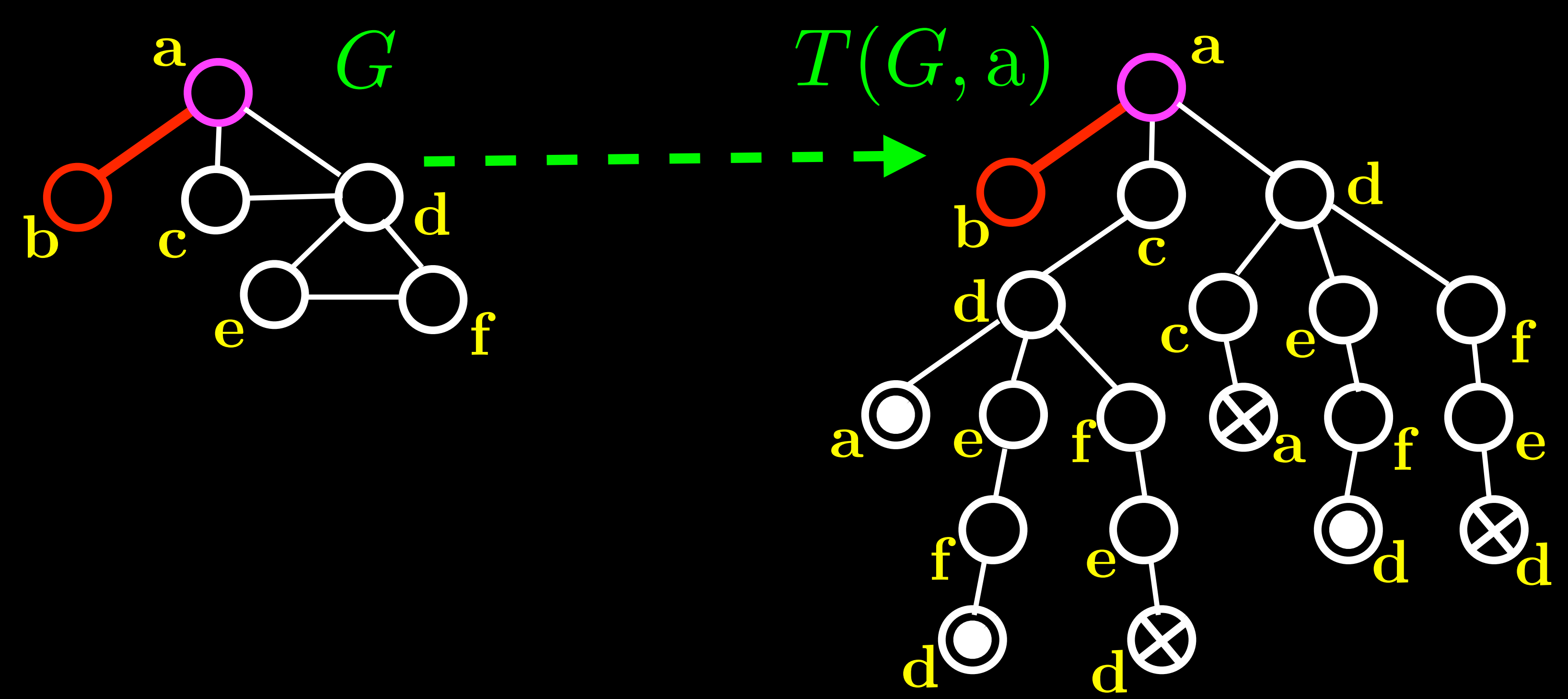
- order the neighbors of each vertex;
- construct the tree of paths originating at $v$;
- vertices that close cycles are fixed to be occupied or unoccupied (determined by the above ordering).



# Construction of $T(G, v)$

Similar to the tree of self-avoiding walks originating at $v$:

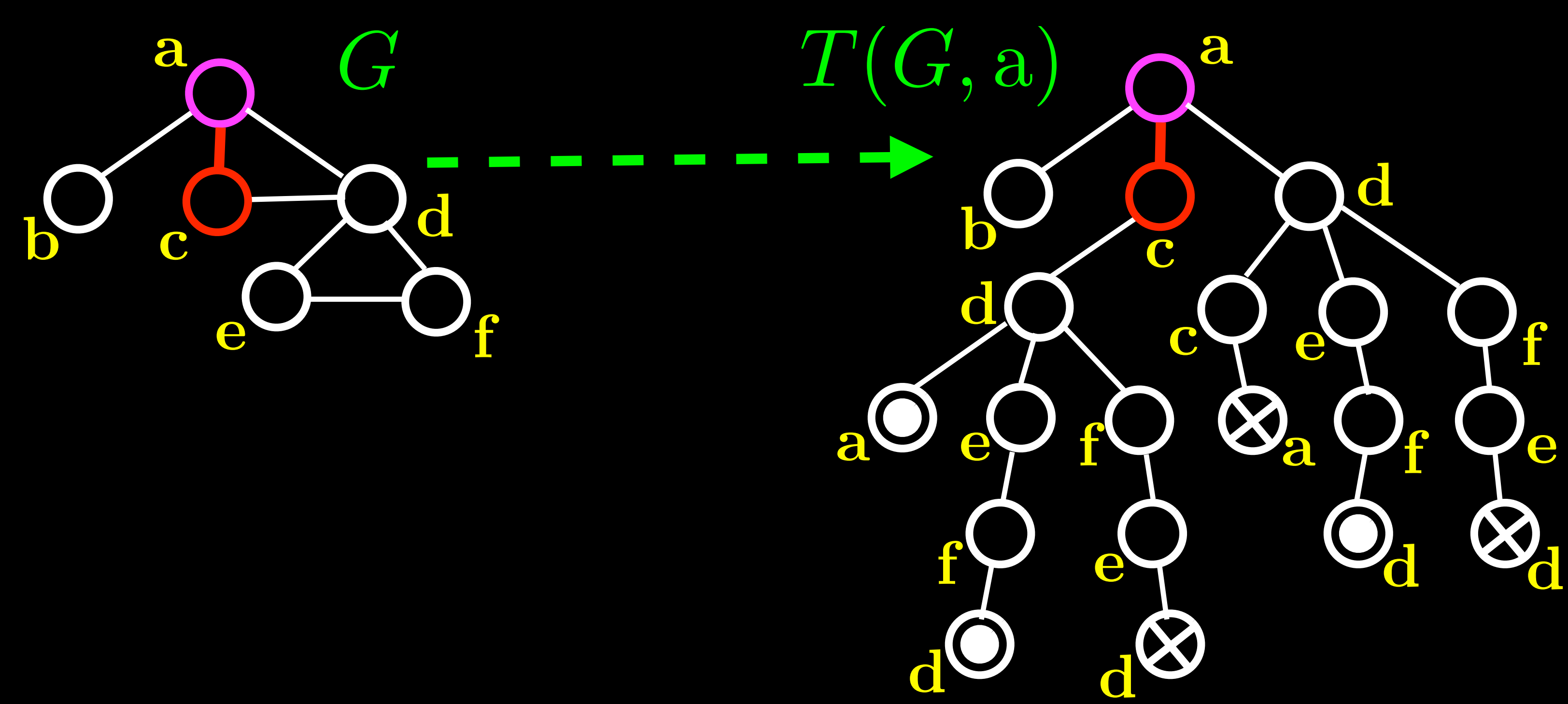
- order the neighbors of each vertex;
- construct the tree of paths originating at $v$;
- vertices that close cycles are fixed to be occupied or unoccupied (determined by the above ordering).



# Construction of $T(G, v)$

Similar to the tree of self-avoiding walks originating at $v$:

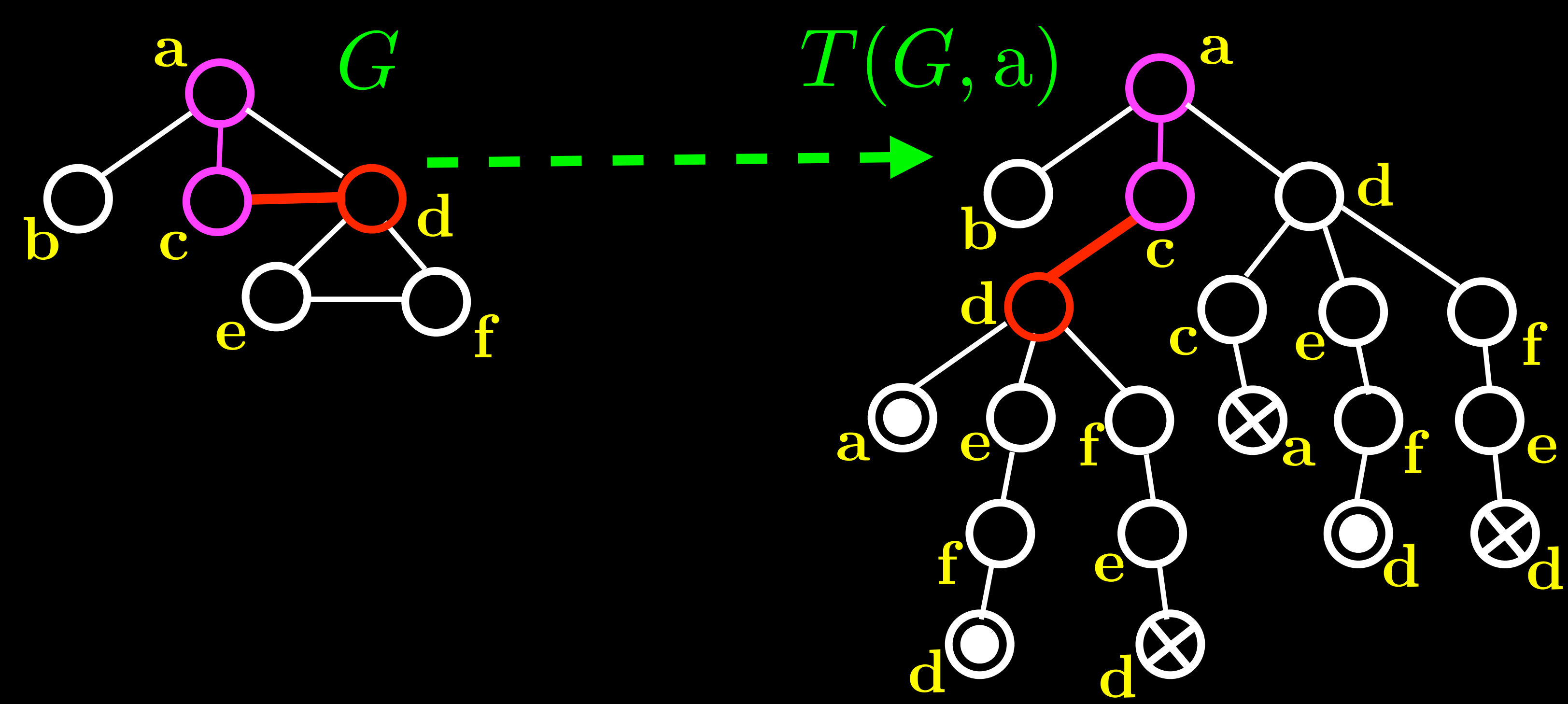
- order the neighbors of each vertex;
- construct the tree of paths originating at $v$;
- vertices that close cycles are fixed to be occupied or unoccupied (determined by the above ordering).



# Construction of $T(G, v)$

Similar to the tree of self-avoiding walks originating at $v$:

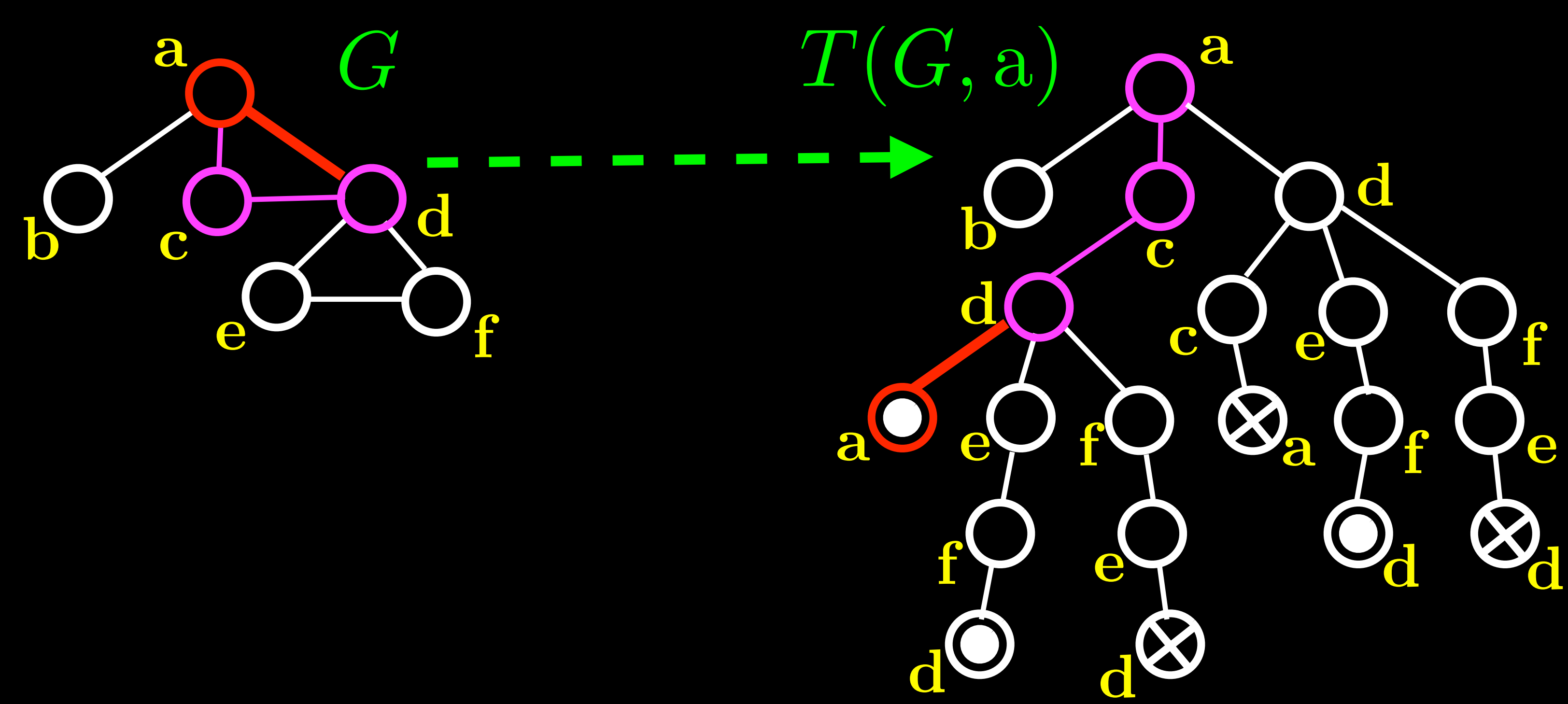
- order the neighbors of each vertex;
- construct the tree of paths originating at $v$;
- vertices that close cycles are fixed to be occupied or unoccupied (determined by the above ordering).



# Construction of $T(G, v)$

Similar to the tree of self-avoiding walks originating at $v$:

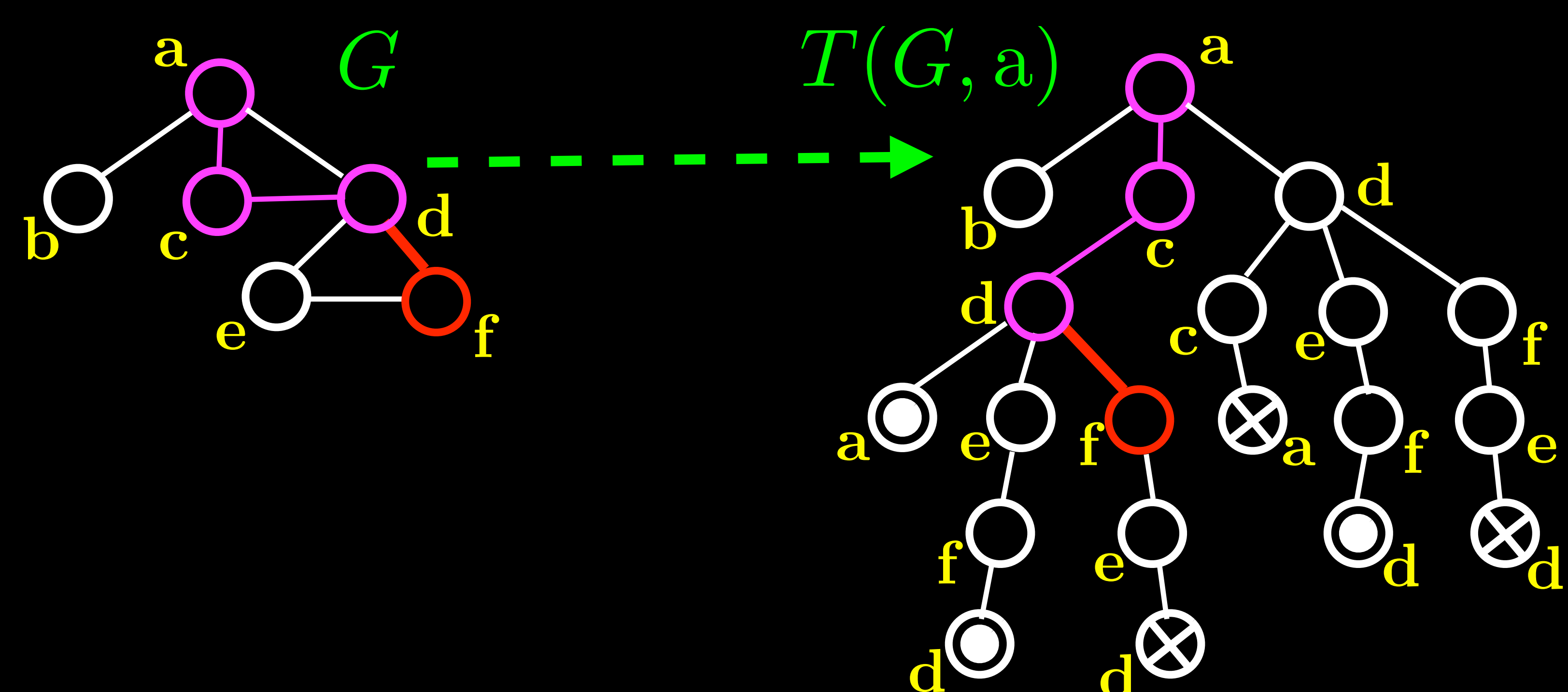
- order the neighbors of each vertex;
- construct the tree of paths originating at $v$;
- vertices that close cycles are fixed to be occupied or unoccupied (determined by the above ordering).



# Construction of $T(G, v)$

Similar to the tree of self-avoiding walks originating at $v$:

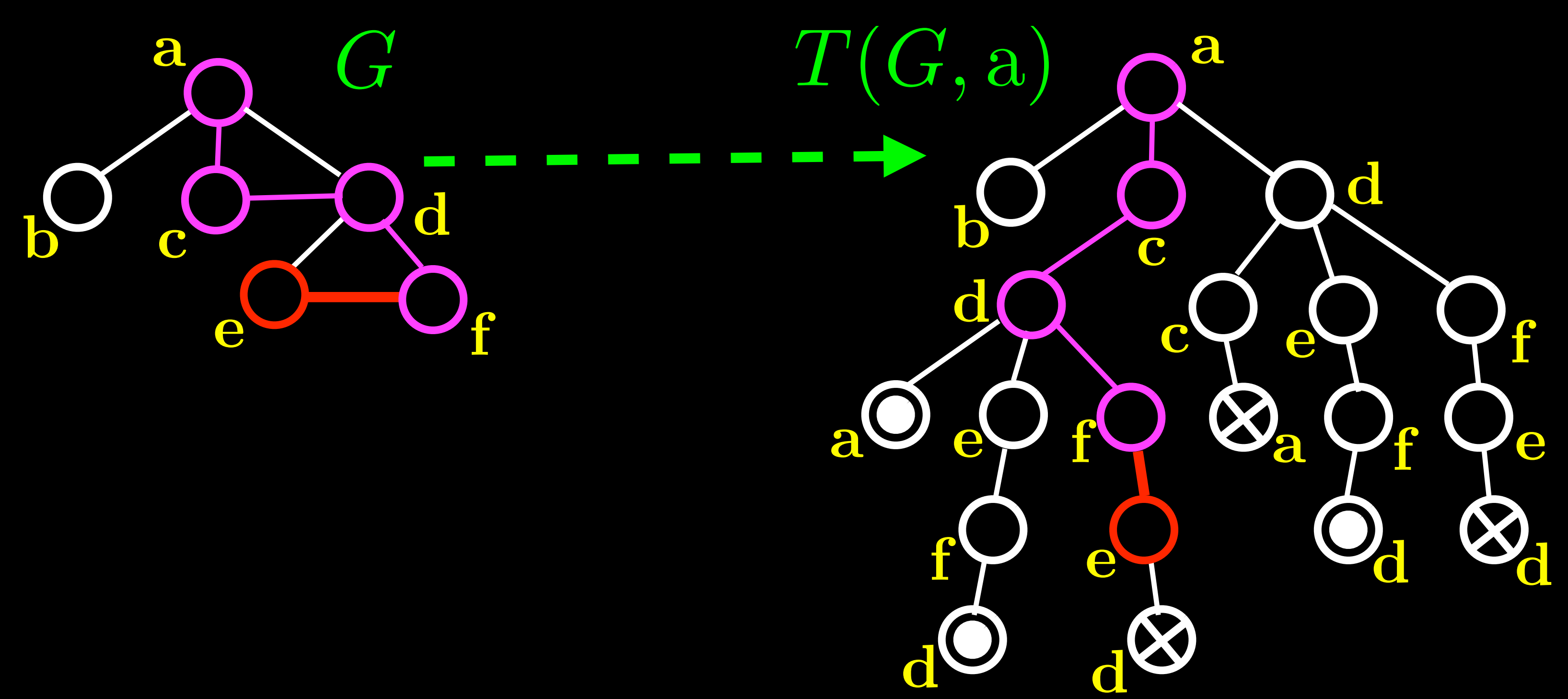
- order the neighbors of each vertex;
- construct the tree of paths originating at $v$;
- vertices that close cycles are fixed to be occupied or unoccupied (determined by the above ordering).



# Construction of $T(G, v)$

Similar to the tree of self-avoiding walks originating at $v$:

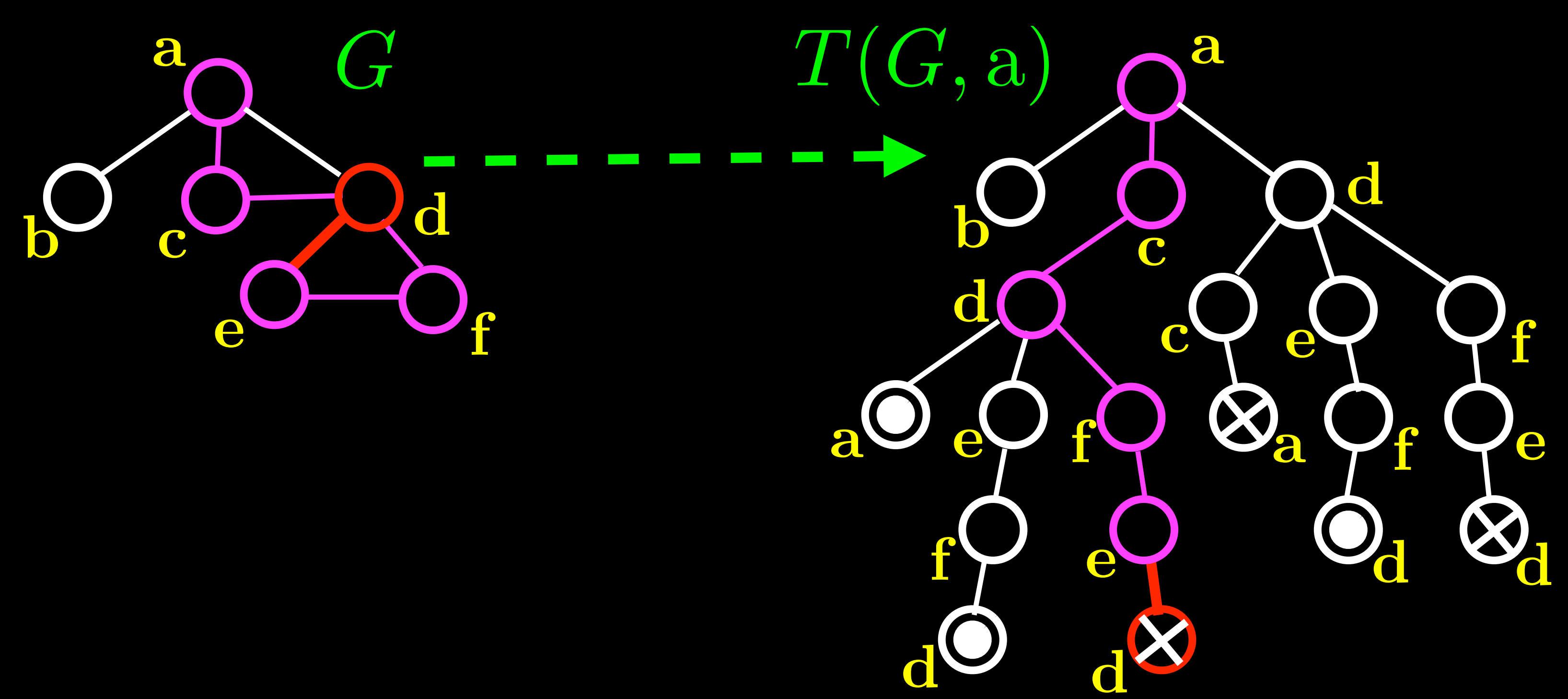
- order the neighbors of each vertex;
- construct the tree of paths originating at $v$;
- vertices that close cycles are fixed to be occupied or unoccupied (determined by the above ordering).



# Construction of $T(G, v)$

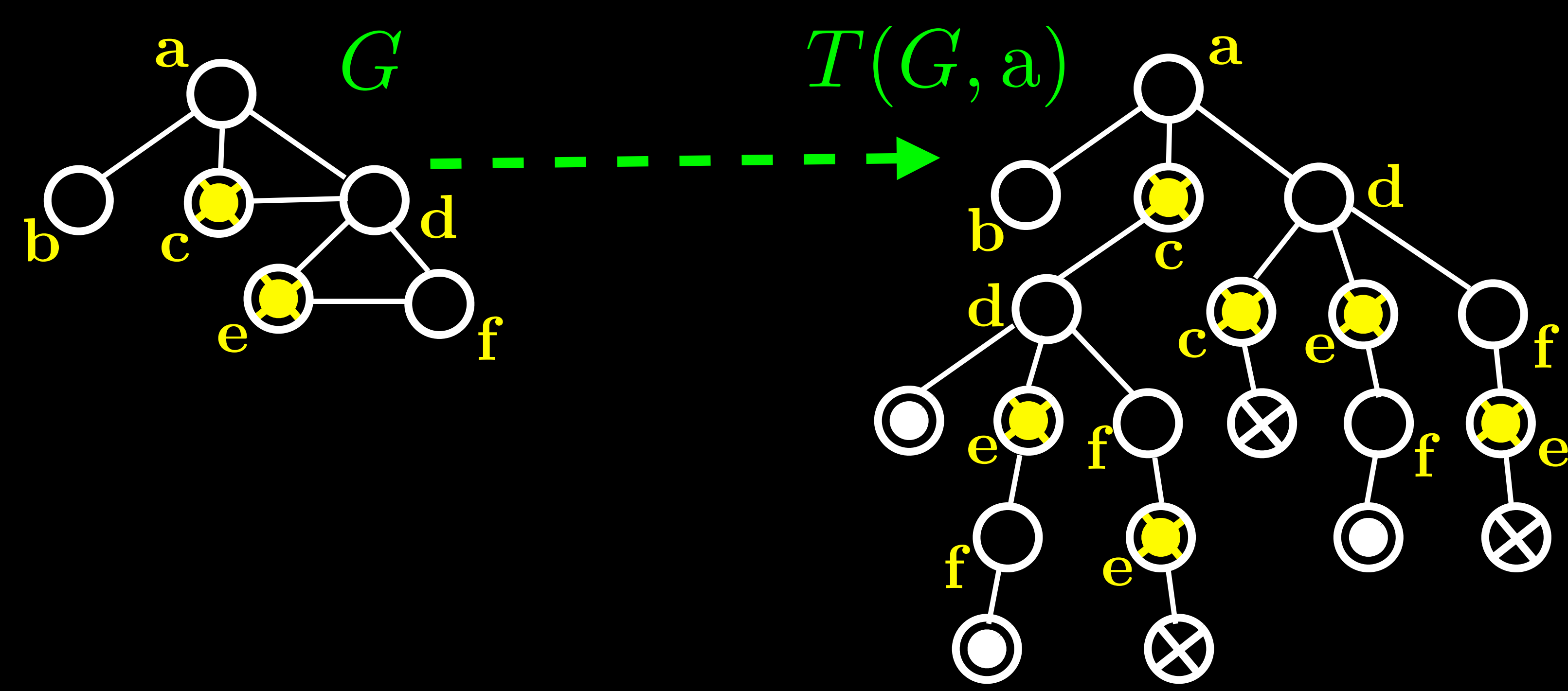
Similar to the tree of self-avoiding walks originating at $v$:

- order the neighbors of each vertex;
- construct the tree of paths originating at $v$;
- vertices that close cycles are fixed to be occupied or unoccupied (determined by the above ordering).



# Construction of $T(G, v)$

Condition on $G$ $\longrightarrow$ Condition on $T(G, v)$



# Calculating Pr(occupation)

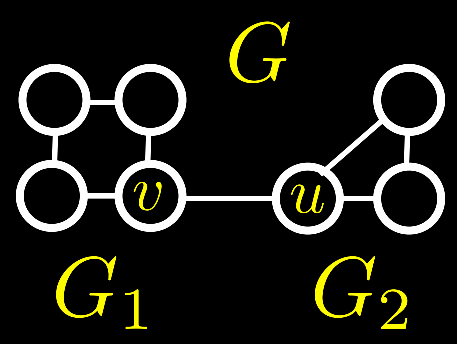
- Notation: $R^{\sigma}_{G,v} = \dfrac{\Pr_G(v \text{ is occupied} \,|\, \sigma)}{\Pr_G(v \text{ is unoccupied} \,|\, \sigma)}$ .

# Calculating Pr(occupation)

- Notation: $R^{\sigma}_{G,v} = \dfrac{\Pr_G(v \text{ is occupied} \mid \sigma)}{\Pr_G(v \text{ is unoccupied} \mid \sigma)}.$

- Basic: when connection two separate graphs -

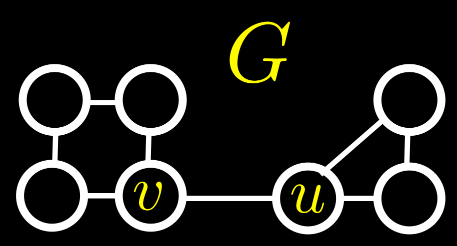
$$R_{G,v} = R_{G_1,v} \cdot \frac{1}{1 + R_{G_2,u}}$$



# Calculating Pr(occupation)

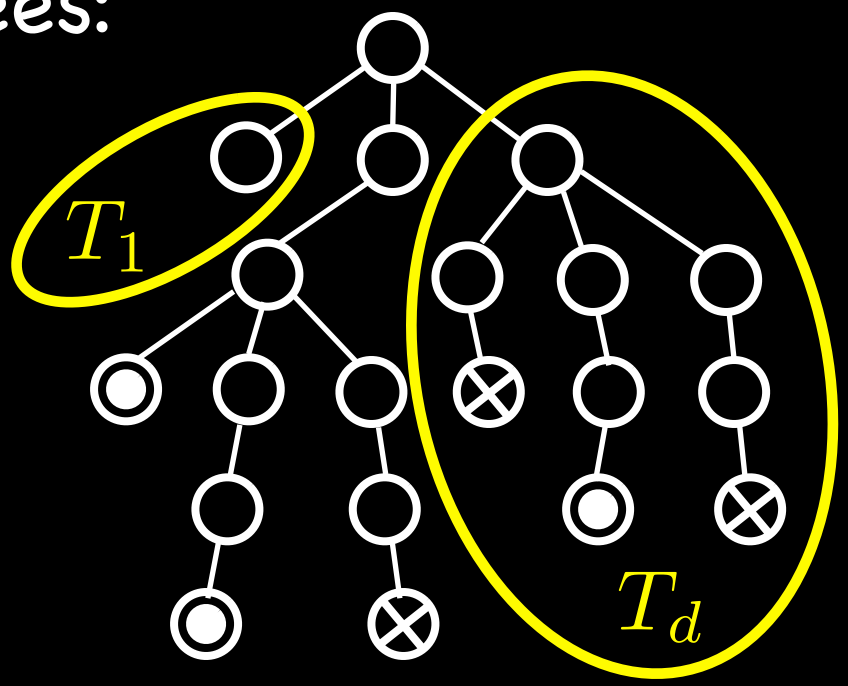
- Notation: $R^{\sigma}_{G,v} \;=\; \dfrac{\Pr_G(v \text{ is occupied} \,|\, \sigma)}{\Pr_G(v \text{ is unoccupied} \,|\, \sigma)}$.
- Basic: when connection two separate graphs -

$$R_{G,v} \;=\; R_{G_1,v} \cdot \frac{1}{1+R_{G_2,u}}$$




- Standard recursive procedure for trees:

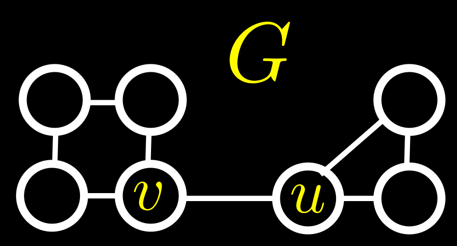
$$R_T \;=\; \lambda \prod_{i=1}^{d} \left( \frac{1}{1+R_{T_i}} \right)$$



# Calculating Pr(occupation)

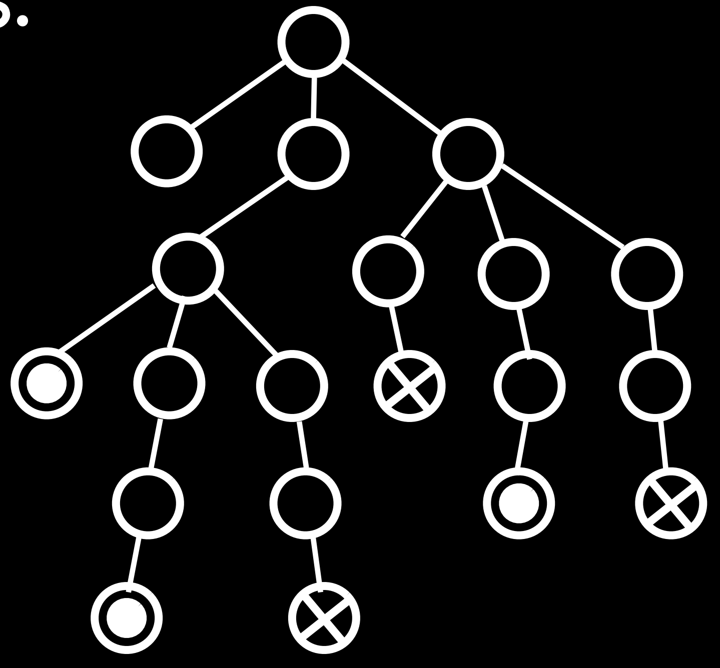
- Notation: $R^{\sigma}_{G,v} = \dfrac{\Pr_G(v \text{ is occupied} \mid \sigma)}{\Pr_G(v \text{ is unoccupied} \mid \sigma)}$.
- Basic: when connection two separate graphs -

$$R_{G,v} = R_{G_1,v} \cdot \frac{1}{1 + R_{G_2,u}}$$




- Standard recursive procedure for trees:

$$R_T = \lambda \prod_{i=1}^{d} \left( \frac{1}{1 + R_{T_i}} \right)$$

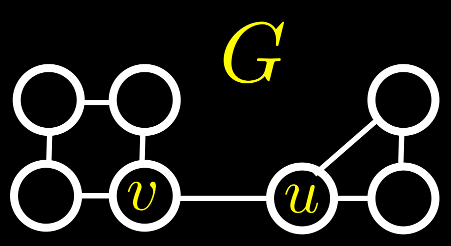


Stopping rules -

# Calculating Pr(occupation)

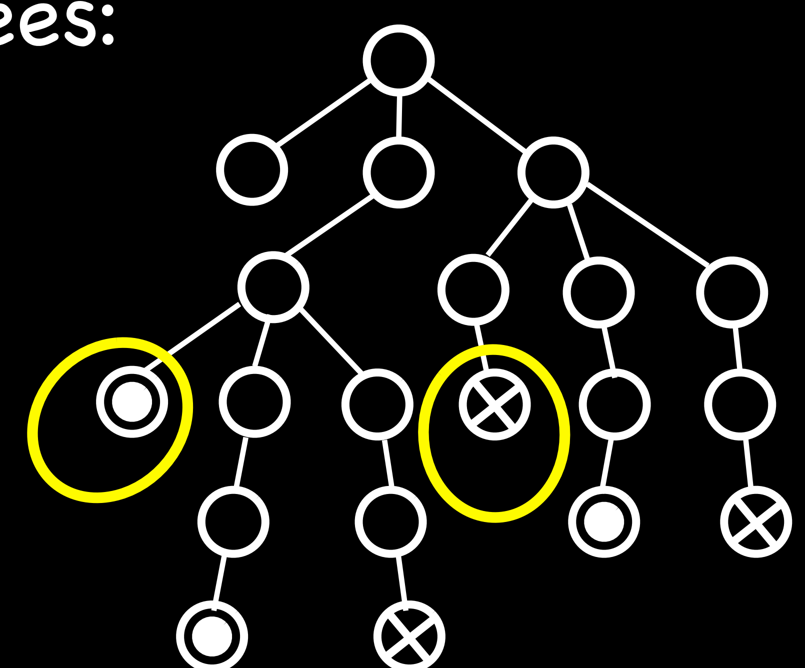
- Notation: $R^{\sigma}_{G,v} = \dfrac{\Pr_G(v \text{ is occupied} \mid \sigma)}{\Pr_G(v \text{ is unoccupied} \mid \sigma)}$.
- Basic: when connection two separate graphs -

$$R_{G,v} = R_{G_1,v} \cdot \frac{1}{1 + R_{G_2,u}}$$




- Standard recursive procedure for trees:

$$R_T = \lambda \prod_{i=1}^{d} \left( \frac{1}{1 + R_{T_i}} \right)$$



Stopping rules -

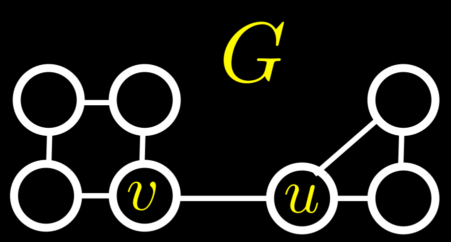
- fixed vertices: $R = \infty$ or $0$;

# Calculating Pr(occupation)

- Notation: $R^{\sigma}_{G,v} = \dfrac{\Pr_G(v \text{ is occupied} \mid \sigma)}{\Pr_G(v \text{ is unoccupied} \mid \sigma)}$.

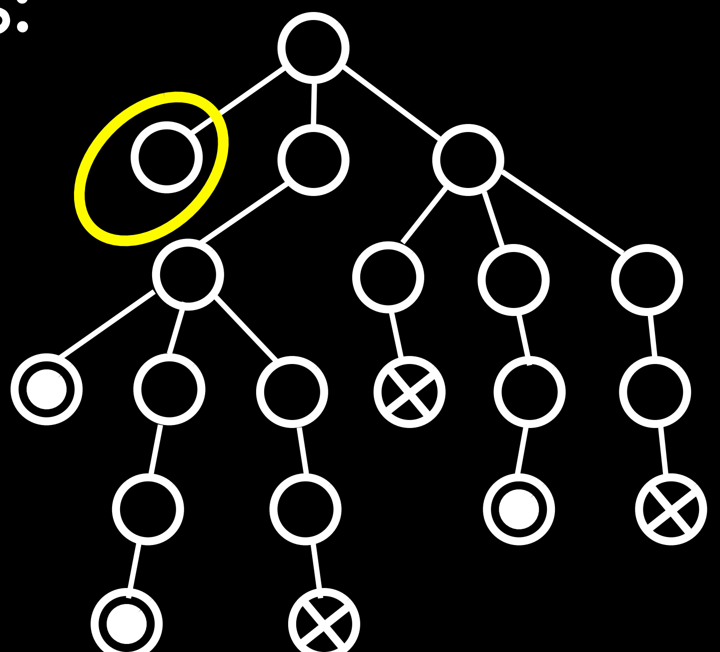
- Basic: when connection two separate graphs –

$$R_{G,v} = R_{G_1,v} \cdot \frac{1}{1 + R_{G_2,u}}$$




- Standard recursive procedure for trees:

$$R_T = \lambda \prod_{i=1}^{d} \left( \frac{1}{1 + R_{T_i}} \right)$$

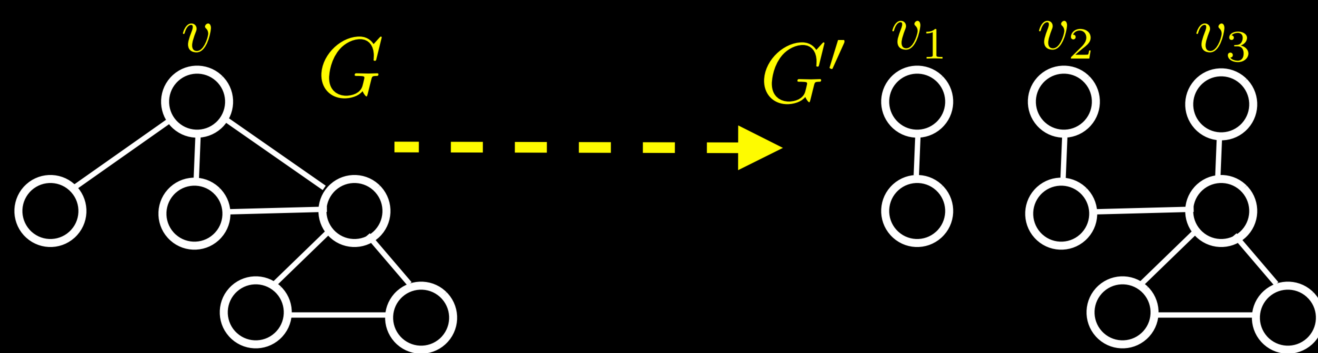


Stopping rules –

- fixed vertices: $R = \infty$ or $0$;
- (unfixed) leaves: $R = \lambda$.

# Calculating $R_{G,v}$

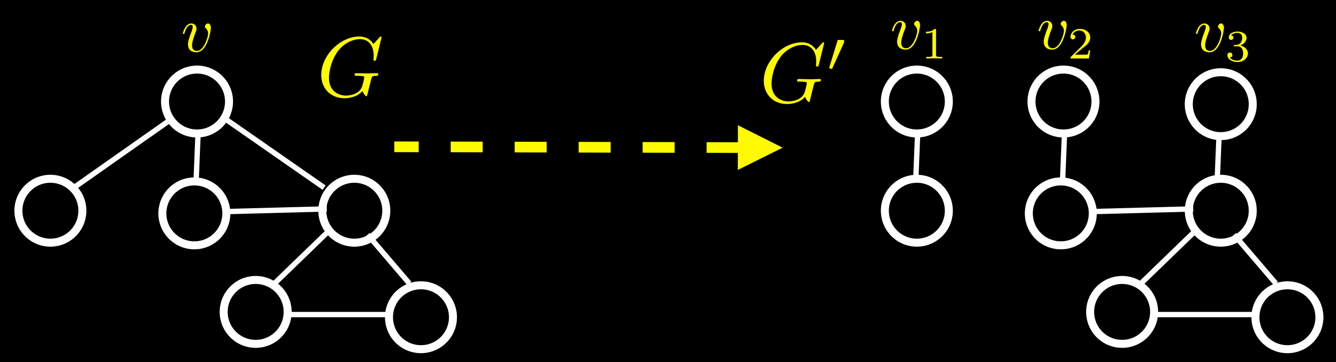
- Split $v$ into $\deg(v)$ degree-one vertices:




associate the activity $\lambda^{1/d}$ with each $v_i$.

# Calculating $R_{G,v}$

- Split $v$ into $\deg(v)$ degree-one vertices:

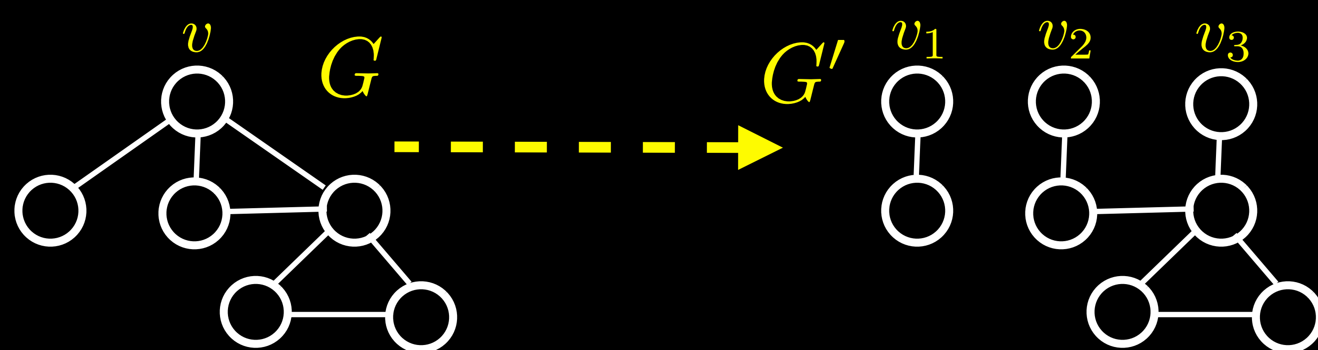



associate the activity $\lambda^{1/d}$ with each $v_i$.

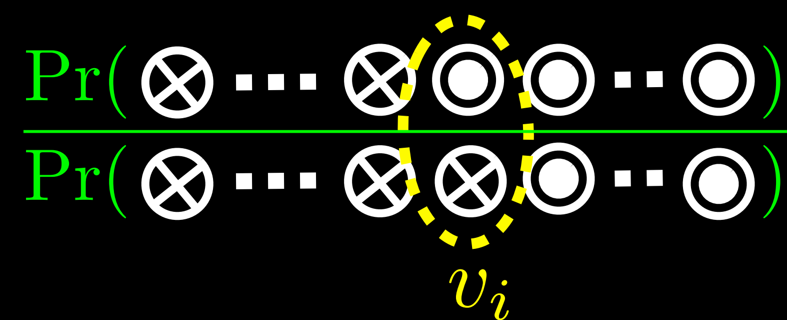
- Observation:

$$R_{G,v} = \frac{\Pr_G(v \text{ is occupied})}{\Pr_G(v \text{ is unoccupied})} = \frac{\Pr_{G'}(\text{all } v_i \text{ are occupied})}{\Pr_{G'}(\text{all } v_i \text{ are unoccupied})}.$$

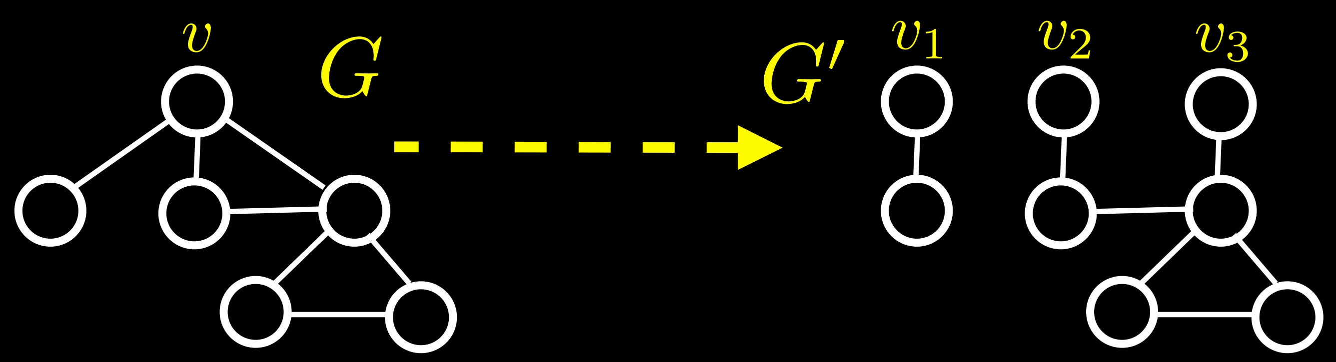
# Telescopic Product




$$\frac{\Pr_{G'}(\text{all } v_i \text{ are occupied})}{\Pr_{G'}(\text{all } v_i \text{ are unoccupied})} = \prod_{i=1}^{d}$$



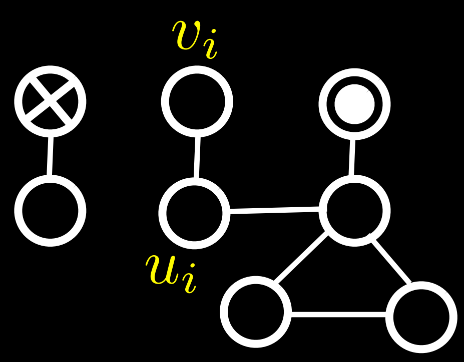
# Conditional Probabilities




$$\frac{\Pr_{G'}(\text{all } v_i \text{ are occupied})}{\Pr_{G'}(\text{all } v_i \text{ are unoccupied})} = \prod_{i=1}^{d} \frac{\Pr(\otimes \cdots \otimes \odot \odot \cdots \odot)}{\Pr(\otimes \cdots \otimes \otimes \odot \cdots \odot)}$$

$$= \prod_{i=1}^{d} R^{\tau_i}_{G',v_i} .$$

# It's all about the Neighbors




$$R^{\tau_i}_{G',v_i} = \frac{\lambda^{1/d}}{1 + R^{\tau_i}_{(G'\backslash v_i),u_i}}$$

# Recursive Procedure for Calculating $R_{G,v}$

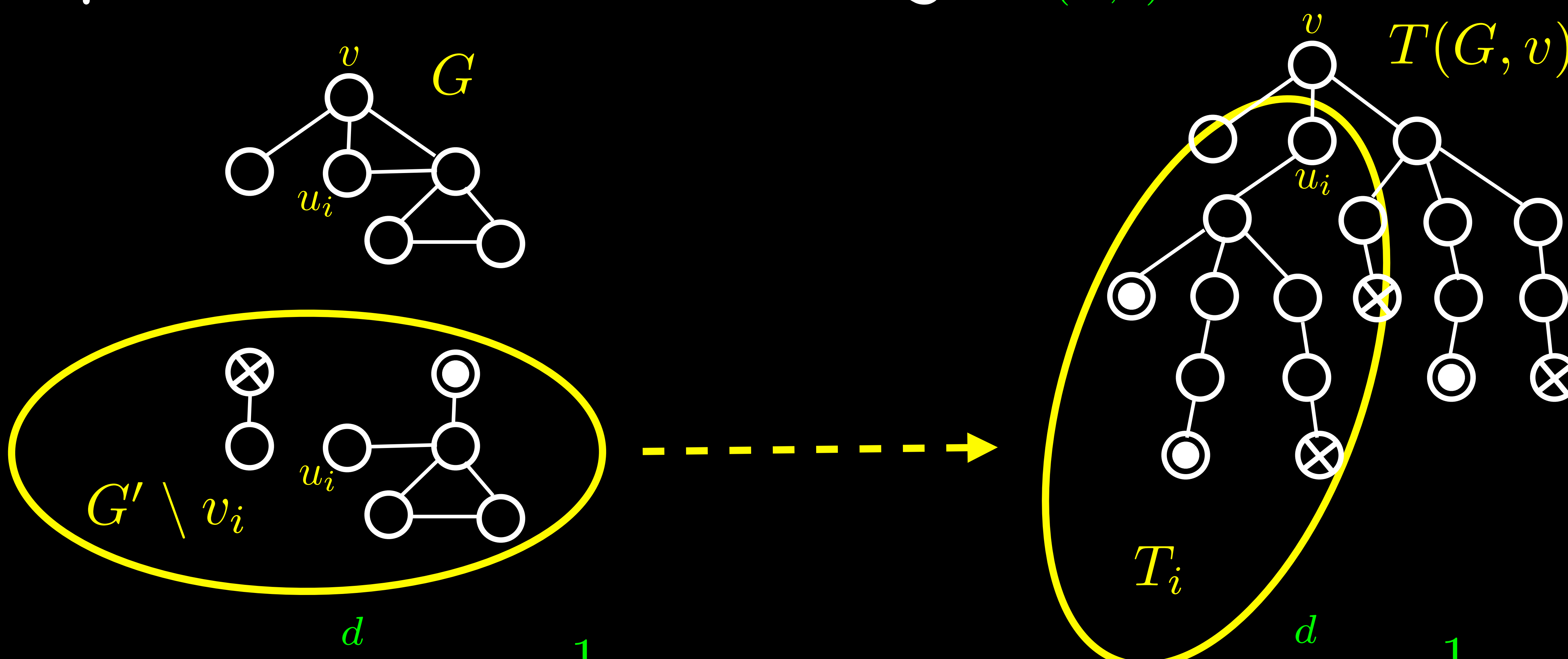

$$R^{\tau_i}_{G',v_i} = \frac{\lambda^{1/d}}{1 + R^{\tau_i}_{(G'\setminus v_i),u_i}}$$

$$\Downarrow$$

$$R_{G,v} = \lambda \prod_{i=1}^{d} \frac{1}{1 + R^{\tau_i}_{(G'\setminus v_i),u_i}}$$

$$R_{G,v} = R_{T(G,v)}$$

The procedure for calculating $R_{G,v}$ makes exactly the same calculations as the tree procedure for calculating $R_{T(G,v)}$.

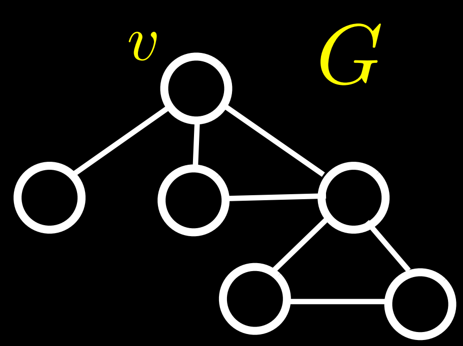


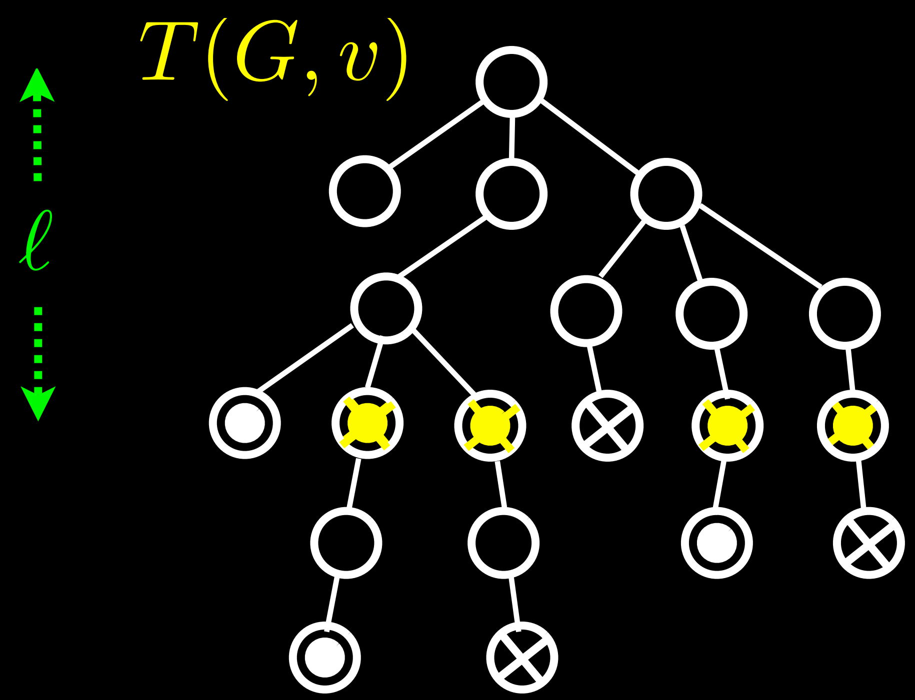

$$R_{G,v} = \lambda \prod_{i=1}^{d} \frac{1}{1 + R^{\tau_i}_{(G' \setminus v_i), u_i}}$$

$$R_T = \lambda \prod_{i=1}^{d} \frac{1}{1 + R_{T_i}}$$

# Approximation Algorithm

- Run the previous recursive procedure, but if the stack of the recursion is $\ell$ levels deep return trivial lower and upper bounds.



# Approximation Algorithm

- Run the previous recursive procedure, but if the stack of the recursion is $\ell$ levels deep return trivial lower and upper bounds.
- Running time is $O((\Delta - 1)^{\ell})$.
- For $\lambda < \lambda_c$ the difference between the resulting lower and upper bounds is $\leq \exp(-\ell)$.

  $\Rightarrow$ $(1+\epsilon)$-approximation for $\Pr(v \text{ is occupied})$ in time $\text{poly}(1/\epsilon)$.

# Summary

- New Tree representation for general graphs.
- Proves that the tree is the "worst-case".
- New tree-like algorithm for approximately counting independent sets (works up to the tree threshold).
- Improved bounds for specific interesting settings:
  - Uniformly weighted independent sets with $\Delta \leq 5$.
  - The square lattice $\mathbb{Z}^2$.

# Open Problems

1. Tree representation is valid for any binary spin system (i.e., Ising models). Is there a tree representation for models with more than two spins (e.g., proper colorings) ?
   - [Gamarnik-Katz, Nair-Tetali]: Tree-like algorithms (branching depends on spins as well, no direct comparison with model on the tree, require stronger and unnatural forms of decay of correlations).
   - Negative result [Sly]: tree is not worst case for uniqueness.

# Open Problems

2. Improve the hardness threshold for approximately counting independent sets.

   – [Mossel-W-Wormald]: Conjecture that $\lambda_c$ is the threshold for the computational probelm. Provide evedince that approximation is hard above $\lambda_c$.

3. More efficient variants of the algorithm (iterative?)

4. Solve other problems using the tree representation:
   – Spin glass Ising on $\mathbb{Z}^d$.
   – SSM down to $T_c$ for Ising on $\mathbb{Z}^d$ for d>2.

# Thanks

- Elchanan Mossel
- Alistair Sinclair & Fabio Martinelli