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COUNTING PSEUDO-HOLOMORPHIC DISCS IN CALABI-YAU 3-FOLDS

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Abstract. In this paper we define an invariant of a pair of a 6 dimensional symplectic manifold with vanishing 1st Chern class and its relatively spin Lagrangian submanifold with vanishing Maslov index. This invariant is a function on the set of the path connected components of bounding cochains (solutions of the A_∞ version of the Maurer-Cartan equation of the filtered A_∞ algebra associated to the Lagrangian submanifold). In the case when the Lagrangian submanifold is a rational homology sphere, it becomes a numerical invariant.

This invariant depends on the choice of almost complex structures. The way how it depends on the almost complex structures is described by a wall crossing formula which involves a moduli space of pseudo-holomorphic spheres.

1. Introduction. This paper is a continuation of [7, Subsection 3.6.4] and [5].

Let (M, ω) be a symplectic manifold of (real) dimension 2×3. We assume that $c^1(M) = 0$ in $H^2(M; \boldsymbol{Q})$. (Here we use a compatible almost complex structure of the tangent bundle to define $c^1(M)$.) Let $L \subset M$ be a relatively spin Lagrangian submanifold and $\mu_L : H_2(M, L; \boldsymbol{Z}) \to 2\boldsymbol{Z}$ its Maslov index homomorphism. (See [7, Subsection 2.1.1].) We assume that μ_L is 0. In this paper we consider such a pair (M, L). A typical example is a pair of a Calabi-Yau 3-fold M and its special Lagrangian submanifold L. This is one of the most interesting cases of (homological) mirror symmetry. Our main purpose of this paper is to define and study an invariant of such (M, L). It is independent of the various choices involved in the construction but depends on the almost complex structures J of M.

We consider $\mathcal{M}(L; J; \Lambda_+)$ the set of 'Λ_+-valued points of a Maurer-Cartan formal scheme' of the filtered A_∞ structure associated to L. This is the set of gauge equivalence classes of bounding cochains and defined in [7, Section 4.3]. (Here we include J in the notation since J dependence is rather crucial in this paper.) We study the cyclic filtered A_∞ algebra $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^J_{k,\beta}\})$ produced in [5] by modifying the construction of [7]. In our case where μ_L is 0 we can reduce the coefficient ring to $\Lambda_0 = \Lambda^{(0)}_{0,\mathrm{nov}}$, that is the degree 0 part of the universal Novikov ring with $\boldsymbol{R}$ coefficient. (See (9).)

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We denote by Λ_+ its maximal ideal. Let $[b] \in \mathcal{M}(L; J; \Lambda_+)$. We define a superpotential (without leading term) by:

$$\Psi'(b; J) = \sum_{k=0}^{\infty} \sum_{\beta \in H_2(M,L;\mathbf{Z})} \frac{T^{\beta \cap \omega}}{k+1} \langle \mathfrak{m}^J_{k,\beta}(b, \dots, b), b \rangle . \tag{1}$$

To obtain a superpotential which is independent of perturbations and other choices involved, we need to add a constant term to $\Psi'(\cdot\,; J)$. Note that $\mathfrak{m}^J_{k,\beta}$ is defined by using the moduli space $\mathcal{M}_{k+1}(\beta; J)$ of pseudo-holomorphic discs with $k+1$ marked points and of homology class $\beta \in H_2(M, L; \mathbf{Z})$. We use $\mathcal{M}_0(\beta; J)$, the moduli space of J holomorphic discs, of homology class β without marked point, to define

$$\mathfrak{m}^J_{-1,\beta} \text{ “=” } \#\mathcal{M}_0(\beta; J) . \tag{2}$$

(See Sections 3 and 4 for precise definition.) And put

$$\Psi(b; J) = \Psi'(b; J) + \sum_{\beta \in H_2(M,L;\mathbf{Z})} T^{\beta \cap \omega} \mathfrak{m}^J_{-1,\beta} . \tag{3}$$

More precisely we assume that our almost complex structure J satisfies the following:

ASSUMPTION 1.1. There exists no nontrival J-holomorphic sphere $v : S^2 \to M$ such that $v(S^2) \cap L \neq \emptyset$.

By dimension counting we find that the set of such J is dense.

THEOREM 1.2. (1) *If J satisfies the Assumption* 1.1, *then there exists a function*

$$\Psi(\cdot\,; J) : H^1(L; \Lambda_+) \to \Lambda_+$$

which depends not only on J but also on the perturbations etc.

(2) *There exists an isomorphism between the set $\mathcal{M}(L; J; \Lambda_+)$ and the set of critical points of $\Psi(\cdot\,; J)$.*

(3) *The restriction of $\Psi(\cdot\,; J)$ to its critical point set $\mathcal{M}(L; J; \Lambda_+)$ depends only on M, L, J and is independent of the choice of the perturbations etc.*

We call $\Psi(\cdot\,; J)$ a *superpotential*. The value $\Psi(b; J)$ depends only on the path connected component of $[b] \in \mathcal{M}(L; J; \Lambda_+)$. See Proposition 2.8.

COROLLARY 1.3. *If L is a rational homology sphere in addition, then $\mathcal{M}(L; J; \Lambda_+)$ is one point. So the value of $\Psi(\cdot\,; J)$ at that point is an invariant of M, L, J.*

In Sections 2 and 3 we develop the theory of the superpotential of cyclic filtered A_∞ algebras of dimension 3 with additional data corresponding to $\mathfrak{m}^J_{-1,\beta}$. In Section 2 we fix our cyclic filtered A_∞ algebras and review the construction of the superpotential and its gauge invariance. We next study its relation to pseudo-isotopies of cyclic filtered A_∞ algebras to complete the algebraic part of the proof of Theorem 1.2 in Section 3. The algebraic structure we assumed in Sections 2 and 3 are realized in Section 4, where the proof of Theorem 1.2 is completed.

We can extend the domain $H^1(L;\Lambda_+)$ of the definition of $\Psi(b;J)$ as follows. Let $\mathbf{e}_i$, $i=1,\dots,b_1$ be a basis of $H^1(L;\mathbf{Z})/\mathrm{Torsion}$. We put $\mathbf{b}=\sum x_i\mathbf{e}_i$ where $x_i\in\Lambda_0$. We put

$$y_i = e^{x_i} = \sum_{k=0}^{\infty}\frac{1}{k!}x_i^k\,.$$

We define the *strongly convergent Laurent power series ring*

$$\Lambda_0\langle\!\langle y_1,\dots,y_{b_1},y_1^{-1},\dots,y_{b_1}^{-1}\rangle\!\rangle$$

as the set of formal sums

$$f(y_1,\dots,y_{b_1})=\sum_{i=1}^{\infty}T^{\lambda_i}P_i(y_1,\dots,y_{b_1}) \tag{4}$$

where $\lambda_i\in\boldsymbol{R}_{\ge 0}$ with $\lim_{i\to\infty}\lambda_i=\infty$ and P_i is a Laurent *polynomial* for all i. (See [2].) We remark that for each f as in (4) and $\mathfrak{y}_1,\dots,\mathfrak{y}_{b_1}\in\Lambda_0$ with $v(\mathfrak{y}_i)=0$, the sum

$$\sum_{i=1}^{\infty}T^{\lambda_i}P_i(\mathfrak{y}_1,\dots,\mathfrak{y}_{b_1})$$

converges in the T adic topology. Here $v(\cdot)$ is defined by

$$v\left(\sum a_iT^{\lambda_i}\right)=\inf\{\lambda_i\,;\,a_i\ne 0\}\,.$$

Therefore $f(\mathfrak{y}_1,\dots,\mathfrak{y}_{b_1})$ is well-defined.

THEOREM 1.4. (1) $\Psi(b;J)\in\Lambda_0\langle\!\langle y_1,\dots,y_{b_1},y_1^{-1},\dots,y_{b_1}^{-1}\rangle\!\rangle$.

(2) *There exists $\delta>0$ such that $\Psi(\cdot\,;J)$ is extended to*

$$\{(y_1,\dots,y_{b_1})\,;\,-\delta<v(y_i)<\delta\}\,. \tag{5}$$

(3) *Its critical point set is identified with $\mathcal{M}(L;J)_\delta$ which is introduced in Theorem* 1.2 [5].

(4) *The restriction of $\Psi(\cdot\,;J)$ to $\mathcal{M}(L;J)_\delta$ is independent of the perturbations etc. and depends only on M, L, J.*

We prove Theorem 1.4 in Section 7.

In Section 5 we use the canonical models constructed in [7, Section 4.5] and [5, Section 10], to rewrite the definition of Ψ.

In Section 6 we discuss the way how the superpotential Ψ depends on almost complex structures. The main result is Theorem 1.5 below. We assume that J_0 and J_1 satisfy Assumption 1.1. We take a path $\mathcal{J}=\{J_t\,;\,t\in[0,1]\}$ of tame almost complex structures joining them. Let $\mathcal{M}_1^{\mathrm{cl}}(\alpha;J)$ be the moduli space of J holomorphic stable maps of genus zero in M of homology class $\alpha\in H_2(M;\mathbf{Z})$ and with one marked point. Here cl in the notation means that this is a moduli space of stable maps from *closed* Riemann surfaces. The moduli space $\mathcal{M}_1^{\mathrm{cl}}(\alpha;J)$ has a Kuranishi structure of (virtual) dimension 2. We put

$$\mathcal{M}_1^{\mathrm{cl}}(\alpha;\mathcal{J})=\bigcup_{t\in[0,1]}\{t\}\times\mathcal{M}_1^{\mathrm{cl}}(\alpha;J_t)\,. \tag{6}$$

Using the evaluation map $ev^{\mathrm{int}} : \mathcal{M}_1^{\mathrm{cl}}(\alpha; \mathcal{J}) \to M$ we obtain a virtual fundamental chain $ev_*^{\mathrm{int}}([\mathcal{M}_1^{\mathrm{cl}}(\alpha; \mathcal{J})])$ of dimension 3. Since J_0 and J_1 satisfy Assumption 1.1 it follows that

$$L \cap ev^{\mathrm{int}}(\partial \mathcal{M}_1^{\mathrm{cl}}(\alpha; \mathcal{J})) = \emptyset .$$

Therefore

$$n(L; \alpha; \mathcal{J}) = [L] \cap ev_*^{\mathrm{int}}([\mathcal{M}_1^{\mathrm{cl}}(\alpha; \mathcal{J})]) \in \boldsymbol{Q} \tag{7}$$

is well-defined. Moreover it depends only on M, L, α, J_0, J_1 and is independent of the path $\mathcal{J}$.

THEOREM 1.5. *Let* $[b] \in \mathcal{M}(L; J_0; \Lambda_+)$. *We take the canonical isomorphism* $I_* : \mathcal{M}(L; J_0; \Lambda_+) \to \mathcal{M}(L; J_1; \Lambda_+)$ *in* [7, Section 4.3]. *Then we have*:

$$\Psi(I_*(b); J_1) - \Psi(b; J_0) = \sum_{\alpha \in H_2(M; \boldsymbol{Z})} T^{\alpha \cap \omega} n(L; \alpha; \mathcal{J}) . \tag{8}$$

Theorem 1.5 is proved in Section 6. In Section 8 we discuss some conjectures, open problems, and relations to various related topics.

REMARK 1.6. (1) The superpotential of the form (1) appears in the physics literature [18, 22].

(2) The idea to include the 2nd term of (3) to obtain a numerical invariant of Lagrangian submanifold is due to D. Joyce. It was communicated to the author by P. Seidel around 2002, who also explained him the importance of cyclic symmetry for this purpose. (However the appearance of the nontrivial wall crossing by the change of J was unknown at that time.)

(3) The appearance of the nonzero wall crossing term in the right-hand side of (8) is closely related to the phenomenon discussed in [7, Section 3.8 and Subsection 7.4.1]. Around the same time as the authors of [7] found this phenomenon, a similar observation was done independently by M. Liu [17].

(4) A related homological algebra was discussed before by [3, 15]. The part concerning the second term of (3) is not discussed there.

(5) All the A_∞ algebras and pseudo-isotopies between them which appear in the geometric situation in this paper, are unital. We omit the argument on unitality since it is a straightforward analog of the one in [5].

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NOTATIONS. The Novikov ring Λ_0 is defined by:

$$\Lambda_0 = \left\{ \sum_{i=0}^{\infty} a_i T^{\lambda_i} ; a_i \in \boldsymbol{R}, \lambda_i \in \boldsymbol{R}_{\geq 0}, \lim_{i \to \infty} \lambda_i = \infty \right\} , \tag{9}$$

where T is a formal variable.

We consider a free graded Λ_0 module C such that

$$C = \overline{C} \otimes_{\boldsymbol{R}} \Lambda_0\,, \tag{10}$$

where $\overline{C}$ is an $\boldsymbol{R}$ vector space. We define $C[1]$ by shifting the grading. Namely:

$$C[1]^k = C^{k+1}\,. \tag{11}$$

We denote by deg, deg$'$, the degree and shifted degree of elements of C, $C[1]$, respectively. Namely:

$$\deg' = \deg - 1\,. \tag{12}$$

We put

$$B_k(C[1]) = \underbrace{C[1] \otimes_{\Lambda_0} \cdots \otimes_{\Lambda_0} C[1]}_{k \text{ times}} \tag{13}$$

and

$$B_k(\overline{C}[1]) = \underbrace{\overline{C}[1] \otimes \cdots \otimes \overline{C}[1]}_{k \text{ times}}\,. \tag{14}$$

2. Superpotentials and their gauge invariance. Let $(C, \langle\cdot\rangle, \{\mathfrak{m}_{k,\beta}\})$ be a G-gapped cyclic filtered A_∞ algebra of dimension 3. (See [5, Definition 6.1].) Namely $\mathfrak{m}_{k,\beta} : B_k(\overline{C}[1]) \to \overline{C}[1]$ is an $\boldsymbol{R}$ linear map for each $\beta \in G$ and k, with $(\beta, k) \neq (0, 0)$, such that

$$\mathfrak{m}_k = \sum_{\beta\in G} T^{E(\beta)} \mathfrak{m}_{k,\beta}$$

(we define E below) satisfies A_∞ relation

$$0 = \sum_{k_1+k_2=k+1} \sum_{i=1}^{k_1} (-1)^* \mathfrak{m}_{k_1}(x_1, \dots, \mathfrak{m}_{k_2}(x_i, \dots, x_{i+k_2-1}), \dots, x_k)$$

where $* = \deg' x_1 + \cdots + \deg' x_{i-1}$.

$\langle\cdot,\cdot\rangle : \overline{C}^k \otimes \overline{C}^{3-k} \to \boldsymbol{R}$ is a symmetric nondegenerate inner product such that

$$\langle \mathfrak{m}_{k,\beta}(x_1, \dots, x_k), x_0\rangle = (-1)^{(\deg' x_0)(\deg' x_1 + \cdots + \deg' x_k)} \langle \mathfrak{m}_{k,\beta}(x_0, \dots, x_{k-1}), x_k\rangle\,.$$

In the general situation of [7, Condition 3.1.6], [5, Definition 6.2], G is a discrete submonoid of $\boldsymbol{R}_{\ge 0} \times 2\boldsymbol{Z}$. In this paper we always assume

$$G \subset \boldsymbol{R}_{\ge 0} \times \{0\}\,. \tag{15}$$

Namely $G \subset \boldsymbol{R}_{\ge 0}$. The map $E : G \to \boldsymbol{R}$ is the identity. In our case

$$\mathfrak{m}_{k,\beta} : B_k(\overline{C}[1]) \to \overline{C}[1]$$

is always of degree 1 (after degree shift). We put $C_+ = \overline{C} \otimes_{\boldsymbol{R}} \Lambda_+$.

In Sections 2 and 3 where we discuss A_∞ algebras we assume either $\overline{C}$ is finite dimensional or $\overline{C}$ is a de Rham complex.

DEFINITION 2.1. We define

$$\Psi' : C^1_+ \to \Lambda_0$$

by

$$\Psi'(b) = \sum_{k=0}^{\infty} \sum_{\beta \in G} \frac{T^{E(\beta)}}{k+1} \langle \mathfrak{m}_{k,\beta}(b, \dots, b), b \rangle . \tag{16}$$

REMARK 2.2. (1) More precisely the right-hand side of (16) converges in the T adic topology. In various cases, it converges in the topology of [5, Definition 13.1]. (It converges in the case of filtered A_∞ algebras of Lagrangian Floer theory by [5, Theorem 1.2] .) See Section 7 on the convergence.

(2) Since $\deg' b = \deg b - 1 = 0$. We have

$$\deg' \mathfrak{m}_{k,\beta}(b, \dots, b) = 1 .$$

Namely $\deg \mathfrak{m}_{k,\beta}(b, \dots, b) + \deg b = 3$. Therefore in the case when the dimension of our cyclic filtered A_∞ algebra is 3, the inner product in the right-hand side of (16) is well-defined.

We fix a basis $\mathbf{e}_i \in \overline{C}$ and put $b = \sum x_i \mathbf{e}_i$. Then $\Psi'(b) = \sum_{\beta \in G} P_\beta(x_1, \dots)$ where P_β is a formal power series. Therefore we can differentiate Ψ' formally. We have:

PROPOSITION 2.3. *If $b \in C^1_+$ then the differential of Ψ' vanishes at b if and only if*

$$\sum_{k=0}^{\infty} \sum_{\beta \in G} T^{E(\beta)} \mathfrak{m}_{k,\beta}(b, \dots, b) = 0 . \tag{17}$$

This is [7, Proposition 3.6.50] . (17) is called the A_∞ Maurer-Cartan equation.

DEFINITION 2.4. $\widetilde{\mathcal{M}}(C; \Lambda_+)$ is the set of all $b \in C^1_+$ satisfying (17).

We next review the definition of the gauge equivalence from [7, Section 4.3] . We consider

$$b(t) = \sum_{\beta : E(\beta) > 0} T^{E(\beta)} b_\beta(t) , \quad c(t) = \sum_{\beta : E(\beta) > 0} T^{E(\beta)} c_\beta(t) , \tag{18}$$

where $b_\beta(t)$, $c_\beta(t)$ are polynomials with coefficients in $\overline{C}^1$, $\overline{C}^0$, respectively.

DEFINITION 2.5. (See [7, Proposition 4.3.5].) We say $b_0 \in \widetilde{\mathcal{M}}(C; \Lambda_+)$ is *gauge equivalent* to $b_1 \in \widetilde{\mathcal{M}}(C; \Lambda_+)$ if there exist $b(t)$, $c(t)$ as in (18) such that:

(1) $b(0) = b_0$, $b(1) = b_1$.

(2)

$$\frac{d}{dt} b(t) + \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \sum_{\beta \in G} T^{E(\beta)} \mathfrak{m}_{k+l+1,\beta}(\overbrace{b(t), \dots, b(t)}^{k}, c(t), \overbrace{b(t), \dots, b(t)}^{l}) = 0 . \tag{19}$$

It is proved in [7, Lemma 4.3.4] that the gauge equivalence is an equivalence relation. We denote by $\mathcal{M}(C; \Lambda_+)$ the set of gauge equivalence classes.

REMARK 2.6. It follows from (1), (2) that $b(t) \in \widetilde{\mathcal{M}}(C; \Lambda_+)$ for any t. (See [7, Lemma 4.3.7] .)

PROPOSITION 2.7. *If $b_0 \in \widetilde{\mathcal{M}}(C; \Lambda_+)$ is gauge equivalent to $b_1 \in \widetilde{\mathcal{M}}(C; \Lambda_+)$ then*

$$\Psi'(b_0) = \Psi'(b_1)\,.$$

PROOF. We have

$$\begin{aligned}
\frac{d}{dt}\Psi'(b) &= \frac{d}{dt}\sum_{k=0}^{\infty}\sum_{\beta\in G}\frac{T^{E(\beta)}}{k+1}\langle\mathfrak{m}_{k,\beta}(b(t),\dots,b(t)),b(t)\rangle \\
&= \sum_{k=0}^{\infty}\sum_{\beta\in G}\frac{T^{E(\beta)}}{k+1}\left\langle\mathfrak{m}_{k,\beta}\left(b(t),\dots,\frac{db(t)}{dt},\dots,b(t)\right),b(t)\right\rangle \\
&\quad+\sum_{k=0}^{\infty}\sum_{\beta\in G}\frac{T^{E(\beta)}}{k+1}\left\langle\mathfrak{m}_{k,\beta}(b(t),\dots,b(t)),\frac{db(t)}{dt}\right\rangle \\
&= \sum_{k=0}^{\infty}\sum_{\beta\in G}T^{E(\beta)}\left\langle\mathfrak{m}_{k,\beta}(b(t),\dots,b(t)),\frac{db(t)}{dt}\right\rangle\,.
\end{aligned}\tag{20}$$

Since $b(t) \in \widetilde{\mathcal{M}}(C; \Lambda_+)$, it follows that (20) is zero. □

By Proposition 2.7 we obtain

$$\Psi' : \mathcal{M}(C; \Lambda_+) \to \Lambda_0\,.\tag{21}$$

We remark that in the proof of Proposition 2.7 we only use the existence of families $b(t)$ in $\widetilde{\mathcal{M}}(C; \Lambda_+)$ joining b_0 and b_1. In other words, we did not use the existence of $c(t)$. Therefore we have:

PROPOSITION 2.8. *If the map $t \mapsto b(t) \in \widetilde{\mathcal{M}}(C; \Lambda_+)$ is a C^1 map then*

$$\Psi'(b(0)) = \Psi'(b(1))\,.$$

REMARK 2.9. Proposition 2.8 may imply that the superpotential is locally constant on $\mathcal{M}(C; \Lambda_+)$ and so Ψ' depends only on 'irreducible components' of $\mathcal{M}(C; \Lambda_+)$. Since the property of $\mathcal{M}(C; \Lambda_+)$ as a topological space can be rather complicated, we do not try to study this point in this paper.

3. Pseudo-isotopy invariance. In [5, Definition 8.5], it is defined that $(C, \langle\cdot\rangle, \{\mathfrak{m}^t_{k,\beta}\}, \{\mathfrak{c}^t_{k,\beta}\})$ is a pseudo-isotopy of cyclic filtered A_∞ algebras if:

(1) $\mathfrak{m}^t_{k,\beta}$ and $\mathfrak{c}^t_{k,\beta}$ are smooth. Namely

$$t \mapsto \mathfrak{m}^t_{k,\beta}(x_1, \dots, x_k)$$

is smooth. (That is the coefficients are smooth functions of $t \in [0, 1]$.)

(2) For each (but fixed) t, the triple $(C, \langle\cdot\rangle, \{\mathfrak{m}^t_{k,\beta}\})$ defines a cyclic fitered A_∞ algebra.

(3) For each (but fixed) t, and $x_i \in \overline{C}[1]$, we have

$$\langle \mathfrak{c}^t_{k,\beta}(x_1, \dots, x_k), x_0 \rangle = (-1)^* \langle \mathfrak{c}^t_{k,\beta}(x_0, x_1, \dots, x_{k-1}), x_k \rangle \,, \tag{22}$$

where $* = (\deg' x_0)(\deg' x_1 + \cdots + \deg' x_k)$.

(4) For each $x_i \in \overline{C}[1]$

$$\begin{aligned} &\frac{d}{dt}\mathfrak{m}^t_{k,\beta}(x_1, \dots, x_k) \\ &\quad + \sum_{k_1+k_2=k+1} \sum_{\beta_1+\beta_2=\beta} \sum_{i=1}^{k_1} (-1)^* \mathfrak{c}^t_{k_1,\beta_1}(x_1, \dots, \mathfrak{m}^t_{k_2,\beta_2}(x_i, \dots), \dots, x_k) \\ &\quad - \sum_{k_1+k_2=k+1} \sum_{\beta_1+\beta_2=\beta} \sum_{i=1}^{k_1} \mathfrak{m}^t_{k_1,\beta_1}(x_1, \dots, \mathfrak{c}^t_{k_2,\beta_2}(x_i, \dots), \dots, x_k) = 0 \,. \end{aligned} \tag{23}$$

Here $* = \deg' x_1 + \cdots + \deg' x_{i-1}$.

(5) $\mathfrak{m}^t_{k,0}$ is independent of t. $\mathfrak{c}^t_{k,0} = 0$.

(6) $\mathfrak{m}^t_{k,\beta}$ has degree 1. $\mathfrak{c}^t_{k,\beta}$ has degree 0.

REMARK 3.1. In case $\overline{C}$ is the de Rham complex of a manifold L, we define the smoothness in (1) above in a slightly different way. Namely it means that if x_i are smooth differential forms on L then $\mathfrak{m}^t_{k,\beta}(x_1, \dots, x_k)$ is a smooth differential form on $[0, 1] \times L$.

DEFINITION 3.2. (1) $(C, \langle\cdot\rangle, \{\mathfrak{m}_{k,\beta}\}, \{\mathfrak{m}_{-1,\beta}\})$ is said to be an *inhomogeneous cyclic filtered A_∞ algebra* if $(C, \langle\cdot\rangle, \{\mathfrak{m}_{k,\beta}\})$ is a cyclic filtered A_∞ algebras and $\mathfrak{m}_{-1,\beta} \in \boldsymbol{R}$. We also assume $\mathfrak{m}_{-1,0} = 0$.

(2) $(C, \langle\cdot\rangle, \{\mathfrak{m}^t_{k,\beta}\}, \{\mathfrak{c}^t_{k,\beta}\}, \{\mathfrak{m}^t_{-1,\beta}\})$ is said to be a *pseudo-isotopy of inhomogeneous cyclic filtered A_∞ algebras* if $(C, \langle\cdot\rangle, \{\mathfrak{m}^t_{k,\beta}\}, \{\mathfrak{c}^t_{k,\beta}\})$ is a pseudo-isotopy of cyclic filtered A_∞ algebras,

$$t \mapsto \mathfrak{m}^t_{-1,\beta}$$

is a real valued smooth function and if

$$\frac{d}{dt}\mathfrak{m}^t_{-1,\beta} + \sum_{\beta_1+\beta_2=\beta} \langle \mathfrak{c}^t_{0,\beta_1}(1), \mathfrak{m}^t_{0,\beta_2}(1) \rangle = 0 \,. \tag{24}$$

Let $(C, \langle\cdot\rangle, \{\mathfrak{m}^t_{k,\beta}\}, \{\mathfrak{c}^t_{k,\beta}\})$ be a pseudo-isotopy of cyclic filtered A_∞ algebras. We consider cyclic filtered A_∞ algebras $(C, \langle\cdot\rangle, \{\mathfrak{m}^0_{k,\beta}\})$ and $(C, \langle\cdot\rangle, \{\mathfrak{m}^1_{k,\beta}\})$. By [5, Theorem 8.2] there exists an isomorphism

$$\mathfrak{c} = \mathfrak{c}(1; 0) : (C, \langle\cdot\rangle, \{\mathfrak{m}^0_{k,\beta}\}) \to (C, \langle\cdot\rangle, \{\mathfrak{m}^1_{k,\beta}\}) \tag{25}$$

of cyclic filtered A_∞ algebras. (Namely $\mathfrak{c}$ is a filtered A_∞ homomorphism which has an inverse, and which preserves inner product in the sense of [5, Definition 8.3].) It induces

$$\mathfrak{c}_* : \mathcal{M}(C, \{\mathfrak{m}^0_{k,\beta}\}) \to \mathcal{M}(C, \{\mathfrak{m}^1_{k,\beta}\})$$

by [7, Theorem 4.3.22]. The main result of this section is as follows.

THEOREM 3.3. *We have*

$$\Psi'(\mathfrak{c}_*(b)) + \sum_{\beta\in G} T^{E(\beta)}\mathfrak{m}^1_{-1,\beta} = \Psi'(b) + \sum_{\beta\in G} T^{E(\beta)}\mathfrak{m}^0_{-1,\beta}\,. \tag{26}$$

PROOF. We also constructed

$$\mathfrak{c}(t;0) : (C, \langle\cdot\rangle, \{\mathfrak{m}^0_{k,\beta}\}) \to (C, \langle\cdot\rangle, \{\mathfrak{m}^t_{k,\beta}\})$$

in [5, Definition 9.4] . It is an isomorphism and depends smoothly on t. We put

$$b(t) = \mathfrak{c}(t;0)_*(b) = \sum_{k=0}^{\infty}\sum_{\beta\in G} T^{E(\beta)}\mathfrak{c}_{k,\beta}(t;0)(\underbrace{b,\dots,b}_{k})$$

and

$$\begin{aligned} f(t) &= \Psi'(b(t)) + \sum_{\beta\in G} T^{E(\beta)}\mathfrak{m}^t_{-1,\beta} \\ &= \sum_{k=0}^{\infty}\frac{1}{k+1}\langle \mathfrak{m}^t_k(b(t),\dots,b(t)), b(t)\rangle + \sum_{\beta\in G} T^{E(\beta)}\mathfrak{m}^t_{-1,\beta}\,, \end{aligned} \tag{27}$$

where $\mathfrak{m}^t_k = \sum_{\beta\in G} T^{E(\beta)}\mathfrak{m}^t_{k,\beta}$. (We will also use $\mathfrak{c}^t_k = \sum_{\beta\in G} T^{E(\beta)}\mathfrak{c}^t_{k,\beta}$.) We calculate the derivative of $f(t)$. The derivative of the first term is:

$$\begin{aligned} &\sum_{k=0}^{\infty}\frac{1}{k+1}\left\langle \frac{d\mathfrak{m}^t_k}{dt}(b(t),\dots,b(t)), b(t)\right\rangle \\ &\quad+\sum_{k=0}^{\infty}\frac{1}{k+1}\left\langle \mathfrak{m}^t_k\Big(b(t),\dots,\frac{db(t)}{dt},\dots,b(t)\Big), b(t)\right\rangle \\ &\quad+\sum_{k=0}^{\infty}\frac{1}{k+1}\left\langle \mathfrak{m}^t_k(b(t),\dots,b(t)), \frac{db(t)}{dt}\right\rangle . \end{aligned} \tag{28}$$

The sum of the 2nd and the 3rd terms of (28) is:

$$\sum_{k=0}^{\infty}\left\langle \mathfrak{m}^t_k(b(t),\dots,b(t)), \frac{db(t)}{dt}\right\rangle = 0$$

by cyclic symmetry and the Maurer-Cartan equation of $b(t)$.

We calculate the 1st term by using (23) and obtain:

$$\begin{aligned} &-\sum_{k=0}^{\infty}\sum_{k_1+k_2=k+1}\sum_{i=0}^{k_1-1}\frac{1}{k+1}\left\langle \mathfrak{c}^t_{k_1}(\underbrace{b(t),\dots,b(t)}_{i}, \mathfrak{m}^t_{k_2}(b(t),\dots),\dots), b(t)\right\rangle \\ &+\sum_{k=0}^{\infty}\sum_{k_1+k_2=k+1}\sum_{i=0}^{k_2-1}\frac{1}{k+1}\left\langle \mathfrak{m}^t_{k_2}(\underbrace{b(t),\dots,b(t)}_{i}, \mathfrak{c}^t_{k_1}(b(t),\dots),\dots), b(t)\right\rangle . \end{aligned} \tag{29}$$

We have

$$\langle \mathfrak{c}^t_{k_1}(\dots, \mathfrak{m}^t_{k_2}(b(t),\dots),\dots), b(t)\rangle = \langle \mathfrak{c}^t_{k_1}(b(t),\dots), \mathfrak{m}^t_{k_2}(b(t),\dots)\rangle$$

and

$$\langle \mathfrak{m}^t_{k_2}(\dots, \mathfrak{c}^t_{k_1}(b(t),\dots),\dots), b(t)\rangle = \langle \mathfrak{m}^t_{k_2}(b(t),\dots), \mathfrak{c}^t_{k_1}(b(t),\dots)\rangle$$
$$= -\langle \mathfrak{c}^t_{k_1}(b(t),\dots), \mathfrak{m}^t_{k_2}(b(t),\dots)\rangle$$

by cyclic symmetry and [5, (56)]. Therefore (29) is equal to

$$-\sum_{k=0}^{\infty} \sum_{k_1+k_2=k+1} \langle \mathfrak{c}^t_{k_1}(b(t),\dots), \mathfrak{m}^t_{k_2}(b(t),\dots)\rangle \,. \tag{30}$$

Using the Maurer-Cartan equation for $b(t)$ we find that (30) is equal to

$$\langle \mathfrak{c}^t_0(1), \mathfrak{m}^t_0(1)\rangle \,.$$

By (24) this cancels with the derivative of the 2nd term of (27). Namely $f(t)$ is independent of t. □

DEFINITION 3.4. Let $(C, \langle\cdot\rangle, \{\mathfrak{m}_{k,\beta}\}, \{\mathfrak{m}_{-1,\beta}\})$ be an inhomogeneous cyclic filtered A_∞ algebra. We call the function $\Psi : \mathcal{M}(C; \Lambda_+) \to \Lambda_+$, defined by

$$\Psi(b) = \Psi'(b) + \sum_{\beta\in G} T^{E(\beta)}\mathfrak{m}_{-1,\beta}$$

its *superpotential.*

4. Geometric realization. Let M be a 3×2 dimensional symplectic manifold with $c^1(M) = 0$ and L its relatively spin Lagrangian submanifold with vanishing Maslov index.

For $\beta \in H_2(M, L; \boldsymbol{Z})$ let $\mathcal{M}_k(\beta; J)$ be the moduli space of stable J holomorphic maps $v : (\Sigma, \partial\Sigma) \to (M, L)$ from a bordered Riemann surface Σ of genus 0 with connected nonempty boundary $\partial\Sigma$, and with k boundary marked points that respect the counter clockwise cyclic order of the boundary, such that v is of homology class β. (See [7, Definition 2.1.27].)

In [5, Theorem 1.1], we defined a G-gapped cyclic filtered A_∞ algebra $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^J_{k,\beta}\})$ on its de Rham complex. Here G is the monoid generated by the subset $\{\beta\cap\omega\,;\ \beta \in H_2(M, L; \boldsymbol{Z}),\ \mathcal{M}_k(\beta; J) \neq \emptyset\}$ of $\boldsymbol{R}_{\ge 0}$. In this and the later sections, in the geometric situation, we denote by G the submonoid of $H_2(M, L; \boldsymbol{Z})$ generated by $\{\beta \in H_2(M, L; \boldsymbol{Z})\,;\ \mathcal{M}_k(\beta; J) \neq \emptyset\}$, and put $E(\beta) = \beta\cap\omega$, by abuse of notation.

We also proved [5] that the pseudo-isotopy type of $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^J_{k,\beta}\})$ is independent of the choice of J, perturbations, etc. The main result of this section is as follows.

THEOREM 4.1. *If J satisfies Assumption* 1.1, *then there exists $\mathfrak{m}^J_{-1,\beta} \in \boldsymbol{R}$ such that $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^J_{k,\beta}\}, \{\mathfrak{m}^J_{-1,\beta}\})$ is an inhomogeneous cyclic and gapped filetered A_∞ algebra.*

Moreover the pseudo-isotopy type of it depends only on M, L, J and is independent of other choices involved in the definition.

PROOF. Let $ev = (ev_0, \dots, ev_{k-1}) : \mathcal{M}_k(\beta; J) \to L^k$ be the evaluation maps at the boundary marked points. (See [7, Subsection 2.1.1].)

In [5, Theorem 3.1 and Corollary 3.1], we proved the existence of a Kuranishi structure of $\mathcal{M}_k(\beta; J)$ with the following properties:

(1) It is compatible with the forgetful map

$$\mathfrak{forget}_{k,0} : \mathcal{M}_k(\beta; J) \to \mathcal{M}_0(\beta; J)\,. \tag{31}$$

(See [5, Section 3] for the precise definition of this compatibility.)

(2) For $k \geq 1$ the evaluation map $ev_0 : \mathcal{M}_k(\beta; J) \to L$ is weakly submersive, in the sense of [7, Definition A1.13].

(3) It is invariant under the cyclic permutation of the boundary marked points.

(4) We consider the decomposition of the boundary:

$$\begin{aligned}\partial\mathcal{M}_{k+1}(\beta; J) = &\bigcup_{1\leq i\leq j+1\leq k+1}\ \bigcup_{\beta_1+\beta_2=\beta} \\ &\qquad \mathcal{M}_{j-i+2}(\beta_2; J)_{ev_0} \times_{ev_i} \mathcal{M}_{k-j+i+1}(\beta_1; J)\,.\end{aligned} \tag{32}$$

(See [7, Subsection 7.1.1.]) Then the restriction of the Kuranishi structure of $\mathcal{M}_{k+1}(\beta; J)$ to the left-hand side coincides with the fiber product Kuranishi structure in the right-hand side.

(5) We consider the decomposition

$$\partial\mathcal{M}_0(\beta; J) = \Big(\bigcup_{\beta_1+\beta_2=\beta} (\mathcal{M}_1(\beta_2; J)\,_{ev_0} \times_{ev_0} \mathcal{M}_1(\beta_1; J))\Big)\Big/\mathbf{Z}_2\,. \tag{33}$$

Then, the fiber product Kuranishi structure on $\mathcal{M}_1(\beta_2; J)\,_{ev_0} \times_{ev_0} \mathcal{M}_1(\beta_1; J)$ (which is well-defined by (2) coincides with the pull back of the Kuranishi structure to $\partial\mathcal{M}_0(\beta; J)$.

We remark that in general the decomposition of the boundary of $\partial\mathcal{M}_0(\beta; J)$ is given by

$$\begin{aligned}\partial\mathcal{M}_0(\beta; J) =& \Big(\bigcup_{\beta_1+\beta_2=\beta} (\mathcal{M}_1(\beta_2; J)\,_{ev_0} \times_{ev_0} \mathcal{M}_1(\beta_1; J))\Big)\Big/\mathbf{Z}_2 \\ &\cup \bigcup_{\tilde\beta} \mathcal{M}_1^{\mathrm{cl}}(\tilde\beta; J)\,_{ev^{\mathrm{int}}} \times_M L\,.\end{aligned} \tag{34}$$

Here $\mathcal{M}_1^{\mathrm{cl}}(\tilde\beta; J)$ is the moduli space of stable maps of genus zero without boundary, one marked point and of homology class $\tilde\beta \in H_2(M; \mathbf{Z})$. The sum is taken over all $\tilde\beta \in H_2(M; \mathbf{Z})$ which goes to β by $i_* : H_2(M; \mathbf{Z}) \to H_2(M, L; \mathbf{Z})$. (See [7, Proposition 3.8.27] for the second term of the right-hand side.) By Assumption 1.1 the 2nd term of the right-hand side of (34) is an empty set.

Let $E_0 > 0$. Then in [5, Theorem 5.1 and Corollary 5.1], we proved the existence of a system of continuous families of multisections on the above Kuranishi spaces $\mathcal{M}_k(\beta; J)$ with $\beta \cap [\omega] < E_0$ with the following properties:

(1) The families of multisections are transversal to 0.

(2) It is compatible with the forgetful map (31). (See [5, Section 5] for the precise definition of this compatibility.)

(3) For $k \geq 1$ the evaluation map ev_0 induces a submersion of its zero set, in the sense of [5, Definition 4.1.4].

(4) It is invariant under the cyclic permutation of the boundary marked points.

(5) It is compatible with the identification (32).

(6) It is compatible with the identification (33).

Let $\rho_i \in \Lambda(L)$ $(i = 1, \ldots, k)$ be differential forms on L. In [5, Section 6] we defined

$$\mathfrak{m}^{J,\mathfrak{s}}_{k,\beta}(\rho_1, \ldots, \rho_k) = \mathrm{Corr}(\mathcal{M}_{k+1}(\beta; J); ((ev_1, \ldots, ev_k), ev_0))(\rho_1 \times \cdots \times \rho_k)\,. \tag{35}$$

Here the right-hand side is the smooth correspondence associated to the above continuous family of perturbations. (See [5, Section 4].) (Note that (35) depends on the choice of family of multisections. The symbol $\mathfrak{s}$ is put to clarify this dependence.)

We next define $\mathfrak{m}^{J,\mathfrak{s}}_{-1,\beta}$. Let pt be the space consisting of one point. We have an obvious map tri : $\mathcal{M}_0(\beta; J) \to \mathrm{pt}$. Note $\Lambda(\mathrm{pt}) = \boldsymbol{R}$. Moreover

$$\dim \mathcal{M}_0(\beta; J) = \dim L - 3 + \mu(\beta) = 0\,.$$

Therefore we have an $\boldsymbol{R}$ linear map:

$$\mathrm{Corr}(\mathcal{M}_0(\beta; J); (\mathrm{tri}, \mathrm{tri})) : \boldsymbol{R} \to \boldsymbol{R}\,.$$

DEFINITION 4.2. For $\beta \cap [\omega] < E_0$, we put

$$\mathfrak{m}^{J,\mathfrak{s}}_{-1,\beta} = \mathrm{Corr}(\mathcal{M}_0(\beta; J); (\mathrm{tri}, \mathrm{tri}))(1) \in \boldsymbol{R}\,.$$

In [5, Definition 6.5] we defined a notion of cyclic filtered A_∞ algebra modulo T^{E_0}. It is defined in a similar way as cyclic filtered A_∞ algebra but we require filtered A_∞ relation and cyclic symmetry only modulo T^{E_0}.

DEFINITION 4.3. (1) An *inhomogeneous cyclic filtered A_∞ algebra modulo T^{E_0}* is

$$(C, \langle\cdot\rangle, \{\mathfrak{m}_{k,\beta}\,;\ E(\beta) < E_0\}, \{\mathfrak{m}_{-1,\beta}\,;\ E(\beta) < E_0\})$$

such that $(C, \langle\cdot\rangle, \{\mathfrak{m}_{k,\beta}\,;\ E(\beta) < E_0\})$ is a cyclic filtered A_∞ algebra modulo T^{E_0} and $\mathfrak{m}_{-1,\beta} \in \boldsymbol{R}$.

(2) A *pseudo-isotopy of inhomogeneous cyclic filtered A_∞ algebra modulo T^{E_0}* is

$$(C, \langle\cdot\rangle, \{\mathfrak{m}^t_{k,\beta}\,;\ E(\beta) < E_0\}, \{\mathfrak{c}^t_{k,\beta}\,;\ E(\beta) < E_0\}, \{\mathfrak{m}^t_{-1,\beta}\,;\ E(\beta) < E_0\})\,,$$

where

$$(C, \langle\cdot\rangle, \{\mathfrak{m}^t_{k,\beta}\,;\ E(\beta) < E_0\}, \{\mathfrak{c}^t_{k,\beta}\,;\ E(\beta) < E_0\})$$

is a pseudo-isotopy of cyclic filtered A_∞ algebras modulo T^{E_0} (namely (22), (23) hold for $E(\beta) < E_0$) and (24) holds for $E(\beta) < E_0$.

The modulo T^{E_0} version of Proposition 2.7 and Theorem 3.3 can be proved by the same proof.

$(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^{J,\mathfrak{s}}_{k,\beta}\}, \{\mathfrak{m}^{J,\mathfrak{s}}_{-1,\beta}\})$ which we defined above is an inhomogeneous cyclic filtered A_∞ algebra modulo T^{E_0}.

PROPOSITION 4.4. $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^{J,\mathfrak{s}}_{k,\beta}\}, \{\mathfrak{m}^{J,\mathfrak{s}}_{-1,\beta}\})$ *is independent of the choice of a Kuranishi structure and a family of multisections $\mathfrak{s}$ satisfying the properties listed in this section, up to pseudo-isotopy of inhomogeneous cyclic filtered A_∞ algebras modulo T^{E_0}.*

PROOF. Let us take two different choices of a system of Kuranishi structures and of families of multisections. We consider $[0, 1] \times \mathcal{M}_k(\beta; J)$ and evaluation maps

$$ev = (ev_0, \dots, ev_{k-1}) : [0, 1] \times \mathcal{M}_k(\beta; J) \to L^k, \quad ev_t : [0, 1] \times \mathcal{M}_k(\beta; J) \to [0, 1] .$$

As in [5, Section 11 Lemmas 11.1 and 11.2], we have a system of Kuranishi structures and continuous families of multisections on $[0, 1] \times \mathcal{M}_k(\beta; J)$ with the following properties:

(1) The families of multisections are transversal to 0.

(2) It is compatible with the forgetful map $[0, 1]\times$ (31).

(3) For $k \geq 1$ the evaluation map

$$(ev_t, ev_0) : [0, 1] \times \mathcal{M}_k(\beta; J) \to [0, 1] \times L$$

is weakly submersive and induces a submersion of the zero set of family of multisections, in the sense of [5, Definition 4.1.4].

(4) They are invariant under the cyclic permutation of the boundary marked points.

(5) It is compatible with the identification

$$[0, 1] \times \partial\mathcal{M}_{k+1}(\beta; J) = \bigcup_{1\leq i\leq j+1\leq k+1} \bigcup_{\beta_1+\beta_2=\beta} [0, 1] \times \left(\mathcal{M}_{j-i+2}(\beta_2; J)_{ev_0} \times_{ev_i} \mathcal{M}_{k-j+i+1}(\beta_1; J)\right)$$

which is induced from (32).

(6) It is compatible with the similar identification induced from (33).

(7)

$$ev_t : [0, 1] \times \mathcal{M}_0(\beta; J) \to [0, 1]$$

is weakly submersive and induces a submersion on the zero set of family of multisections, in the sense of [5, Definition 4.1.4].

(8) At $t_0 = 0, 1$ the induced Kuranishi structures and families of multisecitons on $\{t_0\} \times \mathcal{M}_k(\beta; J)$ coincide with the given two choices of Kuranishi structures and of families of multisections.

In [5, Section 11], we defined a pseudo-isotopy of cyclic filtered A_∞ algebras as follows. Let $\rho_1, \dots, \rho_k \in \Lambda(L)$. We put

$$\begin{aligned}&\mathrm{Corr}_*([0, 1] \times \mathcal{M}_{k+1}(\beta; J); ((ev_1, \dots, ev_k), (ev_t, ev_0)))(\rho_1 \times \cdots \times \rho_k)\\ &= \rho(t) + dt \wedge \sigma(t) ,\end{aligned} \tag{36}$$

and define

$$\mathfrak{m}^t_{k,\beta}(\rho_1, \dots, \rho_k) = \rho(t) , \quad \mathfrak{c}^t_{k,\beta}(\rho_1, \dots, \rho_k) = \sigma(t) . \tag{37}$$

We next define $\mathfrak{m}^t_{-1,\beta}$. Let tri : $[0, 1] \times \mathcal{M}_0(\beta; J) \to$ pt be an obvious map to a point. We take $1 \in \Lambda^0(\mathrm{pt}) = \boldsymbol{R}$ and put the 0-form on $[0, 1]$

$$\mathrm{Corr}_*([0, 1] \times \mathcal{M}_0(\beta; J); (\mathrm{tri}, ev_t))(1) = \rho(t) . \tag{38}$$

We then define

$$\mathfrak{m}^t_{-1,\beta} = \rho(t) . \tag{39}$$

LEMMA 4.5. $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^t_{k,\beta}\}, \{\mathfrak{c}^t_{k,\beta}\}, \{\mathfrak{m}^t_{-1,\beta}\})$ *above defines a pseudo-isotopy of inhomogeneous cyclic filtered A_∞ algebras modulo T^{E_0}.*

PROOF. In [5, Section 11] it is proved that $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^t_{k,\beta}\}, \{\mathfrak{c}^t_{k,\beta}\})$ is a pseudo-isotopy of cyclic filtered A_∞ algebras modulo T^{E_0}. Therefore it suffices to check (24).

Let $0 \le t_1 < t_2 \le 1$. We have:

$$\begin{aligned}
&\partial([t_1, t_2] \times \mathcal{M}_0(\beta; J)) \\
&\quad = (\{t_1, t_2\} \times \mathcal{M}_0(\beta; J)) \\
&\quad\quad \cup \Bigg(\bigcup_{\beta_1+\beta_2=\beta} \big(([t_1, t_2] \times \mathcal{M}_1(\beta_2; J))\ {}_{(ev_t, ev_0)} \times_{(ev_t, ev_0)} ([t_1, t_2] \times \mathcal{M}_1(\beta_1; J))\big)\Bigg)\Big/\mathbf{Z}_2 .
\end{aligned}$$

We now apply Stokes' formula ([5, Proposition 4.2]) to a closed 0-form 1 on the zero set of multisections on $[t_1, t_2] \times \mathcal{M}_0(\beta; J)$ and obtain:

$$\mathfrak{m}^{t_2}_{-1,\beta} - \mathfrak{m}^{t_1}_{-1,\beta} = \sum_{\beta_1+\beta_2=\beta} \int_{t_1}^{t_2} \langle \mathfrak{c}^t_{0,\beta_1}(1), \mathfrak{m}^t_{0,\beta_2}(1)\rangle dt .$$

By taking t_2 derivative we obtain (24). □

The proof of Proposition 4.4 is now complete. □

We thus proved modulo T^{E_0} version of Theorem 4.1. We next prove the following inhomogeneous version of [5, Theorem 8.1].

LEMMA 4.6. *Let $0 < E_0 < E_1$ and $(C, \langle\cdot\rangle, \{\mathfrak{m}^i_{k,\beta}\}, \{\mathfrak{m}^i_{-1,\beta}\})$ be a G-gapped inhomogeneous cyclic filtered A_∞ algebra modulo T^{E_i}, for $i = 0, 1$. Let $(C, \langle\cdot\rangle, \{\mathfrak{m}^t_{k,\beta}\}, \{\mathfrak{c}^t_{k,\beta}\}, \{\mathfrak{m}^t_{-1,\beta}\})$ be a pseudo-isotpy of G-gapped inhomogeneous cyclic filtered A_∞ algebras modulo T^{E_0} between them.*

Then, $(C, \langle\cdot\rangle, \{\mathfrak{m}^0_{k,\beta}\}, \{\mathfrak{m}^0_{-1,\beta}\})$ can be extended to a G-gapped inhomogeneous cyclic filtered A_∞ algebra modulo T^{E_1} and $(C, \langle\cdot\rangle, \{\mathfrak{m}^t_{k,\beta}\}, \{\mathfrak{c}^t_{k,\beta}\}, \{\mathfrak{m}^t_{-1,\beta}\})$ can be extended to a pseudo-isotpy of G-gapped inhomogeneous cyclic filtered A_∞ algebras modulo T^{E_1} between them.

PROOF. We may assume that $G \cap [E_0, E_1) = \{E_0\}$. In [5, Theorem 8.1] the extension to the cyclic filtered A_∞ algebra modulo T^{E_1} and the extension to the pseudo-isotopy of cyclic filtered A_∞ algebras modulo T^{E_1} are obtained. So it suffices to find $\mathfrak{m}^t_{-1,\beta}$ for $E(\beta) = E_1$. We define

$$\mathfrak{m}^t_{-1,\beta} = \mathfrak{m}^1_{-1,\beta} + \sum_{\beta_1+\beta_2=\beta} \int_t^1 \langle \mathfrak{c}^t_{0,\beta_1}(1)\, , \mathfrak{m}^t_{0,\beta_2}(1)\rangle dt .$$

It is easy to check (24). □

We next construct a gapped inhomogeneous cyclic filtered A_∞ algebra

$$(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}_{k,\beta}\}, \{\mathfrak{m}_{-1,\beta}\}) .$$

Let E_i be a sequence $0 < \cdots < E_i < E_{i+1} < \cdots$. We obtain a sequence

$$(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^i_{k,\beta}\}, \{\mathfrak{m}^i_{-1,\beta}\}), \quad i = 1, 2, \ldots$$

of inhomogeneous cyclic filtered A_∞ algebras modulo T^{E_i}. By Lemma 4.5 we have a pseudo-isotopy of inhomogeneous cyclic filtered A_∞ algebras modulo T^{E_i}

$$(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^{i,t}_{k,\beta}\}, \{\mathfrak{c}^{i,t}_{k,\beta}\}, \{\mathfrak{m}^{i,t}_{-1,\beta}\}) \tag{40}$$

between $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^i_{k,\beta}\}, \{\mathfrak{m}^i_{-1,\beta}\})$ and $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^{i+1}_{k,\beta}\}, \{\mathfrak{m}^{i+1}_{-1,\beta}\})$.

We can then use Lemma 4.6 in the same way as [5, Section 12] and [7, Section 7.2], to extend $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^i_{k,\beta}\}, \{\mathfrak{m}^i_{-1,\beta}\})$ to an inhomogenuous cyclic filtered A_∞ algebra and to extend $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^{i,t}_{k,\beta}\}, \{\mathfrak{c}^{i,t}_{k,\beta}\}, \{\mathfrak{m}^{i,t}_{-1,\beta}\})$ to a pseudo-isotopy of inhomogeneous cyclic filtered A_∞ algebras between

$$\text{extension of } (\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^i_{k,\beta}\}, \{\mathfrak{m}^i_{-1,\beta}\}) \quad \text{and} \quad (\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^{i+1}_{k,\beta}\}, \{\mathfrak{m}^{i+1}_{-1,\beta}\})\,.$$

These two extensions are isomorphic to each other.

Therefore we have $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^{J,\mathfrak{s}}_{k,\beta}\}, \{\mathfrak{m}^{J,\mathfrak{s}}_{-1,\beta}\})$.

We can prove that the pseudo-isotopy type of $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^{J,\mathfrak{s}}_{k,\beta}\}, \{\mathfrak{m}^{J,\mathfrak{s}}_{-1,\beta}\})$ is independent of the choice of system of Kuranishi structures and continuous families of multisections in the same way as [5, Section 14] by working out the inhomogeneous version of pseudo-isotopy of pseudo-isotopies. We omit the details of it. Instead, we complete the proof of Theorem 1.2 directly without using the inhomogeneous version of pseudo-isotopy of pseudo-isotopies but using only the result of [5, Section 14] and ones of this paper.

Let $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^{i\prime}_{k,\beta}\}, \{\mathfrak{m}^{i\prime}_{-1,\beta}\})$ be an inhomogeneous cyclic filtered A_∞ algebra modulo T^{E_i} obtained by alternative choices. This sequence induces $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^{J,\mathfrak{s}'}_{k,\beta}\}, \{\mathfrak{m}^{J,\mathfrak{s}'}_{-1,\beta}\})$.

By Proposition 4.4, we can show that for each i we have a pseudo-isotopy modulo T^{E_i} between two inhomogeneous cyclic filtered A_∞ algebras $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^i_{k,\beta}\}, \{\mathfrak{m}^i_{-1,\beta}\})$ and $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^{i\prime}_{k,\beta}\}, \{\mathfrak{m}^{i\prime}_{-1,\beta}\})$. We denote it by $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^{i,t}_{k,\beta}\}, \{\mathfrak{c}^{i,t}_{k,\beta}\}, \{\mathfrak{m}^{i,t}_{-1,\beta}\})$.

By [5, Theorem 14.1], this pseudo-isotopy modulo T^{E_i} extends to a pseudo-isotopy of cyclic A_∞ algebra. (We do not use the fact that it extends to a pseudo-isotpy of *inhomogeneous* cyclic A_∞ algebras here.) Therefore by [7, Theorem 4.3.22], we have an isomorphism

$$(\mathfrak{f}_i)_* : \mathcal{M}(\Lambda(L), \{\mathfrak{m}^i_{k,\beta}\}; \Lambda_+) \cong \mathcal{M}(\Lambda(L), \{\mathfrak{m}^{i\prime}_{k,\beta}\}; \Lambda_+)\,.$$

On the other hand, since $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^{i,t}_{k,\beta}\}, \{\mathfrak{c}^{i,t}_{k,\beta}\}, \{\mathfrak{m}^{i,t}_{-1,\beta}\})$ is a pseudo-isotopy modulo T^{E_i} of *inhomogeneous* cyclic filtered A_∞ algebras, the modulo T^{E_i} version of Theorem 3.3 implies

$$\Psi((\mathfrak{f}_i)_*(b_i); J) \equiv \Psi(b_i; J) \mod T^{E_i}\,. \tag{41}$$

For $i > j$, we have isomorphisms

$$(\mathfrak{c}_{i,j})_* : \mathcal{M}(\Lambda(L), \{\mathfrak{m}^j_{k,\beta}\}; \Lambda_+) \cong \mathcal{M}(\Lambda(L), \{\mathfrak{m}^i_{k,\beta}\}; \Lambda_+)$$

and

$$(\mathfrak{c}'_{i,j})_* : \mathcal{M}(\Lambda(L), \{\mathfrak{m}^{j'}_{k,\beta}\}; \Lambda_+) \cong \mathcal{M}(\Lambda(L), \{\mathfrak{m}^{i'}_{k,\beta}\}; \Lambda_+)\,.$$

They are induced by the pseudo-isotopies of filtered A_∞ algebras. (We obtain it from (40) by the method of [5, Section 14]). Using the fact that (40) is a pseudo-isotopy modulo T^{E_i} of *inhomogeneous* cyclic filtered A_∞ algebras, we have

$$\Psi((\mathfrak{c}'_{i,j})_*(b_i); J) \equiv \Psi(b; J) \mod T^{E_i}\,,\quad \Psi((\mathfrak{c}_{i,j})_*(b'_i); J) \equiv \Psi(b'; J) \mod T^{E_i}\,, \tag{42}$$

where $b_i \in \mathcal{M}(\Lambda(L), \{\mathfrak{m}^{i}_{k,\beta}\}; \Lambda_+)$ and $b'_i \in \mathcal{M}(\Lambda(L), \{\mathfrak{m}^{i'}_{k,\beta}\}; \Lambda_+)$.

Furthermore the construction of pseudo-isotopy of pseudo-isotopies in [5, Section 14] implies

$$(\mathfrak{f}_i)_* \circ (\mathfrak{c}_{i,j})_* = (\mathfrak{c}'_{i,j})_* \circ (\mathfrak{f}_j)_*\,. \tag{43}$$

We remark that there exists an isomorphism

$$(\mathfrak{c}_i)_* : \mathcal{M}(\Lambda(L), \{\mathfrak{m}^{j}_{k,\beta}\}; \Lambda_+) \cong \mathcal{M}((\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^{J,\mathfrak{s}}_{k,\beta}\}, \{\mathfrak{m}^{J,\mathfrak{s}}_{-1,\beta}\}); \Lambda_+)$$

such that $(\mathfrak{c}_i)_* \circ (\mathfrak{c}_{i,j})_* = (\mathfrak{c}_j)_*$. We also have

$$(\mathfrak{c}'_i)_* : \mathcal{M}(\Lambda(L), \{\mathfrak{m}^{j}_{k,\beta}\}; \Lambda_+) \cong \mathcal{M}((\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^{J,\mathfrak{s}'}_{k,\beta}\}, \{\mathfrak{m}^{J,\mathfrak{s}'}_{-1,\beta}\}); \Lambda_+)$$

such that $(\mathfrak{c}'_i)_* \circ (\mathfrak{c}'_{i,j})_* = (\mathfrak{c}'_j)_*$.

Moreover by (42) we have

$$\Psi((\mathfrak{c}_i)_*(b_i); J) \equiv \Psi(b_i; J) \mod T^{E_i}\,,\quad \Psi((\mathfrak{c}'_i)_*(b'_i); J) \equiv \Psi(b'_i; J) \mod T^{E_i}\,. \tag{44}$$

Suppose $\mathfrak{f}(b) = b'$. (Namely $b = (\mathfrak{c}_i)_*(b_i)$, $b' = (\mathfrak{c}'_i)_*((\mathfrak{f}_i)_*(b_i))$.)

Then (41), (43), (44) immediately imply

$$\Psi(b'; J) \equiv \Psi(b; J) \mod T^{E_i}\,.$$

Since this holds for any E_i, we proved Theorem 1.2.(3). The proof of Theorem 1.2 is now complete. □

5. Relations to canonical models. In [7, Subsection 5.4.4] and [5, Section 10], we defined a canonical model $(H, \langle\cdot\rangle, \{\mathfrak{m}^{\text{can}}_{k,\beta}\})$ of a G-gapped cyclic filtered A_∞ algebra $(C, \langle\cdot\rangle, \{\mathfrak{m}_{k,\beta}\})$. Here $\overline{H}$ is the $\mathfrak{m}_{1,0}$ cohomology of $\overline{C}$ and $H = \overline{H} \otimes_{\boldsymbol{R}} \Lambda_0$. We regard $\overline{H} \subset \overline{C}$ by taking an appropriate representative. Then $H \subset C$. (Note we assumed $\overline{C}$ is either finite dimensional or the de Rham complex $\Lambda(L)$.) We also constructed a G-gapped cyclic filtered A_∞ homomorphism $\mathfrak{f} : H \to C$, which is a homotopy equivalence. Suppose that $(C, \langle\cdot\rangle, \{\mathfrak{m}_{k,\beta}\}, \{\mathfrak{m}_{-1,\beta}\})$ is an *inhomegeneous* G-gapped cyclic filtered A_∞ algebra. In this section, we will define $\mathfrak{m}^{\text{can}}_{-1,\beta}$ so that $\mathfrak{f}_* : \mathcal{M}(H; \Lambda_+) \to \mathcal{M}(C; \Lambda_+)$ preserves the superpotentials.

To define $\mathfrak{m}^{\text{can}}_{-1,\beta}$ we need some notations. We use results and notations of [5, Sections 9 and 10] in this section.

Let $\mathcal{T}$ be a ribbon tree. (We always assume that $\mathcal{T}$ is connected.) Let $C_0(\mathcal{T})$ be the set of vertices. We assume that we have its decomposition $C_0(\mathcal{T}) = C_0^{\text{int}}(\mathcal{T}) \sqcup C_0^{\text{ext}}(\mathcal{T})$ to interior vertices and exterior vertices. Let $\beta(\cdot) : C_0^{\text{int}}(\mathcal{T}) \to G$ be a map to a discrete submonoid G of $\boldsymbol{R}_{\geq 0}$.

DEFINITION 5.1. We denote by $Gr^-(k, \beta)$ the set of $\Gamma = (\mathcal{T}, C_0^{\text{int}}(\mathcal{T}), C_0^{\text{ext}}(\mathcal{T}), \beta(\cdot))$ such that: (1) $\sum_{v \in C_0^{\text{int}}(\mathcal{T})} \beta(v) = \beta$. (2) $\# C_0^{\text{ext}}(\mathcal{T}) = k + 1$. (3) If $\beta(v) = 0$, then v has at least 3 edges. (4) If $v \in C_0^{\text{ext}}(\mathcal{T})$ then v has exactly one edge.

The automorphism group $\text{Aut}(\Gamma)$ of an element $\Gamma = (\mathcal{T}, C_0^{\text{int}}(\mathcal{T}), C_0^{\text{ext}}(\mathcal{T}), \beta(\cdot))$ of $Gr^-(k, \beta)$ is the set of isomorphisms $\phi : \mathcal{T} \to \mathcal{T}$ of ribbon trees which preserve the decomposition $C_0(T) = C_0^{\text{int}}(\mathcal{T}) \sqcup C_0^{\text{ext}}(\mathcal{T})$ and satisfies the relation $\beta(\phi(v)) = \beta(v)$.

We define $Gr(k, \beta)$ as in [5, Definition 9.1]. Namely its element is an element of $Gr^-(k, \beta)$ together with a choice of a base point which is an exterior vertex.

We remark that $k = -1, 0, 1, \dots$ in $Gr^-(k, \beta)$. The case $k = -1$ is included. We also remark that the automorphism of a *rooted* ribbon tree is trivial.

Let (v, e) be a flag of Γ, that is a pair of an interior vertex v and an edge e containing v. Let $b \in \overline{H}^1$. We are going to define $\mathfrak{m}(\Gamma; b) \in \boldsymbol{R}$.

Let $\mathcal{T}_0, \dots, \mathcal{T}_l$ be the irreducible components of $\Gamma \setminus v$. We number them so that $e \in \mathcal{T}_0$ and they respect the counter clockwise cyclic order of $\boldsymbol{R}^2$. Together with the data induced from Γ, the tree $\mathcal{T}_i$ defines an element $(\Gamma_i, v) \in Gr(k_i, \beta_i)$. Here the base point of (Γ_i, v) is v for all i.

DEFINITION 5.2.

$$\mathfrak{m}(\Gamma, v, e; b) = \langle \mathfrak{m}_{l,\beta(v)}(\mathfrak{f}_{(\Gamma_1,v)}(b, \dots, b), \dots, \mathfrak{f}_{(\Gamma_l,v)}(b, \dots, b)), \mathfrak{f}_{(\Gamma_0,v)}(b, \dots, b) \rangle \, .$$

Here $\mathfrak{f}_{(\Gamma,v)}$ is defined in [5, Section 10]. We put $\mathfrak{m}(\Gamma, v, e; b) = 0$ if Γ has no interior vertex.

We remark that there is no sign in Definition 5.2, since the degree of b after shifted is even.

LEMMA 5.3. $\mathfrak{m}(\Gamma, v, e; b)$ *is independent of* v *and* e *and depends only on* Γ *and* b.

This is [5, Proposition 10.1]. Hereafter we write $\mathfrak{m}(\Gamma; b)$ in place of $\mathfrak{m}(\Gamma, v, e; b)$.

In case $\mathfrak{T}$ has exactly one interior vertex v, no exterior vertex, and $\beta(v) = \beta$, we define

$$\mathfrak{m}(\Gamma; b) = \mathfrak{m}_{-1,\beta} \tag{45}$$

where $\Gamma = (\mathfrak{T}, \{v\}, \emptyset, \beta(\cdot))$.

DEFINITION 5.4.

$$\mathfrak{m}^{\text{can}}_{-1,\beta} = \sum_{\Gamma \in Gr^-(-1,\beta)} \frac{\mathfrak{m}(\Gamma)}{\#\text{Aut}(\Gamma)} \, .$$

We remark that we write $\mathfrak{m}(\Gamma)$ instead of $\mathfrak{m}(\Gamma; b)$, since in the case of $\Gamma \in Gr^-(-1, \beta)$ there is no exterior vertex and hence b never appears.

$(H, \langle \cdot \rangle, \{\mathfrak{m}^{\text{can}}_{k,\beta}\}, \{\mathfrak{m}^{\text{can}}_{-1,\beta}\})$ is an inhomegeneous G-gapped cyclic filtered A_∞ algebra. Let

$$\Psi^{\text{can}} : \mathcal{M}(H; \Lambda_+) \to \Lambda_+$$

be its superpotential. The filtered A_∞ homomorphism $\mathfrak{f} : H \to C$ induces $\mathfrak{f}_* : \mathcal{M}(H; \Lambda_+) \to \mathcal{M}(C; \Lambda_+)$ by

$$\mathfrak{f}_*(b) = \sum_{k=0}^{\infty} \sum_{\beta \in G} T^{E(\beta)} \mathfrak{f}_{k,\beta}(b, \dots, b)\,. \tag{46}$$

The main result of this section is:

THEOREM 5.5.

$$\Psi(\mathfrak{f}_*(b)) = \Psi^{\mathrm{can}}(b)\,. \tag{47}$$

REMARK 5.6. We consider the case of $\overline{C} = \Lambda(L)$ with $H^1(L; \boldsymbol{R}) = 0$. Then since $H^1 = 0$, the set $\mathcal{M}(H; \Lambda_+)$ consists of one point 0. Therefore $\mathcal{M}(C; \Lambda_+)$ also consists of one point. The invariant of Corollary 1.3 is the value of the superpotential at this point.

Theorem 5.5 implies that this invariant is

$$\sum_{\beta \in G} T^{E(\beta)} \mathfrak{m}^{\mathrm{can}}_{-1,\beta} = \sum_{\beta \in G} \sum_{\Gamma \in Gr^-(-1,\beta)} \frac{T^{E(\beta)}}{\#\mathrm{Aut}(\Gamma)} \mathfrak{m}(\Gamma)\,. \tag{48}$$

PROOF OF THEOREM 5.5. Let $b \in H^1_+ = \overline{H}^1 \otimes_{\boldsymbol{R}} \Lambda_+$. We define

$$\Phi(b) = \sum_{k=-1}^{\infty} \sum_{\beta \in G} \sum_{\Gamma \in Gr^-(k,\beta)} \frac{T^{E(\beta)}}{\#\mathrm{Aut}(\Gamma)} \mathfrak{m}(\Gamma; b) \in \Lambda_+\,. \tag{49}$$

LEMMA 5.7.

$$\Phi(b) = \Psi^{\mathrm{can}}(b)\,.$$

PROOF. In view of Definition 5.4 it suffices to prove:

$$\langle \mathfrak{m}^{\mathrm{can}}_{k,\beta}(b, \dots, b), b \rangle = (k+1) \sum_{\Gamma \in Gr^-(k,\beta)} \frac{\mathfrak{m}(\Gamma; b)}{\#\mathrm{Aut}(\Gamma)}\,. \tag{50}$$

We will prove (50) below.

Let $\Gamma \in Gr^-(k, \beta)$. Let $\{v_0, \dots, v_k\} = C^{\mathrm{ext}}_0(\Gamma)$ such that $v_0, \dots, v_k$ respect the counterclockwise cyclic order of $\boldsymbol{R}^2$. Let e_i be the unique edge containing v_i. We define v'_i by $\partial e_i = \{v_i, v'_i\}$. Then (Γ, v_i) is a rooted ribbon tree with root v_i, for each i.

By the definition of $\mathfrak{m}^{\mathrm{can}}_{k,\beta}$ given in [7, Subsection 5.4.4], we have:

$$\mathfrak{m}^{\mathrm{can}}_{k,\beta}(b, \dots, b) = \sum_{\Gamma \in Gr^-(k,\beta)} \sum_{i=0}^{k} \frac{\mathfrak{m}_{(\Gamma, v_i)}(b, \dots, b)}{\#\mathrm{Aut}(\Gamma)}\,.$$

Here $\mathfrak{m}_{(\Gamma, v_i)}$ is as in [7, Section 10].

This is because $(\Gamma, v_i) \in Gr(k, \beta)$ and (Γ, v_i) is the same element as (Γ, v_j) in $Gr(k, \beta)$ if and only if there exists an element of $\phi \in \mathrm{Aut}(\Gamma)$ such that $\phi(v_i) = v_j$.

Moreover

$$\langle \mathfrak{m}_{(\Gamma, v_i)}(b, \dots, b), b \rangle = \mathfrak{m}(\Gamma, v'_i, e_i; b)\,,$$

where the right-hand side is defined in Definition 5.2. (See also [5, Definition 10.1].) By Lemma 5.3, that is [5, Proposition 10.1], $\mathfrak{m}(\Gamma, v_i, e_i; b)$ is independent of i and is $\mathfrak{m}(\Gamma; b)$. This implies (50). The proof of Lemma 5.7 is complete. □

The next proposition completes the proof of Theorem 5.5. □

PROPOSITION 5.8. *If* $b \in \widetilde{\mathcal{M}}(H; \Lambda_+)$, *then we have*:

$$\Phi(b) = \Psi(\mathfrak{f}_*(b)) \,. \tag{51}$$

PROOF.

LEMMA 5.9.

$$\begin{aligned}\sum_{\beta\neq 0} T^{E(\beta)}\mathfrak{m}_{-1,\beta} + &\sum_{(l,\beta)\neq(1,0)} \frac{T^{E(\beta)}}{l+1}\langle \mathfrak{m}_{l,\beta}(\mathfrak{f}_*(b), \dots, \mathfrak{f}_*(b)), \mathfrak{f}_*(b)\rangle \\ &= \sum_{k=-1}^{\infty}\sum_{\beta\in G}\sum_{\Gamma\in Gr^-(k,\beta)} \frac{T^{E(\beta)}}{\#\mathrm{Aut}(\Gamma)}\#C_0^{\mathrm{int}}(\Gamma)\mathfrak{m}(\Gamma; b)\,.\end{aligned} \tag{52}$$

PROOF. Let $\Gamma \in Gr^-(k, \beta)$ and (v, e) be its flag. (We are considering the case when Γ has at least one edge.) We obtain the irreducible components $\Gamma_0, \dots, \Gamma_l$ of $\Gamma \setminus v$ as before. By definition we have

$$\langle \mathfrak{m}_{l,\beta(v)}(\mathfrak{f}_{(\Gamma_1,v)}(b, \dots, b), \dots, \mathfrak{f}_{(\Gamma_l,v)}(b, \dots, b))\,, \mathfrak{f}_{(\Gamma_0,v)}(b, \dots, b)\rangle = \mathfrak{m}(\Gamma; b)\,. \tag{53}$$

We remark that the left-hand side of (53) is independent of (v, e) by Lemma 5.3 or [5, Proposition 10.1]. If we take the sum of the right-hand side of (53) over all Γ, v with weight $T^{E(\beta)}/\#\mathrm{Aut}(\Gamma)$ then we obtain the sum of the terms $k \geq 0$ in the right-hand side of (52). (Note (Γ, v) may be isomorphic to (Γ, v') by some element of $\mathrm{Aut}(\Gamma)$. We take the sum over $\Gamma \in Gr^-(k+1, \beta)$ and $v \in C_0^{\mathrm{int}}(\Gamma)$, and *not* over the isomorphism classes of (Γ, v).)

We claim that the sum of the left-hand side of (53) over the pair of $\Gamma \in Gr^-(k+1, \beta)$ and $v \in C_0^{\mathrm{int}}(\Gamma)$, with weight $T^{E(\beta)}/\#\mathrm{Aut}(\Gamma)$ is

$$\sum_{(l,\beta)\neq(1,0)} \frac{T^{E(\beta)}}{l+1}\langle \mathfrak{m}_{l,\beta}(\mathfrak{f}_*(b), \dots, \mathfrak{f}_*(b)), \mathfrak{f}_*(b)\rangle\,. \tag{54}$$

To prove this claim, we first remark that the automorphism of (Γ, v, e) is trivial. Therefore, for each given $\Gamma \in Gr^-(k+1, \beta)$ and its frag (v, e), we have

$$\#\{(v', e');\, (\Gamma, v', e') \cong (\Gamma, v, e)\} = \#\mathrm{Aut}(\Gamma)\,.$$

On the other hand, the number of choices of e for given Γ, v is $l+1$. Hence (54).

Finally we remark that the first term of (52) is equal to the sum of the right-hand side in the case $k = -1$ and Γ has no edge, by (45).

We thus obtain (52). □

LEMMA 5.10.

$$\langle \mathfrak{m}_{1,0}(\mathfrak{f}_*(b)), \mathfrak{f}_*(b)\rangle = -2\sum_{k=-1}^{\infty}\sum_{\beta\in G}\sum_{\Gamma\in Gr^-(k,\beta)}\frac{T^{E(\beta)}}{\#\mathrm{Aut}(\Gamma)}\#C_1^{\mathrm{int}}(\Gamma)\mathfrak{m}(\Gamma;b)\,. \tag{55}$$

Here $C_1^{\mathrm{int}}(\Gamma)$ is the set of interior edges. (We say an edge is an interior edge if it does not contain an exterior vertex.)

PROOF. Let (v, e) be a flag of $\Gamma \in Gr^-(k,\beta)$ such that e is an interior edge. We define $\mathfrak{m}'(\Gamma, e, v; b)$ as follows. Let $\mathcal{T}'_{(0)}$, $\mathcal{T}'_{(1)}$ be the irreducible components of $\Gamma \setminus e$ such that $\mathcal{T}'_{(0)}$ contains v. We put $\mathcal{T}_{(0)} = \mathcal{T}'_{(0)} \cup e$, $\mathcal{T}_{(1)} = T'_{(1)} \cup e$. Using the data induced from Γ, the trees $\mathcal{T}_{(0)}$, $\mathcal{T}_{(1)}$ induce $(\Gamma_{(0)}, v') \in Gr(k_{(0)}, \beta_{(0)})$, $(\Gamma_{(1)}, v) \in Gr(k_{(1)}, \beta_{(1)})$. (The roots of $\Gamma_{(0)}$, $\Gamma_{(1)}$ are v' and v, respectively.) We define

$$\mathfrak{m}'(\Gamma, e, v; b) = \langle \mathfrak{m}_{1,0}(\mathfrak{f}_{(\Gamma_{(1)},v)}(b,\dots,b)), \mathfrak{f}_{(\Gamma_{(0)},v')}(b,\dots,b))\rangle\,. \tag{56}$$

SUBLEMMA 5.11.

$$\mathfrak{m}'(\Gamma, e, v; b) = -\mathfrak{m}(\Gamma; b)\,. \tag{57}$$

PROOF. We use [5, Lemma 10.1], its proof and notations there, during the proof of Sublemma 5.11.

Let Γ, v, e be as in Sublemma 5.11. We put $\partial e = \{v, v'\}$. Let $\mathcal{T}_0, \dots, \mathcal{T}_m$ be the irreducible components of $\Gamma \setminus v'$. We number them so that $v \in \mathcal{T}_0$ and it respects the counter-clockwise cyclic order of $\boldsymbol{R}^2$. $\mathcal{T}_i$ together with the data induced from Γ becomes (Γ_i, v'), whose root is v'. By definition

$$\Gamma_{(1)} = \Gamma_1 \cup \cdots \cup \Gamma_m \cup e\,.$$

Therefore by the definition in [5, Section 10], we have

$$\mathfrak{f}_{(\Gamma_{(1)},v)}(b,\dots,b) = (G\circ \mathfrak{m}_{m,\beta(v')})(\mathfrak{f}_{(\Gamma_1,v')}(b,\dots,b),\dots,\mathfrak{f}_{(\Gamma_m,v')}(b,\dots,b))\,. \tag{58}$$

Therefore

$$\begin{aligned}\mathfrak{m}'(\Gamma, e, v; b) = \langle(&\mathfrak{m}_{1,0}\circ G\circ \mathfrak{m}_{m,\beta(v')})(\mathfrak{f}_{(\Gamma_1,v')}(b,\dots,b),\dots,\\ &\mathfrak{f}_{(\Gamma_m,v')}(b,\dots,b)), \mathfrak{f}_{(\Gamma_{(0)},v')}(b,\dots,b)\rangle\,.\end{aligned} \tag{59}$$

By [5, Lemma 10.1] we have

$$\mathfrak{m}_{1,0}\circ G = -G\circ\mathfrak{m}_{1,0} + \Pi - \mathrm{identity}\,. \tag{60}$$

Since v is an interior vertex, then either $\beta(v) \neq 0$ or v has three or more edges. Therefore $(\Gamma_{(0)}, v') \in Gr(k_{(0)}, \beta_{(0)})$ with $(k_{(0)}, \beta_{(0)}) \neq (1, 0)$. It follows that $\mathfrak{f}_{(\Gamma_{(0)},v)}(b,\dots,b) \in \mathrm{Im}\, G$. We remark that

$$\langle \mathrm{Im}\, G, \mathrm{Im}\, G + \mathrm{Im}\, \Pi\rangle = 0\,.$$

Therefore

$$\begin{aligned}\mathfrak{m}'(\Gamma, e, v; b) &= -\langle \mathfrak{m}_{m,\beta(v')}(\mathfrak{f}_{(\Gamma_1,v')}(b,\dots,b),\dots,\mathfrak{f}_{(\Gamma_m,v')}(b,\dots,b)), \mathfrak{f}_{(\Gamma_{(0)},v)}(b,\dots,b)\rangle\\ &= -\mathfrak{m}(\Gamma; b),\end{aligned}$$

as required. The proof of the sublemma is complete. □

Lemma 5.10 now follows from Sublemma 5.11. □

Since Γ is a tree we have $\#C_0^{\mathrm{int}}(\Gamma) - \#C_1^{\mathrm{int}}(\Gamma) = 1$. Therefore Lemmas 5.9 and 5.10 imply Proposition 5.8. □

Using the proof of Theorem 5.5 and [5, Section 9], we can prove the following:

THEOREM 5.12. *If two gapped inhomogeneous cyclic filtered A_∞ algebras are pseudo-isotopic to each other, then so are their canonical models.*

We omit the proof since it is a straightforward analog and we do not use Theorem 5.12 in this paper.

6. Wall crossing formulas. In this section we prove Theorem 1.5. We first review the definition of the numbers $n(L; \alpha; \mathcal{J})$ in (7) in more detail.

We remark that $n(L; \alpha; \mathcal{J})$ is a rational number since we can use multi (but finitely many) valued section of $\mathcal{M}_1^{\mathrm{cl}}(\alpha; \mathcal{J})$ to define it. (The argument to do so is the same as [10].)

On the other hand, to prove Theorem 1.5 we need to choose a perturbation of $\mathcal{M}_1^{\mathrm{cl}}(\alpha; \mathcal{J})$ so that it is compatible with the one of $\mathcal{M}_k(\beta; \mathcal{J})$. Here

$$\mathcal{M}_k(\beta; \mathcal{J}) = \bigcup_{t \in [0,1]} \{t\} \times \mathcal{M}_k(\beta; J_t)\,. \tag{61}$$

Since we use a *continuous family of* multisections to perturb $\mathcal{M}_k(\beta; \mathcal{J})$, we need to use a continuous family of multisections also for $\mathcal{M}_1^{\mathrm{cl}}(\alpha; \mathcal{J})$. Actually this is the way taken in [5, Sections 3 and 5].

There exists a Kuranishi structure and a continuous family of multisections on $\mathcal{M}_1^{\mathrm{cl}}(\alpha; \mathcal{J})$ with the following properties:

(1) The evaluation map

$$(ev_t, ev^{\mathrm{int}}) : \mathcal{M}_1^{\mathrm{cl}}(\alpha; \mathcal{J}) \to [0, 1] \times M \tag{62}$$

is weakly submersive.

(2) The continuous family of multisections is transversal to 0 and (62) induces a submersion on its zero set.

(3) The image of the restriction of $(ev_t, ev^{\mathrm{int}})$ to the zero set of the continuous family of multisections is disjoint from $\{0, 1\} \times L$.

This is [5, Lemmas 3.2 and 5.3]. Let $\mathrm{tri} : \mathcal{M}_1^{\mathrm{cl}}(\alpha; \mathcal{J}) \to \mathrm{pt}$ be the trivial map. We use the above continuous family of multisections and define

$$\mathrm{Corr}(\mathcal{M}_1^{\mathrm{cl}}(\alpha; \mathcal{J}); (\mathrm{tri}, ev^{\mathrm{int}}))(1) \in \Lambda(M)\,. \tag{63}$$

(63) is a smooth differential form of degree

$$\dim_{\boldsymbol{R}} M - \dim_{\boldsymbol{R}} \mathcal{M}_1^{\mathrm{cl}}(\alpha; \mathcal{J}) = 6 - (6 + c^1(M) \cap [\alpha] + 2 - 6 + 1) = 3\,.$$

DEFINITION 6.1. We put:

$$n(L; \alpha; \mathcal{J}) = \int_L \mathrm{Corr}(\mathcal{M}_1^{\mathrm{cl}}(\alpha; \mathcal{J}); (\mathrm{tri}, ev^{\mathrm{int}}))(1) \in \boldsymbol{R}\,.$$

We also define:

$$n(L; \alpha; \mathcal{J}; t) = \int_L \mathrm{Corr}(\mathcal{M}_1^{\mathrm{cl}}(\alpha; \mathcal{J}) \cap ev_t^{-1}([0, t]); (\mathrm{tri}, ev^{\mathrm{int}}))(1) \in \boldsymbol{R}\,.$$

The submersivity of $(ev_t, ev^{\mathrm{int}})$ implies that $n(L; \alpha; \mathcal{J}; t)$ is a smooth function of t.

THEOREM 6.2. *In the situation of Theorem* 1.5, $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}_{k,\beta}^{J_0}\}, \{\mathfrak{m}_{-1,\beta}^{J_0}\})$ *is pseudo-isotopic to* $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}_{k,\beta}^{J_1}\}, \{\mathfrak{m}_{-1,\beta}^{J_1} + \Delta(\beta)\})$ *as inhomogeneous gapped cyclic filtered* A_∞ *algebras. Here*

$$\Delta(\beta) = \sum_{\tilde\beta : i_*(\tilde\beta) = \beta} n(L; \tilde\beta; \mathcal{J})\,.$$

PROOF. We consider the moduli space (61) and the evaluation map

$$(ev_t, ev) = (ev_t, ev_0, \dots, ev_{k-1}) : \mathcal{M}_k(\beta; \mathcal{J}) \to [0, 1] \times L^k\,.$$

By [5, Section 11] we have a system of Kuranishi structures and families of multisections on $\mathcal{M}_k(\beta; \mathcal{J})$ for $\beta \cap \omega < E_0$, with the following properties:

(1) The families of multisections are transversal to 0.

(2) They are compatible with the forgetful map

$$\mathfrak{forget}_{k,0} : \mathcal{M}_k(\beta; \mathcal{J}) \to \mathcal{M}_0(\beta; \mathcal{J})\,. \tag{64}$$

(3) For $k \geq 1$ the evaluation map

$$(ev_t, ev_0) : \mathcal{M}_k(\beta; \mathcal{J}) \to [0, 1] \times L$$

is weakly submersive and induces a submersion of the zero set of the family of multisections, in the sense of [5, Definition 4.1.4].

(4) They are invariant under the cyclic permutation of the boundary marked points.

(5) It is compatible with the identification

$$\partial\mathcal{M}_{k+1}(\beta; \mathcal{J}) \supset \bigcup_{t\in[0,1]} \bigcup_{1\leq i\leq j+1\leq k+1} \bigcup_{\beta_1+\beta_2=\beta} \{t\} \times (\mathcal{M}_{j-i+2}(\beta_2; J_t)_{ev_0} \times_{ev_i} \mathcal{M}_{k-j+i+1}(\beta_1; J_t))\,,$$

which is induced from (32). (Note the difference between the right and left hand sides is the part $t = 0, 1$, which appears in (8).)

(6) We consider the inclusion:

$$\begin{aligned}\partial\mathcal{M}_0(\beta; \mathcal{J}) \supset &\Big(\bigcup_{\beta_1+\beta_2=\beta} (\mathcal{M}_1(\beta_2; \mathcal{J})\,{}_{(ev_t, ev_0)} \times_{(ev_t, ev_0)} \mathcal{M}_1(\beta_1; \mathcal{J}))\Big)/\boldsymbol{Z}_2 \\ &\cup \bigcup_{t\in[0,1]} \bigcup_{\tilde\beta : i_*(\tilde\beta)=\beta} \{t\} \times (\mathcal{M}_1^{\mathrm{cl}}(\tilde\beta; J_t)_{ev^{\mathrm{int}}} \times_M L)\,.\end{aligned} \tag{65}$$

Then the Kuranishi structures and the families of multisections are compatible with (65). We use the Kuranishi structure and families of multisections on $\mathcal{M}_1^{\mathrm{cl}}(\tilde{\beta}; J_t)$ which is explained in this section for the second term of the right-hand side of (65).

(7) The evaluation map, $ev_t : \mathcal{M}_0(\beta; \mathcal{J}) \to [0, 1]$ is weakly submersive and induces a submersion of the zero set of the family of multisections, in the sense of [5, Definition 4.1.4].

(8) At $t_0 = 0, 1$ the induced Kuranishi structure and families of multisections on $\mathcal{M}_k(\beta; \mathcal{J}) \cap ev_t^{-1}(\{t_0\})$ coincides with the given choices of the Kuranishi structures and the families of multisections on $\mathcal{M}_k(\beta; J_{t_0})$.

This is mostly the same as the one we used in the proof of Proposition 4.4. The only difference is the second term of (65). It appears since the fiber product $\mathcal{M}_1^{\mathrm{cl}}(\tilde{\beta}; \mathcal{J})_{ev^{\mathrm{int}}} \times_M L$ can be nonempty in the situation where we consider a one parameter family of almost complex structures.

We now define $\mathfrak{m}^t_{k,\beta}$, $\mathfrak{c}^t_{k,\beta}$ for $k \geq 0$ in the same way as (36), (37) using $\mathcal{M}_k(\beta; \mathcal{J})$ in place of $[0, 1] \times \mathcal{M}_k(\beta; J)$.

We finally define $\mathfrak{m}^t_{-1,\beta}$ as follows. We take $1 \in \Lambda^0(pt) = \boldsymbol{R}$ and put the 0-form on $[0, 1]$

$$\mathrm{Corr}_*(\mathcal{M}_0(\beta; \mathcal{J}); (\mathrm{tri}, ev_t))(1) = \rho(t) \tag{66}$$

and define

$$\mathfrak{m}^t_{-1,\beta} = \rho(t) + \sum_{\tilde{\beta}: i_*(\tilde{\beta})=\beta} n(L; \tilde{\beta}; \mathcal{J}; t)\,. \tag{67}$$

We can prove $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^t_{k,\beta}\}, \{\mathfrak{c}^t_{k,\beta}\})$ is a pseudo-isotopy of gapped cyclic filtered A_∞ algebras modulo T^{E_0} in the same way as [5, Section 11].

To prove $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^t_{k,\beta}\}, \{\mathfrak{c}^t_{k,\beta}\}, \{\mathfrak{m}^t_{-1,\beta}\})$ is an *inhomogeneous* pseudo-isotopy of gapped cyclic filtered A_∞ algebras modulo T^{E_0} it suffices to prove (24). Let $0 \leq t_1 < t_2 \leq 1$. We have:

$$\begin{aligned}
&\partial(\mathcal{M}_0(\beta; \mathcal{J}) \cap ev_t^{-1}([t_1, t_2])) \\
&\quad = (\{t_1\} \times \mathcal{M}_0(\beta; J_{t_1})) \cup (\{t_2\} \times \mathcal{M}_0(\beta; J_{t_2})) \\
&\qquad \cup \frac{\bigcup_{\beta_1+\beta_2=\beta} (\mathcal{M}_1(\beta_2; \mathcal{J}) \,{}_{(ev_t, ev_0)}\times_{(ev_t, ev_0)} \mathcal{M}_1(\beta_1; \mathcal{J})) \cap ev_t^{-1}([t_1, t_2])}{\boldsymbol{Z}_2} \\
&\qquad \cup \bigcup_{t\in[t_1,t_2]} \bigcup_{\tilde{\beta}: i_*(\tilde{\beta})=\beta} \{t\} \times (\mathcal{M}_1^{\mathrm{cl}}(\tilde{\beta}; J_t)_{ev^{\mathrm{int}}} \times_M L)\,.
\end{aligned} \tag{68}$$

We apply Stokes' theorem ([5, Proposition 4.2]) to obtain:

$$\mathfrak{m}^{t_2}_{-1,\beta} - \mathfrak{m}^{t_1}_{-1,\beta} = \sum_{\beta_1+\beta_2=\beta} \int_{t_1}^{t_2} \langle \mathfrak{c}^t_{0,\beta_1}(1), \mathfrak{m}^t_{0,\beta_2}(1)\rangle dt\,. \tag{69}$$

Here the sum of the 1st and the 3rd terms of (68) gives the left-hand side of (69).

We obtain (24) by differentiating (69).

We remark

$$\mathfrak{m}^1_{-1,\beta} = \mathfrak{m}^{J_1}_{-1,\beta} + \sum_{\tilde\beta : i_*(\tilde\beta)=\beta} n(L; \tilde\beta; J)\,.$$

The proof of Theorem 6.2 is complete. (Actually we need to go from modulo T^{E_0} version to Theorem 6.2 itself. We omit the proof of this part since it is the same as the one for Theorems 1.2 and 4.1.) □

7. Convergence. In this section we prove Theorem 1.4. Actually most of the ideas of the proof is in [5, Section 13]. Let $b = \sum_{i=1}^{b_1} x_i \mathbf{e}_i$, where $\mathbf{e}_i$ is a basis of $H^1(L; \boldsymbol{R})$. We put $y_i = e^{x_i}$. For $\beta \in H_2(M, L; \mathbf{Z})$ we define $\partial_i \beta \in \mathbf{Z}$ by $\partial_i \beta = \partial\beta \cap \mathbf{e}_i$ and define

$$y^{\partial\beta} = \prod_{i=1}^{b_1} y_i^{\partial_i \beta}\,. \tag{70}$$

THEOREM 7.1. *We regard the superpotential $\Psi(b; J)$ as a function of x_i then we have*:

$$\Psi(b; J) = \sum_{\beta \in G} T^{\beta\cap[\omega]} \mathfrak{m}^J_{-1,\beta} y^{\partial\beta}\,. \tag{71}$$

Theorem 1.4 (1) follows immediately from Theorem 7.1.

PROOF. Let ρ be a closed one form on L. By definition we have

$$\begin{aligned} &\langle \mathfrak{m}^J_{k,\beta}(\rho, \dots, \rho), \rho\rangle \\ &\quad = \mathrm{Corr}(\mathcal{M}_k(\beta; J); (\mathrm{tri}, (ev_1, \dots, ev_k, ev_0)))(\rho \times \cdots \times \rho) \in \Lambda^0(\mathrm{pt}) = \boldsymbol{R}\,. \end{aligned}$$

Then, by the same argument as the proof of [5, Lemma 13.1], we have

$$\langle \mathfrak{m}^J_{k,\beta}(\rho, \dots, \rho), \rho\rangle = \frac{1}{k!}(\rho \cap \partial\beta)^{k+1} \mathfrak{m}^J_{-1,\beta}\,.$$

Theorem 7.1 follows easily. □

We turn to the proof of Theorem 1.4 (2). We take a Weinstein neighborhood U of L. Namely U is symplectomorphic to a neighborhood U' of the zero section in T^*L. We choose δ_1 so that for $c = (c_1, \dots, c_{b_1}) \in [-\delta_1, +\delta_1]^{b_1}$ the graph of the closed one form $\sum_{i=1}^{b_1} c_i \mathbf{e}_i$ is contained in U'. (Here we take a de Rham representative of $\mathbf{e}_i \in H^1(L; \boldsymbol{R})$ and regard $\mathbf{e}_i$ as a closed one form.) We send it by the symplectomorphism to U and denote it by $L(c)$. We may take $\delta_2 < \delta_1$ so that if $c = (c_1, \dots, c_{b_1}) \in [-\delta_2, +\delta_2]^{b_1}$ then there exists a diffeomorphism $F_c : M \to M$ so that

$$F_c(L) = L(c)\,, \tag{72}$$

$$(F_c)_* J \text{ is tamed by } \omega\,. \tag{73}$$

Then we have an isomorphism

$$\mathcal{M}_0(L(c); (F_c)_*(\beta); (F_c)_* J) \cong \mathcal{M}_0(L; \beta; J)\,. \tag{74}$$

We can extend this isomorphism to their Kuranishi structures and families of multisections on them. We can then use Proposition 5.8 and (74) to obtain:

$$\mathfrak{m}^{J}_{-1,\beta,L} = \mathfrak{m}^{(F_c)_*J}_{-1,(F_c)_*(\beta),L(c)}. \tag{75}$$

Here we include L and $L(c)$ in the notation to clarify the Lagrangian submanifold we study. Theorem 7.1 and

$$\beta \cap [\omega] = (F_c)_*(\beta) \cap [\omega] - \sum_i c_i \partial_i \beta$$

(See [5, Lemma 13.5].) then imply:

$$\Psi(y; L(c); (F_c)_*(J)) = \Psi(y(c); L; J), \tag{76}$$

where we put $y(c)_i = T^{-c_i} y_i$. In (76) we include L in the notation of the superpotentials to clarify the Lagrangian submanifold we study. We regard the superpotential as a function of y_i by using Theorem 7.1.

Since the right-hand side converges in $\Lambda\langle\langle y_1, \dots, y_{b_1}, y_1^{-1}, \dots, y_{b_1}^{-1}\rangle\rangle$, it follows that $\Psi(y(c); L; J)$ converges for $c = (c_1, \dots, c_{b_1})$ with $|c_i| < \delta$. This implies Theorem 1.4 (2).

Theorem 1.4 (3), (4) follows from Theorem 1.2. The proof of Theorem 1.4 is complete. □

Once the convergence is established, Propositions 2.3, 2.7 and Theorems 3.3, 4.1, 5.5 are generalized in the same way to our larger domain of convergence.

8. Concluding remarks.

8.1. Rationality and integrality.

CONJECTURE 8.1. *In the situation of Corollary* 1.3 *we have* $\Psi^{\text{can}}(0; J) \in \Lambda_0^{\boldsymbol{Q}}$.

We remark that a filtered A_∞ structure on $H(L)$ is constructed in [7] over $\Lambda_{0,\text{nov}}^{\boldsymbol{Q}}$. In [5] and in this paper, we work over $\boldsymbol{R}$ coefficient to use a continuous family of multisections and the de Rham theory for construction. This is the reason why we can not prove Conjecture 8.1 by the method of this paper. A possible way to approach this conjecture is to use the theory of Kuranishi homology by Joyce [14].

CONJECTURE 8.2. *There exist integers* $\mathfrak{o}^J_\beta \in \mathbf{Z}$ *for each* $\beta \in H_2(M, L; \mathbf{Z})$ *such that*

$$\Psi^{\text{can}}(0; J) = \sum_{d \in \mathbf{Z}_+ : \beta/d \in H_2(M,L;\mathbf{Z})} d^{-2} T^{\beta \cap \omega} \mathfrak{o}^J_{\beta/d}. \tag{77}$$

This is an anolog of the corresponding conjecture for Gromov-Witten invariants of genus zero. (See [11].) The factor d^{-2} is discussed in [17]. There is a discussion on this conjecture in [19].

8.2. Bulk deformations and generalizations to non Calabi-Yau cases etc. In this paper we assumed $\dim_{\boldsymbol{C}} M = 3$, $c^1(M) = 0$, $\mu_L = 0$. This assumption is used to define $\mathfrak{m}^J_{-1,\beta}$. Namely it is used to show that the (virtual) dimension of $\mathcal{M}_0(\beta; J)$ is 0. We may use bulk deformations ([7, Section 3.8]) to obtain a numerical invariant in some other cases, as follows.

We consider the moduli space $\mathcal{M}_{l,k}(\beta; J)$ of bordered stable J-holomorphic curves of genus zero with l interior marked points and k boundary marked points, one boundary component and of homology class β. Let $\sigma_1, \dots, \sigma_l$ be closed forms on M. We may consider

$$\mathrm{Corr}(\mathcal{M}_{l,0}(\beta; J); (ev^{\mathrm{int}}, \mathrm{tri}))(\sigma_1, \dots, \sigma_l) \in \Lambda^*(\mathrm{pt}) = \boldsymbol{R}$$

if $* = n + \mu(\beta) - 3 + 2l - \sum \deg \sigma_i = 0$.

We obtain similar numbers by considering $\mathcal{M}_{l,k}(\beta; J)$ and differential forms on L. The algebraic structure behind this 'invariant' is not yet clear to the author. So the study of them is a problem for future research. Another case where a numerical invariant is defined is the case when M is a toric manifold and L is its T^n orbit. In that case $\mathcal{M}_{l,k}(\beta; J)$ for $\beta \in H_2(M, L; \mathbf{Z})$ with Maslov index ≥ 2 only is related to our structures. See [8] and references therein for this case.

8.3. The case of real points. We assume $\dim_{\mathbf{C}} M = 3$ and let $\tau : M \to M$ be a J-anti holomorphic involution. We assume that $L = \{x \in M\, ;\ \tau(x) = x\}$ is nonempty. Then it becomes a Lagrangian submanifold. We assume L is τ-relatively spin. (See [9] and [6, Chapter 8] for its definition.) (If L is spin then it is τ-relatively spin.) Then in [9] and [6, Chapter 8 Sections 34 and 38], we constructed $\mathfrak{m}^J_{k,\beta}$ such that

$$\mathfrak{m}^J_{k,\tau_*(\beta)}(x_1, \dots, x_k) = (-1)^{k+1+*}\mathfrak{m}^J_{k,\beta}(x_k, \dots, x_1)\,, \tag{78}$$

where $* = \sum_{0 \leq i < j \leq k} \deg' x_i \deg' x_j$. (See [6, Theorem 34.20] and [9, Theorem 1.5].) We can combine the construction of [9] with the one in [5] and can define an inhomogeneous cyclic filtered A_∞ algebra $(\Lambda(L), \langle\cdot\rangle, \{\mathfrak{m}^J_{k,\beta}\}, \{\mathfrak{m}^J_{-1,\beta}\})$ satisfying (78). Moreover $\mathfrak{m}^J_{-1,\beta}$ satisfies

$$\mathfrak{m}^J_{-1,\tau_*(\beta)} = \mathfrak{m}^J_{-1,\beta}\,. \tag{79}$$

Then its superpotential satisfies

$$\Psi(-b; J) = \Psi(b; J)\,. \tag{80}$$

In particular $b = 0$ is a critical point.

CONJECTURE 8.3. *The critial value $\Psi(0; J)$ is equivalent to a special case of the invariant by Solomon* [20].

We can prove $\Psi(0; J_0) = \Psi(0; J_1)$ if there exists a family of almost complex structures J_t such that $\tau_* J_t = -J_t$. In fact we can show

$$n(L; \tilde\beta; \mathcal{J}) = -n(L; \tau_*\tilde\beta; \mathcal{J}) \tag{81}$$

for such $\mathcal{J} = \{J_t\}$. (See [9, Subsection 6.4].)

If we can generalize this construction in the way suggested in Subsection 8.2, it seems likely that we can reproduce the invariants of Solomon and Welschinger [24] and Pandharipande-Solomon-Walcher [19].

The superpotential we defined in this paper is also likely to be related to the numbers studied by Walcher [23]. (For such a purpose we need to include flat bundle on L. In fact in [23] it seems that several flat connections are used to cancel the wall crossing term which appears in (8).)

8.4 Generalizations to higher genus and Chern-Simons perturbation theory. The right-hand side of the formula (48) has obvious similarity with the invariant of Chern-Simons perturbation theory ([1]). It seems very likely that we can combine two stories to obtain an invariant counting the number of stable maps from bordered Riemann surface with arbitrary many boundary components and of arbitrary genus. Its rigorous definition is not known at the time of writing of this paper. The author is unable to do it at the time of writing this paper because of the transversality problem. Here we describe some ideas and explain the difficulty to make it rigorous.

Let $\mathcal{T}$ be a ribbon graph. Namely it is a graph together with a choice of cyclic order of the sets of edges containing each vertex. It uniquely determines a compact oriented 2 dimensional manifold $\Sigma(\mathcal{T})$ without boundary and an embedding $i : \mathcal{T} \to \Sigma(\mathcal{T})$ such that the cyclic order of the edges is induced by the orientation of $\Sigma(\mathcal{T})$ and that the connected components of $\Sigma(\mathcal{T}) \setminus \mathcal{T}$ are all discs. (We do not assume that $\mathcal{T}$ or $\Sigma(\mathcal{T})$ is connected.)

Let $C_0(\mathcal{T})$ be the set of vertices and let $l = \#C_0(\mathcal{T})$. For $v_i \in C_0(\mathcal{T})$, let k_i be the number of edges containing v_i. Let $e_{i,1}, \dots, e_{i,k_i}$ be the set of such edges. The set of the pair $(v_i, e_{i,j})$ where $i = 1, \dots, l$, $j = 1, \dots, k_i$ is called a flag. Let $\mathrm{Fl}(\mathcal{T})$ be the set of flags.

We next consider a compact oriented 2 dimensional manifold Σ with boundary $\partial\Sigma$. We assume $\partial\Sigma$ has at least l connected components $\partial_i\Sigma$, $i = 1, \dots, l$ and on $\partial_i\Sigma$ we put k_i boundary marked points. There may be other components of $\partial\Sigma$, on which we do not put boundary marked points. (We remark that we do not assume that Σ is connected.) Each of the boundary marked points thus corresponds to an element of $\mathrm{Fl}(\mathcal{T})$.

Let $\beta \in H_2(M, L; \boldsymbol{Z})$ where M is a 6 dimensional symplectic manifold with $c^1(M) = 0$ and L its relatively spin Lagrangian submanifold such that $H^1(L; \boldsymbol{Q}) = 0$. We consider the pair (j, v) where j is a complex structure on Σ and $v : (\Sigma, \partial\Sigma) \to (M, L)$ is a j-J holomorphic map. Let $\mathcal{M}(\Sigma; \beta; L; J)$ be the moduli space of such pairs. (We take the stable map compactification. It has a Kuranishi structure of dimension $\#\mathrm{Fl}(\mathcal{T})$.) The evaluation map at each boundary marked points gives

$$ev : \mathcal{M}(\Sigma; \beta; L; J) \to L^{\#\mathrm{Fl}(\mathcal{T})}\,. \tag{82}$$

We next consider the operator $G : \Lambda(L) \to \Lambda(L)$ of degree $+1$ as in [5, Lemma 10.1]. We can associate a distributional form $\tilde{G}$ on $L \times L$ of degree 2 such that

$$\langle G(u), v\rangle = \int \tilde{G} \wedge (u \times v)\,.$$

(See [1].) For each edge e of $\mathcal{T}$ we have $\pi_e : L^{\#\mathrm{Fl}(\mathcal{T})} \to L^2$, that is the projection to the factors corresponding to (v, e), (v', e) where $\partial e = \{v, v'\}$. We now 'define'

$$\mathfrak{m}(\mathcal{T}; \Sigma; \beta; L; J) = \int_{\mathcal{M}(\Sigma;\beta;L;J)} ev^*\Bigg(\prod_{e\in C_1(\mathcal{T})} \pi_e^*(\tilde{G})\Bigg)\,. \tag{83}$$

To define the right-hand side of (83) rigorously, we need to take an appropriate perturbation of our moduli space $\mathcal{M}(\Sigma; \beta; L; J)$ and use it to define its virtual fundamental chain.

The case when the genus of Σ is 0, Σ has only one boundary component, and $\mathcal{T}$ is a tree, is worked out in this paper and [5]. In that case, it is important to find a perturbation so that it is compatible with the process to forget boundary marked points. As we remarked in [5, Remark 3.2], the way we constructed such a continuous family of multisections in this paper and in [5] uses the fact that the genus of Σ is 0. So it can not be directly generalized to higher genus case.

If we can find an appropriate way to rigorously define (83), we then put

$$\Psi(S, T; L; J) = \sum_{\mathcal{T},\Sigma} S^{2-\chi(\Sigma\#\Sigma(\mathcal{T}))} T^{\beta\cap[\omega]} \mathfrak{m}(\mathcal{T}; \Sigma; \beta; L; J)\,. \tag{84}$$

This is expected to become an invariant of M, L, J.

Here $\Sigma\#\Sigma(\mathcal{T})$ is defined as follows. For each $v \in C_0(\mathcal{T})$ we remove a small ball $B(v)$ centered at v from $\Sigma(\mathcal{T})$. We then glue $\partial B(v_i)$ with the i-th boundary component of Σ. We thus obtain $\Sigma\#\Sigma(\mathcal{T})$ which is a compact oriented 2 dimensional manifold with or without boundary. $\chi(\Sigma\#\Sigma(\mathcal{T}))$ is its Euler number. We take the sum for $\mathcal{T}$, Σ such that $\Sigma\#\Sigma(\mathcal{T})$ is connected. (Here the sum is over the topological types of Σ and $\mathcal{T}$. We actually need to divide each term by the order of the appropriate automorphism group in a way similar to (48).) S is a formal parameter which is called the string coupling constant in physics literature.

PROBLEM 8.4. Let M, L, J be a triple of symplectic manifold M, its relatively spin Lagrangian submanifold L and its tame almost complex structure J, such that $\dim L = 3$, $c^1(M) = 0 = \mu_L$, $H^1(L; \boldsymbol{Q}) = 0$. Define an invariant $\Psi(S, T; L; J)$ so that at $T = 0$ it becomes perturbative Chern-Simons invariant and at $S = 0$ it becomes the invariant of Corollary 1.3.

REMARK 8.5. The study of Chern-Simons perturbation theory suggests that we need to fix a framing of L in order to obtain an appropriate perturbation.

When we generalize the story to the case $H^1(L; \boldsymbol{Q}) \neq 0$, we need to consider the case when $\mathcal{T}$ has exterior vertices and Σ has a boundary marked point on the component other than l components $\partial_i\Sigma$. In that case we expect to obtain a certain algebraic structure on $H^1(L; \Lambda_0)$. We believe that involutive-bi-Lie infinity structure [4] is appropriate for this purpose. More precisely this is the case when at least one element of $H^1(L; \boldsymbol{Q})$ is assigned to each of the connected component of the boundary. (In genus 0 it corresponds to $\mathfrak{m}_{k,\beta}$ with $k \geq 0$.) If we restrict to such cases, the wall crossing phenomenon (the J dependence) does not seem to occur. Namely the algebraic structure is expected to be independent of J up to homotopy equivalence. (This is certainly the case of genus zero as is proved in [5].)

8.5. Mirror to Donaldson-Thomas invariants. Let M be a symplectic manifold of dimension 6 and $c^1(M) = 0$. We consider the set $\widetilde{\mathfrak{Lag}}(M)$ of pairs $(L, [b])$ such that L is a relatively spin Lagrangian submanifold with $\mu_L = 0$ and $[b] \in \mathcal{M}(L; J; \Lambda_0)$.

We say $(L, [b]) \sim (L', [b'])$ if there exists a Hamiltonian diffeomorphism $F : M \to M$ such that $L' = F(L)$ and $F_*(b)$ is gauge equivalent to b'. Let $\mathfrak{Lag}(M)$ be the quotient space. The quotient topology on $\mathfrak{Lag}(M)$ is rather pathological. Namely it is likely to be non-Hausdorff in general. We also need to take an appropriate compactification of this moduli space by including singular Lagrangian submanifolds, for example. (Such a compactification is not known at the time of writing this paper.)

On the other hand, we can define a 'local chart' of $\mathfrak{Lag}(M)$ as follows. Let $(L, [b_0]) \in \mathfrak{Lag}(M)$. We take $\delta > 0$ small so that for $L(c)$ with $c = \sum c_i \mathbf{e}_i$, $|c_i| < \delta$, there exists F_c as in (72), (73). We consider

$$A(\delta) = \{(y_1, \dots, y_{b_1})\,;\ y_i \in \Lambda, |\mathfrak{v}(y_i)| < \delta\}.$$

Then a neighborhood of $(L, [b_0])$ is identified with the set of $(y_1 \dots, y_m) \in A(\delta)$ satisfying the Maurer-Cartan equation

$$\text{(85)} \quad \sum_{k=0}^{\infty} \sum_{\beta \in H_2(M,L;\mathbf{Z})} T^{E(\beta)} \mathfrak{m}^J_{k,\beta}(b, \dots, b) = 0\,, \quad \text{where } b = \log(y_1)\mathbf{e}_1 + \cdots + \log(y_m)\mathbf{e}_m\,.$$

We remark the equation (85) is well-defined by Theorem 1.4.

$y_i = e^{x_i} = T^{c_i} y_i'$ with $c_i = \mathfrak{v}(y_i)$ then $b' = \sum \log y_i' \mathbf{e}_i$ and $L(c)$ defines an element of $\mathfrak{Lag}(M;\,(F_c)_*J)$. (See Section 7 and [5, Section 13].) Using the independence of the Maurer-Cartan scheme of almost complex structures, we obtain an element of $\mathfrak{Lag}(M) = \mathfrak{Lag}(M;\,J)$. In this way the set $\mathcal{M}(L;\,J)_\delta$ in Theorem 1.4 can be glued. Its union can be identified with $\mathfrak{Lag}(M)$. Each of $\mathcal{M}(L;\,J)_\delta$ is a rigid analytic space and the glueing maps are morphisms of rigid analytic spaces. Thus, one may regard $\mathfrak{Lag}(M)$ as a kind of 'non-separated rigid analytic stack'.

We remark that the equation (85) is equivalent to

$$\nabla_y \Psi = 0\,.$$

Thus our situation is similar to the one which appears in Donaldson-Thomas invariants. (Thomas [21], Joyce [13], Kontsevich-Soibelman [16].) There the role of the superpotential is taken by the holomorphic Chern-Simons invariant.

PROBLEM 8.6. (1) Find an appropriate stability condition for the pair $(L, [b])$ and use it to construct a moduli space $\mathfrak{Lag}^{\mathrm{st}}(M)$ of stable pairs $(L, [b])$ which has better properties than $\mathfrak{Lag}(M)$.

(2) Define an invariant which is the 'order' of $\mathfrak{Lag}^{\mathrm{st}}(M)$ in the sense of virtual fundamental cycle.

(3) Prove that it coincides with Donaldson-Thomas invariant of the Mirror manifold of M.

It seems to the author that this problem is very difficult to study at this stage.

REMARK 8.7. After [5] had been put on the arXiv and at the time of final stage of writing this article, a paper [12] was put on the arXiv, where a different construction of a similar invariant as the one in Corollary 1.3 (over $\boldsymbol{Q}$) is sketched.

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