Counting rational curves on K3 surfaces

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Introduction

The aim of these notes is to explain the remarkable formula found by Yau and Zaslow [Y-Z] to express the number of rational curves on a K3 surface. Projective K3 surfaces fall into countably many families $(\mathcal{F}_g)_{g\geq 1}$; a surface in $\mathcal{F}_g$ admits a g-dimensional linear system of curves of genus g . A naïve count of constants suggests that such a system will contain a positive number, say $n(g)$, of rational (highly singular) curves. The formula is

$$\sum_{g\geq 0} n(g)q^g = \frac{q}{\Delta(q)} \ ,$$

where $\Delta(q) = q\prod_{n\geq 1}(1-q^n)^{24}$ is the well-known modular form of weight 12 , and we put by convention $n(0)=1$.

To explain the idea in a nutshell, take the case $g=1$. We are thus looking at K3 surfaces with an elliptic fibration $f:\mathrm{S}\rightarrow\mathbf{P}^1$, and we are asking for the number of singular fibres. The (topological) Euler-Poincaré characteristic of a fibre C_t is 0 if C_t is smooth, 1 if it is a rational curve with one node, 2 if it has a cusp, etc. From the standard properties of the Euler-Poincaré characteristic, we get $e(\mathrm{S})=\sum_t e(\mathrm{C}_t)$; hence $n(1)=e(\mathrm{S})=24$, and this number counts nodal rational curves with multiplicity 1 , cuspidal rational curves with multiplicity 2 , etc.

The idea of Yau and Zaslow is to generalize this approach to any genus. Let S be a K3 surface with a g-dimensional linear system Π of curves of genus g . The role of f will be played by the morphism $\bar{\mathcal{J}}\mathcal{C}\rightarrow\Pi$ whose fibre over a point $t\in\Pi$ is the compactified Jacobian $\bar{\mathrm{J}}\mathrm{C}_t$. To apply the same method, we would like to prove the following facts:

1) The Euler-Poincaré characteristic $e(\bar{\mathcal{J}}\mathcal{C})$ is the coefficient of q^g in the Taylor expansion of $q/\Delta(q)$.

2) $e(\bar{\mathrm{J}}\mathrm{C}_t)=0$ if C_t is not rational.

3) $e(\bar{\mathrm{J}}\mathrm{C}_t)=1$ if C_t is a rational curve with nodes as only singularities. Moreover $e(\bar{\mathrm{J}}\mathrm{C}_t)$ is positive when C_t is rational, and can be computed in terms of the singularities of C_t .

4) For a generic K3 surface S in $\mathcal{F}_g$, all rational curves in Π are nodal.

The first statement is proved in § 1, by comparing $e(\bar{\mathcal{J}}\mathcal{C})$ with the Euler-Poincaré characteristic of the Hilbert scheme $\mathrm{S}^{[g]}$ which has been computed by

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Göttsche. The assertion 2) is proved in § 2. We prove part of 3) in § 3 and 4: we express $e(\bar{\mathrm{J}}\mathrm{C})$, for a rational curve C, in terms of a local invariant of the singularities of C, and compute this local invariant in a number of cases. This invariant has been recently identified by Fantechi, Göttsche and van Straten as the multiplicity of the δ-constant stratum in the semi-universal deformation of the singularity [F-G-S]; this implies in particular the positivity of $e(\bar{\mathrm{J}}\mathrm{C})$. This approach also provides an alternate proof for most of our results in § 3 and 4.

Unfortunately 4) is of a different nature, and seems to be widely open.

The outcome (see Cor. 2.3) is that the coefficient of q^g in $q/\Delta(q)$ counts the rational curves in Π with a certain multiplicity, which is 1 for a nodal curve and can be computed explicitely in many cases; the only missing point (equivalent to 4)) is the fact that for a generic surface in $\mathcal{F}_g$ this coefficient is simply the number of rational curves in Π.

1. The compactified relative Jacobian

(1.1) Let X be a complex variety; we denote by $e(\mathrm{X})$ its Euler-Poincaré characteristic, defined by $e(\mathrm{X}) = \sum_p (-1)^p \dim_{\mathbf{Q}} \mathrm{H}^p_c(\mathrm{X}, \mathbf{Q})$. Recall that this invariant is additive, that is satisfies $e(\mathrm{X}) = e(\mathrm{U}) + e(\mathrm{X} - \mathrm{U})$ whenever U is an open subset of X.

(1.2) We consider a projective K3 surface S with a complete linear system $(\mathrm{C}_t)_{t\in\Pi}$ of curves of genus $g \geq 1$ (so Π is a projective space of dimension g). We will assume that *all the curves* C_t *are integral* (that is irreducible and reduced). This is a simplifying assumption, which can probably be removed at the cost of various technical complications. It is of course satisfied if the class of C_t generates Pic(S).

Let $\mathcal{C} \to \Pi$ be the morphism with fibre C_t over $t \in \Pi$. For each integer $d \in \mathbf{Z}$, we denote by $\bar{\mathcal{J}}\mathcal{C} = \coprod_{d\in\mathbf{Z}} \bar{\mathcal{J}}^d\mathcal{C}$ the compactified Picard scheme of this family. $\bar{\mathcal{J}}^d\mathcal{C}$ is a projective variety of dimension $2g$, which parameterizes pairs $(\mathrm{C}_t, \mathcal{L})$ where $t \in \Pi$ and $\mathcal{L}$ is a torsion free, rank 1 coherent sheaf on C_t of degree d (which means by definition $\chi(\mathcal{L}) = d + 1 - g$). According to Mukai ([M], example 0.5), $\bar{\mathcal{J}}^d\mathcal{C}$ can be viewed as a connected component of the moduli space of simple sheaves on S, and therefore is smooth, and admits a (holomorphic) symplectic structure.

The simplest symplectic varieties associated to the K3 surface S are the Hilbert schemes $\mathrm{S}^{[d]}$, which parameterize finite subschemes of length d of S. The birational comparison of the symplectic varieties $\bar{\mathcal{J}}^d\mathcal{C}$, for various values of d, with $\mathrm{S}^{[g]}$ is an interesting problem, about which not much seems to be known. There is one easy case:

Proposition 1.3 .– *The compactified Jacobian* $\bar{\mathcal{J}}^g\mathcal{C}$ *is birationally isomorphic to* $\mathrm{S}^{[g]}$.

Proof: Let U be the open subset of $\bar{\mathcal{J}}^g\mathcal{C}$ consisting of pairs $(\mathrm{C}_t,\mathrm{L})$ where C_t is smooth, L is invertible and $\dim \mathrm{H}^0(\mathrm{C}_t,\mathrm{L})=1$. To such a pair corresponds a unique effective divisor D on C_t of degree g , which can be viewed as a length g subscheme of S ; since $\dim \mathrm{H}^0(\mathrm{C}_t,\mathcal{O}_{\mathrm{C}_t}(\mathrm{D}))=1$ it is contained in a unique curve of Π , namely C_t . This provides an isomorphism between U and the open subset of $\mathrm{S}^{[g]}$ parameterizing finite subschemes of S contained in a unique smooth curve of Π . ∎

Corollary 1.4 .– *Write* $\dfrac{q}{\Delta(q)}=\sum_{g\geq 0} e(g)\,q^g$. *Then* $e(\bar{\mathcal{J}}^g\mathcal{C})=e(g)$.

Proof: We can either use a recent result of Batyrev [B] saying that two birationally equivalent projective Calabi-Yau manifolds have the same Betti numbers, or a more precise result of Huybrechts [H]: two birationally equivalent projective symplectic manifolds are diffeomorphic. It remains to apply Göttsche's formula $e(\mathrm{S}^{[g]})=e(g)$ [G]. ∎

2. The compactified Jacobian of a non-rational curve

Let C be an integral curve. By a *rank* 1 *sheaf* on C we will mean a torsion free, rank 1 coherent sheaf. The rank 1 sheaves $\mathcal{L}$ on C of degree d are parameterized by the compactified Jacobian $\bar{\mathrm{J}}^d\mathrm{C}$. If L is an invertible sheaf of degree d on C , the map $\mathcal{L}\mapsto\mathcal{L}\otimes\mathrm{L}$ is an isomorphism of $\bar{\mathrm{J}}\mathrm{C}$ onto $\bar{\mathrm{J}}^d\mathrm{C}$, so we can restrict our study to degree 0 sheaves.

Let $\mathcal{L}\in\bar{\mathrm{J}}\mathrm{C}$; the endomorphism ring of $\mathcal{L}$ is an $\mathcal{O}_\mathrm{C}$-subalgebra of the sheaf of rational functions on C . It is finitely generated as a $\mathcal{O}_\mathrm{C}$-module, hence contained in $\mathcal{O}_{\widetilde{\mathrm{C}}}$. It is thus of the form $\mathcal{O}_{\mathrm{C}'}$, where $f:\mathrm{C}'\to\mathrm{C}$ is some partial normalization of C . The sheaf $\mathcal{L}$ is a $\mathcal{O}_{\mathrm{C}'}$-module, which amounts to say that it is the direct image of a rank 1 sheaf $\mathcal{L}'$ on C' .

Lemma 2.1 .– *Let* $\mathrm{L}\in\mathrm{JC}$. *Then* $\mathcal{L}\otimes\mathrm{L}$ *is isomorphic to* $\mathcal{L}$ *if and only if* $f^*\mathrm{L}$ *is trivial.*

Proof: The sheaf $\mathcal{L}\otimes\mathrm{L}$ is isomorphic to $f_*(\mathcal{L}'\otimes f^*\mathrm{L})$, hence to $\mathcal{L}$ if $f^*\mathrm{L}$ is trivial. On the other hand we have

$$\mathcal{H}om_{\mathcal{O}_\mathrm{C}}(\mathcal{L},\mathcal{L}\otimes\mathrm{L})\cong\mathcal{E}nd_{\mathcal{O}_\mathrm{C}}(\mathcal{L})\otimes_{\mathcal{O}_\mathrm{C}}\mathrm{L}\cong f_*\mathcal{O}_{\mathrm{C}'}\otimes\mathrm{L}\cong f_*f^*\mathrm{L}\ ,$$

so if $f^*\mathrm{L}$ is non-trivial, the space $\mathrm{Hom}(\mathcal{L},\mathcal{L}\otimes\mathrm{L})$ is zero, and $\mathcal{L}\otimes\mathrm{L}$ cannot be isomorphic to $\mathcal{L}$. ∎

Proposition 2.2 .– *Let* C *be an integral curve whose normalization* $\widetilde{\mathrm{C}}$ *has genus* ≥ 1 *. Then* $e(\bar{\mathrm{J}}^d\mathrm{C})=0$ *.*

Proof: We have an exact sequence

$$0\rightarrow \mathrm{G}\longrightarrow \mathrm{JC}\longrightarrow \mathrm{J}\widetilde{\mathrm{C}}\rightarrow 0\ ,$$

where G is a product of additive and multiplicative groups. In particular, G is a divisible group, hence this exact sequence splits as a sequence of abelian groups. For each integer n, we can therefore find a subgroup of order n in JC which maps injectively into $\mathrm{J}\widetilde{\mathrm{C}}$. By Lemma 2.1, this group acts freely on $\bar{\mathrm{J}}\mathrm{C}$, which implies that n divides $e(\bar{\mathrm{J}}\mathrm{C})$; since this holds for any n the Proposition follows. ∎

Corollary 2.3 .– *Write* $\dfrac{q}{\Delta(q)}=\sum\limits_{g\geq 0} e(g)\,q^g$ *; let* $\Pi_{\mathrm{rat}}\subset\Pi$ *be the (finite) subset of rational curves. Then* $e(g)=\sum\limits_{t\in\Pi_{\mathrm{rat}}} e(\bar{\mathrm{J}}\mathrm{C}_t)$ *.*

Proof: We first make a general observation: let $f:\mathrm{X}\rightarrow\mathrm{Y}$ be a surjective morphism of complex algebraic varieties whose fibres have Euler characteristic 0; then $e(\mathrm{X})=0$. This is well known (and easy) if f is a locally trivial fibration; the general case follows using (1.1), because there exists a stratification of Y such that f is locally trivial above each stratum [V].

The set Π_{rat} is finite because otherwise it would contain a curve, so S would be ruled. Consider the morphism $p:\bar{\mathcal{J}}^g\mathcal{C}\rightarrow\Pi$ above $\Pi-\Pi_{\mathrm{rat}}$; by the above remark, we have $e(p^{-1}(\Pi-\Pi_{\mathrm{rat}}))=0$, hence the result using again (1.1). ∎

In other words, $e(g)$ counts the number of rational curves with multiplicity, the multiplicity of a curve C being $e(\bar{\mathrm{J}}\mathrm{C})$. In the next two sections we will try to show that this is indeed a reasonable notion of multiplicity.

3. The compactified Jacobian of a rational curve

Lemma 3.1 .– *Let* $f:\mathrm{C}'\rightarrow\mathrm{C}$ *be a partial normalization of* C*. The morphism* $f_*:\bar{\mathrm{J}}\mathrm{C}'\rightarrow\bar{\mathrm{J}}\mathrm{C}$ *is a closed embedding.*

Proof: Let $\mathcal{L},\mathcal{M}$ be two rank 1 sheaves on C'. We claim that any $\mathcal{O}_{\mathrm{C}}$-homomorphism $u:f_*\mathcal{L}\rightarrow f_*\mathcal{M}$ is actually $f_*\mathcal{O}_{\mathrm{C}'}$-linear. Let U be a Zariski open subset of C, $\varphi\in\Gamma(\mathrm{U},f_*\mathcal{O}_{\mathrm{C}'})$, $s\in\Gamma(\mathrm{U},f_*\mathcal{L})$; the rational function φ can be written as a/b, with $a,b\in\Gamma(\mathrm{U},\mathcal{O}_{\mathrm{C}})$ and $b\neq 0$. Then the element $u(\varphi s)-\varphi u(s)$ of $\Gamma(\mathrm{U},f_*\mathcal{M})$ is killed by b, hence is zero since $f_*\mathcal{M}$ is torsion-free.

Therefore if $f_*\mathcal{L}$ and $f_*\mathcal{M}$ are isomorphic as $\mathcal{O}_{\mathrm{C}}$-modules, they are also isomorphic as $f_*\mathcal{O}_{\mathrm{C}'}$-modules, which means that $\mathcal{L}$ and $\mathcal{M}$ are isomorphic: this proves the injectivity of f_* (which would be enough for our purpose). Now if S is any base scheme, the same argument applies to sheaves $\mathcal{L}$, $\mathcal{M}$ on $\mathrm{C}\times\mathrm{S}$, flat over S, whose restrictions to each fibre $\mathrm{C}\times\{s\}$ are torsion free rank 1 (observe that a local section b of $\mathcal{O}_{\mathrm{C}}$ is $\mathcal{M}$-regular because it is on each fibre, and $\mathcal{M}$ is flat over S). This proves that f_* is a monomorphism; since it is proper, it is a closed embedding. ∎

(3.2) Recall that the curve C is said to be *unibranch* if its normalization $\widetilde{\mathrm{C}}\to\mathrm{C}$ is a homeomorphism. Any curve C admits a unibranch partial normalization $\check{\pi}:\check{\mathrm{C}}\to\mathrm{C}$ which is minimal, in the sense that any unibranch partial normalization $\mathrm{C}'\to\mathrm{C}$ factors through $\check{\pi}$. To see this, let $\mathcal{C}$ be the conductor of C, and let $\widetilde{\Sigma}$ be the inverse image in $\widetilde{\mathrm{C}}$ of the singular locus $\Sigma\in\mathrm{C}$. The finite-dimensional k-algebra $\mathrm{A}:=\mathcal{O}_{\widetilde{\mathrm{C}}}/\mathcal{C}$ is a product of local rings $(\mathrm{A}_x)_{x\in\widetilde{\Sigma}}$; let $(e_x)_{x\in\widetilde{\Sigma}}$ be the corresponding idempotent elements of A. A sheaf of algebras $\mathcal{O}_{\mathrm{C}'}$ with $\mathcal{O}_{\mathrm{C}}\subset\mathcal{O}_{\mathrm{C}'}\subset\mathcal{O}_{\widetilde{\mathrm{C}}}$ is unibranch if and only if $\mathcal{O}_{\mathrm{C}'}/\mathcal{C}$ contains each e_x, or equivalently $\mathcal{O}_{\mathrm{C}'}$ contains the classes $e_x+\mathcal{C}$ for each $x\in\widetilde{\Sigma}$; clearly there is a smallest such algebra, namely the algebra $\mathcal{O}_{\check{\mathrm{C}}}$ generated by $\mathcal{O}_{\mathrm{C}}$ and the classes $e_x+\mathcal{C}$. The completion of the local ring of $\check{\mathrm{C}}$ at a point y is the image of $\widehat{\mathcal{O}}_{\mathrm{C},\check{\pi}(y)}$ in $\widehat{\mathcal{O}}_{\widetilde{\mathrm{C}},y}$.

Proposition 3.3.– *With the above notation,* $e(\bar{\mathrm{J}}\mathrm{C})=e(\bar{\mathrm{J}}\check{\mathrm{C}})$.

Proof: In view of Prop. 2.2, we may suppose that $\widetilde{\mathrm{C}}$ is rational. As before we denote by Σ the singular locus of C, and by $\widetilde{\Sigma}$ its inverse image in $\check{\mathrm{C}}$. The cohomology exact sequence associated to the short exact sequence

$$1\to\mathcal{O}^*_{\mathrm{C}}\longrightarrow\mathcal{O}^*_{\widetilde{\mathrm{C}}}\longrightarrow\mathcal{O}^*_{\widetilde{\mathrm{C}}}/\mathcal{O}^*_{\mathrm{C}}\to 1$$

provides a bijective homomorphism (actually an isomorphism of algebraic groups) $\mathcal{O}^*_{\widetilde{\mathrm{C}}}/\mathcal{O}^*_{\mathrm{C}}\xrightarrow{\sim}\mathrm{JC}$.

The evaluation maps $\mathcal{O}^*_{\widetilde{\mathrm{C}}}\to(\mathbf{C}^*)^{\widetilde{\Sigma}}$ and $\mathcal{O}^*_{\mathrm{C}}\to(\mathbf{C}^*)^{\Sigma}$ give rise to a surjective homomorphism $\mathcal{O}^*_{\widetilde{\mathrm{C}}}/\mathcal{O}^*_{\mathrm{C}}\to(\mathbf{C}^*)^{\widetilde{\Sigma}}/(\mathbf{C}^*)^{\Sigma}$; its kernel is unipotent, that is isomorphic to a vector space. If n is any integer $\geq\mathrm{Card}(\widetilde{\Sigma})$, it follows that we can find a section φ of $\mathcal{O}^*_{\widetilde{\mathrm{C}}}$ in a neighborhood of $\widetilde{\Sigma}$ such that the numbers $\varphi(\tilde{x})$ for $\tilde{x}\in\widetilde{\Sigma}$ are all distinct, but φ^n belongs to $\mathcal{O}_{\mathrm{C}}$. Let L be the line bundle on C associated to the class of φ in $\mathcal{O}^*_{\widetilde{\mathrm{C}}}/\mathcal{O}^*_{\mathrm{C}}$.

Let U be the complement of $\check{\pi}_*(\bar{\mathrm{J}}\check{\mathrm{C}})$ in $\bar{\mathrm{J}}\mathrm{C}$; according to 1.1 and Lemma 3.1, our assertion is equivalent to $e(\mathrm{U})=0$. We claim that the line bundle L acts freely on U; since the order of L in JC is finite and arbitrary large, this will finish the proof. Let $\mathcal{L}\in\mathrm{U}$, and let C' be the partial normalization of C such that $\mathcal{E}nd(\mathcal{L})=\mathcal{O}_{\mathrm{C}'}$; by definition of U, C' is not unibranch, hence there are two points of $\widetilde{\Sigma}$ mapping to the same point of C' ; this implies that the function φ does not belong to $\mathcal{O}^*_{C'}$. From the commutative diagram

$$\begin{array}{ccc}
\widetilde{\mathcal{O}}^*_{\widetilde{C}}/\mathcal{O}^*_{C} & \xrightarrow{\ \sim\ } & JC \\
\big\downarrow & & \big\downarrow \\
\mathcal{O}^*_{\widetilde{C}}/\mathcal{O}^*_{C'} & \xrightarrow{\ \sim\ } & JC'
\end{array}$$

we conclude that the pull back of L to JC' is non-trivial; by Lemma 2.1 this implies that $\mathcal{L}\otimes L$ is not isomorphic to $\mathcal{L}$. ■

Corollary 3.4 .– *For a rational nodal curve* C *, we have* $e(\bar{J}C)=1$ *.* ■

Remark 3.5 .– Consider a rational curve C whose singularities are all of type A_{2l-1} , that is locally defined by an equation $u^2-v^{2l}=0$. Locally around such a singularity, the curve C is the union of two smooth branches with a high order contact, so by 3.3 $e(\bar{J}C)$ is equal to 1 . The fact that some highly singular curves count with multiplicity one looks rather surprising. The case $g=2$ provides a (modest) confirmation: the surface S is a double covering of $\mathbf{P}^2$ branched along a sextic curve B ; the curves C_t are the inverse images of the lines in $\mathbf{P}^2$, and they become rational when the line is bitangent to B . We get an A_3-singularity when the line has a contact of order 4 ; thus our assertion in this case follows from the (certainly classical) fact that a line with a fourth order contact counts as a simple bitangent.

(3.6) Prop. 3.3 reduces the computation of the invariant $e(\bar{J}C)$ to the case of a unibranch (rational) curve. To understand this invariant we will use a construction of Rego ([R], see also [G-P]). For each $x\in C$, we put $\delta_x=\dim\mathcal{O}_{\widetilde{C},x}/\mathcal{O}_{C,x}$ and we denote by $\mathcal{C}$ the ideal $\mathcal{O}_{\widetilde{C}}(-\sum_x(2\delta_x)\,x)$; it is contained in the conductor of C (but the inclusion is strict unless C is Gorenstein).

For $x\in C$, we denote by A_x and $\widetilde{A}_x$ the finite dimensional algebras $\mathcal{O}_{C,x}/\mathcal{C}_x$ and $\mathcal{O}_{\widetilde{C},x}/\mathcal{C}_x$. Let $\mathbf{G}(\delta_x,\widetilde{A}_x)$ be the Grassmannian of codimension δ_x subspaces of $\widetilde{A}_x$, and $\mathbf{G}_x$ the closed subvariety of $\mathbf{G}(\delta_x,\widetilde{A}_x)$ consisting of sub-A_x-modules. We can also view $\mathbf{G}_x$ as parameterizing the sub-$\mathcal{O}_{C,x}$-modules $\mathcal{L}_x$ of codimension δ_x in $\mathcal{O}_{\widetilde{C},x}$, because any such sub-module contains $\mathcal{C}_x$ ([G-P], lemma 1.1 (iv)). Since $\mathcal{O}_{\widetilde{C}}/\mathcal{C}$ is a skyscraper sheaf with fibre $\widetilde{A}_x$ at x , the product $\prod\limits_{x\in\Sigma}\mathbf{G}_x$ parameterizes sub-$\mathcal{O}_C$-modules $\mathcal{L}\subset\mathcal{O}_{\widetilde{C}}$ such that $\dim\mathcal{O}_{\widetilde{C},x}/\mathcal{L}_x=\delta_x$ for all x . This implies

$\chi(\mathcal{O}_{\widetilde{\mathrm{C}}}/\mathcal{L}) = \sum_x \delta_x = \chi(\mathcal{O}_{\widetilde{\mathrm{C}}}/\mathcal{O}_{\mathrm{C}})$, hence $\mathcal{L} \in \bar{\mathrm{J}}\mathrm{C}$. We have thus defined a morphism $e : \prod_{x\in\Sigma} \mathbf{G}_x \to \bar{\mathrm{J}}\mathrm{C}$.

Proposition 3.7 .– *The map e is a homeomorphism.*

Note that e is not an isomorphism, already when C is a rational curve with one ordinary cusp s: the Grassmannian $\mathbf{G}_s$ is isomorphic to $\mathbf{P}^1$, while $\bar{\mathrm{J}}\mathrm{C}$ is isomorphic to C .

Since we are dealing with compact varieties, it suffices to prove that e is bijective.

Injectivity: Let $\mathcal{L}$, $\mathcal{M}$ be two sub-$\mathcal{O}_{\mathrm{C}}$-modules of $\mathcal{O}_{\widetilde{\mathrm{C}}}$ containing $\mathcal{C}$. If $\mathcal{L}$ and $\mathcal{M}$ give the same element in $\bar{\mathrm{J}}\mathrm{C}$, there exists a rational function φ on $\widetilde{\mathrm{C}}$ such that $\mathcal{M} = \varphi\mathcal{L}$. But the equalities $\dim \mathcal{O}_{\widetilde{\mathrm{C}},x}/\mathcal{M}_x = \dim \mathcal{O}_{\widetilde{\mathrm{C}},x}/\mathcal{L}_x = \dim \varphi_x\mathcal{O}_{\widetilde{\mathrm{C}},x}/\mathcal{M}_x$ imply $\varphi_x\mathcal{O}_{\widetilde{\mathrm{C}},x} = \mathcal{O}_{\widetilde{\mathrm{C}},x}$ for all x , which means that φ is constant.

Surjectivity: Let $f : \widetilde{\mathrm{C}} \to \mathrm{C}$ be the normalization morphism, and $\mathcal{L} \in \bar{\mathrm{J}}\mathrm{C}$. Let us denote by $\widetilde{\mathcal{L}}$ the line bundle on $\widetilde{\mathrm{C}}$ quotient of $f^*\mathcal{L}$ by its torsion subsheaf. We claim that its degree is ≤ 0 : we have an exact sequence

$$0 \to \mathcal{L} \longrightarrow f_*\widetilde{\mathcal{L}} \longrightarrow \mathcal{T} \to 0$$

where $\mathcal{T}$ is a skyscrapersheaf supported on the singular locus of C , such that $\dim \mathcal{T}_x \leq \delta_x$ for all $x \in \mathrm{C}$ ([G-P], lemma 1.1); this implies $\chi(\widetilde{\mathcal{L}}) - \chi(\mathcal{L}) \leq \chi(\mathcal{O}_{\widetilde{\mathrm{C}}}) - \chi(\mathcal{O}_{\mathrm{C}})$, from which the required inequality follows. Since $\widetilde{\mathrm{C}}$ is rational, it follows that $\widetilde{\mathcal{L}}^{-1}$ admits a global section whose zero set is contained in Σ .

Because of the canonical isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_{\mathrm{C}}}(\mathcal{L}, \mathcal{O}_{\widetilde{\mathrm{C}}}) \cong \mathrm{Hom}_{\mathcal{O}_{\widetilde{\mathrm{C}}}}(f^*\mathcal{L}, \mathcal{O}_{\widetilde{\mathrm{C}}}) \cong \mathrm{Hom}_{\mathcal{O}_{\widetilde{\mathrm{C}}}}(\widetilde{\mathcal{L}}, \mathcal{O}_{\widetilde{\mathrm{C}}}) \ ,$$

we conclude that there exists a homomorphism $i : \mathcal{L} \to \mathcal{O}_{\widetilde{\mathrm{C}}}$ which is bijective outside Σ . Put $n_x = \dim \mathcal{O}_{\widetilde{\mathrm{C}},x}/i(\mathcal{L}_x)$ for each $x \in \Sigma$. Since

$$\sum_{x\in\Sigma} n_x = \dim \mathcal{O}_{\widetilde{\mathrm{C}}}/i(\mathcal{L}) = \chi(\mathcal{O}_{\widetilde{\mathrm{C}}}) - \chi(\mathcal{L}) = g = \sum_{x\in\Sigma} \delta_x \ ,$$

there exists a rational function φ on $\widetilde{\mathrm{C}}$ with divisor $\sum_x (\delta_x - n_x)\, x$. Replacing $\mathcal{L}$ by $\varphi\mathcal{L}$, we may assume $n_x = \delta_x$ for all x , which means that $\mathcal{L}$ belongs to the image of e . ■

The variety $\mathbf{G}_x$ depends only on the local ring $\mathcal{O}$ of C at x (even only on its completion); we will also denote it by $\mathbf{G}_{\mathcal{O}}$. Recall that $\mathbf{G}_{\mathcal{O}}$ parameterizes the sub-$\mathcal{O}$-modules L of the normalization $\widetilde{\mathcal{O}}$ of $\mathcal{O}$ with $\dim \widetilde{\mathcal{O}}/\mathrm{L} = \dim \widetilde{\mathcal{O}}/\mathcal{O}$. We put $\varepsilon(x) = e(\mathbf{G}_x)$ (or $\varepsilon(\mathcal{O}) = e(\mathbf{G}_{\mathcal{O}})$). The above Proposition gives:

Proposition 3.8 .– *Let* C *be a rational unibranch curve; then* $e(\bar{\mathrm{J}}\mathrm{C}) = \prod_{x\in\mathrm{C}} \varepsilon(x)$. ∎

Of course $\varepsilon(x)$ is equal to 1 for a smooth point, so we could as well consider the product over the singular locus Σ of C . Note that in view of Prop. 3.3, we may define $\varepsilon(x)$ for a non-unibranch singularity by taking the product of the ε-invariants of each branch; Prop. 3.8 remains valid.

4. Examples

(4.1) *Singularities with* $\mathbf{C}^*$*-action*

Assume that the local, unibranch ring $\mathcal{O}$ admits a $\mathbf{C}^*$-action. This action extends to its completion, so we will assume that $\mathcal{O}$ is complete. The $\mathbf{C}^*$-action also extends to the normalization $\widetilde{\mathcal{O}}$ of $\mathcal{O}$, and there exists a local coordinate $t \in \widetilde{\mathcal{O}}$ such that the line $\mathbf{C}t$ is preserved (this is because the pro-algebraic group $\mathrm{Aut}(\widetilde{\mathcal{O}})$ is an extension of $\mathbf{C}^*$ by a pro-unipotent group, hence all subgroups of $\mathrm{Aut}(\widetilde{\mathcal{O}})$ isomorphic to $\mathbf{C}^*$ are conjugate). It follows that the graded subring $\mathcal{O}$ is associated to a semi-group $\Gamma \subset \mathbf{N}$, in other words $\mathcal{O}$ is the ring $\mathbf{C}[[\Gamma]]$ of formal series $\sum_{\gamma\in\Gamma} a_\gamma t^\gamma$.

The $\mathbf{C}^*$-actions on $\mathcal{O}$ and $\widetilde{\mathcal{O}}$ give rise to a $\mathbf{C}^*$-action on $\mathbf{G}_{\mathcal{O}}$. The fixed points of this action are the submodules of $\widetilde{\mathcal{O}}$ which are graded, that is of the form $\mathbf{C}[[\Delta]]$, where Δ is a subset of $\mathbf{N}$; the condition $\dim \widetilde{\mathcal{O}}/\mathbf{C}[[\Delta]] = \dim \widetilde{\mathcal{O}}/\mathcal{O}$ means $\mathrm{Card}(\mathbf{N} - \Delta) = \mathrm{Card}(\mathbf{N} - \Gamma)$, and the condition that $\mathbf{C}[[\Delta]]$ is a $\mathcal{O}$-module means $\Gamma + \Delta \subset \Delta$. The first condition already implies that there are only finitely many such fixed points. According to [BB], the number of these fixed points is equal to $e(\mathbf{G}_{\mathcal{O}})$. We conclude:

Proposition 4.2 .– *Let* $\Gamma \subset \mathbf{N}$ *be a semi-group with finite complement. The number* $\varepsilon(\mathbf{C}[[\Gamma]])$ *is equal to the number of subsets* $\Delta \subset \mathbf{N}$ *such that* $\Gamma + \Delta \subset \Delta$ *and* $\mathrm{Card}(\mathbf{N} - \Delta) = \mathrm{Card}(\mathbf{N} - \Gamma)$. ∎

I do not know whether there exists a closed formula computing this number, say in terms of a minimal set of generators of Γ. This turns out to be the case in the situation we were originally interested in, namely planar singularities. The semi-group Γ is then generated by two coprime integers p and q, which means that the local ring $\mathcal{O}$ is of the form $\mathbf{C}[[u,v]]/(u^p - v^q)$.

Proposition 4.3 .– *Let* p, q *be two coprime integers. Then*

$$\varepsilon(\mathbf{C}[[u,v]]/(u^p - v^q)) = \frac{1}{p+q}\binom{p+q}{p} \ .$$

Proof: The following proof has been shown to me by P. Colmez.

(4.3.1) We first observe that if a subset Δ satisfies $\Gamma+\Delta\subset\Delta$, all its translates $n+\Delta$ $(n\in\mathbf{Z})$ contained in $\mathbf{N}$ have the same property; moreover, among all these translates, there is exactly one with $\mathrm{Card}(\mathbf{N}-\Delta)=\mathrm{Card}(\mathbf{N}-\Gamma)$. Thus the number we want to compute is the cardinal of the set $\mathcal{D}$ of subsets $\Delta\subset\mathbf{N}$ such that $\Gamma+\Delta\subset\Delta$, modulo the identification of a subset and its translates.

(4.3.2) For such a subset Δ, let us introduce the generating function $\mathrm{F}_\Delta(\mathrm{T})=\sum_{\delta\in\Delta}\mathrm{T}^\delta\in\mathbf{Z}[[\mathrm{T}]]$. Since $p+\Delta\subset\Delta$, we can write, in a unique way, $\Delta=\bigcup_{i=1}^{p}(a(i)+p\mathbf{N})$; then $(1-\mathrm{T}^p)\,\mathrm{F}_\Delta(\mathrm{T})=\sum_{i=1}^{p}\mathrm{T}^{a(i)}$. Writing similarly $\Delta=\bigcup_{j=1}^{q}(b(j)+q\mathbf{N})$, we get $(1-\mathrm{T}^q)\,\mathrm{F}_\Delta(\mathrm{T})=\sum_{j=1}^{q}\mathrm{T}^{b(j)}$. Put $a(j)=b(j-p)+p$ for $p+1\leq j\leq p+q$; the equality $(1-\mathrm{T}^p)\sum_{j=p+1}^{p+q}\mathrm{T}^{a(j)-p}=(1-\mathrm{T}^q)\sum_{i=1}^{p}\mathrm{T}^{a(i)}$ reads

$$\sum_{i=1}^{p+q}\mathrm{T}^{a(i)}=\sum_{i=1}^{p}\mathrm{T}^{a(i)+q}+\sum_{j=p+1}^{p+q}\mathrm{T}^{a(j)-p}\quad . \tag{4.3 a}$$

Conversely, given a function $a:[1,p+q]\to\mathbf{N}$ satisfying (4.3 a), the set $\Delta=\bigcup_{i=1}^{p}(a(i)+p\mathbf{N})$ is equal to $\bigcup_{j=p+1}^{p+q}(a(j)-p+q\mathbf{N})$, and therefore satisfies $\Gamma+\Delta\subset\Delta$ (note that (4.3 a) implies that the classes (mod. p) of the $a(i)$'s, for $1\leq i\leq p$, are all distinct).

The equality (4.3 a) means that there exists a permutation $\sigma\in\mathfrak{S}_{p+q}$ such that $a(\sigma i)$ is equal to $a(i)+q$ if $i\leq p$ and to $a(i)-p$ if $i>p$. This implies that $a(\sigma^m(i))$ is of the form $a(i)+\alpha q-\beta p$, with $\alpha,\beta\in\mathbf{N}$ and $\alpha+\beta=m$; since p and q are coprime, it follows that σ is of order $p+q$, that is a circular permutation. It also follows that the numbers $a(i)$ are all distinct; hence the permutation σ is uniquely determined. Let τ be a permutation such that $\tau\sigma\tau^{-1}$ is the permutation $i\mapsto i+1$ (mod. $p+q$), and let $\mathrm{S}_\Delta=\tau([1,p])$. Replacing a by $a\circ\tau^{-1}$, our function a satisfies

$$a(i+1)=\begin{cases}a(i)+q & \text{if } i\in\mathrm{S}_\Delta,\\ a(i)-p & \text{if } i\notin\mathrm{S}_\Delta\ .\end{cases} \tag{4.3 b}$$

Since τ is determined up to right multiplication by a power of σ, the set $\mathrm{S}_\Delta\subset[1,p+q]$ is well determined up to a translation (mod. $p+q$). Note that replacing Δ by $n+\Delta$ amounts to add the constant value n to the function a, hence does not change S_Δ.

(4.3.3) Conversely, let us start from a subset $\mathrm{S}\subset[1,p+q]$ with p elements. We define inductively a function a_S on $[1,p+q]$ by the relations (4.3 b), giving to $a_\mathrm{S}(1)$ an arbitrary value, large enough so that a_S takes its values in

$\mathbf{N}$. By construction the function $a_{\rm S}$ satisfies (4.3 *b*), so by (4.3.2) the subset $\Delta_{\rm S}=\bigcup_{s\in {\rm S}}(a_{\rm S}(s)+p\mathbf{N})$ satisfies $\Gamma+\Delta_{\rm S}\subset\Delta_{\rm S}$.

An easy computation gives $a_{{\rm S}+1}(i+1)=a_{\rm S}(i)$ and therefore $\Delta_{{\rm S}+1}=\Delta_{\rm S}$. Let $\mathcal{S}$ be the set of subsets of $[1,p+q]$ with p elements, modulo translation; the maps $\Delta\mapsto {\rm S}_\Delta$ from $\mathcal{D}$ to $\mathcal{S}$ and ${\rm S}\mapsto\Delta_{\rm S}$ from $\mathcal{S}$ to $\mathcal{D}$ are inverse of each other. Since $\mathrm{Card}(\mathcal{S})=\dfrac{1}{p+q}\dbinom{p+q}{p}$, the Proposition follows. ∎

(4.4) *Simple singularities*

We now consider the case where the singularities of C are *simple*, that is of A, D, E type. The local ring of such a singularity has only finitely many isomorphism classes of torsion free rank 1 modules, and this property characterizes these singularities among all plane curves singularities [G-K].

Proposition 4.5.– *Let $\mathcal{O}$ be the local ring of a simple singularity. Then $\varepsilon(\mathcal{O})$ is the number of isomorphism classes of torsion free rank 1 $\mathcal{O}$-modules. It is given by:*

- $\varepsilon(\mathcal{O})=l+1$ *if* $\mathcal{O}$ *is of type* A_{2l};
- $\varepsilon(\mathcal{O})=1$ *if* $\mathcal{O}$ *is of type* A_{2l+1};
- $\varepsilon(\mathcal{O})=1$ *if* $\mathcal{O}$ *is of type* D_{2l} $(l\geq 2)$;
- $\varepsilon(\mathcal{O})=l$ *if* $\mathcal{O}$ *is of type* D_{2l+1} $(l\geq 2)$;
- $\varepsilon(\mathcal{O})=5$ *if* $\mathcal{O}$ *is of type* E_6;
- $\varepsilon(\mathcal{O})=2$ *if* $\mathcal{O}$ *is of type* E_7;
- $\varepsilon(\mathcal{O})=7$ *if* $\mathcal{O}$ *is of type* E_8.

Proof: Let C be a rational curve having only one simple singularity with local ring $\mathcal{O}$; the action of JC on $\bar{\mathrm{J}}\mathrm{C}$ has finitely many orbits, corresponding to the different isomorphism classes of rank 1 $\mathcal{O}$-modules. Since each orbit is an affine space, its Euler characteristic is 1, hence by (1.1) $\varepsilon(\mathcal{O})=e(\bar{\mathrm{J}}\mathrm{C})$ is equal to the number of these orbits.

If $\mathcal{O}$ is unibranch, its completion is of the form $\mathbf{C}[[u,v]]/(u^p-v^q)$, with $p=2$, $q=2l+1$ for the type A_{2l}, $p=3$, $q=4$ for the type E_6 and $p=3$, $q=5$ for the type E_8; in these cases the result follows from 4.3. We have already observed that $\varepsilon=1$ for a A_{2l+1} singularity (Remark 3.5). A D_l singularity is the union of a A_{l-3} branch and a transversal smooth branch, hence the result by 3.3. Finally an E_7 singularity is the union of an ordinary cusp and its tangent, hence has $\varepsilon=2$. ∎

Remark 4.6 .– Let $\mathcal{D}$ be the set of graded sub-$\mathcal{O}$-modules $\mathrm{L}\subset\widetilde{\mathcal{O}}$ with $\dim\widetilde{\mathcal{O}}/\mathrm{L}=\dim\widetilde{\mathcal{O}}/\mathcal{O}$. Two modules L, M in $\mathcal{D}$ are isomorphic if and only if $\mathrm{M}=t^n\mathrm{L}$ for some $n\in\mathbf{Z}$, but the dimension condition forces $n=1$. It follows that *each torsion free rank 1 $\mathcal{O}$-module is isomorphic to exactly one element of $\mathcal{D}$*. It is quite easy

that way to write down the list of isomorphism classes of rank 1 $\mathcal{O}$-modules (which is of course well-known, see e.g. [G-K]). For instance if $\mathcal{O}$ is of type E_8, we get the following modules (with the notation of 4.1):
$\mathcal{O}$, $\mathcal{O}t + \mathcal{O}t^8$, $\mathcal{O}t^2 + \mathcal{O}t^6$, $\mathcal{O}t^2 + \mathcal{O}t^4$, $\mathcal{O}t^3 + \mathcal{O}t^4$, $\mathcal{O}t^3 + \mathcal{O}t^5 + \mathcal{O}t^7$, $\widetilde{\mathcal{O}}t^4$.

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