COUNTING ROOTED SPANNING FORESTS FOR CIRCULANT FOLIATION OVER A GRAPH

L. A. Grunwald, Y. S. Kwon and I. A. Mednykh

Abstract

In this paper, we present a new method to produce explicit formulas for the number of rooted spanning forests $f(n)$ for the infinite family of graphs $H_n = H_n(G_1, G_2, \ldots, G_m)$ obtained as a circulant foliation over a graph H on m vertices with fibers $G_1, G_2, \ldots, G_m$. Each fiber $G_i = C_n(s_{i,1}, s_{i,2}, \ldots, s_{i,k_i})$ of this foliation is the circulant graph on n vertices with jumps $s_{i,1}, s_{i,2}, \ldots, s_{i,k_i}$. This family includes the family of generalized Petersen graphs, I-graphs, sandwiches of circulant graphs, discrete torus graphs and others.

The formulas are expressed through Chebyshev polynomials. We prove that the number of rooted spanning forests can be represented in the form $f(n) = p\, f(H)a(n)^2$, where $a(n)$ is an integer sequence and p is a prescribed natural number depending on the number of odd elements in the set of $s_{i,j}$. Finally, we find an asymptotic formula for $f(n)$ through the Mahler measure of the associated Laurent polynomial.

Introduction

Let G be a finite connected graph. One of the most characteristic of the graph is its complexity. This notion can be defined in a few different ways. It can be just the number of edges or vertices, the number of spanning trees or rooted spanning forests,

2010 *Mathematics Subject Classification*. Primary 05C30; Secondary 39A10.

Key words and phrases. Spanning forest, rooted forest, circulant graph, I-graph, Y-graph, H-graph, Laplacian matrix, Chebyshev polynomial.

The first and the third authors were supported by the Mathematical Center in Akademgorodok, agreement no. 075-15-2019-1613 with the Ministry of Science and Higher Education of the Russian Federation. The second author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2018R1D1A1B05048450).

the Kirchhoff index defined as the sum of resistance distances between vertices or some other important geometrical characteristic. All the above mentioned characteristics can be expressed in terms of Laplacian eigenvalues of a graph or more precisely, through coefficients of its Laplacian characteristic polynomials. The structure of these coefficients is given by the Chelnokov-Kelmans theorem [14] which states that each coefficient, up to sign, is the number of rooted spanning forests in the graph with prescribed number of the trees. The complexity of a graph plays an important role in statistic physics, where the graphs with arbitrary large number of vertices are considered ([28], [13]). In this case, the structure of the Laplacian characteristic polynomial becomes quite complicated and the most interesting invariants are given by its asymptotic.

Recently, it was discovered in paper [22], that the number of spanning forests in circulant graph coincides with the order of homology group for some manifold arising as a cyclic coverings of the three dimensional sphere branched over a knot or link.

A *tree* is a connected undirected graph without cycles. A *spanning tree* in a graph G is a subgraph that is a tree containing all the vertices of G. A *rooted tree* is a tree with one marked vertex called root. A *rooted forest* is a graph whose connected components are rooted trees. A *rooted spanning forest* F in a graph G is a subgraph that is a rooted forest containing all the vertices of G. According to [14] (see also papers [5], [15]) this value can be found as $\det(I+L)$, where L is the Laplacian matrix of the graph. This leads to the question: how to find the product of eigenvalues of the matrix $I+L$? If the size (number of vertices) of a graph is small, it is easy to do. However, the most interesting cases deal with the family of graphs with increasing number of vertices. The direct calculation of this product became quite complicated problem, when the number of vertices n of the graph is going to the infinity. To solve this problem, we use the technique developed in our previous papers [16] and [23].

As a result, we find a closed formula which is the product of a bounded number of factors, each given by the n-th Chebyshev polynomial of the first kind evaluated at the roots of some polynomial of prescribed degree. This gives a possibility to investigate arithmetical properties of the number of rooted spanning forests and find its asymptotic. Up to our knowledge, the obtained results are new and never appeared before in the literature.

There are enormous number of papers devoted to counting of the number of spanning trees. See survey papers [3] and [31]. At the same time it is known very little about analytic formulas for the number of spanning forests. One of the first results was obtained by O. Knill [15] who proved that the number of rooted spanning forests in the complete graph K_n on n vertices is equal to $(n+1)^{n-1}$. Explicit formulas for the number of rooted spanning forests in cycle, star, line graph and some other graphs were given in ([26], [15]).

The number of rooted forests in circulant graphs has been calculated in [11]. As for the number of unrooted forests, it has much more complicated structure [4], [27].

The aim of the present paper is to investigate analytical, arithmetical and asymptotic properties of function counting the number of rooted spanning forests in circulant foliation over a given graph. This notion was introduced in paper [17]. We note that this family of such graphs is quite rich. It includes circulant graphs, generalized Petersen graphs, I-, Y-, H-graphs, discrete tori and others.

The structure of the paper is as follows. Some preliminary results and basic definitions are given in Section 1. In Section 2 we define the notion of circulant foliation over a graph. In Section 3, we present explicit formulas for the number $f(n)$ of rooted spanning forests in graphs $H_n = H_n(G_1, G_2, \ldots, G_m)$ obtained as a circulant foliation over a graph H on m vertices with fibers $G_1, G_2, \ldots, G_m$. Each fiber $G_i = C_n(s_{i,1}, s_{i,2}, \ldots, s_{i,k_i})$ of this foliation is the circulant graph on n vertices with jumps $s_{i,1}, s_{i,2}, \ldots, s_{i,k_i}$. The formulas will be given in terms of Chebyshev polynomials. In Section 4, we provide some arithmetical properties of function $f(n)$ for the family H_n. More precisely, we show that the number of rooted spanning forests in the graph H_n can be represented in the form $f(n) = p\, f(H)\, a(n)^2$, where $a(n)$ is an integer sequence and p is a prescribed natural number depending on jumps and the parity of n. In Section 5, we use explicit formulas for the complexity in order to produce its asymptotic. In the last section, we illustrate the obtained results by a series of examples.

Acknowlegments. The authors are grateful to anonymous referees for thorough review and helpful suggestions.

1 Basic definitions and preliminary facts

Consider a connected finite graph G, allowed to have multiple edges but without loops. We denote the vertex and edge set of G by $V(G)$ and $E(G)$, respectively. Given $u, v \in V(G)$, we set a_{uv} to be the number of edges between vertices u and v. The matrix $A = A(G) = \{a_{uv}\}_{u,v\in V(G)}$ is called *the adjacency matrix* of the graph G. The degree d_v of a vertex $v \in V(G)$ is defined by $d_v = \sum_{u\in V(G)} a_{uv}$. Let $D = D(G)$ be the diagonal matrix indexed by the elements of $V(G)$ with $d_{vv} = d_v$. The matrix $L = L(G) = D(G) - A(G)$ is called *the Laplacian matrix*, or simply *Laplacian*, of the graph G.

In this paper we prefer to deal with more general definition of Laplacian. Let $X = \{x_v,\, v \in V(G)\}$ be the set of variables and let $X(G)$ be the diagonal matrix indexed by the elements of $V(G)$ with diagonal elements x_v. The *generalized Laplacian matrix* of G, denoted by $L(G, X)$, is given by $L(G, X) = X(G) - A(G)$. In the particular case $x_v = d_v$, we have $L(G, X) = L(G)$.

We call an $n \times n$ matrix *circulant,* and denote it by $circ(a_0, a_1, \ldots, a_{n-1})$ if it is of the form

$$circ(a_0, a_1, \ldots, a_{n-1}) = \begin{pmatrix} a_0 & a_1 & a_2 & \ldots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \ldots & a_{n-2} \\ & \vdots & & & \vdots \\ a_1 & a_2 & a_3 & \ldots & a_0 \end{pmatrix}.$$

Recall [6] that the eigenvalues of matrix $\mathtt{C} = circ(a_0, a_1, \ldots, a_{n-1})$ are given by the following simple formulas $\lambda_j = p(\varepsilon_n^j)$, $j = 0, 1, \ldots, n-1$ where $p(x) = a_0 + a_1 x + \cdots + a_{n-1}x^{n-1}$ and ε_n is an order n primitive root of the unity. Moreover, the circulant matrix $\mathtt{C} = p(\mathtt{T}_n)$, where $\mathtt{T}_n = circ(0, 1, 0, \ldots, 0)$ is the matrix representation of the shift operator $\mathtt{T}_n : (x_0, x_1, \ldots, x_{n-2}, x_{n-1}) \to (x_1, x_2, \ldots, x_{n-1}, x_0)$. We note that all $n \times n$ circulant matrices share the same set of eigenvectors. Hence, any set of $n \times n$ circulant matrices can be simultaneously diagonalizable.

Let $s_1, s_2, \ldots, s_k$ be integers such that $1 \le s_1 < s_2 < \cdots < s_k < \frac{n}{2}$. The graph $C_n(s_1, s_2, \ldots, s_k)$ with n vertices $0, 1, 2, \ldots, n-1$ is called *circulant graph* if the vertex $i, 0 \le i \le n-1$ is adjacent to the vertices $i \pm s_1, i \pm s_2, \ldots, i \pm s_k \pmod{n}$. All vertices of the graph are of even degree $2k$. In this paper, we also allow the empty circulant graph $C_n(\emptyset)$ consisting of n isolated vertices.

2 Circulant foliation over a graph

Let H be a finite graph with vertices $v_1, v_2, \ldots, v_m$, allowed to have multiple edges but without loops. Denote by $a_{i\,j}$ the number of edges between vertices v_i and v_j. Since H has no loops, we have $a_{i\,i} = 0$. To define the circulant foliation $H_n = H_n(G_1, G_2, \ldots, G_m)$ we prescribe to each vertex v_i a circulant graph $G_i = C_n(s_{i,1}, s_{i,2}, \ldots, s_{i,k_i})$. In the case $G_i = C_n(\emptyset)$ we set $k_i = 1$ and $s_{i,1} = 0$. The *circulant foliation* $H_n = H_n(G_1, G_2, \ldots, G_m)$ over H with fibers $G_1, G_2, \ldots, G_m$ is a graph with the vertex set $V(H_n) = \{(k, v_i) \mid k = 1, 2, \ldots n, \ i = 1, 2, \ldots, m\}$, where for a fixed k the vertices (k, v_i) and (k, v_j) are connected by $a_{i\,j}$ edges, while for a fixed i, the vertices (k, v_i), $k = 1, 2, \ldots n$ form a graph $C_n(s_{i,1}, s_{i,2}, \ldots, s_{i,k_i})$ in which the vertex (k, v_i) is adjacent to the vertices $(k \pm s_{i,1}, v_i), (k \pm s_{i,2}, v_i), \ldots, (k \pm s_{i,k_i}, v_i) \pmod{n}$. Along the text we will refer to H as a *base graph* for the circulant foliation H_n.

There is a projection $\varphi : H_n \to H$ sending the vertices (k, v_i), $k = 1, \ldots, n$ and edges between them to the vertex v_i and for given k each edge between the vertices (k, v_i) and (k, v_j), $i \ne j$ bijectively to an edge between v_i and v_j. For each vertex v_i of graph H we have $\varphi^{-1}(v_i) = G_i$, $i = 1, 2, \ldots, m$.

Consider an action of the cyclic group $\mathbb{Z}_n$ on the graph H_n defined by the rule $(k, v_i) \to$

$(k+1,\, v_i)$, $k \mod n$. Then the group $\mathbb{Z}_n$ acts free on the set of vertices and the set of edges and the factor graph $H_n/\mathbb{Z}_n$ is an *equipped graph* $\widehat{H}$ obtained from the graph H by attaching k_i loops to each i-th vertex of H.

By making use of the voltage technique [10], one can construct the graph H_n in the following way. We put an orientation to all edges of $\widehat{H}$ including loops. Then we prescribe the voltage 0 to all edges of subgraph H of $\widehat{H}$ and the voltage $s_{i,j} \mod n$ to the j-th loop attached to i-th vertex of H. The respective voltage covering is the graph H_n.

Recall that the adjacency matrix of the non-empty circulant graph $C_n(s_1, s_2, \ldots, s_k)$ on the vertices $1, 2, \ldots, n$ has the form $\sum\limits_{p=1}^{k}(\mathtt{T}_n^{s_p} + \mathtt{T}_n^{-s_p})$. Empty circulant graph has zero matrix as its adjacency matrix. Suppose that the adjacency matrix of the graph H is

$$A(H) = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \ldots & a_{1,m} \\ a_{2,1} & 0 & a_{2,3} & \ldots & a_{2,m} \\ & \vdots & & & \vdots \\ a_{m,1} & a_{m,2} & a_{m,3} & \ldots & 0 \end{pmatrix}.$$

Then, the adjacency matrix of the circulant foliation $H_n = H_n(G_1,\, G_2, \ldots, G_m)$ over a graph H with fibers $G_i = C_n(s_{i,1},\, s_{i,2}, \ldots, s_{i,k_i})$, $i = 1, 2, \ldots, n$ is given by

$$A(H_n) = \begin{pmatrix} \sum\limits_{p=1}^{k_1}(\mathtt{T}_n^{s_{1,p}} + \mathtt{T}_n^{-s_{1,p}}) & a_{1,2}I_n & a_{1,3}I_n & \ldots & a_{1,m}I_n \\ a_{2,1}I_n & \sum\limits_{p=1}^{k_2}(\mathtt{T}_n^{s_{2,p}} + \mathtt{T}_n^{-s_{2,p}}) & a_{2,3}I_n & \ldots & a_{2,m}I_n \\ & \vdots & & & \vdots \\ a_{m,1}I_n & a_{m,2}I_n & a_{m,3}I_n & \ldots & \sum\limits_{p=1}^{k_m}(\mathtt{T}_n^{s_{m,p}} + \mathtt{T}_n^{-s_{m,p}}) \end{pmatrix}.$$

As the first example, we consider the *sandwich graph* $SW_n = H_n(G_1,\, G_2, \ldots, G_m)$. The graph SW_n has the path graph on m vertices $v_1, v_2, \ldots, v_m$ as its base graph H, with prescribed circulant graphs G_i, $i = 1, 2, \ldots, m$. A very particular case of this construction, known as I-graph $I(n, k, l)$, occurs by taking $m = 2$, $G_1 = C_n(k)$ and $G_2 = C_n(l)$. Also, the generalized Petersen graph [25] arises as $GP(n, k) = I(n, k, 1)$. The sandwich of two circulant graphs $H_n(G_1, G_2)$ was investigated in [1].

As the second example, we consider the *generalized Y-graph* $Y_n = Y_n(G_1, G_2, G_3)$, where G_1, G_2, G_3 are given circulant graphs on n vertices. To construct Y_n, we consider a Y-shape graph H consisting of four vertices v_1, v_2, v_3, v_4 and three edges v_1v_4, v_2v_4, v_3v_4. Let $G_4 = C_n(\emptyset)$ be the empty graph of n on vertices. Then, by definition, we put $Y_n = H_n(G_1, G_2, G_3, G_4)$. In a particular case, $G_1 = C_n(k), G_2 = C_n(l)$, and $G_3 = C_n(m)$, the graph Y_n coincides with the Y-graph $Y(n; k, l, m)$ defined earlier in [2, 12].

The third example is the *generalized H-graph* $H_n(G_1, G_2, G_3, G_4, G_5, G_6)$, where G_1, G_2, G_3, G_4 are given circulant graphs and $G_5 = G_6 = C_n(\emptyset)$ are the empty graphs on n vertices. In this case, we take H to be the graph with vertices $v_1, v_2, v_3, v_4, v_5, v_6$ and edges $v_1v_5, v_5v_3, v_2v_6, v_6v_4, v_5v_6$. In the case $G_1 = C_n(i), G_2 = C_n(j), G_3 = C_n(k), G_4 = C_n(l)$, we get the graph $H(n; i, j, k, l)$ investigated in the paper [12]. Shortly, we will write $H_n(G_1, G_2, G_3, G_4)$ ignoring the last two empty graph entries.

3 Counting the number of spanning rooted forests in the graph H_n

Let H be a finite connected graph with the vertex set $V(H) = \{v_1, v_2, \ldots, v_m\}$. Consider the circulant foliation $H_n = H_n(G_1, G_2, \ldots, G_m)$, where $G_i = C_n(s_{i,1}, s_{i,2}, \ldots, s_{i,k_i})$, $i = 1, 2, \ldots, m$. Let $L(H, X)$ be the generalized Laplacian of graph H with the set of variables $X = (x_1, x_2, \ldots, x_m)$. We specify X by setting $x_i = 2k_i + d_i + 1 - \sum\limits_{j=1}^{k_i}(z^{s_{i,j}} + z^{-s_{i,j}})$, where d_i is the degree of v_i in H. By definition of circulant graphs, if $G_i \neq C_n(\emptyset)$ we have $1 \leq s_{i,1} < s_{i,2} < \cdots < s_{i,k_i}$. In this case, x_i is a Laurent polynomial with leading term $-z^{s_{i,k_i}}$. For $G_i = C_n(\emptyset)$ we have $x_i = d_i + 1$. We set $P(z) = \det(L(H, X))$. We note that $P(z)$ is an integer Laurent polynomial.

For further considerations, we need to investigate the structure of leading term of the polynomial $P(z)$. This term comes from the product of polynomials $x_i = 2k_i + d_i + 1 - \sum\limits_{j=1}^{k_i}(z^{s_{i,j}} + z^{-s_{i,j}})$ with $s_{i,k_i} > 0$ and determinant of the matrix, obtained from $L(H, X)$ removing all rows and columns corresponding to the non-empty circulant fibers. One can use the following lemma to calculate the leading term explicitly.

LEMMA 3.1. *Let $V' = (v_1, v_2, \ldots, v_{m'})$ be the subset (possibly empty) of vertices of the graph H with trivial circulant fibers $C_n(\emptyset)$. Define H' as a vertex-induced subgraph of graph H formed by V'. Then leading term of the polynomial $P(z)$ is given by the following formula $(-1)^{m-m'}\eta\, z^s$, where $\eta = \det(L(H', X'))$, $s = \sum\limits_{j=1}^{m} s_{j,k_j}$, $X' = (d_1 + 1, d_2 + 1, \ldots, d_{m'} + 1)$, and d_j is a valency of vertex v_j in graph H.*

PROOF. By definition, $P(z) = \det(L(H, X))$, where $X = (x_1, x_2, \ldots, x_m)$, $x_j = 2k_i + d_i + 1 - \sum\limits_{j=1}^{k_i}(z^{s_{i,j}} + z^{-s_{i,j}})$. For $j = 1, 2, \ldots, m'$ we have $x_j = d_j + 1$ as $k_j = 1, s_{j,1} = 0$. If $j = m' + 1, \ldots, m$, the leading terms of the polynomial x_j is the $-z^{s_{j,k_j}}$. Since all other entries of matrix $L(H, W)$ are integer constants, by making use of basic properties of determinants, we get the statement of lemma. We note that $L(H', X')$ is a symmetric and strictly diagonally dominant matrix. By [8] we always have $\eta > 0$. In the case of $m' = 0$, we set $\eta = 1$. $\square$

Now, we consider one more specification $L(H, W)$ for generalized Laplacian of H with the set $W = (w_1, w_2, \ldots, w_m)$, where $w_i = 2k_i + d_i + 1 - \sum\limits_{j=1}^{k_i} 2\,T_{s_{i,j}}(w)$ and $T_k(w) = \cos(k \arccos w)$ is the Chebyshev polynomial of the first kind. See [19] for the basic properties of the Chebyshev polynomials. We set $Q(w) = \det(L(H, W))$. Then $Q(w)$ is an integer polynomial of degree $s = s_{1,k_1} + s_{2,k_2} + \cdots + s_{m,k_m}$. Since $\frac{1}{2}(z^n + z^{-n}) = T_n(\frac{1}{2}(z + z^{-1}))$, we have $P(z) = Q(w)$, where $w = \frac{1}{2}(z + z^{-1})$. By definition of $Q(w)$ we have

$$Q(w) = \det \begin{pmatrix} w_1 & -a_{1,2} & -a_{1,3} & \ldots & -a_{1,m} \\ -a_{2,1} & w_2 & -a_{2,3} & \ldots & -a_{2,m} \\ & \vdots & & & \vdots \\ -a_{m,1} & -a_{m,2} & -a_{m,3} & \ldots & w_m \end{pmatrix}, \tag{1}$$

where $w_i = 2k_i + d_i + 1 - \sum\limits_{j=1}^{k_i} 2T_{s_{i,j}}(w)$, $i = 1, 2, \ldots, m$.

Remark 3.2. In notation of Lemma 3.1, the leading term of the polynomial $Q(w)$ is $(-1)^{m-m'} 2^s \eta\, w^s$.

The main result of this section is the following theorem.

Theorem 3.3. *The number of spanning rooted forests $f(n)$ in the graph $H_n(G_1, G_2, \ldots, G_m)$ is given by the formula*

$$f(n) = \eta^n \prod_{p=1}^{s} |2T_n(w_p) - 2|,$$

where $s = s_{1,k_1} + s_{2,k_2} + \cdots + s_{m,k_m}$, w_p, $p = 1, 2, \ldots, s$ are all the roots of the equation $Q(w) = 0$ and η is the same as in Lemma 3.1.

Proof. By ([14], [5]), the number of rooted spanning forests $f(n)$ of a graph $H_n(G_1, G_2, \ldots, G_m)$ is equal to the product of eigenvalues of $I + L$ where L is the Laplacian of a graph H_n. To investigate the spectrum of matrix $I + L$, we consider the shift operator $\mathtt{T}_n = circ(0, 1, \ldots, 0)$. Then the matrix $I + L$ is given by the formula

$$I + L = \begin{pmatrix} A_1(\mathtt{T}_n) & -a_{1,2}I_n & -a_{1,3}I_n & \ldots & -a_{1,m}I_n \\ -a_{2,1}I_n & A_2(\mathtt{T}_n) & -a_{2,3}I_n & \ldots & -a_{2,m}I_n \\ & \vdots & & & \vdots \\ -a_{m,1}I_n & -a_{m,2}I_n & -a_{m,3}I_n & \ldots & A_m(\mathtt{T}_n) \end{pmatrix},$$

where $A_i(z) = 2k_i + d_i + 1 - \sum\limits_{j=1}^{k_i}(z^{s_{i,j}} + z^{-s_{i,j}}),\ i = 1, \ldots, m.$

The eigenvalues of circulant matrix $\mathtt{T}_n$ are ε_n^j, $j = 0, 1, \ldots, n-1$, where $\varepsilon_n = e^{\frac{2\pi i}{n}}$. Since all of them are distinct, the matrix $\mathtt{T}_n$ is conjugate to the diagonal matrix $\mathbb{T}_n = diag(1, \varepsilon_n, \ldots, \varepsilon_n^{n-1})$ with diagonal entries $1, \varepsilon_n, \ldots, \varepsilon_n^{n-1}$. To find spectrum of $I + L$, without loss of generality, one can replace $\mathtt{T}_n$ with $\mathbb{T}_n$. Then all $n \times n$ blocks of $I + L$ are diagonal matrices. This essentially simplifies the problem of finding eigenvalues of the block matrix $I + L$. Indeed, let λ be an eigenvalue of $I + L$ and let $(u_1, u_2, \ldots, u_m)$ with $u_i = (u_{i,1}, u_{i,2} \ldots, u_{i,n})^t$, $i = 1, \ldots, m$ be the respective eigenvector. Then we have the following system of equations

$$(2)\qquad \begin{pmatrix} A_1(\mathbb{T}_n) - \lambda\, I_n & -a_{1,2}I_n & -a_{1,3}I_n & \ldots & -a_{1,m}I_n \\ -a_{2,1}I_n & A_2(\mathbb{T}_n) - \lambda\, I_n & -a_{2,3}I_n & \ldots & -a_{2,m}I_n \\ & \vdots & & & \vdots \\ -a_{m,1}I_n & -a_{m,2}I_n & -a_{m,3}I_n & \ldots & A_m(\mathbb{T}_n) - \lambda\, I_n \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = 0.$$

Recall that all blocks in the matrix under consideration are diagonal $n \times n$-matrices and the (j, j)-th entry of $\mathbb{T}_n$ is equal to ε_n^{j-1}.

Hence, the equation (2) splits into n equations

$$(3)\qquad \begin{pmatrix} A_1(\varepsilon_n^j) - \lambda & -a_{1,2} & -a_{1,3} & \ldots & -a_{1,m} \\ -a_{2,1} & A_2(\varepsilon_n^j) - \lambda & -a_{2,3} & \ldots & -a_{2,m} \\ & \vdots & & & \vdots \\ -a_{m,1} & -a_{m,2} & -a_{m,3} & \ldots & A_m(\varepsilon_n^j) - \lambda \end{pmatrix} \begin{pmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{m,j+1} \end{pmatrix} = 0,$$

$j = 0, 1, \ldots, n-1$. Each equation gives m eigenvalues of $I + L$, say $\lambda_{1,j}, \lambda_{2,j}, \ldots, \lambda_{m,j}$. To find these eigenvalues we set

$$(4)\qquad P(z, \lambda) = \det \begin{pmatrix} A_1(z) - \lambda & -a_{1,2} & -a_{1,3} & \ldots & -a_{1,m} \\ -a_{2,1} & A_2(z) - \lambda & -a_{2,3} & \ldots & -a_{2,m} \\ & \vdots & & & \vdots \\ -a_{m,1} & -a_{m,2} & -a_{m,3} & \ldots & A_m(z) - \lambda \end{pmatrix}.$$

Then $\lambda_{1,j}, \lambda_{1,j}, \ldots, \lambda_{m,j}$ are roots of the equation

$$(5)\qquad P(\varepsilon_n^j, \lambda) = 0.$$

In particular, by Vieta's theorem, the product $p_j = \lambda_{1,j}\lambda_{2,j} \cdots \lambda_{m,j}$ is given by the formula $p_j = P(\varepsilon_n^j, 0) = P(\varepsilon_n^j)$, where $P(z) = Q(\frac{1}{2}(z + z^{-1}))$ is defined at the beginning of this section.

Now, for any $j = 0, \ldots, n-1$, matrix $I + L$ has m eigenvalues $\lambda_{1,j}, \lambda_{2,j}, \ldots, \lambda_{m,j}$ satisfying the order m algebraic equation $P(\varepsilon_n^j, \lambda) = 0$. One can see that the polynomial $P(1, \lambda)$ is the characteristic polynomial for $I + L(H)$, where $L(H)$ is the Laplacian matrix of graph H and its roots are eigenvalues of H.

Now we have

$$f(n) = \prod_{j=0}^{n-1} \lambda_{1,j}\lambda_{2,j}\cdots\lambda_{m,j} = \prod_{j=0}^{n-1} p_j = \prod_{j=0}^{n-1} P(\varepsilon_n^j). \tag{6}$$

Note that the product of eigenvalues $p_0 = P(1) = \lambda_{1,0}\lambda_{2,0}\cdots\lambda_{m,0}$ is equal to $f(H)$, where $f(H)$ is the number of rooted spanning forests in graph H. Hence

$$f(n) = f(H) \prod_{j=1}^{n-1} P(\varepsilon_n^j). \tag{7}$$

To continue the proof we replace the Laurent polynomial $P(z)$ by $\widetilde{P}(z) = (-1)^{m-m'} \frac{z^s}{\eta} P(z)$, where m, m' and η are given by Lemma 3.1. Then $\widetilde{P}(z)$ is a monic polynomial of the degree $2s$ with the same roots as $P(z)$. We note that

$$\eta^n \prod_{j=0}^{n-1} \widetilde{P}(\varepsilon_n^j) = (-1)^{(m-m')n} \varepsilon_n^{\frac{(n-1)n}{2}s} \prod_{j=0}^{n-1} P(\varepsilon_n^j) = (-1)^{(m-m')n+s(n-1)} \prod_{j=0}^{n-1} P(\varepsilon_n^j). \tag{8}$$

All roots of polynomials $\widetilde{P}(z)$ and $Q(w)$ are $z_1, 1/z_1, z_2, 1/z_2, \ldots, z_s, 1/z_s$ and $w_j = \frac{1}{2}(z_j + z_j^{-1})$, $j = 1, \ldots, s$, respectively. Also, we can recognize the complex numbers ε_n^j, $j = 0, \ldots, n-1$ as the roots of polynomial $z^n - 1$. By the basic properties of resultant ([24], Ch. 1.3) we have

$$\begin{aligned} &\prod_{j=0}^{n-1} \widetilde{P}(\varepsilon_n^j) = \mathrm{Res}(\widetilde{P}(z), z^n - 1) = \mathrm{Res}(z^n - 1, \widetilde{P}(z)) \\ &= \prod_{z:\widetilde{P}(z)=0} (z^n - 1) = \prod_{j=1}^{s} (z_j^n - 1)(z_j^{-n} - 1) \\ &= \prod_{j=1}^{s} (2 - z_j^n - z_j^{-n}) = (-1)^s \prod_{j=1}^{s} (2T_n(w_j) - 2). \end{aligned} \tag{9}$$

Combine (6), (8) and (9) we have the following formula for the number of spanning trees

$$f(n) = (-1)^{(m-m')n+s\,n} \eta^n \prod_{j=1}^{s} (2T_n(w_j) - 2). \tag{10}$$

Since $f(n)$ is a positive number, by (10) we obtain

$$f(n)=\eta^n\prod_{j=1}^{s}|2T_n(w_j)-2|. \tag{11}$$

We finish the proof of the theorem. □

Claim 3.4. *The number of rooted spanning forests $f(n)$ is a multiple of $f(H)$.*

Proof. Indeed, by formula (7) we have to show that the number $R=\prod\limits_{j=1}^{n-1}P(\varepsilon_n^j)$ is an integer. This value can be found as a resultant of two polynomials with integer coefficients, namely $\frac{z^n-1}{z-1}$ and $\eta\,\widetilde{P}(z)$. So R is an integer. □

4 Arithmetical properties of $f(n)$ for the graph H_n

Let H be a finite connected graph on m vertices. Consider the circulant foliation $H_n=H_n(G_1,\,G_2,\ldots,G_m)$, where $G_i=C_n(s_{i,1},\,s_{i,2},\ldots,s_{i,k_i})$, $i=1,2,\ldots,m$. Recall that any positive integer s can be uniquely represented in the form $s=p\,r^2$, where p and r are positive integers and p is square-free. We will call p the *square-free part* of s.

Theorem 4.1. *Let $f(n)$ be the number of rooted spanning forests in the graph H_n. Denoted by p is the square free parts of $Q(-1)$. Then there exists an integer sequence $a(n)$ such that*

1° $f(n)=f(H)\,a(n)^2$, *if n is odd,*

2° $f(n)=p\,f(H)\,a(n)^2$, *if n is even.*

Proof. Recall that $f(H)=\lambda_{1,0}\lambda_{2,0}\cdots\lambda_{m,0}$, where $\lambda_{i,j}$ are eigenvalues of $I+L$. By formula (7) we have $f(n)=f(H)\prod_{j=1}^{n-1}\lambda_{1,j}\lambda_{2,j}\cdots\lambda_{m,j}$. Here, $\lambda_{i,j}$, $i=1,2,\ldots,m$, $j=1,2,\ldots,n-1$ are all the roots of the quotient of characteristic polynomials for $I_{mn}+L(H_n)$ and $I_m+L(H)$.

Note that $\lambda_{1,j}\lambda_{2,j}\ldots\lambda_{m,j}=P(\varepsilon_n^j)=P(\varepsilon_n^{n-j})=\lambda_{1,n-j}\lambda_{2,n-j}\cdots\lambda_{m,n-j}$. Define $c(n)=\prod\limits_{j=1}^{\frac{n-1}{2}}\lambda_{1,j}\lambda_{2,j}\cdots\lambda_{m,j}$, if n is odd and $d(n)=\prod\limits_{j=1}^{\frac{n}{2}-1}\lambda_{1,j}\lambda_{2,j}\cdots\lambda_{m,j}$, if n is even. Each $\lambda_{i,j}$ is an algebraic number. The product of an algebraic number λ and all its Galois conjugate is called a norm of λ. Up to a sign, it coincides with the constant term of minimal polynomial for λ. Therefore, the norm is always integer. By [18], each algebraic number $\lambda_{i,j}$ comes into the products $\prod_{j=1}^{(n-1)/2}\lambda_{1,j}\lambda_{2,j}\cdots\lambda_{m,j}$ and $\prod_{j=1}^{n/2-1}\lambda_{1,j}\lambda_{2,j}\cdots\lambda_{m,j}$ with all of its Galois conjugate elements. Hence, both products $c(n)$ and $d(n)$ are integers. Moreover, if n is even we get $\lambda_{1,\frac{n}{2}}\lambda_{2,\frac{n}{2}}\cdots\lambda_{m,\frac{n}{2}} = P(-1) = Q(-1)$. We note that $Q(-1)$ is always a positive integer. The precise formula for it is given in Remark 4.2.

Now, we have $f(n) = f(H)\,c(n)^2$ if n is odd, and $f(n) = f(H)\,Q(-1)\,d(n)^2$ if n is even. Let $Q(-1) = p\,r^2$, where p is a square free number. Then

1° $\frac{f(n)}{f(H)} = c(n)^2$ if n is odd,

2° $\frac{f(n)}{f(H)} = p(r\,d(n))^2$ if n is even.

By Claim after the proof of Theorem 3.3, the quotient $\frac{f(n)}{f(H)}$ is an integer. Since p is square free, the squared rational numbers in 1° and 2° are integer. Setting $a(n) = c(n)$ in the first case, and $a(n) = r\,d(n)$ in the second we finish the proof of the theorem. ☐

Remark 4.2. Denoted by t_i the number of odd elements in the sequence $s_{i,1},\ s_{i,2},\ldots,s_{i,k_i}$. Then $Q(-1) = \det L(H,W)$, where $W = (d_1{+}4t_1{+}1, d_2{+}4t_2{+}1,\ldots,d_m{+}4t_m{+}1)$. Indeed, $Q(w) = \det L(H,W)$, where $W = (w_1,w_2,\ldots,w_m)$ and $w_i = 2k_i + d_i + 1 - \sum\limits_{j=1}^{k_i} 2T_{s_{i,j}}(w)$. We have $T_{s_{i,j}}(-1) = \cos(s_{i,j}\arccos(-1)) = \cos(s_{i,j}\pi) = (-1)^{s_{i,j}}$ and $w_i = d_i + 1 + 4\sum\limits_{j=1}^{k_i}\frac{1-(-1)^{s_{i,j}}}{2} = d_i + 4t_i + 1$.

5 Asymptotic formulas for the number $f(n)$ of rooted spanning forests

In this section we obtain the following result.

Theorem 5.1. *The asymptotic behavior for the number of rooted spanning forests $f(n)$ in the graph H_n is given by the formula*

$$f(n) \sim A^n,\ n \to \infty,$$

where $A = \exp(\int\limits_0^1 \log|Q(\cos 2\pi t)|\,dt)$.

To prove the theorem we need the following lemma.

Lemma 5.2. *Let z be a root of the Laurent polynomial $P(z)$ then $|z| \neq 1$.*

Proof. Suppose that $P(z) = 0$ and $|z| = 1$. Then $z = e^{\mathrm{i}\,\varphi}$, $\varphi \in \mathbb{R}$. Since $P(z) = Q(w)$, where $w = \frac{1}{2}(z + z^{-1})$ by formula (1) we have

$$P(e^{\mathrm{i}\,\varphi}) = \det \begin{pmatrix} x_1 & -a_{1,2} & -a_{1,3} & \dots & -a_{1,m} \\ -a_{2,1} & x_2 & -a_{2,3} & \dots & -a_{2,m} \\ & \vdots & & & \vdots \\ -a_{m,1} & -a_{m,2} & -a_{m,3} & \dots & x_m \end{pmatrix},$$

where

$$x_i = 2k_i+d_i+1-\sum_{j=1}^{k_i}(z^{s_{i,j}}+z^{-s_{i,j}}) = 2k_i+d_i+1-\sum_{j=1}^{k_i}(e^{\mathrm{i}\,s_{i,j}\,\varphi}+e^{-\mathrm{i}\,s_{i,j}\,\varphi}) = d_i+1+\sum_{j=1}^{k_i}(2-2\cos(s_{i,j}\,\varphi))$$

and $d_i = \sum\limits_{j=1,\,j\neq i}^{m} a_{i,j}$. Hence $x_i \geq d_i+1 > d_i$, $i = 1,\dots,m$. So, the matrix above is a strictly diagonally dominant. By the Gershgorin circle theorem [8] it is non-singular. That is, $P(z) \neq 0$, which is a contradiction. □

Now we come to the proof of the Theorem 5.1.

Proof. By Theorem 3.3 we have $f(n) = \eta^n \prod\limits_{j=1}^{s} |2T_n(w_j) - 2|$, where w_j, $j = 1, 2, \dots, s$ are all the roots of the polynomial $Q(w)$. Since $P(z) = Q(\frac{1}{2}(z + z^{-1}))$ we have $w_j = \frac{1}{2}(z_j + z_j^{-1})$, where z_j and $1/z_j$, $j = 1,\dots,s$ are all the roots of the polynomial $P(z)$. By Lemma 5.2 we get $|z_j| \neq 1$, $j = 1, 2, \dots, s$.

We have $T_n(w_j) = \frac{1}{2}(z_j^n + z_j^{-n})$. Replacing z_j by $1/z_j$, if necessary, we always can assume that $|z_j| > 1$ for all $j = 1, 2, \dots, s$. Then $T_n(w_j) \sim \frac{1}{2}z_j^n$ and $|2T_n(w_j) - 2| \sim |z_j|^n$ as $n \to \infty$. Hence

$$\eta^n \prod_{j=1}^{s} |2T_n(w_j) - 2| \sim \eta^n \prod_{j=1}^{s} |z_j|^n = \eta^n \prod_{P(z)=0,\,|z|>1} |z|^n = A^n.$$

By Lemma 3.1, η is the absolute value of the leading coefficient of the polynomial $P(z)$. So $A = \eta \prod\limits_{P(z)=0,\,|z|>1} |z|$ is the Mahler measure of the polynomial $P(z)$. By ([7], p. 67), we have $A = \exp\left(\int\limits_0^1 \log|P(e^{2\pi it})|\mathrm{d}t\right)$. By $P(z) = Q(\frac{1}{2}(z + z^{-1}))$ we obtain $A = \exp(\int\limits_0^1 \log|Q(\cos 2\pi t)|\mathrm{d}t)$. The theorem is proved. □

6 Examples

6.1 Circulant graph $C_n(s_1, s_2, \dots, s_k)$

We consider the classical circulant graph $C_n(s_1, s_2, \dots, s_k)$ as a foliation $H_n(G_1)$ on the one vertex graph $H = \{v_1\}$ with the fiber $G_1 = C_n(s_1, s_2, \dots, s_k)$. In this case, $d_1 =$

0, $L(H,X) = (x_1)$, $P(z) = 2k+1-\sum\limits_{p=1}^{k}(z^{s_p}+z^{-s_p})$ and its Chebyshev transform is $Q(w) = 2k+1-\sum\limits_{p=1}^{k}2T_{s_p}(w)$. Different aspects of complexity for circulant graphs were investigated in the papers [29, 30, 9, 21, 20]. The number of rooted spanning forests for circulant graphs is investigated in [11].

6.2 I-graph $I(n,k,l)$ and the generalized Petersen graph $GP(n,k)$

Let H be a path graph on two vertices, $G_1 = C_n(k)$ and $G_2 = C_n(l)$. Then $I(n,k,l) = H_n(G_1,G_2)$ and $GP(n,k) = I(n,k,1)$. We get $P(z) = (4-z^k-z^{-k})(4-z^l-z^{-l})-1$ and $Q(w) = (4-2T_k(w))(4-2T_l(w))-1$. The arithmetical and asymptotical properties of complexity for I-graphs were studied in [23, 1].

6.3 Sandwich of m circulant graphs

Consider a path graph H on m vertices. Then $H_n(G_1,G_2,\dots,G_m)$ is a *sandwich graph* of circulant graphs $G_1,G_2,\dots,G_m$. Here $d_1 = d_m = 1$ and $d_i = 2$, $i = 2,\dots,m-1$. We set

$$D(x_1,x_2,\dots,x_m) = \det\begin{pmatrix} x_1 & -1 & 0 & \dots & 0 & 0 & 0 \\ -1 & x_2 & -1 & \dots & 0 & 0 & 0 \\ & \vdots & & & & \vdots & \\ 0 & 0 & 0 & \dots & -1 & x_{m-1} & -1 \\ 0 & 0 & 0 & \dots & 0 & -1 & x_m \end{pmatrix}.$$

By direct calculation we obtain

$$D(x_1,x_2,\dots,x_m) = x_1D(x_2,\dots,x_m)-D(x_3,\dots,x_m),\; D(x_1) = x_1,\; D(x_1,x_2) = x_1x_2-1.$$

Then $Q(w) = D(w_1,w_2,\dots,w_m)$ and $Q(-1) = D(d_1+4t_1+1,d_2+4t_2+1,\dots,d_m+4t_m+1)$, where w_i and t_i are the same as in Remark 3.2. The particular case $H_n(G_1,G_2)$ is studied in recent paper [1].

6.4 Generalized Y-graph

Consider the generalized Y-graph $Y_n(G_1,G_2,G_3)$ where $G_i = C_n(s_{i,1},\, s_{i,2},\dots,s_{i,k_i})$, $i = 1,2,3$. Here

$$Q(w) = 4A_1(w)A_2(w)A_3(w)-A_1(w)A_2(w)-A_1(w)A_3(w)-A_2(w)A_3(w),$$

where $A_i(w) = 2k_i+2-\sum\limits_{j=1}^{k_i}2T_{s_{i,j}}(w)$.

In the particular case $G_1 = G_2 = G_3 = C_n(1)$ we have Y-graph $Y(n;1,1,1)$. Then $Q(w) = 208 - 336w + 180w^2 - 32w^3$, $\eta = 4$ and the number of rooted spanning forests is given by the formula

$$f(n) = 4^n(2T_n(13/8) - 2)(2T_n(2) - 2)^2.$$

Also, by Theorem 4.1 there exists an integer sequence $a(n)$ such that $f(n) = f(1)a(n)^2$ if n is odd and $f(n) = 21f(1)a(n)^2$ if n is even. Here the multiple 21 is a square-free part of $Q(-1)$ and $f(1) = 20$ is the number of rooted spanning forests in the base Y-graph. By Theorem 5.1 we have the following asymptotic behavior $f(n) \underset{n\to\infty}{\sim} A^n$, where $A = \frac{1}{2}(7 + 4\sqrt{3})(13 + \sqrt{105})$.

6.5 Generalized H-graph

Consider the generalized H-graph $H_n(G_1, G_2, G_3, G_4)$, where $G_i = C_n(s_{i,1},\ s_{i,2}, \ldots, s_{i,k_i})$, $i = 1, 2, 3, 4$. Now we have

$$Q(w) = A_1(w)A_2(w)A_3(w)A_4(w)\left((4 - \frac{1}{A_1(w)} - \frac{1}{A_2(w)})(4 - \frac{1}{A_3(w)} - \frac{1}{A_4(w)}) - 1\right)$$

where $A_i(w)$ are the same as above.

In case $G_1 = G_2 = G_3 = G_4 = C_n(1)$ we have H-graph $H(n;1,1;1,1)$. Then $Q(w) = 16(-2+w)^2(-5+3w)(-9+5w)$ and $\eta = 15$. So, the number of rooted spanning forests in graph $H(n;1,1;1,1)$ is given by the formula

$$f(n) = 15^n(2T_n(5/3) - 2)(2T_n(9/5) - 2)(2T_n(2) - 2)^2.$$

Also, for some integer sequence $a(n)$, we have $f(n) = f(1)a(n)^2$ for odd n and $f(n) = 7f(1)a(n)^2$ for even n. Here 7 is square-free part of $Q(-1)$ and $f(1) = 128$ is the number of rooted spanning forests in the base H-graph. Asymptotic behavior is given by the formula $f(n) \underset{n\to\infty}{\sim} A^n$, where $A = 9(7 + 4\sqrt{3})(9 + 2\sqrt{14})$.

6.6 Discrete torus $T_{n,m} = C_n \times C_m$

We have $T_{n,m} = H_n(\underbrace{C_n(1), \ldots, C_n(1)}_{m\text{ times}})$, where $H = C_m(1)$ is a cycle with n vertices. So, the generalized Laplacian matrix with respect to the set of variables $X = (\underbrace{x, \ldots, x}_{m\text{ times}})$ has the form $L(H,X) = \begin{pmatrix} x & -1 & 0 & \ldots & 0 & -1 \\ -1 & x & -1 & \ldots & 0 & 0 \\ & \vdots & & & & \vdots \\ -1 & 0 & 0 & \ldots & -1 & x \end{pmatrix}$. Then $L(H,X)$ is an $m \times m$ circulant matrix with eigenvalues $\mu_j = x - e^{\frac{2\pi ij}{m}} - e^{-\frac{2\pi ij}{m}} = x - 2\cos(\frac{2\pi j}{m}), j = 1, \ldots, m$. Hence, $\det L(H, X) = \prod_{j=1}^{m} \mu_j = 2T_m(\frac{x}{2}) - 2$. Substituting $x = 5 - z - z^{-1}$ and $w = \frac{1}{2}(z + z^{-1})$, we get $Q(w) = 2T_m(\frac{5}{2} - w) - 2$. So the number of rooted spanning forests in discrete torus $T_{n,m}$ is

$$f(n) = \prod_{j=1}^{m} (2T_n(w_j) - 2),$$

where $w_j = \frac{5}{2} - \cos(\frac{2\pi j}{m})$, $j = 1, 2, \ldots, m$.

6.7 Cartesian product $H \times C_n(s_1, s_2, \ldots, s_k)$

Let H be an arbitrary d-regular graph on m vertices. The Cartesian product of a graph H and circulant graph $G = C_n(s_1, s_2, \ldots, s_k)$ is exactly $H_n(\underbrace{G, \ldots, G}_{m \text{ times}})$. Let $A(H)$ be an adjacency matrix of H. Consider generalized Laplacian of $H : L(H, W) = u\, I_m - A(H)$, where $W = (u, u, \ldots, u)$ and set $u = 2k + 1 + d - \sum_{j=1}^{k} 2T_{s_j}(w)$. Then $Q(w) = \det(L(H, W))$. One can conclude that $Q(w) = \chi_H(2k + 1 + d - \sum_{j=1}^{k} 2T_{s_j}(w))$. Here $\chi_H(x)$ is characteristic polynomial of graph H. In particular, $Q(-1) = \chi_H(d + 4t + 1)$, where t is the number of odd elements in the set of jumps $\{s_j,\ j = 1, \ldots, k\}$.

References

[1] N. V. Abrosimov, G. A. Baigonakova, L. A. Grunwald and I. A. Mednykh, Counting rooted spanning forests in cobordism of two circulant graphs, Sib. Electron. Mat. Rep. 17 (2020), 814–823.

[2] N. L. Biggs, Three remarkable graphs, Canad. J. Math. 25 (1973), 397–411.

[3] F. T. Boesch and H. Prodinger, Spanning tree formulas and Chebyshev polynomials, Graphs Combin. 2(1) (1986), 191–200.

[4] D. Calan, A combinatorial derivation of the number of labeled forests, J. Integer Seq. 6(4) (2003), Art. 03.4.7.

[5] P. Chebotarev and E. Shamis, Matrix forest theorems, Preprint, (2006), arXiv:math/0602575.

[6] P. J. Davis, Circulant Matrices, AMS Chelsea Publishing, New York, 1994.

[7] G. Everest and T. Ward, Heights of polynomials and entropy in algebraic dynamics, Springer Science & Business Media, 2013.

[8] S. Gerschgorin, Über die Abgrenzung der Eigenwerte einer Matrix, Izv. Akad. Nauk. USSR 7 (1931), 749–754.

[9] M. J. Golin, Yong Xuerong and Zhang Yuanping, The asymptotic number of spanning trees in circulant graphs, Discrete Math. 310 (2010), 792–803.

[10] J. L. Gross, T. W. Tucker, Topological graph theory, Wiley-Interscience, New York, 1987.

[11] L. A. Grunwald and I. A. Mednykh, The number of rooted forests in circulant graphs. Preprint, (2019), arXiv:1907.02635.

[12] J. D. Horton and I. Z. Bouwer, Symmetric Y-graphs and H-graphs, J. Combin. Theory, Ser. B 53 (1991), 114–129.

[13] J. L. Jacobsen, J. Salas and A. D. Sokal, Spanning forests and the q-State Potts model in the limit $q \to 0$, J. Stat. Phys. 119 (2005), 1153–1281.

[14] A. K. Kel′mans and V. M. Chelnokov, A certain polynomial of a graph and graphs with an extremal number of trees, J. Combin. Theory, Ser. B 16 (1974), 197–214.

[15] O. Knill, Counting rooted forests in a network, Preprint, (2013), arXiv:1307.3810.

[16] Y. S. Kwon, A. D. Mednykh and I. A. Mednykh, On Jacobian group and complexity of the generalized Petersen graph $GP(n, k)$ through Chebyshev polynomials, Linear Algebra Appl. 529 (2017), 355–373.

[17] Y. S. Kwon, A. D. Mednykh and I. A. Mednykh, Complexity of the circulant foliation over a graph, J. Algebr. Comb. 53 (2021), 115–129.

[18] D. Lorenzini, Smith normal form and Laplacians, J. Combin. Theory Ser. B 98(6) (2008), 1271–1300.

[19] J. C. Mason and D. C. Handscomb, Chebyshev Polynomials, CRC Press, Boca Raton, 2003.

[20] A. Mednykh and I. Mednykh, The number of spanning trees in circulant graphs, its arithmetic properties and asymptotic, Discrete Math. 342(6) (2019), 1772–1781.

[21] A. D. Mednykh and I. A. Mednykh, Asymptotics and Arithmetical Properties of Complexity for Circulant Graphs. Dokl. Math. 97(2) (2018), 147–151.

[22] A. D. Mednykh and I. A. Mednykh, Plans periodicity Theorem for Jacobian of circulant graphs. Dokl. Math. 103(3) (2021), 139–142.

[23] I. A. Mednykh, On Jacobian group and complexity of the I-graph $I(n, k, l)$ through Chebyshev polynomials, Ars Math. Contemp. 15 (2018), 467–485.

[24] V. V. Prasolov, Polynomials, Series: Algorithms and Computation in Mathematics, 11, Springer, Berlin Heidelberg, 2004.

[25] A. Steimle and W. Staton, The isomorphism classes of the generalized Petersen graphs, Discrete Math. 309(1) (2009), 231–237.

[26] L. A. Szekely, Counting rooted spanning forests in complete multipartite graphs, Ars Comb. 73 (2004), 97–100.

[27] L. Takacs, On the number of distinct forests, SIAM J. Disc. Math. 3(4) (1990), 574–581.

[28] F. Y. Wu, Number of spanning trees on a lattice, J. Phys. A: Math. Gen. 10 (1977), L113–L115.

[29] Chen Xiebin, Qiuying Lin and Fuji Zhang, The number of spanning trees in odd valent circulant graphs, Discrete Math. 282(1) (2004), 69–79.

[30] Zhang Yuanping, Xuerong Yong and M. J. Golin, Chebyshev polynomials and spanning tree formulas for circulant and related graphs, Discrete Math. 298(1) (2005), 334–364.

[31] Weigen Yan and Fuji Zhang, Enumeration of spanning trees of graphs with rotational symmetry, J. Comb. Theory, Ser. A 118 (2011), 1270–1290.

Sobolev Institute of Mathematics
Novosibirsk 630090
Novosibirsk State University
Novosibirsk 630090
Russia

E-mail address: lfb_o@yahoo.co.uk

Department of Mathematics
Yeungnam University
Gyeongsan, Gyeongbuk 38541
Korea

E-mail address: ysookwon@ynu.ac.kr

Sobolev Institute of Mathematics
Novosibirsk 630090
Novosibirsk State University
Novosibirsk 630090
Russia

E-mail address: ilyamednykh@mail.ru