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Counting solutions of the Bethe equations of the quantum group invariant open XXZ chain at roots of unity

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Abstract

We consider the $U_q sl(2)$-invariant open spin-1/2 XXZ quantum spin chain of finite length N. For the case that q is a root of unity, we propose a formula for the number of admissible solutions of the Bethe ansatz equations in terms of dimensions of irreducible representations of the Temperley-Lieb algebra; and a formula for the degeneracies of the transfer matrix eigenvalues in terms of dimensions of tilting $U_q sl(2)$-modules. These formulas include corrections that appear if two or more tilting modules are spectrum-degenerate. For the XX case ($q = e^{i\pi/2}$), we give explicit formulas for the number of admissible solutions and degeneracies. We also consider the cases of generic q and the isotropic ($q \to 1$) limit. Numerical solutions of the Bethe equations up to $N = 8$ are presented. Our results are consistent with the Bethe ansatz solution being complete.

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1 Introduction

The Hamiltonian of the $U_q sl(2)$-invariant open spin-1/2 XXZ quantum spin chain with length N is given by [1]

$$H = \sum_{k=1}^{N-1} \left[\sigma_k^x \sigma_{k+1}^x + \sigma_k^y \sigma_{k+1}^y + \frac{1}{2}(q+q^{-1}) \sigma_k^z \sigma_{k+1}^z \right] - \frac{1}{2}(q-q^{-1}) \Big(\sigma_1^z - \sigma_N^z \Big) \,, \tag{1.1}$$

where $\vec{\sigma}$ are the usual Pauli spin matrices, and $q = e^{\eta}$ is an arbitrary complex parameter. This model has been the subject of many investigations (see, for example [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13]).

This model is solvable by Bethe ansatz [1, 14, 15]: the energy eigenvalues are given by

$$E = 2 \sinh^2 \eta \sum_{k=1}^{M} \frac{1}{\sinh(\lambda_k - \frac{\eta}{2}) \sinh(\lambda_k + \frac{\eta}{2})} + (N-1) \cosh \eta \,, \tag{1.2}$$

where $\{\lambda_k\}$ are solutions of the Bethe equations

$$\sinh^{2N}\Big(\lambda_k + \frac{\eta}{2}\Big) \prod_{\substack{j \neq k \\ j=1}}^{M} \sinh(\lambda_k - \lambda_j - \eta) \sinh(\lambda_k + \lambda_j - \eta)$$

$$= \sinh^{2N}\Big(\lambda_k - \frac{\eta}{2}\Big) \prod_{\substack{j \neq k \\ j=1}}^{M} \sinh(\lambda_k - \lambda_j + \eta) \sinh(\lambda_k + \lambda_j + \eta) \,,$$

$$k = 1\,, 2\,, \ldots\,, M\,, \qquad M = 0\,, 1\,, \ldots\,, \left\lfloor \frac{N}{2} \right\rfloor \,, \tag{1.3}$$

where $\lfloor k \rfloor$ denotes the largest integer not greater than k. This exact solution owes its existence to the fact that the model is quantum integrable: there are many ($\sim N$) charges that commute with the Hamiltonian (1.1) and with each other, whose generating function is the so-called transfer matrix (A.1).

The main motivation for the present work is to address the problem of completeness, by which we mean here whether the Bethe equations have too many, too few, or just the right number of solutions to describe all the distinct eigenvalues of the transfer matrix. This question is particularly interesting when q is a root of unity, in which case the Hamiltonian is neither Hermitian nor normal, and in fact has Jordan cells [16, 17, 18]; and therefore the number of (ordinary) eigenvectors is less than 2^N – the total number of states.

For the case that q is a root of unity, we propose a formula for the number of admissible solutions of the Bethe equations in terms of dimensions [12, 13] of irreducible representations of the Temperley-Lieb algebra [19], see Eq. (4.10). We also propose a formula for the degeneracies of the transfer matrix eigenvalues in terms of dimensions of tilting $U_q sl(2)$-modules, see Eq. (4.12) These formulas include corrections that appear if two or more tilting modules are degenerate in eigenvalues of the transfer matrix. For the XX case ($q = e^{i\pi/2}$), we give explicit formulas for the number of admissible solutions and degeneracies, see Eqs. (4.15) and (4.16), respectively. These conjectures, which we have checked up to at least $N = 8$, are indeed consistent with the Bethe ansatz solution for this model being complete, or Eq. (4.14) is satisfied.

An important aspect of these conjectures is the definition of an admissible solution. As is the case for the periodic chain (see e.g. [20] and references therein), the Bethe equations (1.3) admit singular solutions (i.e., solutions that contain $\pm\eta/2$). However, such solutions do not correspond to eigenvalues and eigenvectors of the model (1.1), and therefore, are not admissible. In the language of [20], all singular solutions of the model (1.1) are “unphysical”; i.e., there are no “physical” singular solutions.

Moreover, when q is a root of unity, as is the case for the periodic XXZ chain [21, 22, 23, 24], the Bethe equations (1.3) admit continuous solutions (“algebraic variety of positive dimension”), in addition to the usual discrete solutions (“algebraic variety of dimension 0”). However, we restrict our attention to the latter, which are sufficient to obtain all the distinct eigenvalues of the transfer matrix. The former are important only for the construction of the eigenvectors and generalized eigenvectors, which we do not discuss here.

The outline of this paper is as follows. In section 2, we consider the isotropic (XXX) limit $q \to 1$. In section 3, we consider the case of generic values of q. Our main conjectures are in section 4, where we consider the root of unity case. We briefly discuss our results in section 5. Background material, special cases and numerical results are provided in the appendices. Specifically, the construction of the transfer matrix, its important properties, and the algebraic Bethe ansatz are reviewed in appendix A. The Temperley-Lieb algebra and its relation to the model (1.1) are briefly reviewed in appendix B. The case $p = 2$, which can be treated analytically, is analyzed in appendix C. Examples of cases where two or more tilting modules are degenerate are individually analyzed in appendix D. Finally, numerical solutions of the Bethe equations up to $N = 8$ are displayed in tables in appendix E.

2 XXX

In the limit $\eta \to 0$, the Hamiltonian (1.1) becomes $su(2)$-invariant

$$H = \sum_{k=1}^{N-1} \vec{\sigma}_k \cdot \vec{\sigma}_{k+1} \,, \tag{2.1}$$

the expression (1.2) for the eigenvalues becomes[1]

$$E = -2 \sum_{k=1}^{M} \frac{1}{\lambda_k^2 + \frac{1}{4}} + N - 1 \,, \tag{2.2}$$

[1]We rescale the Bethe roots $\lambda_j \mapsto -i\eta\lambda_j$ before taking $\eta \to 0$.

and the hyperbolic Bethe equations (1.3) become rational

$$\left(\lambda_k+\frac{i}{2}\right)^{2N}\prod_{\substack{j\neq k\\ j=1}}^{M}(\lambda_k-\lambda_j-i)(\lambda_k+\lambda_j-i)=\left(\lambda_k-\frac{i}{2}\right)^{2N}\prod_{\substack{j\neq k\\ j=1}}^{M}(\lambda_k-\lambda_j+i)(\lambda_k+\lambda_j+i)\,,$$

$$k=1\,,2\,,\dots\,,M\,,\qquad M=0\,,1\,,\dots\,,\left\lfloor\frac{N}{2}\right\rfloor\,. \tag{2.3}$$

The Bethe equations have the reflection symmetry $\lambda_k\mapsto-\lambda_k$, while keeping the other λ's (i.e. λ_j with $j\neq k$) unchanged. Moreover, any solution with $\lambda_k=0$ must be discarded, since the corresponding Bethe vector is not an eigenvector of the Hamiltonian (see e.g. [25] and Appendix A). Hence, we define a solution $\{\lambda_1\,,\dots\,,\lambda_M\}$ of the Bethe equations (2.3) to be *admissible* if all the λ_k's are finite and pairwise distinct (no two are equal), and if each λ_k satisfies either

$$\Re e(\lambda_k)>0 \tag{2.4}$$

or

$$\Re e(\lambda_k)=0\qquad\text{and}\qquad\Im m(\lambda_k)>0\,. \tag{2.5}$$

Note that, according to this definition, singular solutions $\{i/2\,,-i/2\,,\dots\}$ are not admissible. As usual, due to the permutation symmetry of the system of Bethe equations, the order of the λ's in any solution $\{\lambda_1\,,\dots\,,\lambda_M\}$ is irrelevant.

According to the Clebsch-Gordan theorem for $su(2)$, the Hilbert space of the XXX chain, the N-fold tensor product of spin-1/2 representations $V_{\frac{1}{2}}$, has the decomposition

$$\left(V_{\frac{1}{2}}\right)^{\otimes N}=\bigoplus_{j=0(1/2)}^{N/2}d_jV_j, \tag{2.6}$$

where the sum starts from $j=0$ for even N and $j=1/2$ for odd N. Moreover, V_j denotes a spin-j irreducible representation of $su(2)$, and the multiplicity d_j is given by

$$d_j=\binom{N}{\frac{N}{2}-j}-\binom{N}{\frac{N}{2}-j-1}\,,\qquad d_j=0\quad\text{for}\quad j>\frac{N}{2}\,. \tag{2.7}$$

Each admissible solution $\{\lambda_1\,,\dots\,,\lambda_M\}$ corresponds to a direct summand V_j in the decomposition (2.6), with spin $j=\frac{N}{2}-M$. Indeed, as in the case for the periodic XXX chain [26], the Bethe states are $su(2)$ highest-weight states and they can be constructed within the algebraic Bethe ansatz, see (A.15) and (A.25). Moreover, we expect that there is a one-to-one correspondence between distinct admissible solutions $\{\lambda_1\,,\dots\,,\lambda_M\}$ and distinct highest-weight vectors of spin $j=\frac{N}{2}-M$. Hence, for given values of N and M, we conjecture that the number $\mathcal{N}(N,M)$ of admissible solutions of the Bethe equations is given by

$$\mathcal{N}(N,M)=d_{\frac{N}{2}-M}=\binom{N}{M}-\binom{N}{M-1}\,. \tag{2.8}$$

M \ N	0	1	2	3
2	1	1		
3	1	2		
4	1	3	2	
5	1	4	5	
6	1	5	9	5
7	1	6	14	14

Table 1: The number $\mathcal{N}(N,M)$ of admissible solutions of the XXX Bethe equations (2.3) for given values of N and M.

For the periodic XXX chain, it is generally believed that the number of solutions of the corresponding Bethe equations is also given by (2.8), see e.g. [26, 27]. However, the situation there is actually more subtle due to the existence of physical singular solutions [28].

Since $\dim V_j = 2j+1$, it is also natural to conjecture that the number or degeneracy $\mathcal{D}(N,M)$ of eigenvalues of the transfer matrix, see its definition in (A.1) and (A.5) at $q=1$, corresponding to each admissible solution is given by

$$\mathcal{D}(N,M) = \dim V_{\frac{N}{2}-M} = N - 2M + 1\,. \tag{2.9}$$

The expressions (2.8) and (2.9) satisfy the well-known identity

$$\sum_{M=0}^{\lfloor \frac{N}{2} \rfloor} \mathcal{N}(N,M)\,\mathcal{D}(N,M) = 2^N\,, \tag{2.10}$$

signifying the completeness of the solution.

Using homotopy continuation [29] (see also [28] and references therein for further details), we have solved (2.3) numerically up to $N=7$. The admissible solutions up to $N=6$ are presented in Table 5. The numbers $\mathcal{N}(N,M)$ of admissible solutions that we have found are reported in Table 1. (For $M=0$, there are no Bethe roots but there is nevertheless an eigenvector (A.16), so we define $\mathcal{N}(N,0)=1$.) These numbers coincide with the conjectured values (2.8). As an independent check, starting from the transfer matrix (A.1), (A.5) at $q=1$ we have explicitly determined each of the transfer matrix eigenvalues $\Lambda(u)$ as polynomials in u [2]; then, by solving the T-Q equation (A.18) for $Q(u)$ and finally finding the zeros of $Q(u)$, we have obtained the corresponding Bethe roots. The results match with those obtained by directly solving the Bethe equations. The number of eigenvalues corresponding to each admissible solution also coincide with (2.9).

[2]Direct diagonalization of the (symbolic) transfer matrix $t(u)$ does not yield the eigenvalues as polynomials in u. We instead proceed by first finding the (numerical) eigenvectors $|v\rangle$ of the (numerical) matrix $t(u_0)$ for some generic numerical value u_0. Then, by acting with $t(u)$ (whose matrix elements are polynomials in u) on each $|v\rangle$, we read off the corresponding eigenvalue $\Lambda(u)$ as a polynomial in u. Note that, by virtue of the commutativity property (A.6), the eigenvalues do not depend on the choice of u_0.

3 XXZ: generic q

We now consider the Bethe equations (1.3) for generic values of q, i.e., when q is not a root of unity. These equations have the reflection symmetry $\lambda_k \mapsto -\lambda_k$ (while keeping the other λ's unchanged), and the periodicity $\lambda_k \mapsto \lambda_k + i\pi$. We again exclude $\lambda_k = 0$, as well as $\lambda_k = \frac{i\pi}{2}$. (See Appendix A.) Hence, we define a solution $\{\lambda_1 , \dots , \lambda_M\}$ of the Bethe equations (1.3) to be *admissible* if all the λ_k's are finite and pairwise distinct (no two are equal), and if each λ_k satisfies either

$$\Re e(\lambda_k) > 0 \qquad \text{and} \qquad -\frac{\pi}{2} < \Im m(\lambda_k) \le \frac{\pi}{2} \tag{3.1}$$

or

$$\Re e(\lambda_k) = 0 \qquad \text{and} \qquad 0 < \Im m(\lambda_k) < \frac{\pi}{2}\,. \tag{3.2}$$

The Hamiltonian (1.1) is invariant under the quantum group $U_q sl(2)$, which is the symmetry of the model. This symmetry is generated by the $S^{\pm}$ and S^z operators that now satisfy the quantum-group relations

$$[S^z, S^{\pm}] = \pm S^{\pm}, \qquad [S^+, S^-] = [2S^z]_q, \qquad [x]_q \equiv \frac{q^x - q^{-x}}{q - q^{-1}}\,, \tag{3.3}$$

which are just q-deformed versions of the usual relations of $su(2)$, or rather $Usl(2)$. The Bethe vectors (A.15) are $U_q sl(2)$ highest-weight states (A.25), (A.26). For generic values of q, the irreducible representations of $U_q sl(2)$ are isomorphic to those of $Usl(2)$. (See e.g. [30] and references therein.) The Hilbert space has the same decomposition as in the XXX case (2.6), except that V_j is now a spin-j irreducible representation of $U_q sl(2)$ with dimension $2j + 1$. We similarly expect that there is a one-to-one correspondence between distinct admissible solutions $\{\lambda_1 , \dots , \lambda_M\}$ and distinct direct summands isomorphic to V_j with $j = \frac{N}{2} - M$. Hence, we conjecture that the number $\mathcal{N}(N, M)$ of admissible solutions of the Bethe equations (1.3) for generic values of q is again given by (2.8); and that the number $\mathcal{D}(N, M)$ of eigenvalues of the transfer matrix corresponding to each admissible solution is again given by (2.9).

In order to check these conjectures, it is convenient to rewrite the Bethe equations (1.3) in polynomial form

$$(qx_k - 1)^{2N} \prod_{\substack{j\ne k\\ j=1}}^{M} (x_k - q^2 x_j)(x_k\, x_j - q^2) = (x_k - q)^{2N} \prod_{\substack{j\ne k\\ j=1}}^{M} (q^2 x_k - x_j)(q^2 x_k\, x_j - 1)\,,$$

$$k = 1\,, 2\,, \dots\,, M\,, \qquad M = 0\,, 1\,, \dots\,, \left\lfloor \frac{N}{2} \right\rfloor, \tag{3.4}$$

where

$$x_k = e^{2\lambda_k}\,. \tag{3.5}$$

For admissible solutions, each x_k satisfies either

$$|x_k| > 1 \tag{3.6}$$

or

$$|x_k| = 1 \qquad \text{and} \qquad 0 < \arg(x_k) < \pi\,. \tag{3.7}$$

We have solved this system numerically with $\eta = 0.1$ up to $N = 7$. The admissible solutions up to $N = 6$ are presented in Table 6. The numbers $\mathcal{N}(N, M)$ of admissible solutions that we have found are reported in Table 2. These results are the same as for the XXX case, and therefore coincide with the conjectured values (2.8). We have also confirmed that the degeneracy of the eigenvalues is again given by (2.9). We have obtained similar results for $\eta = i/2$, in which case $|q| = 1$ (and therefore the Hamiltonian is critical; although the Hamiltonian is not Hermitian or even normal, it is nevertheless diagonalizable) but q is not a root of unity.

$N \backslash M$	0	1	2	3
2	1	1		
3	1	2		
4	1	3	2	
5	1	4	5	
6	1	5	9	5
7	1	6	14	14

Table 2: The number $\mathcal{N}(N, M)$ of admissible solutions of the Bethe equations (1.3), (3.4) for given values of N and M and $\eta = 0.1$ (a generic value).

In view of the algebraic Bethe ansatz construction for the eigenstates (A.15), our conjectures say that distinct Bethe states correspond to distinct admissible solutions of the Bethe equations. Moreover, we also assume that, to each eigenvalue of the transfer matrix, there corresponds a unique admissible solution. Indeed, in the case of generic q, for a given eigenvalue $\Lambda(u)$, we expect that the T-Q equation (A.19) has a unique (up to rescaling) solution $Q(u)$, which implies a corresponding unique admissible solution $\{\lambda_k\}$. (We have checked this numerically for small values of N. Indeed, as in the XXX case, the Bethe roots obtained in this way match with those obtained by directly solving the Bethe equations.) If this is true, that would imply that the spectrum of the transfer matrix on the $U_q sl(2)$ highest-weight states is non-degenerate. (For the periodic XXX chain, it has been shown that the spectrum of the transfer matrix on $sl(2)$ highest-weight states is non-degenerate [31].)

4 XXZ: $q = e^{i\pi/p}$

We now consider the Bethe equations (1.3) when q is a primitive $2p^{th}$ root of unity: $q = e^{i\pi/p}$, where $p = 2, 3, 4, \ldots$ Inspection of (3.4) shows that, for such cases, the top degree terms can cancel, suggesting that the system is qualitatively different from the generic q case.

A particularly interesting new feature is that the Bethe equations now admit continuous solutions, in addition to the usual discrete solutions. For example, the following set of p elements

$$\{\lambda_0 + \frac{i\pi}{2p}(p-1)\,, \lambda_0 + \frac{i\pi}{2p}(p-3)\,, \ldots\,, \lambda_0 - \frac{i\pi}{2p}(p-3)\,, \lambda_0 - \frac{i\pi}{2p}(p-1)\} \tag{4.1}$$

is an exact solution of the Bethe equations (1.3) with $\eta = i\pi/p$ and $M = p$, for arbitrary values of λ_0. Such solutions have been discussed in the context of periodic chains [21, 22, 23, 24], and are called "exact complete p-strings." In the parlance of algebraic geometry, such solutions have positive dimension. In contrast, the usual discrete solutions instead have dimension 0. The solutions (4.1) are related to certain degeneracies of the model: the corresponding energy (as well as eigenvalue $\Lambda(u)$ obtained from the T-Q equation (A.19)) is the same as for the reference (pseudovacuum) state. Bethe states corresponding to such solutions are *prima facie* null; a regularization scheme and a suitable limiting procedure are needed to obtain non-null states (see [24] and references therein for the periodic case).

4.1 Admissible solutions

We restrict our attention here to the usual discrete solutions, which are sufficient to obtain all the distinct eigenvalues of the transfer matrix.[3] Indeed, the union $s_1 \cup s_2$ of a discrete solution s_1 and an exact complete p-string solution s_2 is again a solution; hence, adding a p-string does not change the eigenvalue corresponding to the initial discrete solution.

We therefore define an admissible solution of the Bethe equations as before in (3.1) and (3.2), except with the additional requirement that the solution should *not* contain the exact complete p-string (4.1).

4.2 Generalized eigenvalues and tilting modules

As already noted in the Introduction, non-trivial Jordan-block structure for H appears at roots of unity. Therefore, we now consider *generalized* eigenvalues of the transfer matrix (and of the Hamiltonian); i.e., eigenvalues $\Lambda(u)$ corresponding to generalized eigenvectors $|v\rangle$ that are defined as (also called root vectors)

$$\big(t(u) - \Lambda(u)\mathbf{1}\big)^2 |v\rangle = 0\,, \tag{4.2}$$

or equivalently

$$t(u)\,|v\rangle = \Lambda(u)\,|v\rangle + |v'\rangle \quad \text{and} \quad t(u)\,|v'\rangle = \Lambda(u)\,|v'\rangle\,. \tag{4.3}$$

[3] The transfer matrix and the Hamiltonian generally have the same number of distinct eigenvalues. However, there are exceptions, such as the case $p = 4$ and $N = 8$, where the number of distinct eigenvalues is 43 and 41 for the transfer matrix and Hamiltonian, respectively. In other words, for this case there are 43 admissible solutions: 39 solutions give (through Eq.(1.2)) 39 distinct energies, while 2 pairs of solutions give equal values of the energy, for a total of only 41 distinct energies. Another exception is the case $p = 2$ and $N = 9$, where the number of distinct eigenvalues is 81 and 57 for the transfer matrix and Hamiltonian, respectively. We expect that such "mismatches" occur for other values of p and N, but we have not made an effort to study them systematically.

The power in (4.2) is 2 because there are Jordan cells of maximum rank 2, and here $|v\rangle$ and $|v'\rangle$ belong to a Jordan cell of rank 2.[4] So, we have the number of eigenvectors less than 2^N but the number of generalized eigenvectors is exactly 2^N.

For $q = e^{i\pi/p}$, the N-fold tensor product of spin-1/2 representations decomposes into a direct sum of certain indecomposable modules T_j of $U_q sl(2)$ characterized by spin j. More precisely, these direct summands T_j are so-called *tilting* $U_q sl(2)$-modules which are (i) composed of the standard spin modules and (ii) satisfy a self-duality condition or invariance under the adjoint $\cdot^\dagger$ operation (see [32] for a short review in the context of open spin chains.) These two properties usually lead to a complicated structure of indecomposable but reducible modules, i.e., those having invariant subspaces but cannot be split onto a direct sum. The structure of the tilting $U_q sl(2)$-modules was studied in many works [1, 33, 30, 34, 12] and in brief it is the following: if $2j+1$ is bigger than p and not 0 modulo p then each T_j is composed of the spin-j (or V_j in our notations) and the spin-$(j - s(j))$ modules, where[5] $s(j) = (2j+1) \bmod p$, such that the former is a submodule; otherwise, T_j is irreducible. So, in particular we have the dimensions

$$\dim T_j = \begin{cases} 2j+1, & 2j+1 \le p \quad \text{or} \quad s(j) = 0, \\ 2(2j+1-s(j)), & \text{otherwise.} \end{cases} \tag{4.4}$$

Equipped with this information about T_j's we can write a decomposition of the XXZ spin-$\frac{1}{2}$ chain as

$$\left(V_{\frac{1}{2}}\right)^{\otimes N} = \bigoplus_{j=0(1/2)}^{N/2} d_j^0 T_j, \tag{4.5}$$

where the sum starts from $j = 0$ for even N and $j = 1/2$ for odd N. The important point is that the multiplicities d_j^0 of these T_j modules can be explicitly computed using representation theory [12] and are given by the dimensions d_j^0 of irreducible representations of the Temperley-Lieb (TL) algebra with the fugacity or loop parameter $\delta = 2\cos\frac{\pi}{p}$, see Appendix B for definitions:

$$d_j^0 = \sum_{n\ge 0} d_{j+np} - \sum_{n \ge t(j)+1} d_{j+np-1-2(j \bmod p)}\,, \qquad (j \bmod p) \ne p - \frac{1}{2}\,, \frac{p-1}{2}\,, \tag{4.6}$$

where d_j is given by (2.7), and

$$t(j) = \begin{cases} 1 & \text{for } (j \bmod p) > \frac{p-1}{2}\,, \\ 0 & \text{for } (j \bmod p) < \frac{p-1}{2}\,. \end{cases} \tag{4.7}$$

If $(j \bmod p) = p - \frac{1}{2}\,, \frac{p-1}{2}$, then $d_j^0 = d_j$. Note that one can check

$$\sum_{j=0(1/2)}^{N/2} d_j^0 \dim T_j = 2^N. \tag{4.8}$$

[4] The function `Eigenvalues[]` in Mathematica computes generalized eigenvalues.

[5] $(j \bmod p)$ is the remainder on division of j by p.

Since the transfer matrix commutes with the generators of $U_q sl(2)$, see (A.10), all the (generalized) eigenvectors (4.2) in a given (direct summand isomorphic to the) tilting module T_j have the same (generalized) eigenvalue of the transfer matrix. It is the indecomposable but reducible tilting modules that are responsible for the Jordan cells structure in the Hamiltonian and the presence of the generalized eigenvectors $|v\rangle$: they live in heads of the tilting modules while their partners $|v'\rangle$, see (4.3), live in the socle – the irreducible submodule of T_j.

4.3 Main conjectures

Assuming that there is at most one admissible solution of the Bethe equations for the generalized eigenvalue in each direct summand isomorphic to the $U_q sl(2)$-module T_j, the number $\mathcal{N}(N,M)$ of admissible solutions of the Bethe equations (1.3) with $\eta = i\pi/p$ satisfies the inequality

$$\mathcal{N}(N,M) \le d^0_{\frac{N}{2}-M} \,, \tag{4.9}$$

where d^0_j is given by (4.6)-(4.7), and we have used the relation $j = \frac{N}{2} - M$ stated in (A.29). We argue in Appendix D that $\mathcal{N}(N,M) < d^0_{\frac{N}{2}-M}$ when two or more tilting modules become degenerate in the sense that the generalized eigenvalues of the transfer matrix corresponding to direct summands T_j and T_k in (4.5), for distinct j and k, are equal. This suggests that the conjecture can be sharpened to the following:

$$\mathcal{N}(N,M) = d^0_{\frac{N}{2}-M} - n_{\frac{N}{2}-M} \,, \tag{4.10}$$

where n_j is the number of direct summands T_j that are degenerate with other tilting modules T_k with $k > j$ in the decomposition (4.5). We note that exact complete p-string solutions (4.1) are needed to construct the Bethe states corresponding to such degenerate tilting modules.

We can similarly conjecture that the number or degeneracy $\mathcal{D}(N,M)$ of the generalized eigenvalues of the transfer matrix corresponding to each admissible solution satisfies the inequality[6]

$$\mathcal{D}(N,M) \ge \dim T_{\frac{N}{2}-M} \,, \tag{4.11}$$

where $\dim T_j$ is given by (4.4). We can also sharpen this conjecture by introducing n_{jk}, which we define as the number of tilting modules T_k (with $k < j$) in the decomposition (4.5) that are degenerate with T_j.[7] (We define $n_{jk} = 0$ for $k \ge j$.) Then, we conjecture that

[6]We note that the numbers $\mathcal{N}(N,M)$ and $\mathcal{D}(N,M)$ depend also on p, as the dimensions of irreducible TL representations and of tilting modules do, but we do not use this dependence in notations for brevity.

[7]If $d^0_j > 1$ (i.e., there is more than one copy of T_j) and n_{jk} is nonzero for some $k < j$, then it is implicit that *each* copy of T_j is degenerate with n_{jk} copies of T_k. This assumption appears to be satisfied in all the examples that we have considered.

the degeneracy of an eigenvalue of the transfer matrix (corresponding to a given admissible solution $\{\lambda_1, \ldots, \lambda_M\}$) equals[8]

$$\mathcal{D}(N,M) = \dim T_j + \sum_{k<j} n_{jk} \dim T_k \,, \quad \text{with } \ j = \frac{N}{2} - M \,. \tag{4.12}$$

It is not obvious that the degeneracy $\mathcal{D}(N,M)$ is the same for all admissible solutions with a given value of M (as it is in the generic case), but it is so for the cases that we have considered. We therefore further conjecture that the numbers $\mathcal{D}(N,M)$ (and also n_{jk}) do not actually depend on a particular solution $\{\lambda_1, \ldots, \lambda_M\}$.

N \ M	0	1	2	3	4
2	1	0			
3	1	2			
4	1	2	0		
5	1	4	4 [5]		
6	1	4	4 [5]	0	
7	1	6	12 [14]	8 [14]	
8	1	6	12 [14]	8 [14]	0
9	1	8	24 [27]	32 [48]	16 [42]

(a) $q = e^{i\pi/2}$

N \ M	0	1	2	3	4
2	1	1			
3	1	1			
4	1	3	1		
5	1	4	1		
6	1	4	9	1	
7	1	6	13	1	
8	1	7	13	27 [28]	1

(b) $q = e^{i\pi/3}$

N \ M	0	1	2	3	4
2	1	1			
3	1	2			
4	1	2	2		
5	1	4	4		
6	1	5	4	4	
7	1	6	14	8	
8	1	6	20	8	8

(c) $q = e^{i\pi/4}$

N \ M	0	1	2	3	4
2	1	1			
3	1	2			
4	1	3	2		
5	1	3	5		
6	1	5	8	5	
7	1	6	8	13	
8	1	7	20	21	13

(d) $q = e^{i\pi/5}$

Table 3: The number $\mathcal{N}(N,M)$ of admissible solutions of the Bethe equations (1.3), (3.4) for given values of N, M and q. Numbers within brackets are the values of $d^0_{\frac{N}{2}-M}$ (4.6), when different from $\mathcal{N}(N,M)$.

The two sets of integers $\{n_j\}$ and $\{n_{jk}\}$ should be related by

$$\begin{aligned} n_j &= \sum_{m\geq 0} (-1)^m \sum_{j_0,j_1,\ldots,j_m=0(1/2)}^{N/2} d^0_{j_m}\, n_{j_m j_{m-1}}\, n_{j_{m-1}j_{m-2}} \cdots n_{j_0 j} \,, \\ &= \sum_{j_0=0(1/2)}^{N/2} d^0_{j_0}\, n_{j_0 j} - \sum_{j_0,j_1=0(1/2)}^{N/2} d^0_{j_1}\, n_{j_1 j_0}\, n_{j_0 j} + \ldots \end{aligned} \tag{4.13}$$

[8]The sum in (4.12) is over all $k < j$ in the decomposition (4.5); hence, it starts from $k = 0$ for even N and $k = 1/2$ for odd N. Strictly speaking, the restriction $k < j$ is not necessary since $n_{jk} = 0$ for $k \geq j$, but in this way we emphasize the relevant contributions.

The idea is that, if no more than two (non-isomorphic) tilting modules are degenerate, then only the $m = 0$ term in (4.13) is nonzero; however, if 3 tilting modules are degenerate (e.g. the case $p = 2$, $N = 9$, for which the modules $T_{\frac{9}{2}}$, $T_{\frac{5}{2}}$ and $T_{\frac{1}{2}}$ are degenerate, see (D.8)), then the $m = 1$ term in (4.13) provides a nonzero correction, etc. Indeed, one can verify that the sum rule

$$\sum_{M=0}^{\lfloor \frac{N}{2} \rfloor} \mathcal{N}(N, M)\, \mathcal{D}(N, M) = 2^N \tag{4.14}$$

is satisfied using (4.10) for $\mathcal{N}(N, M)$, (4.12) for $\mathcal{D}(N, M)$, and the expression (4.13) for n_j, with *arbitrary* n_{jk}, except that $n_{jk} = 0$ for $k \geq j$ (already noted above), and also that $n_{jk} = 0$ if $(j - k) \bmod p \neq 0$, which is discussed further below.

For the case $p = 2$, we have more explicit results. The number of admissible solutions $\mathcal{N}(N, M)$ for general values of N and M is given by

$$\mathcal{N}(N, M) = \begin{cases} \frac{(N-2)!!}{M!(N-2-2M)!!} & N = \text{even} \\ \frac{(N-1)!!}{M!(N-1-2M)!!} & N = \text{odd} \end{cases} , \tag{4.15}$$

as shown in Appendix C. We conjecture that the degeneracies $\mathcal{D}(N, M)$ for general values of N and M are given by[9]

$$\mathcal{D}(N, M) = \begin{cases} 2^{\lfloor \frac{N}{2} \rfloor - M + 1} , & M < \frac{N}{2} , \\ 0 , & M = \frac{N}{2} \quad \text{and} \quad N = \text{even} . \end{cases} \tag{4.16}$$

Indeed, this formula reproduces the results in Table 4 (a) below; and, together with (4.15) for $\mathcal{N}(N, M)$, satisfies the sum rule (4.14). Moreover, we propose that the integers n_{jk} in (4.12) are given, for $p = 2$ and j and k integers, by

$$n_{jk} = \begin{cases} \binom{j-1}{\frac{1}{2}(j-k)} \frac{2k}{j+k} , & j > k \quad \text{and} \quad (j-k) \bmod 2 = 0 , \\ 0 , & j \leq k , \quad \text{or} \quad (j-k) \bmod 2 \neq 0 , \quad \text{or} \quad k = 0 , \end{cases} \tag{4.17}$$

which do not depend on N. We note, as a curiosity, that $n_{2j,2}$ for $j > 1$ is equal to the j^{th} Catalan number. For j and k half-odd integers, $n_{jk} = n_{j+\frac{1}{2}, k+\frac{1}{2}}$. Indeed, these formulas reproduce all the values of n_{jk} for $p = 2$ found in Appendix D, and satisfy (4.10) (with $\mathcal{N}(N, M)$ and n_j given by (4.15) and (4.13) , respectively)) as well as (4.12) (with $\mathcal{D}(N, M)$ given by (4.16)).

The appearance of the extra degeneracies among different tilting modules at roots of unity is not surprising, as we have an extra symmetry for the whole family of integrable

[9]In terms of the spin $j = \frac{N}{2} - M$, the degeneracies are given by

$$\mathcal{D}_j \equiv \mathcal{D}(N, \frac{N}{2} - j) = \begin{cases} 2^{\lfloor j \rfloor + 1} , & j > 0 , \\ 0 , & j = 0 , \end{cases}$$

which evidently do not depend on N.

M \ N	0	1	2	3	4
2	4	0			
3	4	2			
4	8	4	0		
5	8 [6]	4	2		
6	16 [12]	8	4	0	
7	16 [8]	8 [6]	4	2	
8	32 [16]	16 [12]	8	4	0
9	32 [10]	16 [8]	8 [6]	4	2

(a) $q = e^{i\pi/2}$

M \ N	0	1	2	3	4
2	3	1			
3	6	2			
4	6	3	1		
5	6	6	2		
6	12	6	3	1	
7	12	6	6	2	
8	12 [9]	12	6	3	1

(b) $q = e^{i\pi/3}$

M \ N	0	1	2	3	4
2	3	1			
3	4	2			
4	8	3	1		
5	8	4	2		
6	8	8	3	1	
7	8	8	4	2	
8	16	8	8	3	1

(c) $q = e^{i\pi/4}$

M \ N	0	1	2	3	4
2	3	1			
3	4	2			
4	5	3	1		
5	10	4	2		
6	10	5	3	1	
7	10	10	4	2	
8	10	10	5	3	1

(d) $q = e^{i\pi/5}$

Table 4: The number $\mathcal{D}(N, M)$ of eigenvalues of the transfer matrix corresponding to each admissible solution of the Bethe equations (1.3), (3.4) for given values of N, M and q. Numbers within brackets are the values of $\dim T_{\frac{N}{2}-M}$ (4.4), when different from $\mathcal{D}(N, M)$.

Hamiltonians. For the case $p = 2$, this extra symmetry was identified in [13, Sec. 2.6.2 and 5] with the zero modes of the so-called lattice W-algebra. These modes $W_0^{\pm,r}$, with $r, s \in 2\mathbb{N}_0$, are particular operators that commute with H and change the total spin S^z by ± 2 and mix the distinct tilting $U_q sl(2)$-modules. These operators satisfy relations

$$[W_0^{+,r}, W_0^{-,s}] = 4W_0^{0,r+s+2} - 4W_0^{0,r+s}, \tag{4.18}$$

$$[W_0^{0,r}, W_0^{+,s}] = -8W_0^{+,r+s+2} + 8W_0^{+,r+s}, \tag{4.19}$$

$$[W_0^{0,r}, W_0^{-,s}] = 8W_0^{-,r+s+2} - 8W_0^{-,r+s}, \tag{4.20}$$

where $W_0^{0,r}$ are spinless zero modes of the W-algebra. The relations resemble the loop $sl(2)$ algebra relations and the algebra of the zero modes $W_0^{\alpha,r}$ was indeed identified with a subalgebra in it [13]. For higher roots of unity, there should exist a similar construction of the zero modes of the lattice W-algebra, defined in [13] for all p, and these operators do not commute with S^z but do commute with the Cartan $U_q sl(2)$ generator $K = q^{2S^z}$. So, we might expect a mixing of tilting modules in sectors by S^z equal modulo p.

We have solved the Bethe equations (3.4) with $q = e^{i\pi/p}$ numerically for $p = 3, 4, 5$ up to $N = 8$, see Tables 7-12. The numbers $\mathcal{N}(N, M)$ of admissible solutions that we have found are reported in Table 3.[10] These values are consistent with the conjecture (4.9). Note that $\mathcal{N}(N,M)$ is equal to the dimension $d^0_{\frac{N}{2}-M}$ of the TL irreducible representation for most of the values of N and M that we have considered. For the few cases that $\mathcal{N}(N,M) < d^0_{\frac{N}{2}-M}$, the values of $d^0_{\frac{N}{2}-M}$ appear in the tables within brackets. We analyze these cases individually in Appendix D, and we argue that they are consistent with the sharpened conjecture (4.10).

The numbers $\mathcal{D}(N,M)$ of eigenvalues of the transfer matrix (A.1) corresponding to each admissible solution of the Bethe equations are reported in Table 4. These values are consistent with the conjecture (4.11). Note that $\mathcal{D}(N,M)$ is equal to $\dim T_{\frac{N}{2}-M}$ for most of the values of N and M that we have considered. For the few cases that $\mathcal{D}(N,M) > \dim T_{\frac{N}{2}-M}$, the values of $\dim T_{\frac{N}{2}-M}$ appear in the tables within brackets. We also analyze these cases individually in Appendix D, and we argue that they are consistent with the conjecture (4.12).

5 Discussion

We have proposed formulas (2.8), (4.10), (4.15) for the number of admissible solutions of the Bethe equations (1.3), as well as formulas (2.9), (4.12), (4.16) for the degeneracies of the transfer matrix eigenvalues, including the root of unity cases $q = e^{i\pi/p}$ with $p \geq 2$. These formulas are consistent with the completeness of the solution (2.10), (4.14). We have checked these conjectures up to at least $N = 8$. We emphasize that we consider here *all* the (admissible) solutions of the Bethe equations, not just those corresponding to "good" states [1, 8]. The construction of all the Bethe states remains to be clarified. Work on this and related questions is now in progress.

We have observed at $p = 2$ and $p = 3$ large degeneracies (in the spectrum of the transfer-matrix) that cannot be explained just using the representation theory of the Temperley-Lieb algebra or $U_q sl(2)$ at roots of unity. We expect actually similar degeneracies for all integer $p \geq 2$ starting with sufficiently large N, for example, $p = 4$ and $N \geq 10$. Such degeneracies appear due to a very fine phenomena. It is similar to the periodic case where, at roots of unity, there is a much bigger symmetry of H – the loop $sl(2)$ algebra (at least for $p = 2$ [35]). This symmetry, additionally to the quantum group generators, mixes H-eigenvectors in sectors modulo p. We expect a similar phenomena in the boundary case, and for $p = 2$ we do have such an extra symmetry written explicitly in terms of $W_0^{\pm,r}$ operators satisfying (4.18)-(4.20), see the discussion in Sec. 4.3. For all integer values of $p \geq 2$, we expect that this extra symmetry commutes with the Cartan operator $K = q^{2S^z}$ (and not with S^z). In particular, tilting modules T_j and T_k might be degenerate only if $|j - k| = 0 \mod p$. However, instead of the loop $sl(2)$ symmetry that appears in the periodic case, the extra symmetry in the open case should be a subalgebra in the loop $sl(2)$. This is expected to be in analogy with the q-Onsager approach [36] to the open XXZ spin-chain with diagonal boundary conditions [37], where the generating-spectrum algebra for the finite open chain – the q-Onsager algebra – is a (co-ideal) subalgebra in the generating-spectrum algebra of the closed/periodic chain,

[10] The results for $p = 2$ were obtained using Eq. (C.7). The results for $p \geq 3$ with $N = 8$ and $M = 4$ were obtained only by solving the T-Q equations.

which is the affine quantum algebra $U_q\widehat{sl}(2)$.

This work raises several interesting questions. Assuming that our conjectures are correct, it would be interesting to find proofs and explore more the role of the lattice W-algebra symmetry [13] in our context of open chains that may be responsible for the degeneracies of the tilting modules, which could help to determine the values of n_{jk} in (4.12) for $p > 2$. (For $p = 2$, see (4.17).) It would also be interesting to perform a similar analysis of related models, such as the quantum group invariant XXZ chain with higher spin, and the periodic XXZ chain.

In our view, it is remarkable that a system of polynomial equations can "know" so much representation theory. It is evidence that Bethe ansatz provides deep links between algebraic geometry, representation theory and quantum mechanics.

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A Transfer matrix and algebraic Bethe ansatz

We briefly review here the transfer matrix and algebraic Bethe ansatz for the model (1.1). These results were first obtained for a more general model by Sklyanin [15]. The transfer matrix $t(u)$ is given by

$$t(u) = \mathrm{tr}_a\, K^+_a(u)\, T_a(u)\, K^-_a(u)\, \hat{T}_a(u)\,, \tag{A.1}$$

where $T_a(u)$ and $\hat{T}_a(u)$ are the monodromy matrices

$$T_a(u) = R_{a1}(u) \cdots R_{aN}(u)\,, \qquad \hat{T}_a(u) = R_{aN}(u) \cdots R_{a1}(u)\,, \tag{A.2}$$

the R-matrix is given by

$$R(u) = \begin{pmatrix} \sinh(u+\eta) & 0 & 0 & 0 \\ 0 & \sinh(u) & \sinh(\eta) & 0 \\ 0 & \sinh(\eta) & \sinh(u) & 0 \\ 0 & 0 & 0 & \sinh(u+\eta) \end{pmatrix}\,, \tag{A.3}$$

and the left and right K-matrices are given by the diagonal matrices

$$K^{+}(u) = \mathrm{diag}(e^{-u-\eta}\,, e^{u+\eta})\,, \qquad K^{-}(u) = \mathrm{diag}(e^{u}\,, e^{-u})\,, \tag{A.4}$$

respectively.[11] The transfer matrix commutes for different values of the spectral parameter

$$[t(u)\,, t(v)] = 0\,, \tag{A.6}$$

and it contains the Hamiltonian (1.1) [12]

$$H = \alpha\, t'(0) + \beta\, \mathbb{I}\,, \tag{A.7}$$

where

$$\alpha = \mathrm{csch}(2\eta)\,\mathrm{csch}^{2(N-1)}\,\eta\,, \qquad \beta = -(N+1)\cosh\eta + \mathrm{sech}\,\eta\,. \tag{A.8}$$

By taking higher derivatives of the transfer matrix, we obtain the higher conserved charges, which commute with each other by virtue of (A.6)

$$H_n = \frac{d^n}{du^n} t(u)\Big|_{u=0}\,, \qquad [H_n\,, H_m] = 0\,. \tag{A.9}$$

The transfer matrix has $U_q sl(2)$ symmetry [2]

$$[t(u)\,, S^z] = 0\,, \qquad \left[t(u)\,, S^{\pm}\right] = 0\,, \tag{A.10}$$

where the $U_q sl(2)$ generators S^z and $S^{\pm}$ are given by

$$\begin{aligned} S^z &= \sum_{k=1}^{N} S^z_k\,, \qquad S^z_k = \frac{1}{2}\sigma^z_k\,, \\ S^{\pm} &= \sum_{k=1}^{N} q^{-(S^z_1+\cdots+S^z_{k-1})}\; S^{\pm}_k\; q^{(S^z_{k+1}+\cdots+S^z_N)}\,, \qquad S^{\pm}_k = \frac{1}{2}(\sigma^x_k \pm i\sigma^y_k)\,, \end{aligned} \tag{A.11}$$

and satisfy (3.3). The $U_q sl(2)$ symmetry of the Hamiltonian can therefore be understood as a consequence of the symmetry of the transfer matrix (A.10) and the relation (A.7). The transfer matrix also has the crossing symmetry [5]

$$t(u) = t(-u-\eta)\,. \tag{A.12}$$

[11]For the XXX case, we first rescale $u \mapsto -i\eta u$ and $R \mapsto \frac{1}{-i\eta}R$ before taking the limit $\eta \to 0$. Hence, we have

$$R(u) = \begin{pmatrix} u+i & 0 & 0 & 0 \\ 0 & u & i & 0 \\ 0 & i & u & 0 \\ 0 & 0 & 0 & u+i \end{pmatrix}\,, \tag{A.5}$$

and $K^{+}(u) = K^{-}(u) = \mathbb{I}$.

[12]For the case $p = 2$ (i.e., $\eta = i\pi/2$), the first derivative of the transfer matrix is proportional to the identity matrix; hence, the Hamiltonian is related to the second derivative of the transfer matrix, $H = (-1)^N \frac{1}{4} t''(0)$.

The A, B, C, and D operators of the algebraic Bethe ansatz are obtained from the operator $\mathcal{U}$ given by

$$\mathcal{U}_a(u) = T_a(u)\, K_a^-(u)\, \hat{T}_a(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) + \frac{\sinh\eta}{\sinh(2u+\eta)} A(u) \end{pmatrix} , \tag{A.13}$$

in terms of which the transfer matrix (A.1) is given by

$$t(u) = \mathrm{tr}_a\, K_a^+(u) \mathcal{U}_a(u) \,. \tag{A.14}$$

The Bethe states are defined by

$$|v_1 \ldots v_M\rangle = \prod_{k=1}^{M} B(v_k)|0\rangle \,, \tag{A.15}$$

where $|0\rangle$ is the reference state with all spins up

$$|0\rangle = \binom{1}{0}^{\otimes N} , \tag{A.16}$$

and $v_1\,, \ldots\,, v_M$ remain to be specified. The Bethe states satisfy the off-shell relation[13]

$$t(u)|v_1 \ldots v_M\rangle = \Lambda(u)|v_1 \ldots v_M\rangle + \sum_{m=1}^{M} \Lambda_m |u, v_1 \ldots \hat{v}_m \ldots v_M\rangle \,, \tag{A.17}$$

where $\Lambda(u)$ is given by the so-called T-Q equation[14]

$$\Lambda(u) = \frac{\sinh(2u+2\eta)}{\sinh(2u+\eta)} \sinh^{2N}(u+\eta) \frac{Q(u-\eta)}{Q(u)} + \frac{\sinh(2u)}{\sinh(2u+\eta)} \sinh^{2N}(u) \frac{Q(u+\eta)}{Q(u)} \,, \tag{A.19}$$

with

$$Q(u) = \prod_{k=1}^{M} \sinh\left(u - v_k\right) \sinh\left(u + v_k + \eta\right) = Q(-u-\eta) \,. \tag{A.20}$$

Moreover,

$$\Lambda_m = f(u, v_m) \Bigg[\sinh^{2N}(v_m+\eta) \prod_{\substack{k \neq m \\ k=1}}^{M} \frac{\sinh(v_m - v_k - \eta) \sinh(v_m + v_k)}{\sinh(v_m - v_k) \sinh(v_m + v_k + \eta)} - \sinh^{2N}(v_m) \prod_{\substack{k \neq m \\ k=1}}^{M} \frac{\sinh(v_m - v_k + \eta) \sinh(v_m + v_k + 2\eta)}{\sinh(v_m - v_k) \sinh(v_m + v_k + \eta)} \Bigg] , \tag{A.21}$$

[13] Details of this computation can be found in e.g. [38].

[14] For the XXX case, the T-Q equation is

$$\Lambda(u) = \frac{2}{(2u+i)} (u+i)^{2N+1} \frac{Q(u-i)}{Q(u)} + \frac{2}{(2u+i)} u^{2N+1} \frac{Q(u+i)}{Q(u)} \,, \quad Q(u) = \prod_{k=1}^{M} (u - v_k)\,(u + v_k + i) \,. \tag{A.18}$$

where

$$f(u,v) = \frac{\sinh(2u+2\eta)\sinh(2v)\sinh\eta}{\sinh(u-v)\sinh(u+v+\eta)\sinh(2v+\eta)} . \tag{A.22}$$

It follows from (A.17) that the Bethe state $|v_1 \ldots v_M\rangle$ (A.15) is an eigenstate of the transfer matrix $t(u)$ in (A.1) with eigenvalue $\Lambda(u)$ in (A.19) if all the Λ_m vanish; i.e., according to (A.21), if $v_1, \ldots, v_M$ satisfy

$$\sinh^{2N}(v_m+\eta) \prod_{\substack{k\neq m \\ k=1}}^{M} \sinh(v_m - v_k - \eta)\sinh(v_m+v_k)$$
$$= \sinh^{2N}(v_m) \prod_{\substack{k\neq m \\ k=1}}^{M} \sinh(v_m - v_k + \eta)\sinh(v_m+v_k+2\eta) , \quad m = 1, \ldots, M . \tag{A.23}$$

These equations coincide with the Bethe equations (1.3) upon identifying

$$v_m = \lambda_m - \frac{\eta}{2} , \quad m = 1, \ldots, M . \tag{A.24}$$

The result (1.2) for the energy follows from (A.7) and (A.19).

In passing to (A.23), it was assumed that the factor $f(u, v_m)$ in (A.21) is regular. However, $f(u,v)$ has a pole at $v = -\eta/2$, as can be seen from (A.22). Hence, solutions of the Bethe equations (A.23) containing $v_m = -\eta/2$ must be discarded, since Λ_m will not vanish, and therefore the corresponding Bethe state will not be an eigenstate of the transfer matrix. Similarly, $v_m = -\eta/2 + i\pi/2$ must be excluded.

In other words, solutions of the Bethe equations (1.3) with $\lambda_m = 0$ or $\lambda_m = i\pi/2$ must be discarded, because they do not correspond to eigenstates of the transfer matrix. It has also been argued [25] that such solutions should be discarded because the corresponding coordinate Bethe ansatz wave function [14] vanishes identically.

For generic values of q, the on-shell (i.e., with Bethe equations satisfied) Bethe state (A.15) is an $U_q sl(2)$ highest-weight state [1, 3, 4]

$$S^+ |v_1 \ldots v_M\rangle = 0 , \tag{A.25}$$

with

$$S^z |v_1 \ldots v_M\rangle = \left(\frac{N}{2} - M\right) |v_1 \ldots v_M\rangle . \tag{A.26}$$

The on-shell Bethe state (A.15) is therefore an eigenstate of the Casimir operator (see e.g. [1])

$$S^2 = S^- S^+ + \left(\left[S^z + \tfrac{1}{2}\right]_q\right)^2 - \left[\tfrac{1}{2}\right]_q^2 , \tag{A.27}$$

with corresponding eigenvalue

$$S^2 |v_1 \ldots v_M\rangle = \left(\left[j + \tfrac{1}{2}\right]_q^2 - \left[\tfrac{1}{2}\right]_q^2\right) |v_1 \ldots v_M\rangle , \tag{A.28}$$

where $[x]_q$ is defined in (3.3) and the spin j is given by

$$j = \frac{N}{2} - M\,. \tag{A.29}$$

The requirement $j \ge 0$ implies that $M \le \frac{N}{2}$. The lower-weight states ($S^z < j$) of the spin-j representation of $U_q sl(2)$ can be obtained by repeatedly acting on the highest-weight state ($S^z = j$) by the S^- operator defined in (A.11).

B Temperley-Lieb algebra

The Hamiltonian (1.1) can evidently be re-expressed (up to an additive constant) as [1]

$$H = -2\sum_{k=1}^{N-1} e_k\,, \tag{B.1}$$

where the e_k are given by

$$e_k = -\frac{1}{2}\left(\sigma_k^x\sigma_{k+1}^x + \sigma_k^y\sigma_{k+1}^y\right) - \frac{1}{4}(q+q^{-1})\left(\sigma_k^z\sigma_{k+1}^z - 1\right) + \frac{1}{4}(q-q^{-1})\left(\sigma_k^z - \sigma_{k+1}^z\right)\,. \tag{B.2}$$

The e_k can be shown to satisfy the Temperley-Lieb algebra [19]

$$\begin{aligned} e_k^2 &= \delta e_k\,, \\ e_k e_{k\pm1} e_k &= e_k\,, \\ e_k e_j &= e_j e_k\,, \quad |j-k| > 1\,, \end{aligned} \tag{B.3}$$

where δ (the so-called fugacity or loop parameter) is given by

$$\delta = q + q^{-1}\,. \tag{B.4}$$

For $q = e^{i\pi/p}$, it follows that $\delta = 2\cos\frac{\pi}{p}$. For all values of q, including the roots of unity, the Temperley-Lieb algebra is identified with the maximum algebra commuting with (or centralizer of) $U_q sl(2)$, see [33, 39].

C Bethe solutions at $p = 2$

The case $p = 2$ (i.e., $\eta = i\pi/2$) is sufficiently simple to be analyzed analytically. The Bethe equations (3.4) decouple and reduce to

$$\left(\frac{i x_k - 1}{x_k - i}\right)^{2N} = 1\,, \tag{C.1}$$

since $q^2 = -1$ and therefore the terms with $\prod_{j\neq k}$ cancel. It follows that

$$\frac{ix_k - 1}{x_k - i} = e^{i\omega_l}\,, \qquad \omega_l = \frac{2\pi l}{2N}\,, \qquad l = 0, 1, \ldots, 2N-1, \tag{C.2}$$

and therefore

$$\lambda_k = \frac{1}{2}\ln x_k = \frac{1}{2}\ln\left(\frac{ie^{i\omega_l} - 1}{e^{i\omega_l} - i}\right)\,. \tag{C.3}$$

The admissible Bethe roots (recall (3.1) and (3.2)) appear as $(\lambda\,, \lambda + i\pi/2)$ (i.e., pairs of roots that differ by $i\pi/2$) corresponding to the following pairs of l values

$$(l\,, N-l)\,, \qquad l = 1, 2, \ldots, l_{\max}\,, \tag{C.4}$$

where

$$l_{\max} = \begin{cases} (N-2)/2 & N = \text{even} \\ (N-1)/2 & N = \text{odd} \end{cases}\,. \tag{C.5}$$

It follows that the number of solutions $\mathcal{N}(N, M)$ for $M = 1$ is given by

$$\mathcal{N}(N,1) = 2l_{\max} = \begin{cases} N-2 & N = \text{even} \\ N-1 & N = \text{odd} \end{cases}\,. \tag{C.6}$$

In order to construct Bethe states (A.15) with $M > 1$, one would naively expect to be able to choose any M roots from the $\mathcal{N}(N,1)$ admissible roots. However, a Bethe vector with two roots that differ by $i\pi/2$ is an exact complete 2-string (4.1). Hence, any solution of the Bethe equations that contains a pair of roots that differ by $i\pi/2$ is not admissible. Since the $\mathcal{N}(N,1)$ admissible roots all come in pairs that differ by $i\pi/2$, it follows that the number of solutions for $M \geq 1$ is given by (notice the double factorials)

$$\mathcal{N}(N,M) = \frac{\mathcal{N}(N,1)!!}{M!(\mathcal{N}(N,1) - 2M)!!} = \begin{cases} \frac{(N-2)!!}{M!(N-2-2M)!!} & N = \text{even} \\ \frac{(N-1)!!}{M!(N-1-2M)!!} & N = \text{odd} \end{cases}\,. \tag{C.7}$$

The results for $N = 2, \ldots, 9$ are displayed in Table 3 (a).

D Explanations of deviations

We consider here in detail the cases of the conjecture (4.10) for which $\mathcal{N}(N, M) < d^0_{\frac{N}{2}-M}$, and the cases of the conjecture (4.12) for which $\mathcal{D}(N, M) > \dim T_{\frac{N}{2}-M}$. We argue that these deviations occur when two or more tilting modules become degenerate. The idea is that we count the total degeneracies of generalized eigenvalues of the transfer matrix; and by comparing them with dimensions of the tilting modules and using the S^z values for corresponding generalized eigenstates, we infer which tilting modules are degenerate. As we do not construct a basis in these tilting modules explicitly, our arguments are rather indirect but definitive.

We analyze below only the cases $p = 2$ and $p = 3$, because for higher values of p we would need to go beyond $N = 10$ which exceeds the capabilities of our available computer resources. For $p = 2$, we use known facts on representation theory of both the Temperley–Lieb and $U_q sl(2)$ algebras [33, 34, 13]:

1. For odd N, the TL algebra is semisimple and the Hamiltonian is diagonalizable – it is the only semisimple/diagonalizable case at roots of unity. Hence, for this case all the eigenvalues and eigenvectors are ordinary (i.e. not generalized). For even N, the TL algebra is non-semisimple and the Hamiltonian has Jordan blocks of maximum rank 2.

2. For odd N, the tilting $U_q sl(2)$-modules T_j in (4.5) appear for half-integer j and are irreducible. The S^z spectrum is then usual one $\{j, j-1, \ldots, -j\}$. For even N, each tilting $U_q sl(2)$-module T_j, where j is a positive integer, is indecomposable but reducible and is composed of the spin-j and the spin-$(j-1)$ modules (recall the discussion above (4.4)), where each spin-j module is also reducible but indecomposable and has the unique submodule isomorphic to the head (or irreducible quotient) of the spin-$(j-1)$ module. The dimension of the head of the spin-j module is $j+1$ and we denote the head by $\langle j \rangle$. In total, the sub-quotient structure of T_j in terms of the irreducible modules $\langle j \rangle$ is

$$T_j \quad : \quad \begin{array}{ccc} & \langle j-1\rangle & \\ \swarrow & & \searrow \\ \langle j-2\rangle & & \langle j\rangle \\ \searrow & & \swarrow \\ & \langle j-1\rangle & \end{array} \tag{D.1}$$

where arrows correspond to irreversible action of $U_q sl(2)$ generators and we set $\langle -1 \rangle = 0$. In the decomposition (4.5), a direct summand T_j has S^z spectrum $\{j, 2\times(j-1), 2\times(j-2), \ldots, 2\times(-j+1), -j\}$ while each irreducible sub-quotient $\langle j \rangle$ has S^z spectrum $\{j, j-2, \ldots, -j+2, -j\}$. We also note that it is only the states in the head of T_j – the top sub-quotient $\langle j-1 \rangle$ in (D.1) – on which the Hamiltonian is non-diagonalizable.

D.1 $p = 2\,,\ N = 5$

For $p = 2$ and $N = 5$, the decomposition (4.5) into tilting modules is given by

$$5T_{\frac{1}{2}} \oplus 4T_{\frac{3}{2}} \oplus T_{\frac{5}{2}}\,.$$

We claim that $T_{\frac{5}{2}}$ and one of the $T_{\frac{1}{2}}$ are degenerate, and therefore $n_{\frac{5}{2},\frac{1}{2}} = 1$, $n_{\frac{1}{2}} = 1$ (all others are zero). Indeed, the subspace with energy $E = 0$ [15], which includes the reference

[15] Strictly speaking, we should consider (generalized) eigenvalues of the transfer matrix. We consider in this appendix instead (generalized) eigenvalues of the Hamiltonian, which are easier to report and give the

state ($M=0\,, j=\frac{5}{2}$) belonging to $T_{\frac{5}{2}}$, can be shown to have dimension 8. That is, the number of $E=0$ eigenstates is 8, which is the sum of dimensions of $T_{\frac{5}{2}}$ (6) and $T_{\frac{1}{2}}$ (2), recall (4.4). This implies that

$$\begin{aligned}\mathcal{D}(5,0) &= \dim T_{\frac{5}{2}} + n_{\frac{5}{2},\frac{1}{2}} \dim T_{\frac{1}{2}} = 6 + 1 * 2 = 8\,, \\ \mathcal{N}(5,2) &= d^0_{\frac{1}{2}} - n_{\frac{1}{2}} = 5 - 1 = 4\,,\end{aligned} \tag{D.2}$$

in agreement with Tables 4 and 3, respectively.

D.2 $p=2\,,\ N=6$

For $p=2, N=6$, the decomposition (4.5) into tilting modules is given by

$$5T_1 \oplus 4T_2 \oplus T_3\,.$$

We claim that T_3 and one of the T_1 are degenerate, and therefore $n_{3,1}=1$, $n_1=1$ (all others are zero). Indeed, the subspace with generalized H-eigenvalue $E=0$, which includes the reference state ($M=0\,, j=3$) belonging to T_3, can be shown to have dimension 16, which is the sum of dimensions of T_3 (12) and T_1 (4). This implies that

$$\begin{aligned}\mathcal{D}(6,0) &= \dim T_3 + n_{3,1} \dim T_1 = 12 + 1 * 4 = 16\,, \\ \mathcal{N}(6,2) &= d^0_1 - n_1 = 5 - 1 = 4\,.\end{aligned} \tag{D.3}$$

D.3 $p=2\,,\ N=7$

For $p=2, N=7$, the decomposition (4.5) into tilting modules is given by

$$14T_{\frac{1}{2}} \oplus 14T_{\frac{3}{2}} \oplus 6T_{\frac{5}{2}} \oplus T_{\frac{7}{2}}\,.$$

We claim that $T_{\frac{7}{2}}$ and two of the $T_{\frac{3}{2}}$ are degenerate, and therefore $n_{\frac{7}{2},\frac{3}{2}}=2$, $n_{\frac{3}{2}}=2$. Indeed, the subspace with energy $E=0$, which includes the reference state ($M=0\,, j=\frac{7}{2}$), can be shown to have dimension 16, which can be now either the sum of dimensions of $T_{\frac{7}{2}}$ (8) and two $T_{\frac{3}{2}}$ (2*4=8) or as $\dim T_{\frac{7}{2}} + \dim T_{\frac{5}{2}} + \dim T_{\frac{1}{2}}$ or the sum $\dim T_{\frac{5}{2}} + \dim T_{\frac{3}{2}} + 2\dim T_{\frac{1}{2}}$. Looking at S^z-sectors for these 16 eigenstates:

$$S^z = \Big\{\pm\frac{7}{2}, \pm\frac{5}{2}, 3\times(\pm\frac{3}{2}), 3\times(\pm\frac{1}{2})\Big\}$$

and recalling the discussion above (D.1), we identify precisely the tilting modules they belong to as $T_{\frac{7}{2}} \oplus 2T_{\frac{3}{2}}$. This gives

$$\begin{aligned}\mathcal{D}(7,0) &= \dim T_{\frac{7}{2}} + n_{\frac{7}{2},\frac{3}{2}} \dim T_{\frac{3}{2}} = 8 + 2 * 4 = 16\,, \\ \mathcal{N}(7,2) &= d^0_{\frac{3}{2}} - n_{\frac{3}{2}} = 14 - 2 = 12\,.\end{aligned} \tag{D.4}$$

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same results.

Moreover, we claim that each of the 6 $T_{\frac{5}{2}}$ are degenerate with 6 $T_{\frac{1}{2}}$, and therefore $n_{\frac{5}{2},\frac{1}{2}} = 1$, $n_{\frac{1}{2}} = 6$. Indeed, we find that there are 6 energy eigenvalues (namely, ± 3.60388, ± 2.49396, and ± 0.890084) that are each 8-fold degenerate; while dim $T_{\frac{5}{2}} = 6$ and dim $T_{\frac{1}{2}} = 2$. This implies that

$$\begin{aligned} \mathcal{D}(7,1) &= \dim T_{\frac{5}{2}} + n_{\frac{5}{2},\frac{1}{2}} \dim T_{\frac{1}{2}} = 6 + 1 * 2 = 8\,, \\ \mathcal{N}(7,3) &= d^0_{\frac{1}{2}} - n_{\frac{1}{2}} = 14 - 6 = 8\,. \end{aligned} \tag{D.5}$$

D.4 $p = 2\,,\ N = 8$

For $p = 2, N = 8$, the decomposition (4.5) into tilting modules is given by

$$14T_1 \oplus 14T_2 \oplus 6T_3 \oplus T_4\,.$$

We claim that T_4 and two of the T_2 are degenerate, and therefore $n_{4,2} = 2$, $n_2 = 2$. Indeed, the subspace with generalized H-eigenvalue $E = 0$, which includes the reference state ($M = 0\,, j = 4$), can be shown to have dimension 32, which is either the sum of dimensions of T_4 (16) and two T_2 (2*8=16) or one of these sums $\dim T_4 + \dim T_3 + \dim T_1 = \dim T_4 + 4 \dim T_1 = \dim T_4 + \dim T_2 + 2 \dim T_1$. Looking then at S^z-sectors for these 32 generalized eigenstates, we find that the 24 S^z eigenvalues corresponding to ordinary eigenvectors are

$$S^z = \Big\{\pm 4, \pm 3, 4 \times (\pm 2), 3 \times (\pm 1), 6 \times 0\Big\}$$

and the 8 S^z eigenvalues corresponding to generalized eigenvectors are

$$S^z = \Big\{\pm 3, 3 \times (\pm 1)\Big\}.$$

Recalling the discussion about tilting modules and their S^z spectrum (above (D.1)) we identify precisely the tilting modules the 32 generalized eigenstates belong to as $T_4 \oplus 2T_2$. So, the S^z spectrum we found implies

$$\begin{aligned} \mathcal{D}(8,0) &= \dim T_4 + n_{4,2} \dim T_2 = 16 + 2 * 8 = 32\,, \\ \mathcal{N}(8,2) &= d^0_2 - n_2 = 14 - 2 = 12\,. \end{aligned} \tag{D.6}$$

Moreover, we claim that each of the 6 T_3 are degenerate with 6 T_1, and therefore $n_{3,1} = 1$, $n_1 = 6$. Indeed, we find that there are 6 energy eigenvalues (namely, $\pm 2\sqrt{2+\sqrt{2}}$, $\pm 2\sqrt{2}$, $\pm 2\sqrt{2-\sqrt{2}}$) that are each 16-fold degenerate; while dim $T_3 = 12$ and dim $T_1 = 4$. This implies that

$$\begin{aligned} \mathcal{D}(8,1) &= \dim T_3 + n_{3,1} \dim T_1 = 12 + 1 * 4 = 16\,, \\ \mathcal{N}(8,3) &= d^0_1 - n_1 = 14 - 6 = 8\,. \end{aligned} \tag{D.7}$$

D.5 $p = 2\,,\ N = 9$

For $p = 2$ and $N = 9$, the decomposition (4.5) into tilting modules is given by

$$42T_{\frac{1}{2}} \oplus 48T_{\frac{3}{2}} \oplus 27T_{\frac{5}{2}} \oplus 8T_{\frac{7}{2}} \oplus T_{\frac{9}{2}}\,.$$

Using analysis similar to the previous cases, we claim that the nonzero n_{jk} are

$$n_{\frac{9}{2},\frac{5}{2}} = 3\,, \quad n_{\frac{9}{2},\frac{1}{2}} = 2\,, \quad n_{\frac{7}{2},\frac{3}{2}} = 2\,, \quad n_{\frac{5}{2},\frac{1}{2}} = 1\,, \tag{D.8}$$

and the nonzero n_j are

$$n_{\frac{5}{2}} = 3\,, \quad n_{\frac{3}{2}} = 16\,, \quad n_{\frac{1}{2}} = 26\,. \tag{D.9}$$

Hence,

$$\begin{aligned}
\mathcal{D}(9,0) &= \dim T_{\frac{9}{2}} + n_{\frac{9}{2},\frac{5}{2}} \dim T_{\frac{5}{2}} + n_{\frac{9}{2},\frac{1}{2}} \dim T_{\frac{1}{2}} = 10 + 3*6 + 2*2 = 32\,, \\
\mathcal{D}(9,1) &= \dim T_{\frac{7}{2}} + n_{\frac{7}{2},\frac{3}{2}} \dim T_{\frac{3}{2}} = 8 + 2*4 = 16\,, \\
\mathcal{D}(9,2) &= \dim T_{\frac{5}{2}} + n_{\frac{5}{2},\frac{1}{2}} \dim T_{\frac{1}{2}} = 6 + 1*2 = 8\,,
\end{aligned} \tag{D.10}$$

and

$$\begin{aligned}
\mathcal{N}(9,2) &= d^0_{\frac{5}{2}} - n_{\frac{5}{2}} = 27 - 3 = 24\,, \\
\mathcal{N}(9,3) &= d^0_{\frac{3}{2}} - n_{\frac{3}{2}} = 48 - 16 = 32\,, \\
\mathcal{N}(9,4) &= d^0_{\frac{1}{2}} - n_{\frac{1}{2}} = 42 - 26 = 16\,,
\end{aligned} \tag{D.11}$$

in agreement with Tables 4 and 3, respectively. One can also verify that the n_j's (D.9) can be obtained from the n_{jk}'s (D.8) using (4.13).

D.6 $p = 3\,,\ N = 8$

For $p = 3, N = 8$, the decomposition (4.5) into tilting modules is given by

$$T_0 \oplus 28T_1 \oplus 13T_2 \oplus 7T_3 \oplus T_4\,.$$

We claim that T_4 and one of the T_1 are degenerate, and therefore $n_{4,1} = 1$, $n_1 = 1$ (all others are zero). Indeed, the subspace with energy $E = 0$, which includes the reference state ($M = 0\,, j = 4$), can be shown to have dimension 12, which is the sum of dimensions of T_4 (9) and T_1 (3). This implies that

$$\begin{aligned}
\mathcal{D}(8,0) &= \dim T_4 + n_{4,1} \dim T_1 = 9 + 1*3 = 12\,, \\
\mathcal{N}(8,3) &= d^0_1 - n_1 = 28 - 1 = 27\,.
\end{aligned} \tag{D.12}$$

E Numerical results

Our numerical solutions of the Bethe equations up to $N = 8$ are presented in Tables 5-12. These results were obtained using homotopy continuation [29] (see also [28] and references therein for further details).

N	M	number	λ_1	λ_2	λ_3
2	1	1	0.5		
3	1	1	0.8660254037844386		
		2	0.2886751345948129		
4	1	1	1.207106781186547		
		2	0.5		
		3	0.2071067811865475		
4	2	1	0.7160149594491338 +0.5125206553446844i	0.7160149594491338 -0.5125206553446844i	
		2	0.6683262276726571	0.2309546565991595	
5	1	1	1.538841768587627		
		2	0.6881909602355868		
		3	0.3632712640026804		
		4	0.1624598481164532		
5	2	1	1.115042120183109 +0.5450541101265968i	1.115042120183109 -0.5450541101265968i	
		2	0.9704069411911774	0.1723800721632705	
		3	0.5137119304322965 +0.4996020494993916i	0.5137119304322965 -0.4996020494993916i	
		4	0.9496686956332134	0.3969680639294287	
		5	0.4272945057154192	0.1793374003754359	
6	1	1	1.866025403784439		
		2	0.2886751345948129		
		3	0.1339745962155613		
		4	0.5		
		5	0.8660254037844386		
6	2	1	1.234440793585582	0.5389490693006668	
		2	0.8418003199559988 +0.4947462450429116i	0.8418003199559988 -0.4947462450429116i	
		3	0.3905082158626772 +0.5000053666355159i	0.3905082158626772 -0.5000053666355159i	
		4	1.277389814218266	0.1387104764546264	
		5	1.471796355306884 -0.5824072783212229i	1.471796355306884 +0.5824072783212229i	
		6	0.591788951573015	0.3152209587092826	
		7	0.1457831570066063	0.3239416643695967	
		8	0.5975749352330829	0.1434644688717632	
		9	1.266274529052914	0.3019322716047725	
6	3	1	0.7487200726653173	0.5881061192792989 +0.5011583393895944i	0.5881061192792989 -0.5011583393895944i
		2	0.9677400136112142	0.9476918141366062 +0.9956807427853811i	0.9476918141366062 -0.9956807427853811i
		3	0.774814166699722	0.35438 89174298362	0.1551499348511761
		4	0.7901794336920558 +0.5103219367879035i	0.148626658019744	0.7901794336920558 -0.5103219367879035i
		5	0.7601147488943615 +0.5085412675384237i	0.3341849467072039	0.7601147488943615 -0.5085412675384237i

Table 5: The admissible solutions of the XXX Bethe equations (2.3) up to $N = 6$.

N	M	number	x_1	x_2	x_3
2	1	1	0.9950207489532265 +0.0996679946249558i		
3	1	1	0.9851362571667408 +0.1717747211189849i		
		2	0.9983374903000208 +0.05763900989309033i		
4	1	1	0.9713235174298164 +0.2377616968474302i		
		2	0.9950207489532265 +0.09966799462495615i		
		3	0.9991439299596814 +0.04136915789236264i		
4	2	1	0.9911071329262735 +0.1330663408329167i	0.9989343649801794 +0.04615337974239554i	
		2	1.096607956599227 +0.1576690437914444i	1.096607956599227 -0.1576690437914444i	
5	1	1	0.9538101147863981 +0.3004101611649615i		
		2	0.9905881296732647 +0.1368764309529703i		
		3	0.9973685392726434 +0.07249825424900729i		
		4	0.9994731533039727 +0.03245636801328048i		
5	2	1	1.087533505007 09 +0.2455058557178879i	1.08753350500709 -0.2455058557178879i	
		2	0.9821261824941047 +0.1882237011100266i	0.996853923761068 +0.07926067550912498i	
		3	1.099298174842604 +0.1129555095638716i	1.099298174842604 -0.1129555095638716i	
		4	0.9813430842510684 +0.1922647939499075i	0.9994062436043682 +0.03445519183818567i	
		5	0.9963579374226493 +0.08526934111909155i	0.9993575911189618 +0.03583859752984044i	
6	1	1	0.9328105645067954 +0.3603671055250655i		
		2	0.9851362571667408 +0.1717747211189849i		
		3	0.9983374903000208 +0.05763900989309033i		
		4	0.9950207489532265 +0.09966799462495615i		
		5	0.9996416778201899 +0.02676781583984422i		
6	2	1	1.075224471519206 +0.3236217054614372i	1.075224471519206 -0.3236217054 614372i	
		2	0.9699960698659229 +0.2431205965044173i	0.9942054556525597 +0.1074965671576824i	
		3	0.9930254383362942 +0.1179002918444679i	0.9980164632164529 +0.06295346812466036i	
		4	1.10182556643507 +0.08593996563356507i	1.10182556643507 -0.08593996563356507i	
		5	0.9684452798485436 +0.2492262826009249i	0.9981790461343699 +0.06032074152627307i	
		6	1.088602293651335 +0.1841309747335413i	1.088602293651335 -0.1841309747335413i	
		7	0.9979056525281584 +0.06468623232458623i	0.9995755201710089 +0.02913382012124421i	
		8	0.9678938580855082 +0.2513592637647361i	0.9996155034837294 +0.02772805790115951i	
		9	0.9928890635454395 +0.1190433009113084i	0.9995888688063166 +0.02867217045339162i	
6	3	1	1.094320030741045 +0.167153344511961i	0.9977671423426131 +0.06678869411401678i	1.094320030741045 -0.167153344511961i
		2	1.093678036321473 +0.1737932036777591i	0.9995582494860389 +0.02972046238546064i	1.093678036321473 -0.1737932036777591i
		3	0.9815106975959883 +0.1914072895807168i	1.19864337246897 +0.2289583093421947i	1.19864337246897 -0.2289583093421947i
		4	0.9888241361234739 +0.1490866453431209i	1.097821757712908 +0.1293812108099802i	1.097821757712908 -0.1293812108099802i
		5	0.9880520841223098 +0.1541203395453045i	0.9974912917057552 +0.07078928570895365i	0.9995188710316143 +0.03101655125392326i

Table 6: The admissible solutions of the XXZ Bethe equations (3.4) with $\eta = 0.1$ up to $N = 6$.

N	M	number	x_1	x_2	x_3
2	1	1	3.732050807568877		
3	1	1	2		
4	1	1	1.628626279736931		
		2	-6.078116022520112		
		3	3.732050807568876		
4	2	1	5.551933372263207	1.668669261292973	
5	1	1	1.461818651603003		
		2	2.445124904035096		
		3	-3.574329190217507		
		4	8.73968131822042		
5	2	1	1.495162537164605	2.744357057158133	
6	1	1	1.366025403784439		
		2	2		
		3	-2.732050807568878		
		4	3.732050807568876		
6	2	1	2.121740590131168	1.389071336983916	
		2	2.065764252710569	4.618645451098817	
		3	4.708465577106654	1.37980190200904	
		4	1.94185325885711	-6.956024177556002	
		5	7.06963925607305	-3.289527548618499	
		6	2.033456364524103 $-3.572707112158111i$	2.033456364524103 $+3.572707112158111i$	
		7	-5.515508983877778	3.211606419543336	
		8	-11.72375959717719	-2.585294089372629	
		9	-7.397095726637462	1.354263271640707	
6	3	1	2.184002431728489	7.103070621468425	1.399963454482446
7	1	1	1.303554144675824		
		2	1.770225971730896		
		3	-2.307585402462597		
		4	2.706596741879841		
		5	-11.05630589583649		
		6	6.245703131914746		
7	2	1	1.836863203195035	1.319993390462937	
		2	2.970622089446266	1.818186625013599	
		3	1.316313662306279	2.990958243918946	
		4	9.876567762581274	2.793117991352707	
		5	1.288545404441192	-4.146370060660599	
		6	-3.574329190217507	2.445124904035096	
		7	4.053394950868378	-2.890714726422575	
		8	-2.178060016150124	-6.266169856419983	
		9	1.547869212009354 $+2.679110191382225i$	1.547869212009354 $-2.679110191382225i$	
		10	-3.978767068330315	1.712236436628736	
		11	10.50224778955699	1.308224352206094	
		12	1.788230153989182	10.3328199011193	
		13	-2.40 322747023718	14.47572976817471	
7	3	1	1.332957726595594	3.390479290658175	1.890753798751596

Table 7: The admissible solutions of the XXZ Bethe equations (3.4) with $q = e^{i\pi/3}$ up to $N = 7$.

N	M	number	x_1	x_2	x_3
8	1	1	1.259483895472751		
		2	-2.051228571072364		
		3	2.256125795901561		
		4	3.732050807568876		
		5	1.628626279736931		
		6	-6.078116022520112		
		7	13.7129943488902		
8	2	1	1.271706087581433	1.670954477097469	
		2	2.383355863629706	1.662783111291984	
		3	2.092100093244703	-2.803377889217483	
		4	1.244313600108916	-3.065339745176996	
		5	2.334010482862648	4.308939906210139	
		6	-2.979846773221015	1.57827840091033	
		7	4.370551169701395	1.651842067444984	
		8	5.648765724234609	-2.237015855424188	
		9	1.269930725213365	2.390822553996926	
		10	-1.947287265389223	-4.491074032089162	
		11	-2.520489596445147	3.031731530022549	
		12	4.397977764490689	1.266495115558148	
		13	1.271475306333206 +2.202304943896206i	1.271475306333206 -2.202304943896206i	
8	3	1	2.40873955875332	2.098628559113852 -3.660393653953999i	2.098628559113853 +3.660393653953999i
		2	1.282713105358393	2.554950754236945	1.70924265930685
		3	2.283630889407069 -4.00022304697502i	2.28363088940707 +4.00022304697502i	1.664985843065564
		4	5.529918109991038	1.51881768033327 +2.631616151633382i	1.518817680333269 -2.631616151633382i
		5	7.414183066616086	-2.862682200630386	3.198078127975502
		6	-5.511247175838946	3.650488595458813	2.240655749924518
		7	3.340653884272446 +5.919991271163282i	-2.14255770445547	3.340653884272446 -5.919991271163282i
		8	-5.514779752344294	-2.023919611315855	10.8777019433327
		9	1.68440705741468	5.423847662867905	2.467651140153592
		10	1.213531067160599 -2.101911443901849i	1.213531067160599 +2.101911443901849i	-10.489730959 88394
		11	-6.301965974660797	3.766071521088686	1.26000911004235
		12	4.885235830838998	-2.183128543959034	-9.430910891448477
		13	2.060403901016193	-2.658376699449247	-12.74552761325145
		14	1.278043319153198	1.693253474753915	5.596399575909029
		15	2.149598643270218	7.839163284765999	-3.330352881033191
		16	-11.81310452125254	-2.420711226941534	2.927550314738232
		17	-8.654318296711313	1.652191871107661	1.266379443459761
		18	-18.08314271073251	-1.91571801974974	-4.226460223694949
		19	1.269639528011694	2.349438781357117 -4.1223176079197i	2.349438781357117 +4.1223176079197i
		20	5.471483838358137	2.476041590250572	1.276150039143275
		21	8.069263317405973	-3.594881640653509	1.599026959755484
		22	1.240656230548367	-2.889477062652692	-13.39895741005542
		23	2.30700 4865853058	-7.830240072884703	1.642950953716392
		24	1.264378468655506	2.315531519097134	-8.061533718602336
		25	1.628626279736931	-6.078116022520112	3.732050807568877
		26	1.250953755245463	8.175846591759141	-3.714015014705677
		27	-2.812486479833951	-13.19173425438937	1.56692995698348

Table 8: The admissible solutions of the XXZ Bethe equations (3.4) with $q = e^{i\pi/3}$ and $N = 8$.

N	M	number	x_1	x_2	x_3
2	1	1	2.414213562373095		
3	1	1	1.628626279736931		
		2	6.078116022520107		
4	1	1	2.414213562373095		
		2	1.414213562373095		
4	2	1	1.45314130298278	3.20337817632093	
		2	2.668693617506732 -3.09600332629458i	2.668693617506733 +3.09600332629458i	
5	1	1	1.311033025558219		
		2	1.861000175046023		
		3	-8.277536750907217		
		4	3.652418494292248		
5	2	1	1.339710835264441	2.046891922279901	
		2	7.305968916122279	1.320324940604242	
		3	1.91282417422196	6.940872062317657	
		4	2.064510526379247 -2.02993224151061i	2.064510526379247 +2.02993224151061i	
6	1	1	1.249688897773919		
		2	1.628626279736931		
		3	2.414213562373095		
		4	-4.663902460147015		
		5	6.078116022520107		
6	2	1	1.71311584242839	1.269294065270789	
		2	1.683266313484753	2.811927386738288	
		3	1.554024547007536 +1.554715716529629i	1.554024547007536 -1.554715716529629i	
		4	2.846518894544238	1.263542099556734	
6	3	1	1.262878926307629	3.075549320798162 +3.542806348465251i	3.075549320798161 -3.542806348465251i
		2	1.771351991900205	3.826157503762062	1.281594555446714
		3	3.211911541809374	2.26937708343367 +2.360695277629578i	2.269377083433669 -2.360695277629579i
		4	2.955672286788613 +3.365585528895605i	2.955672286788612 -3.365585528895605i	1.686278114645637
7	1	1	1.208825798344529		
		2	1.498359634541009		
		3	1.986520161938681		
		4	3.161528726585875		
		5	-3.45462565368127		
		6	13.29830956633742		
7	2	1	1.222821216951139	1.547297967647689	
		2	1.220472188780732	2.150322976591122	
		3	2.140605616819436	1.536617282908898	
		4	2.067053540613296	4.01059689676756	
		5	2.879021158753284 +2.956127592936119i	2.879021158753283 -2.956127592936119i	
		6	4.108663475752145	1.520996638277984	
		7	4.148660960002176	1.215637366932319	
		8	6.571169436423152	-4.94785739884573	
		9	-7.468286858137201	2.959302735558985	
		10	1.331990680492124 +1.331978961361137i	1.331990680492124 -1.331978961361137i	
		11	-8.92688858805227	1.488188528857897	
		12	1.205729430210374	-9.095642759167061	
		13	1.951628400529481	-8.521599164762728	
		14	-16.46493396710602	-3.167610405963212	
7	3	1	2.29407542825786 -2.262286996783694i	2.29407542825786 +2.262286996783694i	1.223807335674004
		2	1.591961830774357	2.387325715651191	1.235084659761902
		3	2.208206873312259 -2.183599412028467i	2.20820687331226 +2.183599412028467i	1.552672568630353
		4	2.258703430918202	1.923901682011331 -1.916321113803041i	1.923901682011331 +1.91632111380304i
		5	1.226752347088832	8.364492571528938	1.561591283662463
		6	1.22436156832076	2.214765473037293	7.893846799575592
		7	1.399715240410388 +1.399671218147326i	9.104198650298802	1.399715240410387 -1.399671218147326i
		8	2.204487438555672	7.745185417863576	1.550608398126798

Table 9: The admissible solutions of the XXZ Bethe equations (3.4) with $q = e^{i\pi/4}$ up to $N = 7$.

N	M	number	x_1	x_2	x_3
8	1	1	1.179580427103275		
		2	1.414213562373095		
		3	1.765366864730179		
		4	2.414213562373095		
		5	-2.847759065022574		
		6	4.261972627395668		
8	2	1	1.446403950334763	1.190016384385456	
		2	1.852238654657947	1.188864345375984	
		3	2.514682127062517	6.295143655976973	
		4	2.307514146628479	-4.564170333529112	
		5	-4.028001605777415	3.623598033160472	
		6	5.426177490883765 +6.696352141994065i	5.42617749088376 -6.696352141994067i	
		7	-8.475234192916405	-2.578922392281056	
		8	9.913152718172503	-3.206182813379592	
		9	2.677671679899443	1.82428114777186	
		10	1.18672458699157	2.711553564494324	
		11	1.43539240503802	2.700777985799591	
		12	2.237470143268887 +2.232703480148387i	2.237470143268887 - 2.232703480148388i	
		13	1.199443547942262 +1.199443691222013i	1.199443547942261 -1.199443691222013i	
		14	1.441546045187176	1.848322534982018	
		15	1.423573997535758	6.679481637950619	
		16	-4.822615261912324	1.733667260375862	
		17	1.402547159199412	-4.948633639651279	
		18	6.730585662140516	1.182717881167556	
		19	-5.007962765238115	1.175606675111065	
		20	6.565048430866063	1.791824885481342	
8	3	1	1.933808178839618	1.545539874804185 -1.545666045059988i	1.545539874804185 +1.545666045059988i
		2	1.347234359171865 -1.347240969937556i	1.347234359171865 +1.347240969937556i	3.304419901978651
		3	1.468276599483929	3.220895697334547	1.19694970312494
		4	1.651830400488286 -1.6521877840048i	1.651830400488287 +1.6521877840048i	1.464153355718869
		5	1.200048659530 919	1.961718914285269	1.477882030590439
		6	1.462678390951383	3.155365989103904	1.916232292792105
		7	1.195184858596888	1.694235432921724 -1.694739206592311i	1.694235432921724 +1.694739206592311i
		8	3.174113422492026	1.92094315096371	1.195647766853818

Table 10: The admissible solutions of the XXZ Bethe equations (3.4) with $q = e^{i\pi/4}$ and $N = 8$.

N	M	number	x_1	x_2	x_3
2	1	1	1.962610505505151		
3	1	1	1.461818651603003		
		2	3.574329190217505		
4	1	1	1.96261050550515		
		2	1.311033025558219		
		3	8.27753675090721		
4	2	1	1.344028591243017	2.447657661838913	
		2	2.189055410994626 -1.71493838744667i	2.189055410994627 +1.71493838744667i	
5	1	1	1.23606797749979		
		2	1.618033988749895		
		3	2.618033988749894		
5	2	1	1.749592331415895	1.259572013945332	
		2	1.667690311962022	3.974545391101193	
		3	1.724293088189099 +1.246431541912111i	1.724293088189099 -1.246431541912111i	
		4	4.116005110398541	1.246284665265577	
		5	4.142310683532282 +4.35688110427138i	4.142310683532279 -4.356881104271381i	
6	1	1	1.190729200383194		
		2	1.461818651603003		
		3	1.96261050550515		
		4	3.574329190217505		
		5	-10.40659374764855		
6	2	1	1.206790520429557	1.524710031541862	
		2	2.231459158071619	1.202712882967582	
		3	2.007446763339642	8.411463578743241	
		4	1.505088894154398	2.211779917490494	
		5	3.391097834451792 +2.137260009634191i	3.391097834451791 -2.137260009634192i	
		6	1.429452052607576 +1.038668851613124i	1.429452052607576 -1.038668851613125i	
		7	1.193470251543278	9.123706421008574	
		8	8.943547047241228	1.471698013592016	
6	3	1	1.874104285363227 -1.378223914748844i	1.874104285363228 +1.378223914748843i	2.589407182559046
		2	1.205338345735675	2.435006504125385 +1.895623247953007i	2.435006504125385 -1.895623247953007i
		3	4.792019859888095	1.147122809503764 +4.191977909831017i	1.147122809503762 -4.191977909831016i
		4	1.217923140935344	2.813376407957134	1.572774650848652
		5	2.352950008994888 +1.817348862908779i	2.352950008994887 -1.81734886290878i	1.52038626323149
7	1	1	1.160202334045279		
		2	1.37099723775096		
		3	1.699473589729839		
		4	2.375165198312843		
		5	-5.721812045187025		
		6	5.148221975956501		
7	2	1	1.408432222064447	1.171698991051096	
		2	1.401167728417128	1.805963656505584	
		3	1.812019057106945	1.170007362023291	
		4	1.279250449479749 +0.9294284209179594i	1.279250449479748 -0.9294284209179595i	
		5	2.170715518398742 +1.586182712241336i	2.170715518398741 -1.586182712241336i	
		6	1.762389509249143	2.822111962624334	
		7	1.166632159289219	2.888620018910967	
		8	2.868965063692831	1.390952988340999	
7	3	1	2.618379654246318	3.748323531486734 +3.748017767598491i	3.748323531486732 -3.748017767598492i
		2	0.9827111074314028 -3.262273894557891i	3.402279 883618681	0.982711107431404 +3.262273894557891i
		3	1.44412440163796	1.975097530601085	1.182185195050447
		4	1.639956241140929 +1.19052263024356i	1.639956241140929 -1.19052263024356i	1.92399300966388
		5	1.422694691325217	1.811547625097516 +1.312430925490274i	1.811547625097515 -1.312430925490274i
		6	1.865424819629057 +1.350217351171199i	1.865424819629057 -1.3502173511712i	1.175408810971523
		7	4.278808762679677 +4.435213551496066i	4.278808762679675 -4.435213551496067i	1.735950022329603
		8	4.482031924050006 +4.683990539341568i	4.482031924050003 -4.683990539341569i	1.163787451697995
		9	1.176249499108818	4.567826614400596	1.423871663317115
		10	1.174464705246947	1.872470411706964	4.389889034358894
		11	1.351963489253257 +0.9822482229626871i	4.883627945846047	1.351963489253257 -0.9822482229 626871i
		12	1.382220257632069	4.424719229653134 +4.614269329027579i	4.424719229653131 -4.61426932902758i
		13	1.865730317958417	4.331809157967312	1.416090213898351

Table 11: The admissible solutions of the XXZ Bethe equations (3.4) with $q = e^{i\pi/5}$ up to $N = 7$.

N	M	number	x_1	x_2	x_3
8	1	1	1.138192552062957		
		2	1.311033025558219		
		3	1.96261050550515		
		4	1.554619032420834		
		5	2.893146291465051		
		6	-4.157155527220437		
		7	8.27753675090721		
8	2	1	1.336088158806443	1.146789909509786	
		2	1.614712957660528	1.332716767162854	
		3	2.129163003251931	1.599504552643429	
		4	1.328550709738535	2.142233338886835	
		5	1.145952463660923	1.617263878227611	
		6	2.148429235497258	1.144427236851177	
		7	5.970673342816859	-6.807556417171186	
		8	1.186193082020218 +0.8618197338868376i	1.186193082020218 -0.8618197338868376i	
		9	3.700975871220594	2.044413881921601	
		10	3.22254557647494 +2.442247793000595i	3.22254557647494 -2.442247793000596i	
		11	1.816361993233109 +1.319263515796331i	1.816361993233108 -1.319263515796331i	
		12	1.580119483569546	3.800019441082347	
		13	1.308213900461231	-10.8884292670433	
		14	-10.96882884583839	1.137180644939945	
		15	1.941451195793101	-10.35085068475084	
		16	3.844192024006444	1.320876365901175	
		17	-9.448193511791008	2.79024143505726	
		18	-10.71635263086061	1.547521808575225	
		19	1.141680918344861	3.864338139442635	
		20	-20.75089982455659	-3.747556189954495	
8	3	1	1.152911608122782	1.521182841520322 +1.105269705773082i	1.521182841520321 -1.105269705773082i
		2	1.689476731560921	1.420511692843986 +1.032076474649996i	1.420511692843985 -1.032076474649996i
		3	2.116551646515748	3.190321178102575 +2.114684828886933i	3.190321178102574 -2.114684828886935i
		4	1.152031745303854	1.670604849464544	2.431584410448484
		5	1.49131548027285 +1.083548228240326i	1.49131548027285 -1.083548228240326i	1.355505247355462
		6	1.153006146461386	2.457631476354184	1.354649410334471
		7	1.155264412504946	1.696786869605106	1.361288361070377
		8	1.301230331780993 +0.945400143152111i	1.301230331780993 -0.945400143152111i	2.520770703624886
		9	2.767507419138274	0.8536816467160973 +2.639644349871746i	0.8536816467160964 -2.639644349871746i
		10	1.667446266251631	2.420 923118070947	1.350646860029378
		11	3.60404301030639 +2.298115335328595i	3.604043010306389 -2.298115335328597i	1.14351105004767
		12	2.17970199500867	8.743528394000164	1.611362131547616
		13	9.641933706319058	1.629607621987226	1.147392392511266
		14	3.466847839330259 +2.237018668690406i	3.466847839330258 -2.237018668690407i	1.595747808557541
		15	1.203999000879017 +0.8747565077890832i	10.7917885753781	1.203999000879017 -0.8747565077890832i
		16	1.91597854331014 +1.390808610142858i	9.300125780741034	1.91597854331014 -1.390808610142858i
		17	1.340380214566179	9.874749723055375	1.148237393071579
		18	2.199876642321688	9.032424520308002	1.145845769779947
		19	1.332719524956927	2.193418356684709	8.945501899172173
		20	1.336967053063209	9.546671796790799	1.627009052110716
		21	1.326274265652015	3.563336816528149 +2.279904684704862i	3.5 63336816528148 -2.279904684704864i

Table 12: The admissible solutions of the XXZ Bethe equations (3.4) with $q = e^{i\pi/5}$ and $N = 8$.

References

[1] V. Pasquier and H. Saleur, "Common Structures Between Finite Systems and Conformal Field Theories Through Quantum Groups," *Nucl.Phys.* **B330** (1990) 523.

[2] P. P. Kulish and E. K. Sklyanin, "The general U(q)(sl(2)) invariant X X Z integrable quantum spin chain," *J.Phys.* **A24** (1991) L435–L439.

[3] B.-Y. Hou, K.-J. Shi, Z.-X. Yang, and R.-H. Yue, "Integrable quantum chain and the representation of quantum group SU-q(2)," *J.Phys.* **A24** (1991) 3825–3836.

[4] L. Mezincescu and R. I. Nepomechie, "Quantum algebra structure of exactly soluble quantum spin chains," *Mod.Phys.Lett.* **A6** (1991) 2497–2508.

[5] L. Mezincescu and R. I. Nepomechie, "Analytical Bethe Ansatz for quantum algebra invariant spin chains," *Nucl.Phys.* **B372** (1992) 597–621, `arXiv:hep-th/9110050 [hep-th]`.

[6] F. C. Alcaraz, M. Droz, M. Henkel, and V. Rittenberg, "Reaction - diffusion processes, critical dynamics and quantum chains," *Annals Phys.* **230** (1994) 250–302, `arXiv:hep-th/9302112 [hep-th]`.

[7] C. Destri and H. de Vega, "Bethe ansatz and quantum groups: The Light cone approach. 2. From RSOS(p+1) models to p restricted Sine-Gordon field theories," *Nucl.Phys.* **B385** (1992) 361–394, `arXiv:hep-th/9203065 [hep-th]`.

[8] G. Juttner and M. Karowski, "Completeness of 'good' Bethe Ansatz solutions of a quantum group invariant Heisenberg Model," *Nucl.Phys.* **B430** (1994) 615–632, `arXiv:hep-th/9406183 [hep-th]`.

[9] V. Fridkin, Y. Stroganov, and D. Zagier, "Ground state of the quantum symmetric finite size X X Z spin chain with anisotropy parameter Delta = 1/2," *J.Phys.* **A33** (2000) L121–L125, `arXiv:hep-th/9912252 [hep-th]`.

[10] A. A. Belavin, S. Y. Gubanov, and B. L. Feigin, "Truncation of functional relations in the XXZ model," *Moscow Math. J.* **1** (2001) 145–156, `arXiv:hep-th/0008011 [hep-th]`.

[11] C. Korff and R. A. Weston, "PT symmetry on the lattice: The Quantum group invariant XXZ spin-chain," *J.Phys.* **A40** (2007) 8845–8872, `arXiv:math-ph/0703085 [math-ph]`.

[12] A. M. Gainutdinov and R. Vasseur, "Lattice fusion rules and logarithmic operator product expansions," *Nucl.Phys.* **B868** (2013) 223–270, `arXiv:1203.6289 [hep-th]`.

[13] A. Gainutdinov, H. Saleur, and I. Y. Tipunin, "Lattice W-algebras and logarithmic CFTs," *J.Phys.* **A47** no. 49, (2014) 495401, `arXiv:1212.1378 [hep-th]`.

[14] F. C. Alcaraz, M. N. Barber, M. T. Batchelor, R. J. Baxter, and G. R. W. Quispel, "Surface Exponents of the Quantum XXZ, Ashkin-Teller and Potts Models," *J.Phys.* **A20** (1987) 6397.

[15] E. K. Sklyanin, "Boundary Conditions for Integrable Quantum Systems," *J.Phys.* **A21** (1988) 2375–289.

[16] J. Dubail, J. L. Jacobsen, and H. Saleur, "Conformal field theory at central charge c = 0: A measure of the indecomposability (b) parameters," *Nucl.Phys.* **B834** (2010) 399–422, `arXiv:1001.1151 [math-ph]`.

[17] R. Vasseur, J. L. Jacobsen, and H. Saleur, "Indecomposability parameters in chiral Logarithmic Conformal Field Theory," *Nucl.Phys.* **B851** (2011) 314–345, `arXiv:1103.3134 [math-ph]`.

[18] A. Morin-Duchesne and Y. Saint-Aubin, "The Jordan Structure of Two Dimensional Loop Models," *J.Stat.Mech.* **1104** (2011) P04007, `arXiv:1101.2885 [math-ph]`.

[19] H. Temperley and E. Lieb, "Relations between the 'percolation' and 'colouring' problem and other graph-theoretical problems associated with regular planar lattices: some exact results for the 'percolation' problem," *Proc.Roy.Soc.Lond.* **A322** (1971) 251–280.

[20] R. I. Nepomechie and C. Wang, "Twisting singular solutions of Bethe's equations," *J.Phys.* **A47** (2014) 505004, `arXiv:1409.7382 [math-ph]`.

[21] R. J. Baxter, "Eight vertex model in lattice statistics and one-dimensional anisotropic Heisenberg chain. I. Some fundamental eigenvectors," *Annals Phys.* **76** (1973) 1–24.

[22] K. Fabricius and B. M. McCoy, "Bethe's equation is incomplete for the XXZ model at roots of unity," *J.Statist.Phys.* **103** (2001) 647–678, `arXiv:cond-mat/0009279 [cond-mat.stat-mech]`.

[23] R. J. Baxter, "Completeness of the Bethe ansatz for the six and eight vertex models," *J.Statist.Phys.* **108** (2002) 1–48, `arXiv:cond-mat/0111188 [cond-mat]`.

[24] V. O. Tarasov, "On Bethe vectors for the XXZ model at roots of unity," *J. Math. Sciences* **125** (2005) 242–248, `arXiv:math/0306032 [math.QA]`.

[25] P. Fendley and H. Saleur, "Deriving boundary S matrices," *Nucl.Phys.* **B428** (1994) 681–693, `arXiv:hep-th/9402045 [hep-th]`.

[26] L. D. Faddeev, "How algebraic Bethe ansatz works for integrable models," in *Symétries Quantiques (Les Houches Summer School Proceedings vol 64)*, A. Connes, K. Gawedzki, and J. Zinn-Justin, eds., pp. 149–219. North Holland, 1998. `arXiv:hep-th/9605187 [hep-th]`.

[27] F. H. L. Essler, H. Frahm, F. Gohmann, A. Klumper, and V. E. Korepin, *The One-Dimensional Hubbard Model*. CUP, 2005.

[28] W. Hao, R. I. Nepomechie, and A. J. Sommese, "Completeness of solutions of Bethe's equations," *Phys.Rev.* **E88** no. 5, (2013) 052113, `arXiv:1308.4645 [math-ph]`.

[29] D. J. Bates, J. D. Hauenstein, A. J. Sommese, and C. W. Wampler, *Numerically Solving Polynomial Systems with Bertini*. SIAM, 2013.

[30] V. Chari and A. Pressley, *A guide to quantum groups*. CUP, 1994.

[31] E. Mukhin, V. Tarasov, and A. Varchenko, "Bethe algebra of homogeneous XXX Heisenberg model has simple spectrum," *Commun. Math. Phys.* **288** (2009) 1–42, `arXiv:0706.0688 [math]`.

[32] A. M. Gainutdinov, J. L. Jacobsen, H. Saleur, and R. Vasseur, "A physical approach to the classification of indecomposable Virasoro representations from the blob algebra," *Nucl.Phys.* **B873** (2013) 614–681, `arXiv:1212.0093 [hep-th]`.

[33] P. P. Martin and D. S. Mcanally, "On commutants, dual pairs and nonsemisimple algebras from statistical mechanics," *Int.J.Mod.Phys.* **A7S1B** (1992) 675–705.

[34] N. Read and H. Saleur, "Associative-algebraic approach to logarithmic conformal field theories," *Nucl.Phys.* **B777** (2007) 316–351, `arXiv:hep-th/0701117 [hep-th]`.

[35] T. Deguchi, K. Fabricius, and B. M. McCoy, "The sl(2) loop algebra symmetry of the six vertex model at roots of unity. 1. Jordan-Wigner techniques," *J.Statist.Phys.* **102** (2001) 701–736, `arXiv:cond-mat/9912141 [cond-mat]`.

[36] P. Baseilhac and K. Koizumi, "A Deformed analogue of Onsager's symmetry in the XXZ open spin chain," *J.Stat.Mech.* **0510** (2005) P10005, `arXiv:hep-th/0507053 [hep-th]`.

[37] P. Baseilhac and S. Belliard, "The half-infinite XXZ chain in Onsager's approach," *Nucl.Phys.* **B873** (2013) 550–584, `arXiv:1211.6304 [math-ph]`.

[38] S. Belliard, "Modified algebraic Bethe ansatz for XXZ chain on the segment I: Triangular cases," *Nucl.Phys.* **B892** (2015) 1–20, `arXiv:1408.4840 [math-ph]`.

[39] P. P. Martin, "On Schur–Weyl duality, A_n Hecke algebras and quantum $sl(N)$ on $\otimes^{n+1}\mathbb{C}^N$," *Int.J.Mod.Phys.* **A7S1B** (1992) 645–673.