Counting Spanning Trees and Other Structures in Non-constant-jump Circulant Graphs* (Extended Abstract)

Mordecai J. Golin, Yiu Cho Leung, and Yajun Wang

Department of Computer Science, HKUST, Clear Water Bay,
Kowloon, Hong Kong
{golin, cho, yalding}@cs.ust.hk

Abstract. Circulant graphs are an extremely well-studied subclass of regular graphs, partially because they model many practical computer network topologies. It has long been known that the number of spanning trees in n-node circulant graphs with *constant jumps* satisfies a recurrence relation in n. For the non-constant-jump case, i.e., where some jump sizes can be functions of the graph size, only a few special cases such as the Möbius ladder had been studied but no general results were known.

In this note we show how that the number of spanning trees for *all* classes of n node circulant graphs satisfies a recurrence relation in n even when the jumps are non-constant (but linear) in the graph size. The technique developed is very general and can be used to show that many other structures of these circulant graphs, e.g., number of Hamiltonian Cycles, Eulerian Cycles, Eulerian Orientations, etc., also satisfy recurrence relations.

The technique presented for *deriving* the recurrence relations is very mechanical and, for circulant graphs with small jump parameters, can easily be quickly implemented on a computer. We illustrate this by deriving recurrence relations counting all of the structures listed above for various circulant graphs.

1 Introduction

The purpose of this note is to develop techniques for counting structures, e.g., Spanning Trees, Hamiltonian Cycles, Eulerian Cycles, Eulerian Orientations, Matchings, etc., in circulant graphs with non-constant jumps. We start off by defining circulant graphs and reviewing the large literature on counting spanning trees in constant-jump circulant graphs and then the much lesser literature on counting spanning trees in non-constant-jump ones.

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Definition 1. *The n-node* undirected circulant graph *with jumps* $s_1, s_2, \ldots s_k$, *is denoted by* $C_n^{s_1,s_2,\cdots,s_k}$. *This is the regular graph with* n *vertices labelled* $\{0,\ 1,\ 2,\ \cdots, n-1\}$, *such that each vertex* i $(0 \le i \le n-1)$ *is adjacent to the vertices* $i \pm s_1,\ i \pm s_2,\ \cdots,\ i \pm s_k \bmod n$. *Formally,* $C_n^{s_1,s_2,\ldots,s_k} = (V(n), E_C(n))$ *where*

$$V(n) = \{0, 1, \ldots, n-1\} \text{ and } E_C(n) = \Big\{ \{i,j\} \ :\ i - j \bmod n \in \{s_1, s_2, \ldots, s_k\} \Big\}.$$

The simplest circulant graph is the n vertex cycle C_n^1. The next simplest is the *square of the cycle* $C_n^{1,2}$ in which every vertex is connected to its two neighbors and neighbor's neighbors. The lefthand sides of figures 1, 2 and 3 illustrate various circulant graphs. Circulant graphs (sometimes known as "loop networks") are very well studied structures, in part because they model practical data connection networks [12, 3].

One frequently studied parameter of circulant graphs is the number of spanning trees they have. For connected graph G, $T(G)$ denotes the number of spanning trees in G. $T(G)$ is a examined both for its own sake and because it has practical implications for network reliability, e.g., [8, 9]. For any *fixed* graph G, Kirchhoff's *Matrix-Tree Theorem* [13] efficiently permits calculating $T(G)$ by evaluating a co-factor of the *Kirchoff matrix* of G (this essentially calculates the determinant of a matrix related to the adjacency matrix of G.) The interesting problem, then, is not in calculating the number of spanning trees in a *particular* graph, but in calculating the number of spanning trees as a function of a parameter, in graphs chosen from *classes* defined by a parameter.

The first result in this area for circulant graphs was the fact that $T(C_n^{1,2}) = nF_n^2$, F_n the *Fibonacci* numbers, i.e., $F_n = F_{n-1} + F_{n-2}$ with $F_1 = F_2 = 1$. This result, originally conjectured by Bedrosian [2] was subsequently proven by Kleitman and Golden [14]. (The same formula was also conjectured by Boesch and Wang [5] without knowledge of [14].)

Later proofs of $T(C_n^{1,2}) = nF_n^2$, and analyses of $T\left(C_n^{s_1,s_2,\cdots,s_k}\right)$ as a function of n for special *fixed* values of $s_1, s_2, \cdots, s_k$ can be found in [1, 21, 6, 24, 19, 23]. A general result due to [21] and later [25] is that, for any *fixed, constant* $1 \le s_1 < s_2 < \cdots < s_k$, $T\left(C_n^{s_1,s_2,\cdots,s_k}\right)$ satisfies a constant coefficient linear recurrence relation of order $2^{2s_k-2} - 1$.

Knowing the *existence* and *order* of the recurrence relation permits explicitly constructing it by using Kirchoff's theorem to evaluate $T(C_n^{s_1,s_2,\cdots,s_k})$ for enough values of n to solve for the coefficients of the recurrence relation.

With the exception of [14], which was a combinatorial proof using techniques very specific to the $C_n^{1,2}$ case, all of the results above were algebraic. That is they all worked by evaluating the co-factor of the Kirchoff matrix of the graphs. These co-factors could be expressed in terms of the eigenvalues of the adjacency matrices of the circulant graphs and the eigenvalues of these adjacency matrices (also known as *circulant matrices*) are well known [4]. All of the results mentioned took advantage of these facts and essentially consisted of clever algebraic manipulations of these known terms. The recurrence relations for $T(C_n^{s_1,s_2,\cdots,s_k})$

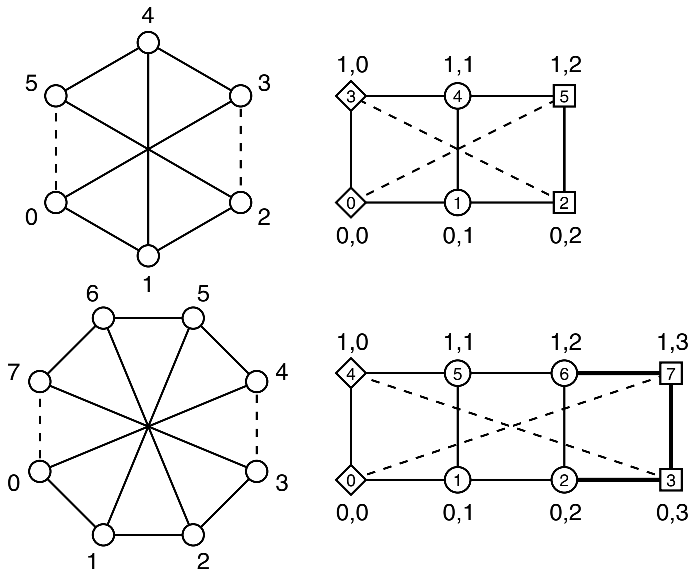

Fig. 1. The Möbius ladder $C_{2n}^{1,n}$ for $n = 3, 4$ drawn in both circulant form and lattice form. The solid edges in the figures on the right are $L_n^{1,2}$. The dashed edges are $E_C(n) - E_L(n)$. The bold edges are $E_L(n) - E_L(n-1)$. The diamond vertices are $L(n)$ while the square vertices are $R(n)$

popped out of these manipulations but did not possess any explicit combinatorial meaning.

In a recent paper [11] two of the authors of this note introduced a new technique, *unhooking*, for counting spanning trees in circulant graphs with *constant* jumps. This technique was purely combinatorial and therefore permitted a combinatorial derivation of the recurrence relations on $T(C_n^{s_1,s_2,\cdots,s_k})$. It also permitted deriving recurrence relations for the number of Hamiltonian Cycles, Eulerian Cycles, Eulerian Orientations and other structures in circulant graphs with constant jumps.

An open question in [11] was whether there was *any* general technique, combinatorial or otherwise, for counting structures in circulant graphs with non-constant jumps, i.e., graphs in which the jumps are a function of the graph size. The canonical example of such graphs is the Möbius ladder $C_{2n}^{1,n}$. (See Figure 1; Two other non-constant-jump circulant graphs, $C_{3n}^{1,n}$ and $C_{3n+1}^{1,n}$, are illustrated in Figures 2 and 3.) It is well known that

$$T\left(C_{2n}^{1,n}\right) = \frac{n}{2}\left[\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n + 2\right] \tag{1}$$

According to [6] this result is due to [18]. Other proofs can be found in [15, 17] (combinatorial) and [6] (algebraic). The combinatorial proofs are very specially crafted for the Möbius ladder and do not seem generalizable to other circulant

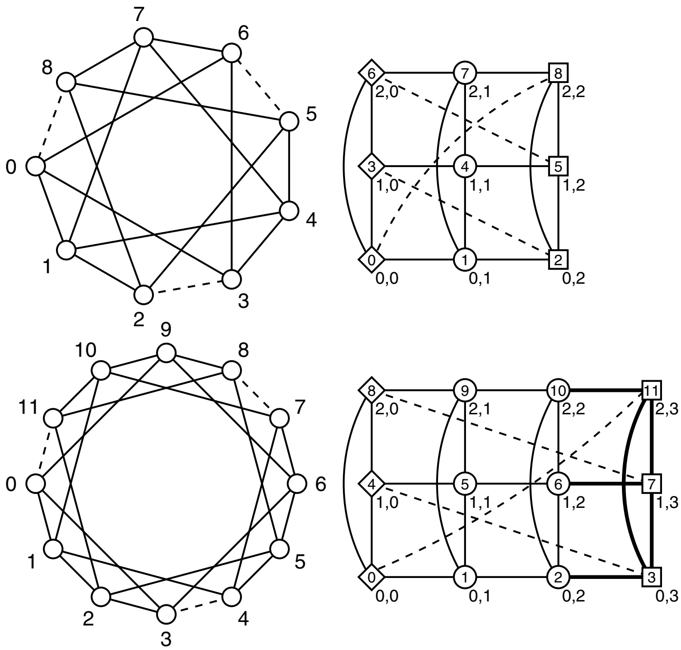

Fig. 2. $C_{3n}^{1,n}$ and $L_{3n}^{1,n}$ for $n = 3, 4$

graphs. The technique of [6] again consisted of using algebraic techniques, this time clever manipulation of Chebyshev polynomials, to evaluate a co-factor of the Kirchoff-Matrix. [10] showed how to push this Chebyshev polynomial technique slightly further to count the number of spanning trees in a small number of very special non-constant-jump circulant graphs.

The major result of this paper is a *general* technique for counting structures in circulant graphs with *non-constant* linear jumps. More specifically,

Theorem 1. *Let*

$\mathcal{A} \in$ {Spanning Trees, Hamiltonian Cycles, Eulerian Cycles, Eulerian Orientations}

For graph G let $\mathcal{A}(G)$ be the set of $\mathcal{A}$ structures in G. Now let $p, s, p_1, p_2, \ldots, p_k$ and $s_1, s_2, \ldots, s_k$ be fixed nonnegative integral constants such that $\forall i,\ p_i < p$. Set

$$C_n = C_{pn+s}^{p_1n+s_1, p_2n+s_2, \cdots, p_kn+s_k}$$

to be the circulant graph with $pn + s$ vertices and the given jumps. Set $T^{\mathcal{A}}(n) = |\mathcal{A}(C_n)|$ to be the number of $\mathcal{A}$ structures in C_n. Then

$T^{\mathcal{A}}(n)$ satisfies a linear recurrence relation with constant coefficients in n.

Note that if $\mathcal{A}$ is Spanning Trees, $p = 1, s = 0$ and $\forall i, p_i = 0$ then this collapses to the known fact [21, 25] that the number of spanning trees in circulant graphs with constant jumps satisfies a recurrence relation.

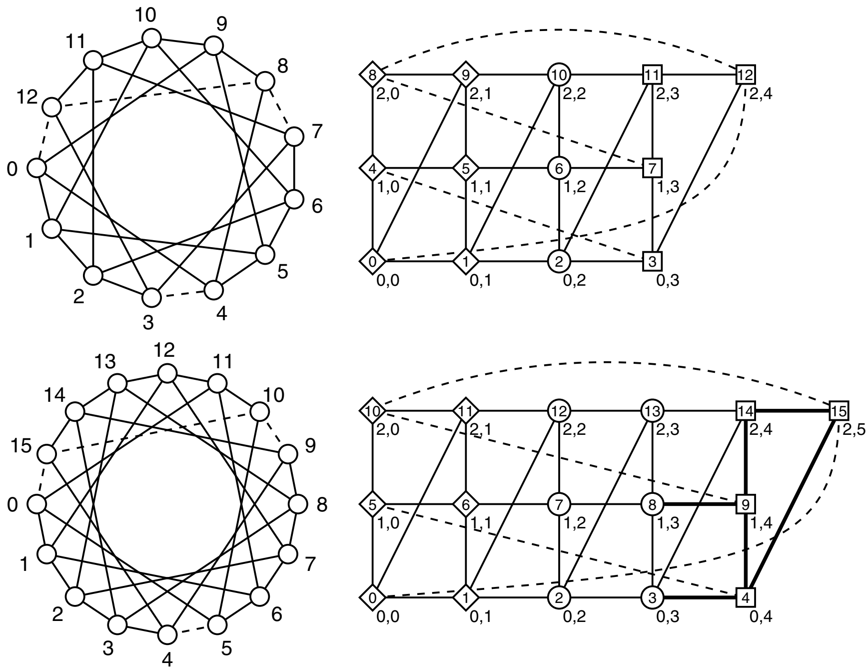

Fig. 3. $C^{1,n}_{3n+1}$ and $L^{1,n}_{3n+1}$ for $n = 4, 5$

All circulant graphs are shown in circulant form and lattice form. The solid edges in the figures on the right are the corresponding lattice graphs. The dashed edges are $E_C(n) - E_L(n)$. The bold edges are $E_L(n) - E_L(n-1)$. The diamond vertices are $L(n)$ while the square vertices are $R(n)$. Note that all edges in $E_C(n) - E_L(n)$ are in $R(n) \times L(n)$. Also note that all edges in $E_L(n+1) - E_L(n)$ are in $(R(n+1) - R(n)) \times (R(n) \cup R(n+1))$.

The proof of Theorem 1 is "constructive". That is, it shows how to build such a recurrence relation. The construction is also mechanical; that is, it is quite easy to program a computer to derive the recurrences. As examples, we have calculated the recurrence relations for various graphs which are presented in Table 1. We point out, though, that the main contribution of Theorem 1 is the *existence* of such recurrence relations. The recurrence relations (and therefore the construction) can grow exponentially in p, s and the s_i and can therefore quickly become infeasible to implement.

Technically, the major contribution of this paper is a new *representation* of non-constant-jump circulant graphs in terms of *lattice graphs*. This new representation will permit using the machinery developed in [11] to derive recurrence relations.

In the next section we introduce lattice graphs and describe how to represent circulant graphs in terms of them. We also prove properties of lattice graphs that will enable us to use them to derive recurrence relations on the number of structures. In Section 3 we then use these properties to prove Theorem 1 when

Table 1. Some sample results derived using Theorem 1. Note that for $C_{2n}^{1,n}$, the *Möbius ladder*, $T(n)$ is equivalent to (1) and was already derived in [18, 15, 17, 6] while $H(n)$ was given in [15]. The other results in the table all new. Note that the number of Eulerian Orientations and Cycles of the Möbius ladder was not calculated. This is because, as a 3-regular graph, it does not have *any* Eulerian Orientations or Cycles

C_n	Number of structures in C_n as function of n
	Spanning trees
$\mathbf{C_{2n}^{1,n}}$	$\mathbf{T(n) = 10T(n-1) - 35T(n-2) + 52T(n-3) - 35T(n-4)}$ $\mathbf{+10T(n-5) - T(n-6)}$ with initial values $16, 81, 392, 1815, 8112, 35301$ for $n = 2, 3, 4, 5, 6, 7$
$\mathbf{C_{2n+1}^{1,n}}$	$\mathbf{T(n) = 16T(n-1) - 80T(n-2) + 130T(n-3) - 80T(n-4)}$ $\mathbf{+16T(n-5) - T(n-6)}$ with initial values $125, 1183, 10404, 87131, 705757, 5581500$ for $n = 2, 3, 4, 5, 6, 7$
$\mathbf{C_{3n}^{1,n}}$	$\mathbf{T(n) = 58T(n-1) - 1131T(n-2) + 8700T(n-3) - 29493T(n-4)}$ $\mathbf{+43734T(n-5) - 29493T(n-6) + 8700T(n-7)}$ $\mathbf{-1131T(n-8) + 58T(n-9) - T(n-10)}$ with initial values $384, 12321, 371712, 10634460, 292771602,$ $7840133364, 205687578624, 5312055930723, 135495271297920,$ 3421537009450692 for $n = 2, 3, 4, 5, 6, 7, 8, 9, 10, 11$
	Hamiltonian cycles
$\mathbf{C_{2n}^{1,n}}$	$\mathbf{H(n) = H(n-1) + H(n-2) - H(n-3)}$ with initial values $3, 6, 5$ for $n = 2, 3, 4$ $= \begin{cases} n+1 \; n \text{ even} \\ n+3 \; n \text{ odd} \end{cases}$
$\mathbf{C_{2n+1}^{1,n}}$	$\mathbf{H(n) = 3H(n-1) - H(n-2) - 2H(n-3) + H(n-5)}$ with initial values $12, 23, 41, 79, 158$ for $n = 2, 3, 4, 5, 6$
	Eulerian cycles
$\mathbf{C_{3n}^{1,n}}$	$\mathbf{EC(n) = 47EC(n-1) - 742EC(n-2) + 4796EC(n-3)}$ $\mathbf{-13144EC(n-4) + 12320EC(n-5)}$ with initial values $372, 8924, 209228, 4798236, 108376972$ for $n = 2, 3, 4, 5, 6$
	Eulerian orientations
$\mathbf{C_{3n}^{1,n}}$	$\mathbf{EO(n) = 7EO(n-1) - 14EO(n-2) + 8EO(n-3)}$ with initial values $38, 142, 542$ for $n = 2, 3, 4$

$\mathcal{A}$ is "Spanning Trees". The proof is actually constructive; in the Appendix we walk through this construction to rederive the number of spanning trees in $C_{2n}^{1,n}$ as a function of n.

Note: In this extended abstract we only derive recurrence relations for the number of Spanning Trees. Derivation of the number of Hamiltonian Cycles, Eulerian Orientations and Eulerian Cycles is very similar but each derivation requires its own set of parallel definitions, lemmas and proofs, tailored to the specific problem being addressed. Due to lack of space in this extended abstract we therefore do not address them here and leave them for the full paper.

We finish this section by pointing out that counting the number of Hamiltonian Cycles, Eulerian Orientations and Eulerian Cycles in general undirected graphs is, in a qualitative way, very different from counting the number of spanning trees. While the Kirchoff matrix technique provides a polynomial time algorithm for counting the number of spanning trees, it is known that counting Hamiltonian Cycles [20], Eulerian Orientations [16] and Eulerian Cycles [7] in a general undirected graph is $\#P$-complete. When the problem is restricted to circulant graphs, with the exception of [15] which counts the number of Hamiltonian Cycles in the Möbius ladder and [22], which analyzes Hamiltonian cycles in constant jump circulant *digraphs* with $k = 2$ we know of no results (other than the previously mentioned [11]) counting these values.

2 Lattice Graphs

The major impediment to deriving recurrence relations relating the number of structures in circulant graphs to the number of structures in smaller circulant graphs is that it is difficult to see how to decompose C_n in terms of C_m where $m < n$, i.e., *big cycles can not be built from small cycles.*

The main result for this section is a way of visualizing a circulant graph $C_n = C_{pn+s}^{p_1 n+s_1, p_2 n+s_2, \cdots, p_k n+s_k}$ as a *lattice graph* L_n plus a constant number of extra edges where the constant *does not depend upon* n.

The reason for taking this approach is that, unlike the circulant graph, the lattice graphs *are* built recursively with L_{n+1} being L_n with the addition of a constant (independent of n) set of edges.

Before starting we will first have to rethink the way that we define circulant graphs.

Definition 2. *Let $p, s, p_1, p_2, \ldots, p_k$ and $s_1, s_2, \ldots, s_k$ be given nonnegative integral constants. For u, v and integer n, set $f(n; u, v) = un + v$. Define*

$$\widehat{C}_n = \left(\widehat{V}_C(n),\ \widehat{E}_C(n)\right)$$

where

$$\widehat{V}(n) = \{\, (u, v) \,:\, 0 \le u \le p-1,\, 0 \le v \le n-1\} \cup \{\, (p-1, v) \,|\, n \le v \le n+s-1\}$$

$$\widehat{E}_C(n) = \bigcup_{i=1}^{k} \left\{ \begin{array}{l} \{(u_1, v_1), (u_2, v_2)\} \,:\, (u_1, v_1),\, (u_2, v_2) \in \widehat{V}(n) \text{ and} \\ \qquad\qquad f(n; u_2, v_2) - f(n; u_1, v_1) \equiv p_i n + s_i \pmod{pn+s} \end{array} \right\}$$

Directly from the definition we see $\widehat{C}_n$ is *isomorphic* to $C_n = C_{pn+s}^{p_1 n+s_1, p_2 n+s_2, \cdots, p_k n+s_k}$. Figures 1 2 and 3 illustrate this isomorphism; the graphs on the left are the circulant graphs drawn in the normal way; the graphs on the right, the new way, with node labelling showing the isomorphism.

Since the graphs are isomorphic, counting structures in C_n is equivalent to counting structures in $\widehat{C}_n$. Therefore, *for the rest of this paper* we will replace C_n by $\widehat{C}_n$. That is C_n will refer to $\widehat{C}_n$, $V(n)$ to $\widehat{V}(n)$ and $E_C(n)$ to $\widehat{E}_C(n)$. Given this new representation we can now define

Definition 3 (Lattice Graphs). *Let $p, s,\ p_1, p_2, \ldots, p_k$ and $s_1, s_2, \ldots, s_k$ be given nonnegative integral constants. Define the* Lattice Graph

$$L_n = L_{pn+s}^{p_1 n+s_1, p_2 n+s_2, \cdots, p_k n+s_k}$$

to be the graph $L_n = (V(n), E_L(n))$ where $V(n)$ is the vertex set defined in Definition 2 and

$$E_L(n) = \bigcup_{i=1}^{k} \left\{ \begin{array}{l} \{(u_1,v_1),(u_2,v_2)\} : (u_1,v_1),\ (u_2,v_2) \in \widehat{V}(n), \\ \qquad f(n;u_2,v_2) - f(n;u_1,v_1) \equiv p_i n + s_i \pmod{pn+s}, \\ \qquad \text{and } u_2 - u_1 \equiv p_i \pmod{p} \end{array} \right\}$$

In Figures 1, 2 and 3 the solid edges in the graphs on the right form the Lattice Graphs. By comparing the definition of $E_L(n)$ in this definition to that of $E_C(n) = \widehat{E}_C(n)$ given in Definition 2 we immediately have that $E_L(n) \subseteq E_C(n)$. Referring back to the examples in Figures 1, 2 and 3 again we see that the edges in $E_C(n) - E_L(n)$, the dashed edges, always seem to connect vertices on the left (diamonds) with vertices on the right (squares). This simple observation will be at the core of our analysis. We first need the following definition:

Definition 4. *Set $s_{max} = \max\{s, s_1, s_2, \cdots, s_k\}$. Then*

$$L(n) = \Big\{ (u,v) \ : \ 0 \le u \le (p-1),\ 0 \le v \le s + s_{max} - 1 \Big\}, \qquad \text{Left Vertices}$$
$$R(n) = \Big\{ (u,v) \ : \ 0 \le u \le (p-1),\ v \ge n - s_{max} \Big\}, \qquad \text{Right Vertices}$$

In order to make our analysis work we will require that $L(n) \cap R(n) = \emptyset$. To ensure this we will, from now on, require that $n > s + 2s_{max} - 1$.

Suppose $e = \{(u_1,v_1),(u_2,v_2)\} \in E_C(n)$ is a jump $p_i n + s_i$ from (u_1,v_1). There are two cases:

1. The jump does not "cross" vertex $(p-1, n+s-1)$:
$$u_2 n + v_2 - (u_1 n + v_1) = p_i n + s_i. \tag{2}$$
If $e \in E_L(n)$, we have $u_2 - u_1 = p_i$ and $v_2 - v_1 = s_i$. If $e \in E_C(n) - E_L(n)$, because $n > s + 2s_{max} - 1$, we will have $v_2 - v_1 - s_i = (p_i - u_2 + u_1)n$, so $u_2 - u_1 = p_i \pm 1$.
2. The jump crosses vertex $(p-1, n+s-1)$:
$$u_2 n + v_2 - (u_1 n + v_1) + pn + s = p_i n + s_i. \tag{3}$$
If $e \in E_L(n)$, we have $v_2 - v_1 + s = s_i$ and $u_2 - u_1 + p = p_i$. If $e \in E_C(n) - E_L(n)$, we have $v_2 - v_1 + s - s_i = (p_i - u_2 + u_1 - p)n$, where $p_i - u_2 + u_1 - p = \pm 1$.

We can now prove a simple structural lemma on the relationship between circulant graphs and lattice graphs:[1]

[1] For sets A, B we use the notation $A \times B$ to denote $\{\{a,b\} : a \in A,\ b \in B\}$.

Lemma 1. *Given the above definitions*

$$E_C(n) - E_L(n) \subseteq R(n) \times L(n)$$

Furthermore, as functions of n, $E_C(n) - E_L(n)$ is constant.

As examples of this lemma note that, for $C_{2n}^{1,n}$ (Figure 1)

$$E_C(n) - E_L(n) = \Big\{ \{(0,0),\ (1,n-1)\},\ \{(1,0),\ (0,n-1)\} \Big\}$$

while for $C_{3n+1}^{1,n}$ (Figure 3)

$$E_C(n)-E_L(n)=\Big\{\{(0,0),\ (2,n)\},\ \{(1,0),\ (0,n-1)\},\ \{(2,0),\ (2,n)\},\ \{(2,0),\ (1,n-1)\}\Big\}$$

Proof. Suppose edge $e = \{(u_1,v_1),(u_2,v_2)\} \in E_C(n) - E_L(n)$. The analysis above gives us $|v_2 - v_1| \geq n - s_{max}$. But on the other hand , from Definition 4, if $e \notin L(n) \times R(n)$, $|v_2 - v_1| \leq n - s_{max} - 1$. Thus $e \in R(n) \times L(n)$.

For the constant property, we show that the condition that an edge $e = \{(u_1,v_1),(u_2,n-v_2)\} \in E_C(n) - E_L(n)$ does not depend on n, where $(u_1,v_1) \in L(n)$ and $(u_2,n-v_2) \in R(n)$. Note that v_2 can be negative and $v_1 \leq s + s_{max} - 1, |v_2| \leq s_{max}$.

There are four cases:

1. e is a jump from (u_1,v_1) by $p_i n + s_i$ and does not cross vertex $(p-1,n+s-1)$: We will have $n \pm n - v_1 - v_2 = s_i$. Because $v_1 \leq s + s_{max} - 1, |v_2| \leq s_{max}$, the $\pm n$ term must be $-n$, $s_i + v_1 + v_2 = 0$ and $u_2 - u_1 = p_i - 1$. The fact that $e \in E_C(n) - E_L(n)$, is therefore independent of n.
2. e is a jump from (u_1,v_1) by $p_i n + s_i$ and crosses vertex $(p-1,n+s-1)$: We will have $n \pm n + s - v_1 - v_2 = s_i$. The $\pm n$ term must be $-n$, $s - v_1 - v_2 = s_i$, and $p_i - u_2 + u_1 - p = 1$. The fact that $e \in E_C(n) - E_L(n)$, is therefore independent of n.
3. e is a jump from $(u_2,n-v_2)$ and does not cross vertex $(p-1,n+s-1)$: We have $n \pm n + v_1 + v_2 = s_i$. The $\pm n$ term must be $-n$, $v_1 + v_2 = s_i$, and $u_1 - u_2 = p_i + 1$. The fact that $e \in E_C(n) - E_L(n)$, is therefore independent of n.
4. e is a jump from $(u-2,n-v_2)$ and crosses vertex $(p-1,n+s-1)$: We have $n \pm n + v_1 + v_2 + s = s_i$. The $\pm n$ term must be $-n$, $v_1 + v_2 + s = s_i$, and $u_1 - u_2 + p = p_i + 1$. The fact that $e \in E_C(n) - E_L(n)$, is therefore independent of n.

All conditions above are independent of n. So the edge set $E_C(n) - E_L(n)$ is constant (independent of n).

We have just seen that C_n can be built from L_n using a constant set of edges. We will now see that L_{n+1} can also be built from L_n using a constant set of edges.

Lemma 2.

$$E_L(n+1) - E_L(n) \subseteq (R(n+1) - R(n)) \times (R(n) \cup R(n+1)).$$

Furthermore, as functions of n, the edge set $E_L(n+1) - E_L(n)$ is constant.

As examples of this lemma note that, for $C_{2n}^{1,n}$ (Figure 1)

$$E_L(n+1) - E_L(n) = \Big\{ \{(0,n-1),\,(0,n)\},\,\{(1,n-1),\,(1,n)\},\,\{(0,n),\,(1,n)\} \Big\}$$

while for $C_{3n+1}^{1,n}$ (Figure 3)

$$E_L(n+1) - E_L(n) = \Big\{ \{(0,n),\,(1,n)\},\,\{(0,n-1),\,(0,n)\},\,\{(0,n),\,(2,n+1)\}$$
$$,\,\{(1,n-1),\,(1,n)\},\,\{(2,n),\,(2,n+1)\},\,\{(1,n),\,(2,n)\} \Big\}$$

Proof. Equations (2),(3) do not depend on n when $e = \{(u_1, v_1), (u_2, v_2)\} \in E_L(n)$. Thus if $e \in E_L(n+1)$ and both vertices of e are in $V_L(n)$, $e \in E_L(n)$. So if $e \in E_L(n+1) - E(n)$, e must contain one vertex in $V_L(n+1) - V_L(n) = R(n+1) - R(n)$. Furthermore we have $|v_2 - v_1| \leq s_{max}$ from the equations, which means the other node is in $R(n) \cup R(n+1)$.

For the constant property, from Equations (2) and Equation (3), $e(n) = \{(u_1, n - v_1), (u_2, n - v_2)\} \in E_L(n)$ does not depend on n. If $e(n) \in E_L(n) - E_L(n-1)$, $e(n)$ contains at least one vertex in $R(n) - R(n-1)$ which directly implies that $e(n+1)$ contains one vertex in $R(n+1) - R(n)$. Thus $e(n) \in E_L(n) - E_L(n-1)$ does not depend on n. So $E_L(n+1) - E_L(n)$ is constant. □

3 Spanning Trees

In this section $\mathcal{A}$ objects are spanning trees and our problem is to count the number of spanning trees in C_n. Recall that, given a graph $G = (V, E)$, a spanning tree $T \subseteq E$ is a subset of the edges that forms a connected acyclic graph.

Let $p, s,\ p_1, p_2, \ldots, p_k$ and $s_1, s_2, \ldots, s_k$ be given nonnegative integral constants with $\forall i,\ p_i \leq p$. Set $C_n = C_{pn+s}^{p_1 n+s_1, p_2 n+s_2, \cdots, p_k n+s_k}$ to be the circular graph, $L_n = L_{pn+s}^{p_1 n+s_1, p_2 n+s_2, \cdots, p_k n+s_k}$ to be the lattice graph, and $T(n) = T(C_n)$ to be the number of spanning trees in C_n.

In [11] tools were developed for constructing recurrence relations on structures of *constant-jump* circulant graphs. The difficulty in extending that result to *non-constant-jump* circulants was the lack of some type of recursive decomposition of non-constant-jump circulants. Given the Lattice graph representation of circulant graphs of Lemma 1 and the recursive construction of Lattice graphs implied by Lemma 2 we can now plug these facts into the tools developed in [11] and develop recurrence formulas for $T(n)$. Since the proofs are rather straightforward and follow those of [11] we do not give them here. In the appendix we will show how to use these techniques to rederive the exact solution for $T\left(C_{2n}^{1,n}\right)$.

Let T be a spanning tree of circulant graph C_n. Removing all edges of $E_C(n) - E_L(n)$ from T leaves a forest $T \cap E_L$ in Lattice graph L_n. Lemma 1 tells us that all endpoints of edges in $E_C(n) - E_L(n)$ are in $L(n) \cup R(n)$, so *every component of the forest $T \cap E_L$ must contain at least one node from $L(n) \cup R(n)$.* This motivates the following definition of *legal forests*:

Definition 5. *Let $n > s + 2s_{max} - 1$. Set $W(n) = L(n) \cup R(n)$. Let $Par(W)$ be the collection of all set partitions of set W, e.g.,*
$Par(\{1,2,3\}) = \{ \{\{1,2\}\{3\}\}, \{\{1,3\}\{2\}\}, \{\{2,3\}\{1\}\}, \{\{1\}\{2\}\{3\}\}, \{\{1,2,3\}\} \}$.
Then

1. *A* legal forest *F in L_n is one in which every connected component of F contains at least one node in $W(n)$.*
2. *$\mathcal{P} = Par(W(n))$ is the collection of all* set partitions *of $W(n)$.*
3. *Let F be a legal forest of L_n. Then $C(F)$, the* classification of *F is $X \in \mathcal{P}$ such that $\forall u, v \in W(n)$, u, v are in the same connected component of F iff u, v are in the same set in X.*
4. *For $X \in \mathcal{P}$ set $T_X(n) = |\{F : F$ is a legal forest of L_n with $C(F) = X\}|$*

Note: The reason for requiring $n > s + 2s_{max} - 1$ is to guarantee that $L(n) \cap R(n) = \emptyset$.

We are now interested in how to reconstruct spanning trees from legal forests. Define

Definition 6. $\mathcal{S} = \{S : S \subseteq (E_C - E_L)\}$.

Note that, given a *legal forest* F, it may not always possible to find $S \in \mathcal{S}$ such that $F \cup S$ is a spanning tree of C_n. We make the following straightforward observation

Lemma 3. *Let F, F' be two legal forests of L_n such that $C(F) = C(F')$ and $S \in \mathcal{S}$. Then $F \cup S$ is a spanning tree of C_n if and only if $F' \cup S$ is a spanning tree of C_n.*

This permits the following definition

Definition 7. *For $X \in \mathcal{P}$ and $S \in \mathcal{S}$ set*

$$\alpha_{S,X} = \begin{cases} 1: \textit{if adding } S \textit{ to forest } F \textit{ with } C(F) = X \textit{ yields a spanning tree of } C_n. \\ 0: \textit{otherwise} \end{cases}$$

$$\beta_X = \sum_{S \in \mathcal{S}} \alpha_{S,X} \quad \textit{and} \quad \boldsymbol{\beta} = (\beta_X)_{X \in \mathcal{P}},$$

where vector $\boldsymbol{\beta}$ is ordered using some fixed arbitrary ordering of the elements $\mathcal{P}$.

The crucial observation in the above definitions is that Lemma 3 implies that $\alpha_{S,X}$ is *independent* of n and can be easily evaluated just by looking at S and X.

As an example, suppose we are given $C_{2n}^{1,n}$ and its $L_{2n}^{1,n}$ and, for some n, F is a legal forest of $L_{2n}^{1,n}$ with $C(F) = X = \{(0,0), (0, n-1), (1,0)\}, \{(1, n-1)\}$. That is, F has exactly two connected components partitioning the nodes in

$W(n) = L(n) \cup R(n)$; one of the components contains $(0,0), (0, n-1), (1, 0)$ and the other contains $(1, n-1)$. Now, if $S = \{\{(0,0), (1, n-1)\}\}$ then $\alpha_{S,X} = 1$ since the single edge in S connects the two components to form a spanning tree while if $S = \{\{(0, n-1), (1, 0)\}\}$ then $\alpha_{S,X} = 0$ since the single edge in S creates a cycle in the component containing $(0,0), (0, n-1), (1, 0)$.

Since, by definition, every spanning tree of C_n is uniquely decomposable into a legal forest F of L_n plus some $S \in \mathcal{S}$ we immediately find

Lemma 4.

$$T(C_n) = \sum_{X \in \mathcal{P}} \left(\sum_{S \in \mathcal{S}} \alpha_{S,X} \right) T_X(n) = \sum_{X \in \mathcal{P}} \beta_X T_X(n). \tag{4}$$

Letting $\boldsymbol{T}(L_n)$ be the column vector $(T_X(n))_{X \in \mathcal{P}}$, this can also be written as $T(C_n) = \boldsymbol{\beta} \cdot \boldsymbol{T}(L_n)$.

So far we have only shown that the number of spanning trees of a circulant graph is a linear combination of the number of different legal forests of the associated lattice graph. We will now show the the number of different legal forests can be written as a system of linear recurrences in n. The main observation is the following lemma:

Lemma 5. *Let F be a legal forest in L_{n+1} and $U = F \cap \big(E_L(n+1) - E_L(n)\big)$. Then $F - U$ is a legal forest in L_n.*

Note that this lemma implies that every legal forest of L_{n+1} can be built from a legal forest of L_n. To continue we will need the following observation:

Lemma 6. *Let F, F' be legal forests in L_n such that $C(F) = C(F')$. Let $U \subseteq E_L(n+1) - E_L(n)$. Then*

- *$F \cup U$ is a legal forest of L_{n+1} if and only if $F' \cup U$ is a legal forest of L_n and*
- *if both $F \cup U$ and $F' \cup U$ are legal forests of L_{n+1} then $C(F \cup U) = C(F' \cup U)$.*

Before continuing we should emphasize a subtle point concerning the classification of a *legal forest* in L_n, which is that *it strongly depends upon n*. For example, in lattice graph $L_{2n}^{1,n}$, when $n = 3$, a *legal forest D* with classification $C(D) = \{(0,0), (0, n-1)\}, \{(1,0), (1, n-1)\}$ implies that D has two components with one containing nodes $(0,0)$ and $(0,2)$ and the other containing nodes $(1,0)$ and $(1,2)$. Now suppose $n = 4$ with no edges added to D, in the new lattice graph, the new forest D' contains four components, which means $C(D') = \{(0,0)\}, \{(0, n-1)\}, \{(1,0)\}, \{(1, n-1)\}$. When calculating how adding vertices and edges to legal forests in L_n changes them into different legal forests in L_{n+1} we must take account of this fact.

Lemma 6 permits the next definition

Definition 8. *For $X, X' \in \mathcal{P}$ and $U \subseteq E_L(n+1) - E_L(n)$ set*

$$\gamma_{X',X,U} = \begin{cases} 1 : \textit{if adding } U \textit{ to forest } F \textit{ with } C(F) = X' \textit{ yields a forest } F' \\ \quad\;\; \textit{with } C(F') = X' \\ 0 : \textit{otherwise} \end{cases}$$

$$\alpha_{X',X} = \sum_{U \subseteq E_L(n+1) - E_L(n)} \gamma_{X',X,U} \quad \textit{and} \quad A = (a_{X',X})_{X',X \in \mathcal{P}}$$

where, in the last equation, A is a square matrix whose columns/rows are ordered using the same ordering as in the definition of $\boldsymbol{\beta}$ in Definition 7.

Note: As in the observation following Lemma 7 we point out that the values of $\gamma_{X',X,U}$ and thus $\alpha_{X',X}$ and A are *independent* of n. It is therefore possible to mechanically calculate all of the β_X and $\alpha_{X',X}$.

Combining Lemmas 5 and 6 then yield

Lemma 7. $\forall X' \in \mathcal{P}$,

$$T_{X'}(n+1) = \sum_{X \in \mathcal{P}} a_{X',X} T_X(n) \quad \textit{or, equivalently,} \quad \boldsymbol{T}(L_{n+1}) = A\boldsymbol{T}(L_n)$$

Combining everything in this section gives our main theorem on spanning trees of circulant graphs, proving Theorem 1 for the case $\mathcal{A} =$ *Spanning Trees.*

Theorem 2. *Let $T(n)$ denote the number of spanning trees in C_n. Let $\mathcal{P} = Par(L(n) \cup R(n))$, $T_X(n)$ denote the number of* legal forests *with classification X and $\boldsymbol{T}(L(n))$ be the column vector $(T_X(n))_{X \in \mathcal{P}}$ Then, for $n \geq s + 2s_{max} - 1$,*

$$T(C(n)) = \boldsymbol{\beta} \cdot \boldsymbol{T}(L(n))$$
$$\boldsymbol{T}(L(n+1)) = A\boldsymbol{T}(L(n))$$

where $\boldsymbol{\beta}$ is the constant vector defined in Definition 7 and A is the constant square matrix defined in Definition 8.

This theorem implies that $T(C(n))$ satisfies a linear recurrence recurrence relation with constant coefficients of order rank of the matrix A.

Since the *size* of matrix A is $|\mathcal{P}| = B(p(s+2s_{max})+s)$ where $B(m)$ is the Bell number[2] of order m the order of the recurrence is at most $B(p(s+2s_{max})+s)$.

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[2] $B(m)$ counts the number of set partitions of m items.

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