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Counting Unbranched Subgraphs

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Abstract. Given an arbitrary finite graph, the polynomial $Q(z) = \sum_{F\in\mathcal{U}} z^{\mathrm{card}F}$ associates a weight $z^{\mathrm{card}F}$ to each unbranched subgraph F of length card$F$. We show that all the zeros of Q have negative real part.

Keywords: counting polynomial, graph, unbranched subgraph

A graph (V, E, v) consists of a finite set V of vertices, a finite set E of edges, and a map v of E to the two-element subsets of V. If $a \in E$ and $v(a) = \{j, k\}$, we say that the edge a joins the vertices j, k. (We impose that $j \neq k$, but allow different edges to join the same two vertices. We assume that each vertex j is in $v(a)$ for some $a \in E$).

For our purposes a *subgraph* of (V, E, v) will be a graph (V, F, ϕ) where $F \subset E$ and $\phi = v \mid F$. We shall now fix (V, E, v), and say that F is a subgraph of E if $F \subset E$ (this defines (V, F, ϕ) uniquely). We define the subset $\mathcal{U}$ of *unbranched* subgraphs of E by

$$\mathcal{U} = \{F \subset E : (\forall j)\,\mathrm{card}\{a \in F : v(a) \ni j\} \leq 2\}$$

Proposition 1 *The polynomial*

$$Q_{\mathcal{U}}(z) = \sum_{F\in\mathcal{U}} z^{\mathrm{card}F}$$

has all its zeros in $\{z : \mathrm{Re}z \leq -2/n(n-1)^2\}$ *where* $n \geq 2$ *is the largest number of edges ending in any vertex* j.

The proof is given below. This result is related to a well-known theorem of Heilman and Lieb [2] on counting dimer subgraphs (for which $\mathrm{card}\{a \in F : v(a) \ni j\} \leq 1$).

Let us consider an edge a as a closed line segment containing the endpoints $j, k \in v(a)$. Also identify a subgraph $F \subset E$ with the union of its edges. Then F is the union of its connected components, and if $F \in \mathcal{U}$, these are homeomorphic to a line segment or to a circle. We call $b(F)$ the number of components homeomorphic to a line segment, therefore

$$2b(F) = \mathrm{card}\{j \in V : v(a) \ni j \text{ for exactly one } a \in F\}$$

Let us define

$$Q_{\mathcal{U}}(z,t) = \sum_{F\in\mathcal{U}} z^{\mathrm{card}F} t^{b(F)}.$$

We see that

$$Q_{\mathcal{U}}(z,1) = Q_{\mathcal{U}}(z).$$

Proposition 2 *If t is real $\geq 2 - 2/n$, then $Q_{\mathcal{U}}(z,t)$ has all its zeros (with respect to z) on the negative real axis.*

The proof is given below. For $t \geq 2$, this is a special case a theorem of Wagner [6] as pointed out by the referee (take $Q_v(y) = 1 + sy + y^2/2$ for each vertex v in Theorem 3.2 of [6]).

We shall use the following two lemmas.

Lemma 1 *Let A, B be closed subsets of the complex plane $\mathbf{C}$, which do not contain 0. Suppose that the complex polynomial*

$$\alpha + \beta z_1 + \gamma z_2 + \delta z_1 z_2$$

can vanish only when $z_1 \in A$ or $z_2 \in B$. Then

$$\alpha + \delta z$$

can vanish only when $z \in -AB$.

This is the key step in an extension (see Ruelle [5]) of the Lee-Yang circle theorem [3]. Note that in applications of the lemma, the coefficients α, β, γ, δ are usually polynomials in variables z_j (different from z_1, z_2, z).

Lemma 2 *Let $Q(z)$ be a polynomial of degree n with complex coefficients and $P(z_1,\ldots,z_n)$ the only polynomial which is symmetric in its arguments, of degree 1 in each, and such that*

$$P(z,\ldots,z) = Q(z).$$

If the roots of Q are all contained in a closed circular region M, and $z_1 \notin M,\ldots,z_n \notin M$, then $P(z_1,\ldots,z_n) \neq 0$).

This is Grace's theorem, see Polya and Szegö [4] V, Exercise 145.

Proof of Proposition 1: If $a \in E$, and $v(a) = \{j,k\}$, we introduce complex variables z_{aj}, z_{ak}. For each $j \in V$, let p_j be the polynomial in $Z^{(j)} = (z_{aj})_{v(a)\ni j}$ such that

$$p_j(Z^{(j)}) = 1 + \sum_a z_{aj} + \sum_{a\neq b} z_{aj} z_{bj}$$

(where we assume $v(a) \ni j$, $v(b) \ni j$). Putting all z_{aj} equal to z, we obtain a polynomial

$$q_j(z) = 1 + n_j z + \frac{n_j(n_j - 1)}{2} z^2$$

where $n_j > 0$ is the number of edges ending in j. Define $\zeta_\pm^{(j)} = -1$ when $n_j = 1$ or 2, and

$$\zeta_\pm^{(j)} = \frac{-n_j \pm \sqrt{2n_j - n_j^2}}{n_j(n_j - 1)}$$

if $n_j \geq 2$. The zeros of q_j, considered as a polynomial of degree n_j are $\zeta_\pm^{(j)}$, and ∞ if $n_j > 2$. They are therefore contained in the closed circular regions (half-planes)

$$H_{\theta+}^{(j)} = \left\{z : \mathrm{Re}\left[e^{-i\theta}\left(z - \zeta_+^{(j)}\right)\right] \leq 0\right\}$$
$$H_{\theta-}^{(j)} = \left\{z : \mathrm{Re}\left[e^{i\theta}\left(z - \zeta_-^{(j)}\right)\right] \leq 0\right\}$$

for $0 \leq \theta < \pi/4$. By Lemma 2, we have thus $p_j(Z^{(j)}) \neq 0$ if $z_{aj} \notin H_{\theta\pm}^{(j)}$ for all $a \in E$ such that $j \in v(a)$.

If a polynomial is separately of first order in two variables z_1, z_2, i.e., it is of the form

$$\alpha + \beta z_1 + \gamma z_2 + \delta z_1 z_2$$

the *Asano contraction* [1] consists in replacing it by the first-order polynomial

$$\alpha + \delta z$$

in one variable z, as in Lemma 1. As already noted, the coefficients $\alpha, \beta, \gamma, \delta$ may depend on variables z_i different from z_1, z_2, z. Let now $Z = (z_a)_{a \in E}$ and

$$P_{\mathcal{U}}(Z) = \sum_{F \in \mathcal{U}} \prod_{a \in F} z_a.$$

If we take the product $\prod_{j \in V} p_j(Z^{(j)})$ and perform the Asano contraction

$$\alpha + \beta z_{aj} + \gamma z_{ak} + \delta z_{aj} z_{ak} \longrightarrow \alpha + \delta z_a$$

for all $a \in E$ we obtain $P_{\mathcal{U}}(Z)$. Using Lemma 1 iteratively, once for each edge $a \in E$, we see thus that $P_{\mathcal{U}}(Z)$ has no zeros when for each $a \in E$

$$z_a \in \mathbf{C} \Big\backslash \left(-H_{\theta\pm}^{(j)} H_{\theta\pm}^{(k)}\right)$$

where $v(a) = \{j, k\}$ and

$$H_{\theta\pm}^{(j)} H_{\theta\pm}^{(k)} = \left\{uv : u \in H_{\theta\pm}^{(j)}, v \in H_{\theta\pm}^{(k)}\right\}.$$

We have

$$\mathbf{C}\setminus\big(-H_{\theta\pm}^{(j)}H_{\theta\pm}^{(k)}\big) \supset \mathbf{C}\setminus(-H_{\theta\pm}H_{\theta\pm})$$

where $H_{\theta\pm}$ is the largest $H_{\theta\pm}^{(j)}$ (obtained by replacing n_j by $n = \max_j n_j$). Note that $\mathbf{C}\setminus(-H_{\theta\pm}H_{\theta\pm})$ is the interior of a parabola passing through $-\zeta_\pm^2$ and with axis passing through 0 and making an angle $\pm 2\theta$ with the positive real axis. When $\pm\theta$ varies between $-\pi/4$ and $\pi/4$, the parabola sweeps the region $\mathrm{Re}z > -\mathrm{Re}\zeta_\pm^2 = -2/n(n-1)^2$. Since $Q_{\mathcal{U}}(z)$ is obtained from $P_{\mathcal{U}}(Z)$ by putting all z_a equal to z, this proves Proposition 1. □

Proof of Proposition 2: We proceed as for Proposition 1, defining here

$$p_j(Z^{(j)}) = 1 + s\sum_a z_{aj} + \sum_{a\neq b} z_{aj}z_{bj},$$

$$q_j(z,t) = 1 + n_j sz + \frac{n_j(n_j-1)}{2}z^2.$$

If $s \geq \sqrt{2-2/n_j}$, the roots of q_j are real negative, and the same type of argument used for theorem 1 shows that all the zeros of $Q_{\mathcal{U}}(z, s^2)$ are real and negative. □

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