Coupled Generalized Nonlinear Stokes Flow with flow through a Porous Media

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Abstract

In this article, we analyze the flow of a fluid through a coupled Stokes-Darcy domain. The fluid in each domain is non-Newtonian, modeled by the generalized nonlinear Stokes equation in the free flow region and the generalized nonlinear Darcy equation in the porous medium. A flow rate is specified along the inflow portion of the free flow boundary. We show existence and uniqueness of a variational solution to the problem. We propose and analyze an approximation algorithm and establish a-priori error estimates for the approximation.

Key words. Generalized nonlinear Stokes flow; coupled Stokes and Darcy flow; defective boundary condition

AMS Mathematics subject classifications. 65N30

1 Introduction

The coupling of Stokes and Darcy flow problems has received significant attention over the past several years due to its importance in modeling problems such as surface fluid flow coupled with flow in a porous media (see, for instance, [4, 9, 12, 14, 16, 20, 21]). As in [12], the investigation in this paper is motivated by industrial filtering applications where a non-Newtonian fluid passes through a filter to remove unwanted particulates. The lifetime of the filter is dictated by the increase in pressure drop across the porous medium. This pressure drop increase occurs as debris, transported into the filter by the free flowing fluid, deposits into the filter. Models of the coupled system are necessary to develop simulators that can aid in the design of filters with extended lifetimes and minimize release of debris into the downstream flow.

In these applications, flow rates are typically specified at the inflow of the filtering apparatus. Our first step in modeling the filtration problem is to consider the case of coupled nonlinear Stokes-Darcy flow problem with defective boundary conditions. Namely, we assume that only flow rates are specified along the inflow boundary. In [12], the authors use the Darcy equation as a boundary condition for the Stokes problem in the free-flow region. We couple the flows across the internal boundary by using conservation of mass and balance of forces across the interface, as in [9, 14, 20, 21].

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For Newtonian fluids the extra stress tensor, $\boldsymbol{\tau}$, is proportional to the deformation tensor, $\mathbf{d}(\mathbf{u})$, with the constant of proportionality being the value of the kinematic viscosity, ν. Our model problem uses generalized power law fluids, which are an extension of Newtonian fluids. Generalized power law fluids have a non-constant viscosity that is a function of the magnitude of the deformation tensor. Models for such viscosity functions include the following [3, 17]:

Carreau model:

$$\nu(\mathbf{d}(\mathbf{u})) \;=\; \nu_\infty \,+\, (\nu_0 - \nu_\infty)/(1 \,+\, K|\mathbf{d}(\mathbf{u})|^2)^{(2-r)/2}\,, \text{ where } r > 1,\ \nu_0,\ \nu_\infty, K > 0 \text{ are constants,} \tag{1.1}$$

Cross model:

$$\nu(\mathbf{d}(\mathbf{u})) \;=\; \nu_\infty \,+\, (\nu_0 - \nu_\infty)/(1 \,+\, K|\mathbf{d}(\mathbf{u})|^{(2-r)})\,, \text{ where } r > 1,\ \nu_0, \nu_\infty, K > 0 \text{ are constants,} \tag{1.2}$$

Power law model:

$$\nu(\mathbf{d}(\mathbf{u})) \;=\; K|\mathbf{d}(\mathbf{u})|^{r-2}\,, \text{ where } r > 1,\ K > 0 \text{ are constants.} \tag{1.3}$$

Many generalized Newtonian fluids exhibit a sheer thinning property; that is, as the magnitude of $\mathbf{d}(\mathbf{u})$ increases the viscosity decreases. For the above models this corresponds to a value for r between 1 and 2. Generalized power law viscosity models have been used in modeling the viscosity of biological fluids, lubricants, paints, and polymeric fluids. In the analysis below we assume a general function for $\nu(\mathbf{d}(\mathbf{u}))$ that satisfies particular continuity and monotonicity properties. (See (2.16),(2.17).)

For non-Newtonian fluid flow in a porous medium, various models for the effective viscosity ν_{eff} have been proposed in the literature. (See for example, [15, 18] and the references cited therein.) Based upon dimensional analysis most models assume that ν_{eff} is a function of $|\mathbf{u}_p|/(\sqrt{\kappa}\, m_c)$, where κ denotes the permability of the porous medium, $\mathbf{u}_p$ the Darcy velocity, and m_c is a constant related to the internal structure of the porous media. Models for ν_{eff} include [15, 18]:

Cross model:

$$\nu_{eff}(\mathbf{u}_p) \;=\; \nu_\infty \,+\, (\nu_0 - \nu_\infty)/(1 \,+\, K\,|\mathbf{u}_p|^{2-r})\,, \text{ where } r > 1,\ \nu_0,\ \nu_\infty, K > 0 \text{ are constants,} \tag{1.4}$$

Power law model:

$$\nu_{eff}(\mathbf{u}_p) \;=\; K\,\left(|\mathbf{u}_p|/(\sqrt{\kappa}\, m_c)\right)^{r-2}\,, \text{ where } r > 1,\ K > 0 \text{ are constants.} \tag{1.5}$$

Again, in the analysis below we assume a general function for $\nu_{eff}(\mathbf{u}_p)$ that satisfies particular continuity and monotonicity properties. (See (2.16),(2.17).)

Remark: In this work we ignore the influence of pressure on viscosity.

The variational formulation presented below for the coupled nonlinear flow problem (ignoring the defective boundary conditions) is analogous to that for the linear coupled problem studied in [9, 14, 20, 21]. However as the function setting for the linear problem is in Hilbert spaces ($H_1(\Omega)$, $L^2(\Omega)$) compared to Banach spaces ($W_{1,r}(\Omega)$, $L^{r'}(\Omega)$) for the nonlinear problem, the analysis used herein is considerably different than that in [9, 14, 20, 21].

2 Modeling equations

Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, denote the flow domain of interest. Additionally, let Ω_f and Ω_p denote bounded Lipschitz domains for the nonlinear generalized Stokes flow and nonlinear generalized Darcy flow, respectively. The interface boundary between the domains we denote by $\Gamma := \partial\Omega_f \cap \partial\Omega_p$. Note that $\Omega := \Omega_f \cup \Omega_p \cup \Gamma$. The outward pointing unit normal vectors to Ω_f and Ω_p are denoted $\mathbf{n}_f$ and $\mathbf{n}_p$, respectively. The tangent vectors on Γ are denoted by $\mathbf{t}_1$ (for n = 2), or $\mathbf{t}_l, l = 1, 2$, (for n = 3).

We assume that there is an inflow boundary Γ_{in}, a subset of $\partial\Omega_f \backslash \Gamma$, which is separated from Γ, and an outflow boundary Γ_{out}, a subset of $\partial\Omega_p \backslash \Gamma$, which is also separated from Γ. See Figure 2.1 for an illustration of the domain of the problem.

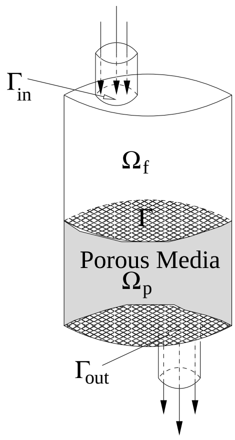

Figure 2.1: Illustration of flow domain.

Define $\Gamma_f := \partial\Omega_f \backslash (\Gamma \cup \Gamma_{in})$, and $\Gamma_p := \partial\Omega_p \backslash (\Gamma \cup \Gamma_{out})$.

Velocities are denoted by $\mathbf{u}_j : \Omega_j \to \mathbb{R}^n$, $j = f, p$, and pressures are denoted by $p_j : \Omega_j \to \mathbb{R}$, $j = f, p$.

In Ω_f, we assume that the flow is governed by the nonlinear generalized Stokes flow, subject to a specified flow rate, $-fr$, across Γ_{in} and no-slip condition on Γ_f:

$$-\nabla \cdot (\boldsymbol{\sigma} - p_f \mathbf{I}) = \mathbf{f}_f \quad \text{in } \Omega_f , \tag{2.1}$$

$$\nabla \cdot \mathbf{u}_f = 0 \quad \text{in } \Omega_f , \tag{2.2}$$

$$\boldsymbol{\sigma} = g_f(\mathbf{d}(\mathbf{u}_f))\mathbf{d}(\mathbf{u}_f) \quad \text{in } \Omega_f , \tag{2.3}$$

$$\mathbf{u}_f = \mathbf{0} \quad \text{on } \Gamma_f , \tag{2.4}$$

$$\int_{\Gamma_{in}} \mathbf{u}_f \cdot \mathbf{n}_f \, ds = -fr , \tag{2.5}$$

where $\boldsymbol{\sigma}$ denotes the fluid's extra stress tensor and $\mathbf{d}(\mathbf{v}) := \frac{1}{2}(\nabla\mathbf{v} + \nabla^T\mathbf{v})$ is the deformation tensor. The particular form for the nonlinear viscosity function $g_f(\cdot)$ is discussed in Section 2.2. For simplicity we consider here the case of a single inflow boundary Γ_{in}. Multiple inflow boundary segments with separately specified *flow rates* can also be modeled [6, 7, 11].

We assume that the flow in the porous domain Ω_p is governed by a generalized Darcy's equation subject to a specified flow rate, fr, across Γ_{out} and a non-penetration condition on Γ_p:

$$\mathbf{u}_p = -\frac{\kappa}{\nu_{eff}} \nabla p_p \quad \text{in } \Omega_p \,, \tag{2.6}$$

$$\nabla \cdot \mathbf{u}_p = 0 \quad \text{in } \Omega_p \,, \tag{2.7}$$

$$\mathbf{u}_p \cdot \mathbf{n}_p = 0 \quad \text{on } \Gamma_p \,, \tag{2.8}$$

$$\int_{\Gamma_{out}} \mathbf{u}_p \cdot \mathbf{n}_f \, ds = fr \,, \tag{2.9}$$

In general κ denotes a symmetric, positive definite tensor. For simplicity, we will assume κ is a positive (scalar) constant.

2.1 Interface conditions

The flows in Ω_f and Ω_p are coupled across the interface Γ. Conditions describing the coupling of the flows are discussed below.

Conservation of mass across Γ: The conservation of mass across Γ imposes the constraint

$$\mathbf{u}_f \cdot \mathbf{n}_f + \mathbf{u}_p \cdot \mathbf{n}_p = 0 \quad \text{on } \Gamma \,, \tag{2.10}$$

Balance of the normal forces across Γ: The balance of the normal forces across Γ imposes the constraint

$$p_f - (\boldsymbol{\sigma} \mathbf{n}_f) \cdot \mathbf{n}_f = p_p \quad \text{on } \Gamma \,, \tag{2.11}$$

Balance of the forces on Γ: For the tangential forces on Γ we use the Beavers–Joseph–Saffman condition [1, 13, 22]

$$\mathbf{u}_f \cdot \mathbf{t}_l = -csr_l \, (\boldsymbol{\sigma} \mathbf{n}_f) \cdot \mathbf{t}_l \quad \text{on } \Gamma, \quad l = 1, \ldots, n-1, \tag{2.12}$$

where $csr_l, \quad l = 1, \ldots, n-1$, denote frictional constants that can be determined experimentally.

2.2 Variational Formulations

Given $r \in \mathbb{R}$, $r > 1$, we denote its unitary conjugate by r', satisfying $r^{-1} + (r')^{-1} = 1$.

For Ω_f, define

$$X_f := \left\{ \mathbf{v} : \mathbf{v} \in (W^{1,r}(\Omega_f))^n \mid \ \mathbf{v}|_{\Gamma_f} = \mathbf{0} \right\}, \quad \text{and} \quad M_f := L^{r'}(\Omega_f) \,.$$

For $\mathbf{v} \in X_f$, $q \in M_f$ we define $\|\mathbf{v}\|_{X_f} := \|\mathbf{v}\|_{(W^{1,r}(\Omega_f))^n}$, and $\|q\|_{M_f} := \|q\|_{L^{r'}(\Omega_f)}$.

For Ω_p, define

$$L^r(div, \Omega_p) := \{ \mathbf{v} : \mathbf{v} \in (L^r(\Omega_p))^n \text{ and } \nabla \cdot \mathbf{v} \in L^r(\Omega_p) \} \,,$$

$$X_p := \left\{ \mathbf{v} : \mathbf{v} \in L^r(div, \Omega_p) \,|\, \mathbf{v} \cdot \mathbf{n}|_{\Gamma_p} = 0 \right\} \,, \quad \text{and} \quad M_p := L^{r'}(\Omega_p) \,.$$

Similarly, for $\mathbf{v} \in X_p$, $q \in M_p$, we define $\|\mathbf{v}\|_{X_p} := \|\mathbf{v}\|_{(L^r(\Omega_p))^n} + \|\nabla \cdot \mathbf{v}\|_{L^r(\Omega_p)}$, and $\|q\|_{M_p} := \|q\|_{L^{r'}(\Omega_p)}$.

We also use the spaces X, and M defined on Ω by

$$X := X_f \times X_p\,, \text{ and } \quad M := \left\{ q \in M_f \times M_p \,|\, \int_\Omega q\, dA = 0 \right\}\,,$$

and denote the dual space of X by X^*.

For $\mathbf{v} = (\mathbf{v}_f\,, \mathbf{v}_p) \in X$ and $q = (q_f\,, q_p) \in M$,

$$\|\mathbf{v}\|_X := \|\mathbf{v}_f\|_{X_f} + \|\mathbf{v}_p\|_{X_p}\,, \text{ and } \quad \|q\|_M := \left(\|q_f\|^{r'}_{L^{r'}(\Omega_f)} + \|q_p\|^{r'}_{L^{r'}(\Omega_p)} \right)^{1/r'}\,.$$

Also, for $f,\, k : \Omega \to \mathbb{R}^m$, $(f, k) := \int_\Omega f \cdot k\, dA$.

Let $g(\mathbf{x}) : \mathbb{R}^N \longrightarrow \mathbb{R}^+ \cup \{0\}$ and $G(\mathbf{x}) : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be given by $G(\mathbf{x}) := g(\mathbf{x})\mathbf{x}$. Further for $\mathbf{x}, \mathbf{h} \in \mathbb{R}^N$, let $G(\cdot)$ satisfy, (for constants C_1, C_2, $C_3 > 0$, and $c \geq 0$)

$$\textbf{A1:} \;\; |G(\mathbf{x}+\mathbf{h}) - G(\mathbf{x})|\,|\mathbf{h}| \;\leq\; C_1\,(G(\mathbf{x}+\mathbf{h}) - G(\mathbf{x}))\cdot\mathbf{h}\,, \tag{2.13}$$

$$\textbf{A2:} \;\; \frac{|\mathbf{h}|^2}{c + |\mathbf{x}|^{2-r} + |\mathbf{x}+\mathbf{h}|^{2-r}} \;\leq\; C_2\,(G(\mathbf{x}+\mathbf{h}) - G(\mathbf{x}))\cdot\mathbf{h}\,,\;, \tag{2.14}$$

$$\textbf{A3:} \;\; |G(\mathbf{x}+\mathbf{h}) - G(\mathbf{x})| \;\leq\; C_3 \frac{|\mathbf{h}|}{c + |\mathbf{x}|^{2-r} + |\mathbf{x}+\mathbf{h}|^{2-r}}\,, \tag{2.15}$$

with the convention that $G(\mathbf{x}) = \mathbf{0}$ if $\mathbf{x} = \mathbf{0}$, and $|\mathbf{h}|/(c + |\mathbf{x}| + |\mathbf{h}|) = 0$ if $c = 0$ and $\mathbf{x} = \mathbf{h} = \mathbf{0}$.

From **A1**, **A2** and **A3** it follows (see [23]) that there exists constants $C_4, C_5 > 0$ such that for $\mathbf{s}, \mathbf{t}, \mathbf{w} \in (L^r(\Omega))^N$

$$\int_\Omega (G(\mathbf{s}) - G(\mathbf{t}))\cdot(\mathbf{s}-\mathbf{t})\, dA \;\geq\; C_4 \left(\int_\Omega |G(\mathbf{s}) - G(\mathbf{t})||\mathbf{s}-\mathbf{t}|\, dA + \frac{\|\mathbf{s}-\mathbf{t}\|^2_{L^r(\Omega)}}{c + \|\mathbf{s}\|^{2-r}_{L^r(\Omega)} + \|\mathbf{t}\|^{2-r}_{L^r(\Omega)}} \right)\,, \tag{2.16}$$

$$\int_\Omega (G(\mathbf{s}) - G(\mathbf{t}))\cdot\mathbf{w}\, dA \;\leq\; C_5 \left\| \frac{|\mathbf{s}-\mathbf{t}|}{c + |\mathbf{s}| + |\mathbf{t}|} \right\|_\infty^{\frac{2-r}{r}} \left(\int_\Omega |G(\mathbf{s}) - G(\mathbf{t})||\mathbf{s}-\mathbf{t}|\, dA \right)^{1/r'} \|\mathbf{w}\|_{L^r(\Omega)}\,. \tag{2.17}$$

In Ω_p, with $\mathbf{x}, \mathbf{h}$ in (2.13)–(2.15) denoting vectors in $\mathbb{R}^n$ and $\cdot$ the usual vector dot product, we assume that $g_p(\mathbf{u}_p) := \nu_{eff}/\kappa$, and let $G_p(\mathbf{v}) = g_p(\mathbf{v})\mathbf{v}$.

In Ω_f we assume that $\boldsymbol{\sigma} = g_f(\mathbf{d}(\mathbf{u}_f))\mathbf{d}(\mathbf{u}_f)$, and let $G_f(\boldsymbol{\tau}) := g_f(\boldsymbol{\tau})\boldsymbol{\tau}$, where we interpret $\mathbf{x}, \mathbf{h}$ in (2.13)–(2.15) as tensors in $\mathbb{R}^{n\times n}$ and $\cdot$ as the usual tensor scalar product :.

Remark: For $\nu_\infty = 0$, conditions (2.13)-(2.15) are satisfied for $G_f(\boldsymbol{\tau})$ and $G_p(\mathbf{v})$ with $g_f(\mathbf{d}(\mathbf{u})) = \nu(\mathbf{d}(\mathbf{u}))$ described in (1.1),(1.2),(1.3), and $g_p(\mathbf{u}_p) = \nu_{eff}(\mathbf{u}_p)$ described in (1.4),(1.5) (see [23]). Different functions spaces are required, that the setting studied herein, for $\nu_\infty > 0$

Multiplying (2.1) through by $\mathbf{v}_1 \in X_f$, integrating over Ω_f, using (2.3) and the fact that $\{\mathbf{n}_f, \mathbf{t}_l, l = 1, \ldots, n-1\}$ form an orthonormal basis along Γ, we have

$$\begin{aligned}
\int_{\Omega_f} \mathbf{f}_f \cdot \mathbf{v}_1\, dA \;&=\; \int_{\Omega_f} \boldsymbol{\sigma} : \mathbf{d}(\mathbf{v}_1)\, dA - \int_{\Omega_f} p_f \nabla\cdot\mathbf{v}_1\, dA - \int_{\Gamma\cup\Gamma_{in}} ((-p_f\mathbf{I} + \boldsymbol{\sigma})\mathbf{n}_f)\cdot\mathbf{v}_1\, ds \\
&=\; \int_{\Omega_f} g_f(\mathbf{d}(\mathbf{u}_f))\mathbf{d}(\mathbf{u}_f) : \mathbf{d}(\mathbf{v}_1)\, dA - \int_{\Omega_f} p_f \nabla\cdot\mathbf{v}_1\, dA + \sum_{l=1}^{n-1} \int_\Gamma -\mathbf{n}_f^T\boldsymbol{\sigma}\mathbf{t}_l\; \mathbf{v}_1\cdot\mathbf{t}_l\, ds \\
&\quad + \int_\Gamma (p_f - \mathbf{n}_f^T\boldsymbol{\sigma}\mathbf{n}_f)\, \mathbf{v}_1\cdot\mathbf{n}_f\, ds - \int_{\Gamma_{in}} ((-p_f\mathbf{I} + \boldsymbol{\sigma})\mathbf{n}_f)\cdot\mathbf{v}_1\, ds\,.
\end{aligned} \tag{2.18}$$

Also, multiplying (2.6) through by $\mathbf{v}_2 \in X_p$ and integrating over Ω_p we obtain

$$0 \;=\; \int_{\Omega_p} g_p(\mathbf{u}_p)\mathbf{u}_p \cdot \mathbf{v}_2 \, dA \;-\; \int_{\Omega_p} p_p \, \nabla \cdot \mathbf{v}_2 \, dA \;+\; \int_{\Gamma_{out}} p_p \, \mathbf{v}_2 \cdot \mathbf{n}_p \, ds$$
$$+ \int_{\Gamma} p_p \, \mathbf{v}_2 \cdot \mathbf{n}_p \, ds \,. \tag{2.19}$$

The coupling of the Stokes and Darcy flows occur through the interface conditions (2.10) and (2.11). Following [14], we introduce a new variable λ representing

$$\lambda := p_f \;-\; (\boldsymbol{\sigma}\mathbf{n}_f) \cdot \mathbf{n}_f \;=\; p_p \,, \tag{2.20}$$

and incorporate (2.11) into (2.18) and (2.19). Equation (2.10) is imposed *weakly* in a separate equation. (See (2.32) below.)

Note that using the Beavers–Joseph–Saffman condition (2.12)

$$\sum_{l=1}^{n-1} \int_{\Gamma} - \mathbf{n}_f^T \, \boldsymbol{\sigma} \mathbf{t}_l \; \mathbf{v}_1 \cdot \mathbf{t}_l \, ds \;=\; \sum_{l=1}^{n-1} \int_{\Gamma} c s r_l^{-1} \, (\mathbf{u}_f \cdot \mathbf{t}_l) \; (\mathbf{v}_1 \cdot \mathbf{t}_l) \, ds \,.$$

To incorporate the specified flow rate conditions into the mathematical formulation we use a Lagrange multiplier approach. In (2.18) and (2.19)

$$\int_{\Gamma_{in}} ((-p_f \mathbf{I} \,+\, \boldsymbol{\sigma})\mathbf{n}_f) \cdot \mathbf{v}_1 \, ds \quad \text{is replaced by} \quad \beta_{in} \int_{\Gamma_{in}} \mathbf{v}_1 \cdot \mathbf{n}_f \, ds \,, \tag{2.21}$$

$$\int_{\Gamma_{out}} p_p \, \mathbf{v}_2 \cdot \mathbf{n}_p \, ds \quad \text{is replaced by} \quad \beta_{out} \int_{\Gamma_{out}} \mathbf{v}_2 \cdot \mathbf{n}_p \, ds \,, \tag{2.22}$$

where β_{in}, $\beta_{out} \in \mathbb{R}$ are undetermined constants. Below we comment on the implicit assumptions induced by using the Lagrange multiplier approach.

For $\mathbf{v} \in W^{0,r}(div, \Omega_p)$, we have that $\mathbf{v} \cdot \mathbf{n}_p \in W^{-1/r,r}(\partial\Omega_p)$, [8] pg. 47.

For $\mathbf{v} \in X_p$ and $\lambda \in W^{1/r,r'}(\Gamma)$ we define the operator $\mathbf{v} \cdot \mathbf{n}_p \in W^{-1/r,r}(\Gamma)$ as

$$\langle \mathbf{v} \cdot \mathbf{n}_p \,,\, \lambda \rangle_{\Gamma} \;:=\; \langle \mathbf{v} \cdot \mathbf{n}_p \,,\, E_{\Gamma}^{r'} \lambda \rangle_{\partial\Omega_p} \,, \tag{2.23}$$

with $E_{\Gamma}^{r'}\lambda$ defined as in Lemma 9 in the **Appendix** (with the association $p = r'$, $\Omega = \Omega_p$, $\Gamma = \Gamma$, $\Gamma_b = \Gamma_p$, $\Gamma_d = \Gamma_{out}$).

Note that for $\mathbf{v} \in X_p$ sufficiently smooth,

$$\langle \mathbf{v} \cdot \mathbf{n}_p \,,\, \lambda \rangle_{\Gamma} \;=\; \langle \mathbf{v} \cdot \mathbf{n}_p \,,\, E_{\Gamma}^{r'} \lambda \rangle_{\partial\Omega_p} \;=\; \int_{\Gamma} \mathbf{v} \cdot \mathbf{n}_p \, \lambda \, ds \,.$$

For $\mathbf{v} \in (W^{1,r}(\Omega_f))^n$ we have that $\mathbf{v} \cdot \mathbf{n}_f \in W^{1/r',r}(\partial\Omega_f)$, hence $\int_{\Gamma} \mathbf{v} \cdot \mathbf{n}_f \, \lambda \, ds$ is well defined.

In order to compactly write the mathematical formulation we introduce the following bilinear forms:

$$a_f(\mathbf{u},\mathbf{v}) \;:=\; \int_{\Omega_f} g_f(\mathbf{d}(\mathbf{u}))\mathbf{d}(\mathbf{u}) : \mathbf{d}(\mathbf{v})\, dA + \sum_{l=1}^{n-1} \int_\Gamma csr_l^{-1}\, (\mathbf{u}\cdot\mathbf{t}_l)\,(\mathbf{v}\cdot\mathbf{t}_l)\, ds\,, \tag{2.24}$$

$$a_p(\mathbf{u},\mathbf{v}) \;:=\; \int_{\Omega_p} g_p(\mathbf{u})\mathbf{u}\cdot\mathbf{v}\, dA\,, \tag{2.25}$$

$$b_f(\mathbf{v},q,\beta) \;:=\; \int_{\Omega_f} q\nabla\cdot\mathbf{v}\, dA \,+\, \beta\int_{\Gamma_{in}} \mathbf{v}\cdot\mathbf{n}_f\, ds\,, \tag{2.26}$$

$$b_p(\mathbf{v},q,\beta) \;:=\; \int_{\Omega_p} q\,\nabla\cdot\mathbf{v}\, dA \,+\, \beta\int_{\Gamma_{out}} \mathbf{v}\cdot\mathbf{n}_p\, ds\,. \tag{2.27}$$

With the above notation, the modeling equations in Ω_f may be written as:

$$a_f(\mathbf{u}_f,\mathbf{v}_1) \;-\; b_f(\mathbf{v}_1,p_f,\beta_{in}) \;+\; \int_\Gamma \mathbf{v}_1\cdot\mathbf{n}_f\,\lambda\, ds \;=\; (\mathbf{f}_f,\mathbf{v}_1)_{\Omega_f} \quad \forall \mathbf{v}_1\in X_f\,, \tag{2.28}$$

$$b_f(\mathbf{u}_f,q_1,\beta_1) \;=\; -\beta_1\, fr \quad \forall (q_1\times\beta_1)\in M_f\times \mathbb{R}\,, \tag{2.29}$$

and in Ω_p

$$a_p(\mathbf{u}_p,\mathbf{v}_2) \;-\; b_p(\mathbf{v}_2,p_p,\beta_{out}) \;+\; \langle\lambda,\mathbf{v}_2\cdot\mathbf{n}_p\rangle_\Gamma \;=\; 0 \quad \forall\mathbf{v}_2\in X_p\,, \tag{2.30}$$

$$b_p(\mathbf{u}_p,q_2,\beta_2) \;=\; \beta_2\, fr \quad \forall (q_2\times\beta_2)\in M_p\times\mathbb{R}\,. \tag{2.31}$$

Together with (2.28)–(2.31) we have the interface condition (2.10). We impose this constraint weakly using

$$\int_\Gamma \mathbf{u}_f\cdot\mathbf{n}_f\,\zeta\, ds \;+\; \langle \mathbf{u}_p\cdot\mathbf{n}_p,\zeta\rangle_\Gamma \;=\; 0\,, \quad \forall\zeta\in W^{1/r,r'}(\Gamma)\,. \tag{2.32}$$

Introduce $\mathbf{f} := (\mathbf{f}_f\,,\,\mathbf{0})$, $b_I(\cdot,\cdot) : X\times W^{1/r,r'}(\Gamma)\to\mathbb{R}$ as

$$b_I(\mathbf{v},\zeta) \;:=\; \int_\Gamma \mathbf{v}_f\cdot\mathbf{n}_f\,\zeta ds \;+\; \langle\mathbf{v}_p\cdot\mathbf{n}_p,\zeta\rangle_\Gamma\,, \tag{2.33}$$

and $a(\cdot,\cdot): X\times X\to\mathbb{R}$, $b(\cdot,\cdot,\cdot) : X\times M\times\mathbb{R}^2\to\mathbb{R}$ as

$$a(\mathbf{u},\mathbf{v}) \;:=\; a_f(\mathbf{u}_f,\mathbf{v}_f) \,+\, a_p(\mathbf{u}_p,\mathbf{v}_p) \;\text{ and }\; b(\mathbf{v},q,\boldsymbol{\gamma}) \;:=\; b_f(\mathbf{v}_f,q_f,\gamma_1) \,+\, b_p(\mathbf{v}_p,q_p,\gamma_2)\,. \tag{2.34}$$

We then state the coupled fluid flow problem as: *Given* $\mathbf{f}\in X^*$*,* $fr\in\mathbb{R}$*, determine* $(\mathbf{u},p,\lambda,\boldsymbol{\beta})\in X\times M\times W^{1/r,r'}(\Gamma)\times\mathbb{R}^2$ *such that*

$$a(\mathbf{u},\mathbf{v}) \,-\, b(\mathbf{v},p,\boldsymbol{\beta}) \,+\, b_I(\mathbf{v},\lambda) \;=\; (\mathbf{f},\mathbf{v}) \quad \forall\mathbf{v}\in X\,, \tag{2.35}$$

$$b(\mathbf{u},q,\boldsymbol{\gamma}) \,-\, b_I(\mathbf{u},\zeta) \;=\; \boldsymbol{\gamma}\cdot\begin{bmatrix} -1 \\ 1\end{bmatrix} fr \quad \forall (q,\zeta,\boldsymbol{\gamma}) \,\in\, M\times W^{1/r,r'}(\Gamma)\times\mathbb{R}^2\,. \tag{2.36}$$

The unique solvability of (2.35)-(2.36) hinges on showing two inf-sup conditions: one for $b(\cdot,\cdot,\cdot)$, and the other for $b_I(\cdot,\cdot)$.

Equivalence of the Differential Equations and Variational Formulations
As demonstrated above, the variational formulation (2.35)-(2.36) is obtained by multiplying the differential equations by sufficiently smooth functions, integrating over the domain and, where appropriate, applying Green's theorem. In addition we used (2.21)-(2.22) to impose the specified flow rate boundary conditions. For a smooth solution the steps used in deriving the variational equations can be reversed to show that equations (2.1)-(2.5), (2.6)-(2.9) are satisfied. In addition we have that a smooth solution of (2.35)-(2.36) satisfies the following additional boundary conditions (see [7]).

For $\mathbf{n}_f$ the outward normal on Γ_{in}, express the extra stress vector on Γ_{in}, $\boldsymbol{\sigma}\mathbf{n}_f$, as

$$\boldsymbol{\sigma}\mathbf{n}_f \;=\; s_n\mathbf{n}_f \,+\, \mathbf{s}_T\,,$$

where $s_n = (\boldsymbol{\sigma}\mathbf{n}_f)\cdot\mathbf{n}_f$ and $\mathbf{s}_T = \boldsymbol{\sigma}\mathbf{n}_f - s_n\mathbf{n}_f$. The scalar s_n represents the magnitude of the extra stress in the outward normal direction to Γ_{in}, and $\mathbf{s}_T$ the component of the extra stress vector which lies in the plane of Γ_{in}.

Lemma 1 *Any smooth solution of (2.35),(2.36) satisfies the following boundary conditions.*

$$\text{on } \Gamma_{in}: \qquad -p_f + s_n = -\beta_{in} \quad \text{and} \quad \mathbf{s}_T = \mathbf{0}\,. \tag{2.37}$$

$$\text{on } \Gamma_{out}: \qquad p_p = -\beta_{out}\,. \tag{2.38}$$

Proof: The proof follows as in [7].

■

Remark: The equations (2.1)-(2.5), (2.6)-(2.9), (2.10)-(2.12), do not uniquely define a solution, but rather a set of solutions. The variational formulation (2.35)-(2.36) chooses a solution from the solution set. Specifically, (2.35)-(2.36) chooses *the solution* which satisfies (2.37)-(2.38) . A different variational formulation may result in the selection of a different solution from the solution set. (See, for example, [7].)

3 Existence and Uniqueness of the Variational Formulation

In order to show the existence and uniqueness of the variational formulation we introduce the following subspaces of X:

$$V := \{\mathbf{v}\in X \mid b_I(\mathbf{v},\zeta) = 0, \ \text{ for all } \zeta\in W^{1/r,r'}(\Gamma)\}\ , \tag{3.1}$$

$$Z := \{\mathbf{v}\in V \mid b(\mathbf{v},q,\boldsymbol{\gamma}) = 0, \ \text{ for all } (q,\boldsymbol{\gamma})\in M\times\mathbb{R}^2\}\ . \tag{3.2}$$

Consider $b(\cdot,\cdot,\cdot) : X\times M\times\mathbb{R}^2 \to \mathbb{R}$ defined in (2.34). Using Hölders inequality together with the definition (2.23), we have that $b(\cdot,\cdot,\cdot)$ is continuous. In addition, $b(\cdot,\cdot,\cdot)$ satisfies the following inf-sup condition.

Lemma 2 *There exists* $C_{MRV} > 0$ *such that*

$$\inf_{(0,\mathbf{0})\neq(q,\boldsymbol{\gamma})\in M\times\mathbb{R}^2}\ \sup_{\mathbf{u}\in V} \frac{b(\mathbf{u},q,\boldsymbol{\gamma})}{\|\mathbf{u}\|_X\|(q,\boldsymbol{\gamma})\|_{M\times\mathbb{R}^2}} \geq C_{MRV}\ , \tag{3.3}$$

where $\|(q,\boldsymbol{\gamma})\|_{M\times\mathbb{R}^2} := \|q\|_M + \|\boldsymbol{\gamma}\|_{\mathbb{R}^2}$.

Proof:
Fix $(q, \boldsymbol{\gamma}) \in M \times \mathbb{R}^2$ and let

$$\hat{q} := \frac{|q|^{r'/r-1} q}{\|q\|_M^{r'-1}} \quad , \quad \hat{\boldsymbol{\gamma}} := \frac{\boldsymbol{\gamma}}{\|\boldsymbol{\gamma}\|_{\mathbb{R}^2}} \, . \tag{3.4}$$

Note that $\int_\Omega q \, \hat{q} \, d\Omega \;=\; \|q\|_M, \;\; \|\hat{q}\|_{L^r(\Omega)} \;=\; 1$, and $\boldsymbol{\gamma} \cdot \hat{\boldsymbol{\gamma}} \;=\; \|\boldsymbol{\gamma}\|_{\mathbb{R}^2}, \;\; \|\hat{\boldsymbol{\gamma}}\|_{\mathbb{R}^2} \;=\; 1$.

Let $\Gamma_i^m \subset \Gamma_i$ such that $meas(\Gamma_i^m) > 0$, and $dist(\Gamma_i^m, \partial\Omega \backslash \Gamma_i) > 0$, for $i = in, out$.

Let $h \in \; C(\partial\Omega) \subset W^{1/r',r}(\partial\Omega)$ be given by

$$\begin{aligned} h|_{\Gamma_i^m} \;&:=\; \hat{\gamma}_i / meas(\Gamma_i^m) \, , i = in, \, out \; , \\ h|_{\partial\Omega \backslash (\Gamma_{in} \cup \Gamma_{out})} \;&:=\; 0 \end{aligned}$$

and on $\Gamma_i \backslash \Gamma_i^m$ h is either a strictly increasing or strictly decreasing function.

Also, let $\delta \in \mathbb{R}$ be given by

$$\delta := \left(\int_{\partial\Omega} h \, ds \;-\; \int_\Omega \hat{q} \, dA \right) / meas(\Omega) \; .$$

From [8] pg. 127, given $f \in L^r(\Omega), \; \mathbf{a} \in W^{1-1/r,r}(\partial\Omega), \; 1 < r < \infty$, satisfying

$$\int_\Omega f \, dA \;=\; \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n} \, ds \, , \tag{3.5}$$

there exists $\mathbf{v} \in W^{1,r}(\Omega)$ such that

$$\nabla \cdot \mathbf{v} \;=\; f \qquad \text{in } \Omega \, , \tag{3.6}$$

$$\mathbf{v} \;=\; \mathbf{a} \qquad \text{on } \partial\Omega \, , \tag{3.7}$$

$$\text{with } \|\mathbf{v}\|_{W^{1,r}(\Omega)} \;\leq\; C \left(\|f\|_{L^r(\Omega)} \,+\, \|\mathbf{a}\|_{W^{1-1/r,r}(\partial\Omega)} \right) \, . \tag{3.8}$$

Let $f \;=\; \hat{q} \,+\, \delta$, and for $\{\mathbf{n}, \mathbf{t}_i, i \;=\; 1, \ldots n-1\}$ denoting an orthonormal system on $\partial\Omega$, let $\mathbf{a}$ be defined by

$$\left\{ \begin{aligned} \mathbf{a} \cdot \mathbf{n} \;&=\; h \\ \mathbf{a} \cdot \mathbf{t}_i \;&=\; 0, \, i = 1, \ldots n-1. \end{aligned} \right.$$

<u>Remark</u>: The choice of the constant δ guarantees that the compatibility condition $\int_\Omega f \, d\Omega \;=\; \int_{\partial\Omega} \mathbf{a} \cdot \mathbf{n} \, ds$ is satisfied.

Note that $\|\mathbf{a}\|_{W^{1/r',r}(\partial\Omega)} \;\leq\; C_1 \, \|\hat{\gamma}\|_{\mathbb{R}^m} \;=\; C_1$.

Also,

$$\int_\Omega \hat{q} \, dA \;\;\leq\;\; \|\hat{q}\|_{L^r(\Omega)} \, \|1\|_{L^{r'}(\Omega)} \;=\; C_2 \; , \tag{3.9}$$

$$\int_{\partial\Omega} h \, ds \;\;\leq\;\; \|\hat{\gamma}\|_{\mathbb{R}^2} \, \|\mathbf{1}\|_{\mathbb{R}^2} \;=\; C_3 \; , \tag{3.10}$$

and thus $\|\delta\|_{L^r(\Omega)} \;\leq\; C_4$.

Let $\mathbf{v}_f = \mathbf{v}|_{\overline{\Omega_f}}$, $\mathbf{v}_p = \mathbf{v}|_{\overline{\Omega_p}}$, where $\mathbf{v}$ denotes the solution of (3.6)–(3.7). From (3.8) we have

$$\|\mathbf{v}\|_X \;\leq\; C\,(1 \,+\, C_4 \,+\, C_1) \;\leq\; C_5 \;. \tag{3.11}$$

Also, note that $\mathbf{v}_f \in W^{1/r',r}(\partial\Omega_f)$, $\mathbf{v}_p \in W^{1/r',r}(\partial\Omega_p)$, and $\mathbf{v}_f \;=\; \mathbf{v}_p$ on Γ. Thus, for $\lambda \in W^{1/r,r'}(\Gamma)$,

$$\int_\Gamma \mathbf{v}_f \cdot \mathbf{n}_f \, \lambda \, ds \;+\; \langle \mathbf{v}_p \cdot \mathbf{n}_p \,,\, \lambda \rangle_\Gamma \;=\; \int_\Gamma \mathbf{v}_f \cdot \mathbf{n}_f \, \lambda \, ds \;+\; \int_\Gamma \mathbf{v}_p \cdot \mathbf{n}_p \, \lambda \, ds \;=\; 0\,,$$

i.e., $\mathbf{v} \in V$.

Now,

$$\begin{aligned} b(\mathbf{v}, q, \boldsymbol{\gamma}) \;&=\; \int_\Omega q \, \nabla \cdot \mathbf{v} dA \;+\; \gamma_1 \int_{\Gamma_{in}} \mathbf{v} \cdot \mathbf{n}_f \, ds \;+\gamma_2 \int_{\Gamma_{out}} \mathbf{v} \cdot \mathbf{n}_p \, ds \\ &\geq\; \int_\Omega q \, (\hat{q} + \delta) \, dA \;+\; \hat{\boldsymbol{\gamma}} \cdot \boldsymbol{\gamma} \\ &=\; \|q\|_M \;+\; \|\boldsymbol{\gamma}\|_{\mathbb{R}^2} \\ &=\; \|(q\,,\boldsymbol{\gamma})\|_{M \times \mathbb{R}^2}\,, \end{aligned}$$

as $\int_\Omega q \, \delta \, dA = 0$ for $q \in M$. Thus,

$$\sup_{\mathbf{u} \in V} \frac{b(\mathbf{u}, (q, \boldsymbol{\gamma}))}{\|(q\,,\beta)\|_{M \times \mathbb{R}^m} \|\mathbf{u}\|_X} \;\geq\; \frac{b(\mathbf{v}, (q, \boldsymbol{\gamma}))}{\|(q\,,\beta)\|_{M \times \mathbb{R}^m} \|\mathbf{v}\|_X} \;\geq\; \frac{1}{C_5}\,,$$

from which (3.3) directly follows.

■

The required inf-sup condition for $b_I(\cdot,\cdot)$ may be stated as follows.

Lemma 3 *The bilinear form $b_I(\cdot,\cdot) \;:\; X \times W^{1/r,r'}(\Gamma) \to \mathbb{R}$ is continuous. Moreover, there exists $C_{X\Gamma} > 0$ such that*

$$\inf_{0 \neq \lambda \in W^{1/r,r'}(\Gamma)} \sup_{\mathbf{u} \in X} \frac{b_I(\mathbf{u}, \lambda)}{\|\mathbf{u}\|_X \, \|\lambda\|_{W^{1/r,r'}(\Gamma)}} \;\geq\; C_{X\Gamma}\,. \tag{3.12}$$

Proof:
The continuity of $b_I(\cdot,\cdot)$ follows from the continuity of the trace operator and definition (2.23).

The proof of this inf-sup condition requires a suitable extension of a functional from $W^{-1/r,r}(\Gamma)$ to $W^{-1/r,r}(\partial\Omega_p)$ be defined. Some of the notation used in this proof is defined in the **Appendix** where suitable extension operators from Γ to $\partial\Omega_p$ are discussed.

To show (3.12), let $\lambda \in W^{1/r,r'}(\Gamma)$. Then, from the definition of the norm, there exists $f_\Gamma \in W^{-1/r,r}(\Gamma)$, $\|f_\Gamma\|_{W^{-1/r,r}(\Gamma)} = 1$, such that

$$\langle f_\Gamma \,,\, \lambda \rangle_\Gamma \;\geq\; \frac{1}{2} \|\lambda\|_{W^{1/r,r'}(\Gamma)}\,. \tag{3.13}$$

Given $f_\Gamma \in W^{-1/r,r}(\Gamma)$ we can extend it to a functional in $W^{-1/r,r}(\partial\Omega_p)$, f, by

$$\langle f\,,\,\xi\rangle_{\partial\Omega_p} \;:=\; \langle f_\Gamma\,,\,\xi|_\Gamma\rangle_\Gamma\,,\ \text{ for } \xi \in W^{1/r,r'}(\partial\Omega_p). \tag{3.14}$$

Note that for $\eta \in W_{00}^{1/r,r'}(\partial\Omega_p\backslash\Gamma)$

$$\langle f\,,\,E^{r'}_{00,\partial\Omega_p\backslash\Gamma}\eta\rangle_{\partial\Omega_p} \;=\; \langle f_\Gamma\,,\,E^{r'}_{00,\partial\Omega_p\backslash\Gamma}\eta|_\Gamma\rangle_\Gamma \;=\; \langle f_\Gamma\,,\,0\rangle_\Gamma \;=\; 0\,.$$

Thus, from Definition 1 (see **Appendix**), $f|_{\partial\Omega_p\backslash\Gamma} = 0$.

Also,

$$\begin{aligned}\|f\|_{W^{-1/r,r}(\partial\Omega_p)} \;&=\; \sup_{\xi\in W^{1/r,r'}(\partial\Omega_p)} \frac{\langle f\,,\,\xi\rangle_{\partial\Omega_p}}{\|\xi\|_{W^{1/r,r'}(\partial\Omega_p)}} \;=\; \sup_{\xi\in W^{1/r,r'}(\partial\Omega_p)} \frac{\langle f_\Gamma\,,\,\xi_\Gamma\rangle_\Gamma}{\|\xi\|_{W^{1/r,r'}(\partial\Omega_p)}} \\ &\leq\; \sup_{\xi\in W^{1/r,r'}(\partial\Omega_p)} \frac{\|f_\Gamma\|_{W^{-1/r,r}(\Gamma)}\,\|\xi_\Gamma\|_{W^{1/r,r'}(\Gamma)}}{\|\xi\|_{W^{1/r,r'}(\partial\Omega_p)}} \;\leq\; \|f_\Gamma\|_{W^{-1/r,r}(\Gamma)} \;=\; 1.\end{aligned} \tag{3.15}$$

Let $\phi \in W^{1,r'}(\Omega_p)$ be given by the weak solution of

$$-\nabla\cdot|\nabla\phi|^{r'-2}\nabla\phi \,+\, |\phi|^{r'-2}\phi \;=\; 0 \quad \text{in } \Omega_p\,, \tag{3.16}$$

$$|\nabla\phi|^{r'-2}\nabla\phi\cdot\mathbf{n}_p \;=\; f \quad \text{on } \partial\Omega_p\,, \tag{3.17}$$

i.e. ϕ satisfies

$$(T(\phi), w) \;:=\; \int_{\Omega_p}\Big(|\nabla\phi|^{r'-2}\nabla\phi\cdot\nabla w \;+\; |\phi|^{r'-2}\phi w\Big)\,dA \;=\; \int_{\partial\Omega_p} f w\,ds\;, \quad \forall w \in W^{1,r'}(\Omega_p)\,.$$

Existence and uniqueness of ϕ follow from the strong monotonicity of $T : W^{1,r'}(\Omega_p) \longrightarrow \Big(W^{1,r'}(\Omega_p)\Big)^*$.

Note that

$$\begin{aligned}(T(\phi),\phi) \;=\; \|\phi\|^{r'}_{W^{1,r'}(\Omega_p)} \;&\leq\; \|f\|_{W^{-1/r,r}(\partial\Omega_p)}\|\phi\|_{W^{1/r,r'}(\partial\Omega_p)} \;\leq C_1\|f\|_{W^{-1/r,r}(\partial\Omega_p)}\|\phi\|_{W^{1,r'}(\Omega_p)} \\ \Longrightarrow\; \|\phi\|^{r'}_{W^{1,r'}(\Omega_p)} \;&\leq\; C_*\|f\|^{r}_{W^{-1/r,r}(\partial\Omega_p)} \;\leq\; C_* \;\;(\text{ as } \|f\|_{W^{-1/r,r}(\partial\Omega_p)} \leq 1).\end{aligned} \tag{3.18}$$

Now, let $\mathbf{v} := |\nabla\phi|^{r'-2}\nabla\phi$. Note that from (3.16) that $\nabla\cdot\mathbf{v} \,=\, |\phi|^{r'-2}\phi$, and

$$\|\mathbf{v}\|^{r}_{W^{0,r}(div,\Omega_p)} \;=\; \|\phi\|^{r'}_{W^{1,r'}(\Omega_p)} \;\leq\; C_*. \tag{3.19}$$

i.e., $\mathbf{v} \in W^{0,r}(div,\Omega_p)$, and $\mathbf{v}\cdot\mathbf{n}_p \in W^{-1/r,r}(\partial\Omega_p)$.

Finally, let $\mathbf{w} = (\mathbf{0}, \mathbf{v}) \in X$. Then, in view of (2.23)

$$\begin{aligned}
\sup_{\mathbf{u}\in X} \frac{b_I(\mathbf{u},\lambda)}{\|\mathbf{u}\|_X} \ \geq \ \frac{b_I(\mathbf{w},\lambda)}{\|\mathbf{w}\|_X} \ &= \ \frac{0 \ + \ \langle \mathbf{v}\cdot\mathbf{n}_p \, , \, \lambda\rangle_\Gamma}{\|\mathbf{v}\|_{W^{0,r}(div,\Omega_p)}} \\
&\geq \ \frac{\langle \mathbf{v}\cdot\mathbf{n}_p \, , \, E_\Gamma^{r'}\lambda\rangle_{\partial\Omega_p}}{C_*^{1/r}} \\
&= \ \frac{1}{C_*^{1/r}} \langle f \, , \, E_\Gamma^{r'}\lambda\rangle_{\partial\Omega_p} \\
&= \ \frac{1}{C_*^{1/r}} \langle f_\Gamma \, , \, \lambda\rangle_\Gamma \quad \text{as } f|_{\partial\Omega_p\setminus\Gamma} = 0 \, , (\text{ see } (A.7)) \\
&\geq \ \frac{1}{2C_*^{1/r}} \|\lambda\|_{W^{1/r,r'}(\Gamma)} \text{ from } (3.13) \, .
\end{aligned}$$

■

We are now in a position to prove the existence and uniqueness of the solution.

Theorem 1 *There exists a unique solution* $(\mathbf{u}, p, \lambda, \boldsymbol{\beta}) \in X \times M \times W^{1/r,r'}(\Gamma) \times \mathrm{I\!R}^2$ *satisfying (2.35)-(2.36). In addition, there exists a constant* $C > 0$ *such that*

$$\|\mathbf{u}\|_X \ \leq \ C \left(\|\mathbf{f}_f\|_{X_f^*} \ + \ |fr| \right) \, . \tag{3.20}$$

Proof: For $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in Z$, note that $\nabla\cdot\mathbf{v}_1 = 0$ a.e. in Ω_f and $\nabla\cdot\mathbf{v}_2 = 0$ a.e. in Ω_p. Hence, for $\mathbf{v} \in Z$, $\|\mathbf{v}_2\|_{X_p} = \|\mathbf{v}_2\|_{L^r(\Omega_p)}$, and $\|\mathbf{v}\|_X = \|\mathbf{v}_1\|_{X_f} + \|\mathbf{v}_2\|_{L^r(\Omega_p)}$.

From the continuity and inf-sup condition for $b(\cdot,\cdot,\cdot)$ ([10], Remark 4.2, pg. 61) there exists $\mathbf{u}_0 \in V$ such that

$$\begin{aligned}
b(\mathbf{u}_0, q, \boldsymbol{\gamma}) \ &= \ \boldsymbol{\gamma}\cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} fr \ , \forall (q,\boldsymbol{\gamma}) \in M \times \mathrm{I\!R}^2 \, , \\
\text{with} \quad \|\mathbf{u}_0\|_X \ &\leq \ C|fr| \, .
\end{aligned} \tag{3.21}$$

Together with the continuity and inf-sup condition of $b_I(\cdot,\cdot)$, the existence and uniqueness of the solution to (2.35)-(2.36) can be equivalently stated as: *Given* $\mathbf{f} \in X^*$, *determine* $\tilde{\mathbf{u}} \in Z$, $\mathbf{u} \ = \ \tilde{\mathbf{u}} + \mathbf{u}_0$, *such that*

$$a(\tilde{\mathbf{u}} + \mathbf{u}_0, \mathbf{v}) \ = \ (\mathbf{f}, \mathbf{v}) \, , \quad \forall \mathbf{v} \in Z \, . \tag{3.22}$$

The existence and uniqueness of the solution to (3.22) follows from the continuity and strict monotonicity of $a(\cdot,\cdot)$ on $Z \times Z$; which follows from assumptions (2.16)-(2.17), and the restriction that for $\Omega \subset \mathrm{I\!R}^2$, $4/3 < r \leq 2$ and $\Omega \subset \mathrm{I\!R}^3$, $3/2 < r \leq 2$. This restriction arises in applying the Sobolev inbedding theorem to verify the continuity of $a(\cdot,\cdot)$. Specifically,

$$\begin{aligned}
\sum_{l=1}^{n-1} \int_\Gamma csr_l^{-1} \left((\mathbf{u}_f - \mathbf{w}_f)\cdot\mathbf{t}_l \right) (\mathbf{v}_f\cdot\mathbf{t}_l) \, ds \ &\leq \ C\|\mathbf{u}_f - \mathbf{w}_f\|_{L^2(\Gamma)} \, \|\mathbf{v}_f\|_{L^2(\Gamma)} \\
&\leq \ C\|\mathbf{u}_f - \mathbf{w}_f\|_{W^{1-1/r,r}(\partial\Omega_f)} \, \|\mathbf{v}_f\|_{W^{1-1/r,r}(\partial\Omega_f)} \\
&\leq \ C\|\mathbf{u} - \mathbf{w}\|_X \, \|\mathbf{v}\|_X \, .
\end{aligned}$$

Also, it follows from (2.16),(2.17) and (3.21), that

$$\|\tilde{\mathbf{u}}\|_X \;\le\; C\left(\|\mathbf{f}\|_{X^*} + |fr|\right) \;=\; C\left(\|\mathbf{f}_f\|_{X_f^*} + |fr|\right) ,$$

and therefore, the estimate

$$\|\mathbf{u}\|_X \;\le\; C\left(\|\mathbf{f}_f\|_{X_f^*} + |fr|\right) .$$

■

4 Finite Element Approximation

In this section we discuss the finite element approximation to the coupled generalized nonlinear Stokes–Darcy system (2.35),(2.36). We focus our attention on conforming approximating spaces $X_{f,h} \subset X_f$, $M_{f,h} \subset M_f$, $X_{p,h} \subset X_p$, $M_{p,h} \subset M_p$, $L_h \subset W^{1/r,r'}(\Gamma)$, where $X_{f,h}, M_{f,h}$ denote velocity and pressure spaces typically used for fluid flow approximations, and $X_{p,h}, M_{p,h}$ denote velocity and pressure spaces typically used for (mixed formulation) Darcy flow approximations.

We begin by describing the finite element approximation framework used in the analysis.

Let $\Omega_j \subset \mathbb{R}^n$, $(n = 2, 3)$ $j = f, p$, be a polygonal domain and let $\mathcal{T}_{j,h}$ be a triangulation of $\overline{\Omega_j}$ made of triangles (in $\mathbb{R}^2$) or tetrahedrals (in $\mathbb{R}^3$). Thus, the computational domain is defined by

$$\overline{\Omega} = \cup K; \;\; K \in \mathcal{T}_{f,h} \cup \mathcal{T}_{p,h}.$$

We assume that there exist constants c_1, c_2 such that

$$c_1 h \le h_K \le c_2 \rho_K$$

where h_K is the diameter of triangle (tetrahedral) K, ρ_K is the diameter of the greatest ball (sphere) included in K, and $h = \max_{K \in \mathcal{T}_{f,h} \cup \mathcal{T}_{p,h}} h_K$.

For simplicity, we assume that the triangulations on $\overline{\Omega_f}$ and $\overline{\Omega_p}$ induce the same partition on Γ, which we denote $\mathcal{T}_{\Gamma,h}$.

Let $P_k(A)$ denote the space of polynomials on A of degree no greater than k. Also, for $\mathbf{x} = [x_1, \ldots, x_n]^T \in \mathbb{R}^n$, let $RT_k(A) := (P_k(A))^n + \mathbf{x} P_k(A)$ denote the kth order Raviart-Thomas elements. Then we define the finite element spaces as follows.

$$X_{f,h} \;:=\; \left\{\mathbf{v} \in X_f \cap C(\overline{\Omega_f})^2 : \mathbf{v}|_K \in P_m(K),\; \forall K \in \mathcal{T}_{f,h}\right\} , \tag{4.1}$$

$$M_{f,h} \;:=\; \left\{q \in M_f \cap C(\overline{\Omega_f}) : q|_K \in P_{m-1}(K),\; \forall K \in \mathcal{T}_{f,h}\right\} . \tag{4.2}$$

$$X_{p,h} \;:=\; \{\mathbf{v} \in RT_k(K),\; \forall K \in \mathcal{T}_{p,h}\} , \tag{4.3}$$

$$M_{p,h} \;:=\; \{q \in M_f : q|_K \in P_k(K),\; \forall K \in \mathcal{T}_{f,h}\} , \tag{4.4}$$

$$L_h \;:=\; \left\{\zeta \in W^{1/r,r'}(\Gamma) \cap C(\Gamma) : \zeta|_K \in P_l(K),\; \forall K \in \mathcal{T}_{\Gamma,h}\right\} . \tag{4.5}$$

Note that as we are assuming $1 < r < 2$, then $1/r > 1/2$, which implies that, for $\Omega \subset \mathbb{R}^2$, $\lambda \in W^{1/r,r'}(\Gamma)$ is continuous. For $m = 2$, $X_{f,h}$ and $M_{f,h}$ denote the Taylor-Hood spaces.

Below we assume that $m \ge 2$, $k \ge 1$, and $l \le k$.

Let

$$X^0_{f,h} := \left\{ \mathbf{v} \in X_{f,h} : \mathbf{v}|_{\partial\Omega_f} = \mathbf{0} \right\} \ , \text{ and } X^0_{p,h} := \left\{ \mathbf{v} \in X_{p,h} : \mathbf{v} \cdot \mathbf{n}_p|_{\partial\Omega_p} = 0 \right\} \ .$$

Lemma 4 *There exists constants $C_{f,h}$, $C_{p,h} > 0$, such that*

$$\inf_{0 \neq q_h \in M_{f,h}} \ \sup_{\mathbf{v}_h \in X^0_{f,h}} \ \frac{\int_{\Omega_f} q_h \, \nabla \cdot \mathbf{v}_h \, dA}{\|q_h\|_{M_f} \, \|\mathbf{v}_h\|_{X_f}} \ \geq \ C_{f,h} \, , \tag{4.6}$$

$$\inf_{0 \neq q_h \in M_{p,h}} \ \sup_{\mathbf{v}_h \in X^0_{p,h}} \ \frac{\int_{\Omega_p} q_h \, \nabla \cdot \mathbf{v}_h \, dA}{\|q_h\|_{M_p} \, \|\mathbf{v}_h\|_{X_p}} \ \geq \ C_{p,h} \, . \tag{4.7}$$

Proof For the case of the pressure spaces having mean value equal to zero the inf-sup conditions (4.6) and (4.7) are well established. As commented in [14], one can extend the inf-sup conditions to the above pressure spaces via a local projector operator argument. (See [2], Section VI.4.) ■

Remark: There are several other suitable choices of approximation spaces. (See discussions in [14, 9].)

<u>Discrete Approximation Problem</u>: *Given $\mathbf{f} \in X^*$, $fr \in \mathbb{R}$, determine $(\mathbf{u}_h, p_h, \lambda_h, \boldsymbol{\beta}_h) \in X_h \times M_h \times L_h \times \mathbb{R}^2$ such that*

$$a(\mathbf{u}_h, \mathbf{v}_h) \ - \ b(\mathbf{v}_h, p_h, \boldsymbol{\beta}_h) \ + \ b_I(\mathbf{v}_h, \lambda_h) \ = \ (\mathbf{f}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in X_h \, , \tag{4.8}$$

$$b(\mathbf{u}_h, q_h, \boldsymbol{\gamma}_h) \ - \ b_I(\mathbf{u}_h, \zeta_h) \ = \ \boldsymbol{\gamma}_h \cdot \begin{bmatrix} -1 \\ 1 \end{bmatrix} fr \quad \forall (q_h, \boldsymbol{\gamma}_h, \zeta_h) \ \in \ M_h \times \mathbb{R}^2 \times L_h(\Gamma) \, . \tag{4.9}$$

For the analysis a more general inf-sup condition than that given by (4.6),(4.7) is needed. This is established using the following two lemmas. (See also [24].)

Corresponding to V and Z as defined in (3.1),(3.2), we have the discrete counterparts

$$V_h := \{ \mathbf{v} \in X_h \mid b_I(\mathbf{v}_h, \zeta) = 0, \ \text{ for all } \zeta \in L_h \} \ , \tag{4.10}$$

$$Z_h := \{ \mathbf{v} \in V_h \mid b(\mathbf{v}, q, \boldsymbol{\gamma}) = 0, \ \text{ for all } (q, \boldsymbol{\gamma}) \in M_h \times \mathbb{R}^2 \} \ . \tag{4.11}$$

Lemma 5 *There exists $C_{RXh} > 0$ such that for h sufficiently small*

$$\inf_{\mathbf{0} \neq \boldsymbol{\beta} \in \mathbb{R}^2} \ \sup_{\mathbf{w}_h \in Z_h} \ \frac{\int_{\Gamma_{in}} \beta_1 \, \mathbf{w}_{f,h} \cdot \mathbf{n}_f \, ds \ + \ \int_{\Gamma_{out}} \beta_2 \, \mathbf{w}_{p,h} \cdot \mathbf{n}_p \, ds}{\|\mathbf{w}_h\|_X \, \|\boldsymbol{\beta}\|_{\mathbb{R}^2}} \ \geq \ C_{RXh} \ . \tag{4.12}$$

Proof: We use (3.5)-(3.8) to construct a suitable function $\mathbf{v}$. Then using a linear interpolant for $\mathbf{v}$ we obtain the stated result.

Assume $\boldsymbol{\beta} = [\beta_1, \beta_2]^T \in \mathbb{R}^2$ is given.

For $i \in \{in, \, out\}$, let $s_i(\mathbf{x})$ denote an arclength parameter on Γ_i, and define $\phi_i : \partial\Omega \to \mathbb{R}$ by

$$\phi_i(\mathbf{x}) \ = \ \begin{cases} \frac{2}{|\Gamma_i|} s_i(\mathbf{x}) \, , & \mathbf{x} \in \Gamma_i, \quad 0 \leq s \leq \frac{|\Gamma_i|}{2} \, , \\ \frac{2}{|\Gamma_i|} (|\Gamma_i| - s_i(\mathbf{x})) \, , & \mathbf{x} \in \Gamma_i, \quad \frac{|\Gamma_i|}{2} < s_i(\mathbf{x}) \leq |\Gamma_i| \, , \\ 0 \, , & \text{otherwise.} \end{cases}$$

Further, let $\mathbf{a} \in W^{1-1/r,r}(\partial\Omega)$, and $f \in L^r(\Omega)$ be given by

$$\mathbf{a}(\mathbf{x}) \;=\; (\beta_1\, \phi_{in}(\mathbf{x}) \,+\, \beta_2\, \phi_{out}(\mathbf{x}))\, \mathbf{n}\;, \qquad f(\mathbf{x}) \;=\; \frac{1}{|\Omega|^{1/r}} \int_{\partial\Omega} \mathbf{a}\cdot\mathbf{n}\, ds\;. \tag{4.13}$$

Note that $\|\mathbf{a}\|_{W^{1-1/r,r}(\partial\Omega)} \;\le\; |\beta_1|\, \|\phi_{in}\mathbf{n}\|_{W^{1-1/r,r}(\partial\Omega)} \,+\, |\beta_2|\, \|\phi_{out}\mathbf{n}\|_{W^{1-1/r,r}(\partial\Omega)} \;\le\; C\|\boldsymbol{\beta}\|_{\mathbb{R}^2}$, and $\|f\|_{L^r(\Omega)} \;\le\; (|\beta_1|\, |\Gamma_{in}| \,+\, |\beta_2|\, |\Gamma_{out}|)\,/2 \;\le\; C\|\boldsymbol{\beta}\|_{\mathbb{R}^2}$.

With $\mathbf{a}$ and f given by (4.13), let $\mathbf{v}$ be given by (3.6), (3.7), and $\mathbf{v}_{f,h} = I_h(\mathbf{v})|_{\overline{\Omega_f}}$, $\mathbf{v}_{p,h} = I_h(\mathbf{v})|_{\overline{\Omega_p}}$, where $I_h(\mathbf{v})$ denotes a continuous linear interpolant of $\mathbf{v}$ with respect to $\mathcal{T}_{f,h} \cup \mathcal{T}_{p,h}$.

Note that $\mathbf{v}_h = (\mathbf{v}_{f,h}, \mathbf{v}_{p,h}) \;\in Z_h$ and

$$\begin{aligned}
\|\mathbf{v} - \mathbf{v}_h\|_{W^{s,r}(\Omega)} \;&\le\; Ch^{1-s}\|\mathbf{v}\|_{W^{1,r}(\Omega)}\,, s = 0, 1,\\
\|\mathbf{v} - \mathbf{v}_h\|_{W^{0,r}(\partial\Omega)} \;&\le\; Ch^{r'}\|\mathbf{v}\|_{W^{1,r}(\Omega)}\,.
\end{aligned}$$

Then, for h sufficiently small,

$$\begin{aligned}
&\sup_{\mathbf{w}_h \in X_h} \frac{\int_{\Gamma_{in}} \beta_1\, \mathbf{w}_{f,h}\cdot\mathbf{n}_f\, ds \;+\; \int_{\Gamma_{out}} \beta_2\, \mathbf{w}_{p,h}\cdot\mathbf{n}_p\, ds}{\|\mathbf{w}_h\|_X} \;\ge\; \frac{\int_{\Gamma_{in}} \beta_1\, \mathbf{v}_{f,h}\cdot\mathbf{n}_f\, ds \;+\; \int_{\Gamma_{out}} \beta_2\, \mathbf{v}_{p,h}\cdot\mathbf{n}_p\, ds}{\|\mathbf{v}_h\|_X}\\
&\ge\; \frac{\int_{\Gamma_{in}} \beta_1\, \mathbf{v}_f\cdot\mathbf{n}_f\, ds \;+\; \int_{\Gamma_{out}} \beta_2\, \mathbf{v}_p\cdot\mathbf{n}_p\, ds \;+\; \int_{\Gamma_{in}} \beta_1\, (\mathbf{v}_{f,h} - \mathbf{v}_f)\cdot\mathbf{n}_f\, ds \;+\; \int_{\Gamma_{out}} \beta_2\, (\mathbf{v}_{p,h} - \mathbf{v}_p)\cdot\mathbf{n}_p\, ds}{C\,\|\mathbf{v}\|_X}\\
&\ge\; C_1\|\boldsymbol{\beta}\|_{\mathbb{R}^2} \;-\; C_2 h^{r'}\|\boldsymbol{\beta}\|_{\mathbb{R}^2}\,,
\end{aligned}$$

from which (4.12) follows.

■

Lemma 6 *For h sufficiently small, there exists $C_{bh} > 0$ such that*

$$\inf_{(0,\mathbf{0})\neq(q_h,\boldsymbol{\beta})\in M_h\times\mathbb{R}^2}\; \sup_{\mathbf{v}_h\in V_h} \frac{b(\mathbf{v}_h, (q_h, \boldsymbol{\beta}))}{\|\mathbf{v}_h\|_X\, \|(q, \boldsymbol{\beta})\|_{M\times\mathbb{R}^2}} \ge C_{bh}\,. \tag{4.14}$$

Proof: Let $(p_h, \boldsymbol{\beta}) \in M_h \times \mathbb{R}^2$. From Lemma 5, there exists $\hat{\mathbf{u}}_h \in X_h$ such that

$$\|\hat{\mathbf{u}}_h\|_X \;=\; \|\boldsymbol{\beta}\|_{\mathbb{R}^m} \quad \text{and} \quad \frac{\int_{\Gamma_{in}} \beta_1\, \mathbf{v}_{f,h}\cdot\mathbf{n}_f\, ds \;+\; \int_{\Gamma_{out}} \beta_2\, \mathbf{v}_{p,h}\cdot\mathbf{n}_p\, ds}{\|\hat{\mathbf{u}}_h\|_X} \ge C_{RXh}\|\boldsymbol{\beta}\|_{\mathbb{R}^2}\,. \tag{4.15}$$

Consider the following two problems.
Problem 1 Discrete power-law problem in Ω_f: *Determine $\tilde{\mathbf{u}}_{f,h} \in X^0_{f,h},\ \tilde{p}_{f,h} \in M_{f,h}$ such that*

$$\begin{aligned}
(|\mathbf{d}(\tilde{\mathbf{u}}_{f,h})|^{r-2}\mathbf{d}(\tilde{\mathbf{u}}_{f,h})\,,\, \mathbf{d}(\mathbf{v})) \;&-\; (\tilde{p}_{f,h}\,,\, \nabla\cdot\mathbf{v}) \;=\; 0\,, \quad \forall \mathbf{v} \in X^0_{f,h}\,, &(4.16)\\
(q\,,\, \nabla\cdot\tilde{\mathbf{u}}_{f,h}) \;&=\; (q\,,\, \|p_{f,h}\|^{1-r'/r}_{M_f}\, |p_{f,h}|^{r'/r-1}\, p_{f,h} \;-\; \nabla\cdot\hat{\mathbf{u}}_{f,h})\,, \quad \forall q \in M_{f,h}\,. &(4.17)
\end{aligned}$$

and
Problem 2 in Ω_p: *Determine $\tilde{\mathbf{u}}_{p,h} \in X^0_{p,h},\ \tilde{p}_{p,h} \in M_{p,h}$ such that*

$$\begin{aligned}
(|\tilde{\mathbf{u}}_{p,h}|^{r-2}\tilde{\mathbf{u}}_{p,h}\,,\, \mathbf{v}) \;&-\; (\tilde{p}_{p,h}\,,\, \nabla\cdot\mathbf{v}) \;=\; 0\,, \quad \forall \mathbf{v} \in X^0_{p,h}\,, &(4.18)\\
(q\,,\, \nabla\cdot\tilde{\mathbf{u}}_{p,h}) \;&=\; (q\,,\, \|p_{p,h}\|^{1-r'/r}_{M_p}\, |p_{p,h}|^{r'/r-1}\, p_{p,h} \;-\; \nabla\cdot\hat{\mathbf{u}}_{p,h})\,, \quad \forall q \in M_{p,h}\,. &(4.19)
\end{aligned}$$

Note that $\|p_{j,h}\|_{M_j}^{1-r'/r}\,|p_{j,h}|^{r'/r-1}\,p_{j,h} \,-\, \nabla\cdot\hat{\mathbf{u}}_{j,h} \;\in\; L^r(\Omega_j),\; j=f,p.$

Existence and uniqueness of $\tilde{\mathbf{u}}_{f,h} \in X^0_{f,h},\; \tilde{p}_{f,h} \in P_{f,h}$ and $\tilde{\mathbf{u}}_{p,h} \in X^0_{p,h},\; \tilde{p}_{p,h} \in P_{p,h}$ satisfying (4.16),(4.17) and (4.18),(4.19), respectively, follow from the inf-sup conditions (4.6),(4.7) and the strong monotonicity of $T : X \longrightarrow X^*$, $(T(\phi),\psi) \,:=\, \int |\phi|^{r-2}\phi\cdot\psi\,dA$.

From (4.16),(4.17) with the choices $\mathbf{v} = \tilde{\mathbf{u}}_{f,h}$, and $q = \tilde{p}_{f,h}$,

$$
\begin{aligned}
\|\tilde{\mathbf{u}}_{f,h}\|^r_{X_f} \;=\; (|\mathbf{d}(\tilde{\mathbf{u}}_{f,h})|^{r-2}\mathbf{d}(\tilde{\mathbf{u}}_{f,h})\,,\,\mathbf{d}(\tilde{\mathbf{u}}_{f,h})) \;&=\; (\tilde{p}_{f,h}\,,\,\nabla\cdot\tilde{\mathbf{u}}_{f,h}) \\
&=\; (\tilde{p}_{f,h}\,,\,\|p_{f,h}\|_{M_f}^{1-r'/r}\,|p_{f,h}|^{r'/r-1}\,p_{f,h} \,-\, \nabla\cdot\hat{\mathbf{u}}_{f,h}) \\
&\le\; \|\tilde{p}_{f,h}\|_{M_f}\left(\|p_{f,h}\|_{M_f}^{1-r'/r}\,\||p_{f,h}|^{r'/r-1}\,p_{f,h}\|_{L^r} \,+\, \|\nabla\cdot\hat{\mathbf{u}}_{f,h}\|_{L^r}\right) \\
&\le\; \|\tilde{p}_{f,h}\|_{M_f}\left(\|p_{f,h}\|_{M_f} \,+\, C\,\|\hat{\mathbf{u}}_{f,h}\|_{X_f}\right) \\
&\le\; C\,\|\tilde{p}_{f,h}\|_{M_f}\left(\|p_{f,h}\|_{M_f} \,+\, \|\boldsymbol{\beta}\|_{\mathbb{R}^2}\right). \qquad (4.20)
\end{aligned}
$$

Also, from the inf-sup condition for spaces $X^0_{f,h}$ and $M_{f,h}$ we have

$$
\begin{aligned}
c\,\|\tilde{p}_{f,h}\|_{M_f} \;&\le\; \sup_{\mathbf{v}\in X^0_{f,h}} \frac{(\tilde{p}_{f,h}\,,\,\nabla\cdot\mathbf{v})}{\|\mathbf{v}\|_{X_f}} \\
&=\; \sup_{\mathbf{v}\in X^0_{f,h}} \frac{(|\mathbf{d}(\tilde{\mathbf{u}}_{f,h})|^{r-2}\mathbf{d}(\tilde{\mathbf{u}}_{f,h})\,,\,\mathbf{d}(\mathbf{v}))}{\|\mathbf{v}\|_{X_f}} \\
&\le\; \sup_{\mathbf{v}\in X^0_{f,h}} \frac{(\||\mathbf{d}(\tilde{\mathbf{u}}_{f,h})|^{r-2}\mathbf{d}(\tilde{\mathbf{u}}_{f,h})\|_{L^{r'}}\,\|\mathbf{d}(\mathbf{v})\|_{L^r}}{\|\mathbf{v}\|_{X_f}} \\
&=\; \||\mathbf{d}(\tilde{\mathbf{u}}_{f,h})|^{r-2}\mathbf{d}(\tilde{\mathbf{u}}_{f,h})\|_{L^{r'}} \\
&=\; \|\tilde{\mathbf{u}}_{f,h}\|^{r/r'}_{X_f}\,. \qquad (4.21)
\end{aligned}
$$

Combining (4.20) and (4.21) we have the estimate

$$
\|\tilde{\mathbf{u}}_{f,h}\|_{X_f} \;\le\; C\,\left(\|p_{f,h}\|_{M_f} \,+\, \|\boldsymbol{\beta}\|_{\mathbb{R}^2}\right). \qquad (4.22)
$$

Proceeding in a similar fashion for $\tilde{\mathbf{u}}_{p,h}$ satisfying **Problem 2** leads to the estimate

$$
\|\tilde{\mathbf{u}}_{p,h}\|_{X_p} \;\le\; C\,\left(\|p_{p,h}\|_{M_p} \,+\, \|\boldsymbol{\beta}\|_{\mathbb{R}^2}\right). \qquad (4.23)
$$

Let $\mathbf{u}_{j,h} \;=\; \tilde{\mathbf{u}}_{j,h} \,+\, \hat{\mathbf{u}}_{j,h},\; j=f,p$. Note that, as $\mathbf{u}_{f,h} = \mathbf{0}$ on Γ and $\mathbf{u}_{p,h}\cdot\mathbf{n}_p = 0$ on Γ, $\mathbf{u}_h \in V_h$.

Then, using (4.17),(4.19) and (4.12)

$$\begin{aligned} b(\mathbf{u}_h,(p_h,\boldsymbol{\beta})) &= \int_{\Omega_f} p_{f,h}\,\nabla\cdot\mathbf{u}_{f,h}\,dA + \int_{\Omega_p} p_{p,h}\,\nabla\cdot\mathbf{u}_{p,h}\,dA + \beta_1\int_{\Gamma_{in}} \mathbf{u}_{f,h}\cdot\mathbf{n}_f\,ds \\ &\quad + \beta_2\int_{\Gamma_{out}} \mathbf{u}_{p,h}\cdot\mathbf{n}_p\,ds \\ &= \int_{\Omega_f} p_{f,h}\,\|p_{f,h}\|_{M_f}^{1-r'/r}\,|p_{f,h}|^{r'/r-1}\,p_{f,h}\,dA + \int_{\Omega_p} p_{p,h}\,\|p_{p,h}\|_{M_p}^{1-r'/r}\,|p_{p,h}|^{r'/r-1}\,p_{p,h}\,dA \\ &\quad + \beta_1\int_{\Gamma_{in}} \hat{\mathbf{u}}_{f,h}\cdot\mathbf{n}_f\,ds + \beta_2\int_{\Gamma_{out}} \hat{\mathbf{u}}_{p,h}\cdot\mathbf{n}_p\,ds \\ &\geq c\left(\|p_h\|_M^2 + \|\boldsymbol{\beta}\|_{\mathbb{R}^2}^2\right). \end{aligned} \tag{4.24}$$

Thus, using (4.24),(4.22) and (4.23), we have

$$\begin{aligned} \sup_{\mathbf{v}_h\in X_h}\frac{b(\mathbf{v}_h,(p_h,\boldsymbol{\beta}))}{\|\mathbf{v}_h\|_X} &\geq \frac{b(\mathbf{u}_h,(p_h,\boldsymbol{\beta}))}{\|\mathbf{u}_h\|_X} \\ &\geq C\left(\|p_h\|_P + \|\boldsymbol{\beta}\|_{\mathbb{R}^2}\right), \end{aligned}$$

from which (4.14) immediately follows.

■

The discrete inf-sup condition for $b_I(\cdot,\cdot)$ follows from the continuous inf-sup condition and the existence of a bounded interpolation operator $I_{p,h}: X_p \to X_{h,p}$ satisfying, for some $\alpha > 0$,

$$\|\mathbf{w} - I_{p,h}(\mathbf{w})\cdot\mathbf{n}_p\|_{W^{-1/r,r}(\partial\Omega_p)} \leq C_{ap}h^{\alpha}\|\mathbf{w}\|_{X_p}, \quad \text{and} \quad \|I_{p,h}(\mathbf{w})\|_{X_p} \leq C_{ip}\,\|\mathbf{w}\|_{X_p}. \tag{4.25}$$

Lemma 7 *There exists $C_{X\Gamma h} > 0$ such that for h sufficiently small*

$$\inf_{0\neq\lambda_h\in L_h}\sup_{\mathbf{u}_h\in X_h}\frac{b_I(\mathbf{u}_h,\lambda_h)}{\|\mathbf{u}_h\|_X\,\|\lambda_h\|_{W^{1/r,r'}(\Gamma)}} \geq C_{X\Gamma h}. \tag{4.26}$$

Proof:
With $\lambda = \lambda_h$, let $\mathbf{v}_p \in W^{0,r}(div,\Omega_p)$ be as defined by (3.16)–(3.19), and let $\mathbf{v}_{p,h} = I_{R-T}(\mathbf{v}_p) \in X_{p,h}$ denote the Raviart-Thomas interpolant of $\mathbf{v}_p$. Further, let $\mathbf{v}_h = (\mathbf{0},\mathbf{v}_{p,h}) \in X_h$. Then,

$$\begin{aligned} \sup_{\mathbf{u}_h\in X_h}\frac{b_I(\mathbf{u}_h,\lambda_h)}{\|\mathbf{u}_h\|_X} \geq \frac{b_I(\mathbf{v}_h,\lambda_h)}{\|\mathbf{v}_h\|_X} &= \frac{0 + \langle\mathbf{v}_{p,h}\cdot\mathbf{n}_p\,,\,\lambda_h\rangle_\Gamma}{\|\mathbf{v}_{p,h}\|_{W^{0,r}(div,\Omega_p)}} \\ &= \frac{\langle\mathbf{v}_p\cdot\mathbf{n}_p\,,\,\lambda_h\rangle_\Gamma}{\|\mathbf{v}_{p,h}\|_{W^{0,r}(div,\Omega_p)}} + \frac{\langle(\mathbf{v}_{p,h}-\mathbf{v}_p)\cdot\mathbf{n}_p\,,\,\lambda_h\rangle_\Gamma}{\|\mathbf{v}_{p,h}\|_{W^{0,r}(div,\Omega_p)}} \\ &\geq \frac{\langle\mathbf{v}_p\cdot\mathbf{n}_p\,,\,\lambda_h\rangle_\Gamma}{C\,\|\mathbf{v}_p\|_{W^{0,r}(div,\Omega_p)}} + \frac{\langle(\mathbf{v}_{p,h}-\mathbf{v}_p)\cdot\mathbf{n}_p\,,\,E_\Gamma^{r'}\lambda_h\rangle_{\partial\Omega_p}}{\|\mathbf{v}_{p,h}\|_{W^{0,r}(div,\Omega_p)}} \\ &\geq \frac{1}{2C}\|\lambda\|_{W^{1/r,r'}(\Gamma)} + \frac{\langle(\mathbf{v}_{p,h}-\mathbf{v}_p)\cdot\mathbf{n}_p\,,\,E_\Gamma^{r'}\lambda_h\rangle_{\partial\Omega_p}}{\|\mathbf{v}_{p,h}\|_{W^{0,r}(div,\Omega_p)}}. \end{aligned}$$

With $\lambda = \lambda_h$ let φ be given by (A.1)–(A.3), and let $\varphi_h = I(\varphi)$ denote a continuous linear interpolant of φ with respect to $\mathcal{T}_{p,h}$. Note that $\lambda_h = \varphi_h$ on Γ and Γ_{out}.

Now,

$$\begin{aligned}
\langle (\mathbf{v}_{p,h} - \mathbf{v}_p)\cdot \mathbf{n}_p \,, E_\Gamma^{r'}\lambda_h\rangle_{\partial\Omega_p} &= \langle (\mathbf{v}_{p,h} - \mathbf{v}_p)\cdot \mathbf{n}_p \,, \varphi_h\rangle_{\partial\Omega_p} \;+\; \langle (\mathbf{v}_{p,h} - \mathbf{v}_p)\cdot \mathbf{n}_p \,, (E_\Gamma^{r'}\lambda_h - \varphi_h)\rangle_{\partial\Omega_p} \\
&= 0 \;+\; \langle \mathbf{v}_{p,h}\cdot \mathbf{n}_p \,, (E_\Gamma^{r'}\lambda_h - \varphi_h)\rangle_{\partial\Omega_p} \;-\; \langle \mathbf{v}_{p}\cdot \mathbf{n}_p \,, (E_\Gamma^{r'}\lambda_h - \varphi_h)\rangle_{\partial\Omega_p} \,.
\end{aligned}$$

As, $E_\Gamma^{r'}\lambda_h - \varphi_h = 0$ on $\partial\Omega_p\backslash\Gamma_p$ and $\mathbf{v}_p\cdot\mathbf{n}_p|_{\Gamma_p} = 0$ then $\langle \mathbf{v}_{p}\cdot \mathbf{n}_p \,, (E_\Gamma^{r'}\lambda_h - \varphi_h)\rangle_{\partial\Omega_p} = 0$. Further, as $\mathbf{v}_{p,h}\cdot\mathbf{n}_p = 0$ on Γ_p, $\langle \mathbf{v}_{p,h}\cdot \mathbf{n}_p \,, (E_\Gamma^{r'}\lambda_h - \varphi_h)\rangle_{\partial\Omega_p} = 0$, from which (4.26) then follows.

■

We now state and prove the existence and uniqueness of solutions to (4.8)-(4.9).

Theorem 2 *There exists a unique solution* $(\mathbf{u}_h, p_h, \lambda_h, \boldsymbol{\beta}_h) \in X_h \times M_h \times L_h \times \mathbb{R}^2$ *satisfying (4.8)-(4.9). In addition, there exists a constant* $C > 0$ *such that*

$$\|\mathbf{u}_h\|_X \;\le\; C\left(\|\mathbf{f}_f\|_{X_f^*} \,+\, |fr|\right) . \tag{4.27}$$

Proof: With the inf-sup conditions given in (4.14) and (4.26), the existence and uniqueness follows exactly as for the continuous problem in Theorem 1. Similarly, the norm estimate for $\mathbf{u}_h$ follows as that for $\mathbf{u}$.

■

4.1 A Priori Error Estimate

Next we investigate the error between the solution of the continuous variational formulation and its discrete counterpart.

Theorem 3 *Let*

$$\begin{aligned}
\mathcal{E}(\mathbf{u},\mathbf{u}_h) &= \left\|\frac{|\mathbf{d}(\mathbf{u}_f) - \mathbf{d}(\mathbf{u}_{f,h})|}{c + |\mathbf{d}(\mathbf{u}_f)| + |\mathbf{d}(\mathbf{u}_{f,h})|}\right\|_{L^\infty(\Omega_f)}^{\frac{2-r}{r}} \;+\; \left\|\frac{|\mathbf{u}_p - \mathbf{u}_{p,h}|}{c + |\mathbf{u}_p| + |\mathbf{u}_{p,h}|}\right\|_{L^\infty(\Omega_f)}^{\frac{2-r}{r}} \qquad \textit{and} \\
\mathcal{G}(\mathbf{u},\mathbf{u}_h) &= \int_{\Omega_f} |g_f(d(\mathbf{u}_f))d(\mathbf{u}_f) - g_f(d(\mathbf{u}_{f,h}))d(\mathbf{u}_{f,h})||d(\mathbf{u}_f) - d(\mathbf{u}_{f,h})|\, dA \\
&\quad + \int_{\Omega_p} |g_p(\mathbf{u}_p)\mathbf{u}_p - g_p(\mathbf{u}_{p,h})\mathbf{u}_{p,h}||\mathbf{u}_p - \mathbf{u}_{p,h}|\, dA \,.
\end{aligned}$$

Then for $(\mathbf{u}, p, \lambda, \boldsymbol{\beta})$ *satisfying (2.35)-(2.36) and* $(\mathbf{u}_h, p_h, \lambda_h, \boldsymbol{\beta}_h)$ *satisfying (4.8)-(4.9), and* h *sufficiently small, there exists a constant* $C > 0$ *such that*

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_X^2 \,+\, \mathcal{G}(\mathbf{u},\mathbf{u}_h) \;&\le\; C\left\{ inf_{\mathbf{v}_h\in X_h}\left(\|\mathbf{u} - \mathbf{v}_h\|_X^2 \,+\, \mathcal{E}(\mathbf{u},\mathbf{u}_h)^r\|\mathbf{u} - \mathbf{v}_h\|_X^r\right)\right. \\
&\qquad \left. + \, inf_{q_h\in M_h}\|p - q_h\|_M^2 \;+\; inf_{\zeta_h\in L_h}\|\lambda - \zeta_h\|_{W^{1/r,r'}(\Gamma)}\right\} , \qquad (4.28) \\
\|p - p_h\|_M \;+\; \|\beta - \beta_h\|_{\mathbb{R}^2} \;+\; \|\lambda - \lambda_h\|_{W^{1/r,r'}(\Gamma)} \;&\le\; C\left\{\mathcal{E}(\mathbf{u},\mathbf{u}_h)\,\mathcal{G}(\mathbf{u},\mathbf{u}_h)^{1/r'}\right. \\
&\qquad \left. + \, inf_{q_h\in M_h}\|p - q_h\|_M \,+\, inf_{\zeta_h\in L_h}\|\lambda - \zeta_h\|_{W^{1/r,r'}(\Gamma)}\right\} . \qquad (4.29)
\end{aligned}$$

Note that the constant C in Theorem 3 may depend upon $\|\mathbf{u}\|_X$.

The following *combined* inf-sup condition is used in the proof of Theorem 3.

Lemma 8 *There exists a constant $C_c > 0$ such that*

$$\inf_{(0,0,\mathbf{0})\neq(q_h,\zeta_h,\boldsymbol{\gamma}_h)\in M_h\times L_h\times\mathrm{I\!R}^2}\ \sup_{\mathbf{v}_h\in X_h} \frac{b(\mathbf{v}_h,q_h,\boldsymbol{\gamma}_h)\ -\ b_I(\mathbf{v}_h,\zeta_h)}{(\|q_h\|_M + \|\zeta_h\|_{W^{1/r,r'}(\Gamma)} + \|\boldsymbol{\gamma}_h\|_{\mathrm{I\!R}^2})\,\|\mathbf{v}_h\|_X}\ \geq\ C_c\,. \tag{4.30}$$

Proof: As $b(\cdot,\cdot,\cdot)$ and $b_I(\cdot,\cdot)$ are continuous and satisfy inf-sup conditions (4.14) and (4.26), the inf-sup condition (4.30) follows immediately. (See Theorem 4 in the Appendix.)

■

Proof of Theorem 3:
Introduce the affine subspace $\tilde{Z}_h \subset Z_h \subset X_h$ defined by

$$\tilde{Z}_h := \left\{(q_h,\zeta_h,\boldsymbol{\gamma}_h)\in M_h\times L_h\times\mathrm{I\!R}^2 :\ -b(\mathbf{v}_h,q_h,\boldsymbol{\gamma}_h)\ +\ b_I(\mathbf{v}_h,\zeta_h)\ =\ (\mathbf{f},\mathbf{v}_h) - a(\mathbf{u}_h,\mathbf{v}_h)\quad \forall\mathbf{v}_h\in X_h\right\}.$$

Note that $(p_h,\lambda_h,\beta_h)\in\tilde{Z}_h$.

For $\mathbf{u}_{f,h}$, from (2.16)

$$\begin{aligned}
&\frac{\|\mathbf{d}(\mathbf{u}_f)-\mathbf{d}(\mathbf{u}_{f,h})\|^2_{L^r(\Omega_f)}}{c+\|\mathbf{d}(\mathbf{u}_f)\|^{2-r}_{L^r(\Omega_f)}+\|\mathbf{d}(\mathbf{u}_{f,h})\|^{2-r}_{L^r(\Omega_f)}}\ +\ \int_{\Omega_f}|g_f(\mathbf{d}(\mathbf{u}_f))\mathbf{d}(\mathbf{u}_f)-g_f(\mathbf{d}(\mathbf{u}_{f,h}))\mathbf{d}(\mathbf{u}_{f,h})||\mathbf{d}(\mathbf{u}_f)-\mathbf{d}(\mathbf{u}_{f,h})|\,dA\\
&\quad\leq\ C\int_{\Omega_f}(g_f(\mathbf{d}(\mathbf{u}_f))\mathbf{d}(\mathbf{u}_f)-g_f(\mathbf{d}(\mathbf{u}_{f,h}))\mathbf{d}(\mathbf{u}_{f,h})):(\mathbf{d}(\mathbf{u}_f)-\mathbf{d}(\mathbf{u}_{f,h}))\,dA\\
&\quad=\ C\int_{\Omega_f}(g_f(\mathbf{d}(\mathbf{u}_f))\mathbf{d}(\mathbf{u}_f)-g_f(\mathbf{d}(\mathbf{u}_{f,h}))\mathbf{d}(\mathbf{u}_{f,h})):(\mathbf{d}(\mathbf{u}_f)-\mathbf{d}(\mathbf{v}_{f,h}))\,dA\\
&\qquad+C\int_{\Omega_f}(g_f(\mathbf{d}(\mathbf{u}_f))\mathbf{d}(\mathbf{u}_f)-g_f(\mathbf{d}(\mathbf{u}_{f,h}))\mathbf{d}(\mathbf{u}_{f,h})):(\mathbf{d}(\mathbf{v}_{f,h})-\mathbf{d}(\mathbf{u}_{f,h}))\,dA\\
&\quad=\ I_1\ +\ I_2\,.
\end{aligned}$$

To estimate I_1 we use (2.17).

$$\begin{aligned}
&\int_{\Omega_f}(g_f(\mathbf{d}(\mathbf{u}_f))\mathbf{d}(\mathbf{u}_f)-g_f(\mathbf{d}(\mathbf{u}_{f,h}))\mathbf{d}(\mathbf{u}_{f,h})):(\mathbf{d}(\mathbf{u}_f)-\mathbf{d}(\mathbf{v}_{f,h}))\,dA\\
&\quad\leq\ C\left(\int_{\Omega_f}|g_f(\mathbf{d}(\mathbf{u}_f))\mathbf{d}(\mathbf{u}_f)-g_f(\mathbf{d}(\mathbf{u}_{f,h}))\mathbf{d}(\mathbf{u}_{f,h})|\ |\mathbf{d}(\mathbf{u}_f)-\mathbf{d}(\mathbf{u}_{f,h})|\,dA\right)^{1/r'}\cdot\\
&\qquad\qquad\left\|\frac{|\mathbf{d}(\mathbf{u}_f)-\mathbf{d}(\mathbf{u}_{f,h})|}{c+|\mathbf{d}(\mathbf{u}_f)|+|\mathbf{d}(\mathbf{u}_{f,h})|}\right\|_\infty^{\frac{2-r}{r}}\|\mathbf{d}(\mathbf{u}_f)-\mathbf{d}(\mathbf{v}_{f,h})\|_{L^r(\Omega_f)}\\
&\quad\leq\ \epsilon_1\int_{\Omega_f}|g_f(\mathbf{d}(\mathbf{u}_f))\mathbf{d}(\mathbf{u}_f)-g_f(\mathbf{d}(\mathbf{u}_{f,h}))\mathbf{d}(\mathbf{u}_{f,h})|\ |\mathbf{d}(\mathbf{u}_f)-\mathbf{d}(\mathbf{u}_{f,h})|\,dA\\
&\qquad+\ C\left\|\frac{|\mathbf{d}(\mathbf{u}_f)-\mathbf{d}(\mathbf{u}_{f,h})|}{c+|\mathbf{d}(\mathbf{u}_f)|+|\mathbf{d}(\mathbf{u}_{f,h})|}\right\|_\infty^{\frac{2-r}{r}r}\|\mathbf{d}(\mathbf{u}_f)-\mathbf{d}(\mathbf{v}_{f,h})\|^r_{L^r(\Omega_f)}\,.
\end{aligned}$$

Thus we have that

$$\frac{\|\mathbf{d}(\mathbf{u}_f)-\mathbf{d}(\mathbf{u}_{f,h})\|^2_{L^r(\Omega_f)}}{c+\|\mathbf{d}(\mathbf{u}_f)\|^{2-r}_{L^r(\Omega_f)}+\|\mathbf{d}(\mathbf{u}_{f,h})\|^{2-r}_{L^r(\Omega_f)}} + \int_{\Omega_f}|g_f(\mathbf{d}(\mathbf{u}_f))\mathbf{d}(\mathbf{u}_f)-g_f(\mathbf{d}(\mathbf{u}_{f,h}))\mathbf{d}(\mathbf{u}_{f,h})||\mathbf{d}(\mathbf{u}_f)-\mathbf{d}(\mathbf{u}_{f,h})|\,dA$$

$$\leq C\left\|\frac{|\mathbf{d}(\mathbf{u}_f)-\mathbf{d}(\mathbf{u}_{f,h})|}{c+|\mathbf{d}(\mathbf{u}_f)|+|\mathbf{d}(\mathbf{u}_{f,h})|}\right\|_\infty^{\frac{2-r}{r}r}\|\mathbf{d}(\mathbf{u}_f)-\mathbf{d}(\mathbf{v}_{f,h})\|^r_{L^r(\Omega_f)} + I_2\,. \tag{4.31}$$

Similarly, we obtain that for $v_{p,h}\in X_{p,h}$

$$\frac{\|\mathbf{u}_p-\mathbf{u}_{p,h}\|^2_{L^r(\Omega_p)}}{c+\|\mathbf{u}_p\|^{2-r}_{L^r(\Omega_p)}+\|\mathbf{u}_{p,h}\|^{2-r}_{L^r(\Omega_p)}} + \int_{\Omega_p}|g_p(\mathbf{u}_p)\mathbf{u}_p-g_p(\mathbf{u}_{p,h})\mathbf{u}_{p,h}|\,|\mathbf{u}_p-\mathbf{u}_{p,h}|\,dA$$

$$\leq \; C\left\|\frac{|\mathbf{u}_p-\mathbf{u}_{p,h}|}{c+|\mathbf{u}_p|+|\mathbf{u}_{p,h}|}\right\|_\infty^{\frac{2-r}{r}r}\|\mathbf{u}_p-\mathbf{v}_{p,h}\|^r_{L^r(\Omega_p)} + I_4\,, \tag{4.32}$$

where I_4 is given by

$$I_4 \;:=\; C\int_{\Omega_p}(g_p(\mathbf{u}_p)\mathbf{u}_p-g_p(\mathbf{u}_{p,h})\mathbf{u}_{p,h}):(\mathbf{v}_{p,h}-\mathbf{u}_{p,h})\;dA\,.$$

Note that, with $\mathbf{v}_h=(\mathbf{v}_{f,h},\mathbf{v}_{p,h})$, $I_2+I_4 \;=\; a(\mathbf{u},\mathbf{v}_h-\mathbf{u}_h)-a(\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_h)$, and for $(q_h,\zeta_h,\boldsymbol{\gamma}_h)\in\tilde{Z}_h$

$$\begin{aligned}
a(\mathbf{u},\mathbf{v}_h-\mathbf{u}_h) - a(\mathbf{u}_h,\mathbf{v}_h-\mathbf{u}_h) \;&=\; b(\mathbf{v}_h-\mathbf{u}_h,p,\boldsymbol{\beta}) - b_I(\mathbf{v}_h-\mathbf{u}_h,\lambda) - b(\mathbf{v}_h-\mathbf{u}_h,p_h,\boldsymbol{\beta}_h)\\
&\quad + b_I(\mathbf{v}_h-\mathbf{u}_h,\lambda_h)\\
&=\; b(\mathbf{v}_h-\mathbf{u}_h,p,\boldsymbol{\beta}) - b_I(\mathbf{v}_h-\mathbf{u}_h,\lambda) \quad (\text{ as } (p_h,\lambda_h,\boldsymbol{\beta}_h)\in\tilde{Z}_h)\\
&=\; b(\mathbf{v}_h-\mathbf{u}_h,p-q_h,\boldsymbol{\beta}-\boldsymbol{\gamma}_h) - b_I(\mathbf{v}_h-\mathbf{u}_h,\lambda-\zeta_h)\\
&=\; b(\mathbf{u}-\mathbf{u}_h,p-q_h,\boldsymbol{\beta}-\boldsymbol{\gamma}_h) - b(\mathbf{u}-\mathbf{v}_h,p-q_h,\boldsymbol{\beta}-\boldsymbol{\gamma}_h)\\
&\quad - b_I(\mathbf{u}-\mathbf{u}_h,\lambda-\zeta_h) + b_I(\mathbf{u}-\mathbf{v}_h,\lambda-\zeta_h)\\
&\leq\; \epsilon\|\mathbf{u}-\mathbf{u}_h\|^2_X\\
&\qquad + C\left(\|\mathbf{u}-\mathbf{v}_h\|^2_X + \|p-q_h\|^2_M + \|\lambda-\zeta_h\|^2_{W^{1/r,r'}(\Gamma)}\right)
\end{aligned} \tag{4.33}$$

In the last step of (4.33) we use the continuity of the operators $b(\cdot,\cdot,\cdot)$ and $b_I(\cdot,\cdot)$.

Combining (4.31)-(4.33) we obtain the estimate (4.28) for $(q_h,\zeta_h,\boldsymbol{\gamma}_h)\in\tilde{Z}_h$. The inf-sup condition (4.30) then enables $(q_h,\zeta_h,\boldsymbol{\gamma}_h)$ to be lifted from $\tilde{Z}_h$ to $M_h\times L_h\times\mathbb{R}_2$. (See [5] for details.)

To establish (4.29) we begin with the inf-sup condition (4.30).

$$\begin{aligned}
\|p_h-q_h\|_M \;+\; \|\boldsymbol{\beta}_h-\boldsymbol{\gamma}_h\|_{\mathbb{R}^2} + \|\lambda_h-\zeta_h\|_{W^{1/r,r'}(\Gamma)} \;&\leq\; C\frac{b(\mathbf{v}_h,(p_h-q_h),(\boldsymbol{\beta}_h-\boldsymbol{\gamma}_h))-b_I(\mathbf{v}_h,\lambda_h-\zeta_h)}{\|\mathbf{v}_h\|_X}\\
&\leq\; C\left(\frac{b(\mathbf{v}_h,(p-q_h),(\boldsymbol{\beta}-\boldsymbol{\gamma}_h))-b_I(\mathbf{v}_h,\lambda-\lambda_h)}{\|\mathbf{v}_h\|_X} - \frac{b(\mathbf{v}_h,(p-p_h),(\boldsymbol{\beta}-\boldsymbol{\beta}_h))-b_I(\mathbf{v}_h,\lambda-\zeta_h)}{\|\mathbf{v}_h\|_X}\right)\\
&\leq\; C\left(\|p-q_h\|_M + \|\boldsymbol{\beta}-\boldsymbol{\gamma}_h\|_{\mathbb{R}^2} + \|\lambda_h-\zeta_h\|_{W^{1/r,r'}(\Gamma)} - \frac{a(\mathbf{u},\mathbf{v}_h) - a(\mathbf{u}_h,\mathbf{v}_h)}{\|\mathbf{v}_h\|_X}\right)\\
&\leq\; C\left(\|p-q_h\|_M + \|\boldsymbol{\beta}-\boldsymbol{\gamma}_h\|_{\mathbb{R}^2} + \|\lambda_h-\zeta_h\|_{W^{1/r,r'}(\Gamma)} + \mathcal{E}(\mathbf{u},\mathbf{u}_h)\,\mathcal{G}(\mathbf{u},\mathbf{u}_h)^{1/r'}\right).
\end{aligned} \tag{4.34}$$

Combining (4.34)with the triangle inequality, we obtain (4.29).

■

Appendix

A Extension operator from Γ to $\partial\Omega$

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^n$ ($n = 2$ or 3), and let $\partial\Omega = \bar{\Gamma} \cup \bar{\Gamma}_b \cup \bar{\Gamma}_d$, where Γ, Γ_b, and Γ_d are pairwise disjoint and $dist(\Gamma, \Gamma_d) > 0$. Additionally, let $\Gamma^c = \partial\Omega \backslash \Gamma$.

We use standard notation to denote the function spaces used, for example $W^{s,p}(\Omega)$, $W^{l,p}(\partial\Omega)$, etc., with $W_{00}^{-l,q}(\partial\Omega)$ denoting the dual space of $W_{00}^{l,p}(\partial\Omega)$, where q is the unitary conjugate of p, i.e. $1/q := 1 - 1/p$.

The expression $A \preceq B$ is used to denote the inequality $A \leq (constant) \cdot B$.

Next we investigate a suitable extension of a function λ defined on Γ to a function defined on $\partial\Omega$.

Assume that $p \geq 2$.

Lemma 9 *Given $\lambda \in W^{1/q,p}(\Gamma)$ define $E_\Gamma^p \lambda := \gamma_0 \varphi$, where γ_0 is the trace operator from $W^{1,p}(\Omega)$ to $W^{1/q,p}(\partial\Omega)$, and $\varphi \in W^{1,p}(\Omega)$ is the weak solution to*

$$-\nabla \cdot |\nabla\varphi|^{p-2}\nabla\varphi \;=\; 0 \quad in\ \Omega, \tag{A.1}$$

$$\varphi \;=\; \begin{cases} \lambda & on\ \Gamma \\ 0 & on\ \Gamma_d \end{cases} \tag{A.2}$$

$$|\nabla\varphi|^{p-2}\partial_{\mathbf{n}}\varphi \;=\; 0 \quad on\ \Gamma_b\,. \tag{A.3}$$

Then $E_\Gamma^p \lambda \in W^{1/q,p}(\partial\Omega)$, and $\|E_\Gamma^p \lambda\|_{W^{1/q,p}(\partial\Omega)} \preceq \|\lambda\|_{W^{1/q,p}(\Gamma)}$.

Proof:
The proof follows from the strong monotonicity [19] of the operator $\mathcal{L} : X \longrightarrow X^*$, $\mathcal{L}(u) := -\nabla \cdot |\nabla u|^{p-2}\nabla u$, where $X = \{f \in W^{1,p}(\Omega) \,:\, f|_{\Gamma \cup \Gamma_d} = 0\}$ [23].

■

For $\lambda \in W^{1/q,p}(\Gamma)$, let $E_{00,\Gamma}^p \lambda$ denote the extension of λ by zero on Γ^c.

Remark: Note that $E_{00,\Gamma}^p \lambda \in W^{1/q,p}(\partial\Omega)$ if and only if $\lambda \in W_{00}^{1/q,p}(\Gamma)$.

Lemma 10 *[9] For $\zeta \in W^{1/q,p}(\partial\Omega)$ there exists $\zeta_\Gamma \in W^{1/q,p}(\Gamma)$ and $\zeta_{\Gamma^c} \in W_{00}^{1/q,p}(\Gamma^c)$ such that $\zeta = E_\Gamma^p \zeta_\Gamma + E_{00,\Gamma^c}^p \zeta_{\Gamma^c}$. Moreover, this decomposition is unique.*

Proof: Let $\zeta \in W^{1/q,p}(\partial\Omega)$. Define, $\zeta_\Gamma := \zeta|_\Gamma$ and $\zeta_{\Gamma^c} := \xi|_{\Gamma^c}$ where $\xi := \zeta - E_\Gamma^p \zeta_\Gamma$. Note that $\zeta|_\Gamma \in W^{1/q,p}(\Gamma)$ and

$$\|E_\Gamma^p \zeta_\Gamma\|_{W^{1/q,p}(\partial\Omega)} \;\preceq\; \|\zeta_\Gamma\|_{W^{1/q,p}(\Gamma)} \;\leq\; \|\zeta\|_{W^{1/q,p}(\partial\Omega)}\,,$$

hence $\xi \in W^{1/q,p}(\partial\Omega)$. Also, $E^p_{00,\Gamma^c}\zeta_{\Gamma^c} = \xi$ as ζ and $E^p_\Gamma \zeta_\Gamma$ agree on Γ. Thus, from the remark above, $\zeta_{\Gamma^c} \in W^{1/q,p}_{00}(\Gamma^c)$.

To show uniqueness of the decomposition, observe that if $0 = E^p_\Gamma \zeta_\Gamma + E^p_{00,\Gamma^c}\zeta_{\Gamma^c}$ then ζ_Γ is the trace of the weak solution of (A.1)-(A.3) for $\lambda = 0$. Hence $\zeta_\Gamma = 0$.

■

Next we introduce the concept of *the restriction of an operator* in $W^{-1/q,q}(\partial\Omega)$ to be equal to zero.

Definition 1 *[9] If $f \in W^{-1/q,q}(\partial\Omega)$, then $f|_{\Gamma^c} = 0$ means by definition that*

$$\langle f \,, E^p_{00,\Gamma^c}\xi \rangle_{\partial\Omega} \;=\; 0\,, \qquad \text{for all} \quad \xi \in W^{1/q,p}_{00}(\Gamma^c)\,. \tag{A.4}$$

The following lemma describes how an operator in $W^{-1/q,q}(\partial\Omega)$ can be decomposed into an operator in $W^{-1/q,q}(\Gamma)$ and an operator in $W^{-1/q,q}(\Gamma^c)$.

Lemma 11 *[9] For $f \in W^{-1/q,q}(\partial\Omega)$ there exists $f_\Gamma \in W^{-1/q,q}(\Gamma)$ and $f_{\Gamma^c} \in W^{-1/q,q}_{00}(\Gamma^c)$ such that for $\zeta \in W^{1/q,p}(\partial\Omega)$, with $\zeta \;=\; E^p_\Gamma \zeta_\Gamma \,+\, E^p_{00,\Gamma^c}\zeta_{\Gamma^c}$, as defined in Lemma 10, we have*

$$\langle f \,,\, \zeta \rangle_{\partial\Omega} \;=\; \langle f_\Gamma \,,\, \zeta_\Gamma \rangle_\Gamma \,+\, \langle f_{\Gamma^c} \,,\, \zeta_{\Gamma^c} \rangle_{\Gamma^c}\,. \tag{A.5}$$

Proof: For $\zeta_\Gamma \in W^{1/q,p}(\Gamma)$ and $\zeta_{\Gamma^c} \in W^{1/q,p}_{00}(\Gamma^c)$, define

$$\langle f_\Gamma \,, \zeta_\Gamma \rangle_\Gamma \;:=\; \langle f \,, E^p_\Gamma \zeta_\Gamma \rangle_{\partial\Omega} \qquad \text{and } \langle f_{\Gamma^c} \,, \zeta_{\Gamma^c} \rangle_{\Gamma^c} \;:=\; \langle f \,, E^p_{00,\Gamma^c}\zeta_{\Gamma^c} \rangle_{\partial\Omega}\,. \tag{A.6}$$

Then,

$$\langle f_\Gamma \,, \zeta_\Gamma \rangle_\Gamma \;\le\; \|f\|_{W^{-1/q,q}(\partial\Omega)} \|E^p_\Gamma \zeta_\Gamma\|_{W^{1/q,p}(\partial\Omega)} \;\preceq\; \|f\|_{W^{-1/q,q}(\partial\Omega)} \|\zeta_\Gamma\|_{W^{1/q,p}(\Gamma)}\,,$$

thus $f_\Gamma \in W^{-1/q,q}(\Gamma)$. Analogously, $f_{\Gamma^c} \in W^{-1/q,q}_{00}(\Gamma^c)$. Additionally,

$$\langle f_\Gamma \,,\, \zeta_\Gamma \rangle_\Gamma \,+\, \langle f_{\Gamma^c} \,,\, \zeta_{\Gamma^c} \rangle_{\Gamma^c} \;=\; \langle f \,,\, E^p_\Gamma \zeta_\Gamma \rangle_{\partial\Omega} \,+\, \langle f \,,\, E^p_{00,\Gamma^c}\zeta_{\Gamma^c} \rangle_{\partial\Omega} \;=\; \langle f \,,\, \zeta \rangle_{\partial\Omega}\,.$$

■

Note that for $f \in W^{-1/q,q}(\partial\Omega)$ with $f|_{\Gamma^c} = 0$ (see Definition 1), from (A.6),

$$\langle f \,,\, \zeta \rangle_{\partial\Omega} \;=\; \langle f_\Gamma \,,\, \zeta_\Gamma \rangle_\Gamma \quad \text{for all } \zeta \in W^{1/q,p}(\partial\Omega)\,. \tag{A.7}$$

Thus functionals in $W^{-1/q,q}(\partial\Omega)$ which are zero when restricted to $\partial\Omega\backslash\Gamma$ can be identified with functionals in $W^{-1/q,q}(\Gamma)$.

B Combined inf-sup conditions

In deriving a priori error estimates for mixed methods, whose analysis relies on several inf-sup conditions, combined inf-sup conditions are needed. In this section we show that the required inf-sup conditions follow readily from the continuity of the bilinear forms and the individual inf-sup conditions.

Theorem 4 *Let V, Q_1, Q_2 be Banach spaces; $b_1(\cdot,\cdot) : V \times Q_1 \longrightarrow \mathbb{R}$, $b_2(\cdot,\cdot) : V \times Q_2 \longrightarrow \mathbb{R}$ and $Z_1 := \{v \in V \,|\, b_1(v,q) = 0, \forall q \in Q_1\}$. Assume that $b_2(\cdot,\cdot)$ is continuous and there exists β_1, $\beta_2 > 0$ such that*

$$\begin{aligned} \sup_{v \in V, \|v\|_V = 1} b_1(v,q_1) &\geq \beta_1 \|q_1\|_{Q_1}\,, \qquad \forall q_1 \in Q_1\,, \\ \sup_{v \in Z_1, \|v\|_V = 1} b_2(v,q_2) &\geq \beta_2 \|q_2\|_{Q_2}\,, \qquad \forall q_2 \in Q_2\,. \end{aligned}$$

Then there exists $\beta > 0$ such that

$$\sup_{v \in V, \|v\|_V = 1} \left(b_1(v,q_1) + b_2(v,q_2)\right) \geq \beta \left(\|q_1\|_{Q_1} + \|q_2\|_{Q_2}\right)\,, \quad \forall (q_1,q_2) \in Q_1 \times Q_2\,.$$

Proof:
By the continuity of $b_2(\cdot,\cdot)$, there exists $C_2 > 0$ such that

$$b_2(v,q_2) \leq C_2 \|v\|_V \|q_2\|_{Q_2}\,, \quad \forall (v,q_2) \in V \times Q_2\,.$$

Let $(q_1,q_2) \in Q_1 \times Q_2$ be given, and $v_1 \in V$ with $\|v_1\|_V = 1$, $v_2 \in Z_1$ with $\|v_2\|_V = 1$, satisfy

$$b_1(v_1,q_1) \geq \frac{\beta_1}{2} \|q_1\|_{Q_1}\,, \quad b_2(v_2,q_2) \geq \frac{\beta_2}{2} \|q_2\|_{Q_2}\,.$$

Then for $u = v_1 + (1 + 2C_2/\beta_2) v_2$ we have

$$\begin{aligned} b_1(u,q_1) &= b_1(v_1,q_1) \geq \frac{\beta_1}{2} \|q_1\|_{Q_1}\,, \\ b_2(u,q_2) &= b_2(v_1,q_2) + (1 + \frac{2C_2}{\beta_2}) b_2(v_2,q_2) \geq \frac{\beta_2}{2} \|q_2\|_{Q_2}\,. \end{aligned}$$

Finally, as $\|u\|_V \leq 2(1 + 2C_2/\beta_2)$, with $u_0 = u/\|u\|_V$

$$b_1(u_0,q_1) + b_2(u_0,q_2) \geq \beta \left(\|q_1\|_{Q_1} + \|q_2\|_{Q_2}\right)\,,$$

where $\beta = \min\{\beta_1, \beta_2\}/(4(1 + C_2/\beta_2))$.

■

Corollary 1 *Let Z_0, Q_i, $i = 1, \ldots, n$ be Banach spaces, $b_i(\cdot,\cdot) : Z_0 \times Q_i \longrightarrow \mathbb{R}\, i = 1, \ldots, n$ and $Z_i := \{v \in Z_{i-1} \,|\, b_i(v,q) = 0, \forall q \in Q_i\}$, $i = 1, \ldots, n-1$. Assume that $b_i(\cdot,\cdot)$ is continuous and there exist β_i such that*

$$\sup_{v \in Z_{i-1}, \|v\|_{Z_0} = 1} b_i(v,q) \geq \beta_i \|q\|_{Q_i}\,, \qquad \forall q \in Q_i\,, \, i = 1, \ldots, n.$$

Then there exists $\beta > 0$ such that

$$\sup_{v \in Z_0, \|v\|_{Z_0} = 1} \sum_{i=1}^{n} b_i(v,q_i) \geq \beta \left(\|q_1\|_{Q_1} + \ldots + \|q_n\|_{Q_n}\right)\,, \quad \forall (q_1, \ldots, q_n) \in Q_1 \times \ldots \times Q_n\,. \qquad \text{(B.8)}$$

Proof: The proof of (B.8) follows from Theorem 4 and induction.

■

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