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Coupled Method of Homotopy Perturbation Method and Variational Approach for Solution to Nonlinear Cubic-Quintic Duffing Oscillator

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Abstract

This paper presents a new approach for solving approximate analytical higher order solutions for strong nonlinear Duffing oscillators with cubic-quintic nonlinear restoring force. The system is conservative and with odd nonlinearity. The new approach couples Homotopy Perturbation Method with Variational method. These approximate solutions are valid for small as well as large amplitudes of oscillation. In addition, it is not restricted to the presence of a small parameter such as in the classical perturbation method. Illustrative examples are presented to verify accuracy and explicitness of the approximate solutions.

Keywords: Nonlinear Oscillators, Variational formulation, Homotopy Perturbation Method, Duffing equation

1- Introduction

We present in this paper a new accurate approach for accurate higher-order approximate analytical solutions of the Duffing oscillator with strong cubic and quintic nonlinearities.

Nonlinear oscillation in engineering and applied mathematics has been a topic to intensive research for many years. Many asymptotic techniques including,variational iteration method [2, 12], homotopy perturbation method [3–15], energy balance method [4, 6, 8, 10] were used to handle strongly nonlinear systems. Coupled Method of Homotopy Perturbation Method and Variational Method was paid attention recently; it is proven this method is very effective to determine the natural frequencies of strongly nonlinear oscillators with high accuracy [9].

In Coupled Method of Homotopy Perturbation Method and Variational Method, the following homotopy is constructed and a variational formulation for the nonlinear oscillation is established, from which the natural frequency and approximate solution can be readily obtained [9].

The Duffing equation is a well-known nonlinear differential equation which is related to many practical engineering systems such as the classical nonlinear spring system with odd nonlinear restoring characteristics and more recently in different physical phenomena. There have been many variations of Duffing equation, for instance, the Duffing-harmonic equation and the cubic-quintic Duffing equation [13].

Due to the presence of fifth power nonlinearity, the cubic-quintic Duffing equation inherits strong nonlinearity and thus accuracy of approximate analytical methods becomes extremely demanding. cubic-quintic Duffing equation can be found in the modeling of free vibration of a restrained uniform beam carrying intermediate lumped mass and undergoing large amplitude of oscillations in the unimodel Duffing type temporal problem [7,11], the nonlinear dynamics of a slender elastica [14], the generalized Pochhammer-Chree (PC) equations [16] and the compound Korteweg-de Vries (KdV) equation [5] in nonlinear wave systems, and the propagation of a short electromagnetic pulse in a nonlinear medium [1]

2- The Homotopy Perturbation Method and Variational Formulation

To illustrate the basic ideas of this method, we consider the following equation [3, 15]:

$$A(u)-f(r)=0 \qquad r\in\Omega, \tag{1}$$

With the boundary condition of:

$$B\left(u,\frac{\partial u}{\partial n}\right)=0 \qquad r\in\Gamma, \tag{2}$$

Where *A* is a general differential operator, *B* a boundary operator, *f (r)* a known analytical function and *Γ* is the boundary of the domain *Ω*.

A can be divided into two parts which are *L* and *N*, where *L* is linear and *N* is nonlinear. Eq. (1) can therefore be rewritten as follows:

$$L(u)+N(u)-f(r)=0 \qquad r\in\Omega, \tag{3}$$

Homotopy perturbation structure is shown as follows:

$$H(v,p)=(1-p)\left[L(v)-L(u_0)\right]+p\left[A(v)-f(r)\right]=0 \tag{4}$$

Where,

$$v(r,p):\ \Omega\times[0,1]\rightarrow R \tag{5}$$

In Eq. (5), $p\in[0\,,1]$ is an embedding parameter and u_0 is the first approximation that satisfies the boundary condition. We can assume that the solution of Eq. (5) can be written as a power series in *p*, as following:

$$v=v_0+pv_1+p^2v_2+... \tag{6}$$

And the best approximation for solution is:

$$u=\lim_{p\to 1} v=v_0+v_1+v_2+... \tag{7}$$

Consider the following generalized nonlinear oscillations without forced terms.

$$u''+\omega_0^{\ 2}\,u+\varepsilon f\,(u)=0 \tag{8}$$

Where f is a nonlinear function of u'',u',u **.**

Its variational functional can be easily obtained as [9]:

$$J(u)=\int_0^t\left\{\ -\frac{1}{2}u'^2+\frac{1}{2}\omega_0^{\ 2}u_2+\varepsilon F(u)\right\}dt \tag{9}$$

Where F is the potential,

$$\frac{dF}{du}=f \tag{10}$$

2- Problem Definition

A cubic-quintic Duffing oscillator of a conservative autonomous system can be described by the following second-order differential equation with cubic-quintic nonlinearities [13]

$$u''+f\,(u)=0 \tag{11}$$

With initial conditions: $u(0)=A$, $u'(0)=0$

Where $f\,(u)=\alpha u+\beta u^3+\gamma u^5$ is an odd function, and u and t are generalized dimensionless displacement and time variables while α , β and γ are positive constant parameters if $\gamma=0$ it is a cubic Duffing oscillator, if $\beta=0$ it is a quintic oscillator otherwise it is a cubic-quintic oscillator.

3- Applications

In order to assess the advantages and the accuracy of the Coupled Method of Homotopy Perturbation Method and Variational Method, we will consider the following two examples.

3.1- Example 1

We consider the quintic nonlinear oscillator [9]:

$$u'' + u + \varepsilon u^5 = 0 \tag{12}$$

With initial condition of: $u(0) = A$, $u'(0) = 0$
Suppose that the frequency of Eq. (12) is ω.
We construct following homotopy by same manipulation as basic idea:

$$u'' + \omega^2 u + p[\varepsilon u^5 + (1-\omega^2)u] = 0 \ , \ p \in [0,1] \tag{13}$$

We assume that the periodic solution to equation Eq. (13) may be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + ... \tag{14}$$

Substituting Eq. (14) into Eq. (13), collecting terms of the same power of p, gives:

$$u_0'' + \omega^2 u_0 + = 0 \ , \ u_0(0) = A \ , \ u_0'(0) = 0 \tag{15}$$

And:

$$u_1'' + \omega^2 u_1 + \varepsilon u_0^5 + (1-\omega^2)u_0 = 0 \ , \ u_1(0) = 0 \ , \ u_1'(0) = 0 \tag{16}$$

The solution of Eq. (15) is $u_0 = A\cos\omega t$, where ω will be identified from the variational formulation for u_1, which reads:

$$J(u_1) = \int_0^T \{-\frac{1}{2}u_1'^2 + \frac{1}{2}\omega^2 u_1^2 + (1-\omega^2)\, u_0 u_1 + u_0^5 u_1\}\, dt \ , \tag{17}$$

To better illustrate the procedure, we choose the simplest trail function:

$$u_1 = B(\cos\omega t - \frac{1}{5}\cos 3\omega t) \tag{18}$$

Substituting u_1 into the functional Eq. (17) results in:

$$J(B,\omega) = \left\{ \frac{1}{1200} \frac{(-192\omega^2 B\pi - 1200A\omega^2\pi + 675\varepsilon A^5\pi + 1200A\pi)B}{\omega} \right\} \tag{19}$$

Setting:

$$\frac{\partial J}{\partial B} = 0 \text{ , and} \frac{\partial J}{\partial \omega} = 0 \tag{20}$$

We obtain:

$$-\omega^2 + 1 + \frac{9}{16}\varepsilon A^4 = 0, \text{ and ,B=0} \tag{21}$$

The first order approximate solution is obtained, which reads:

$$\omega = \sqrt{1 + \frac{9}{16}\varepsilon A^4} \tag{22}$$

In order to compare with energy balance solution, we write Pashaei, Ganji, and Akbarzade's result [6]:

$$\omega = \sqrt{1 + \frac{7}{12}\varepsilon A^4} \tag{23}$$

And we write homotopy perturbation method solution, by J. H. He's result [9]:

$$\omega = \sqrt{\frac{5}{8}\varepsilon A^4 + 1} \tag{24}$$

Table 1. Comparison of coupled method frequency with parameter expanding frequency and energy balance frequency ($\varepsilon = 1$).

A	Coupled method frequency	Homotopy perturbation method	Energy balance frequency
0.01	1.0000	1.0000	1.0000
0.1	1.0000	1.0000	1.0000
0.2	1.0004	1.0005	1.0005
0.3	1.0023	1.0025	1.0024
0.4	1.0072	1.0080	1.0074
0.5	1.0174	1.0193	1.0181
1	1.2500	1.2747	1.2583
5	18.7766	19.7895	19.1202

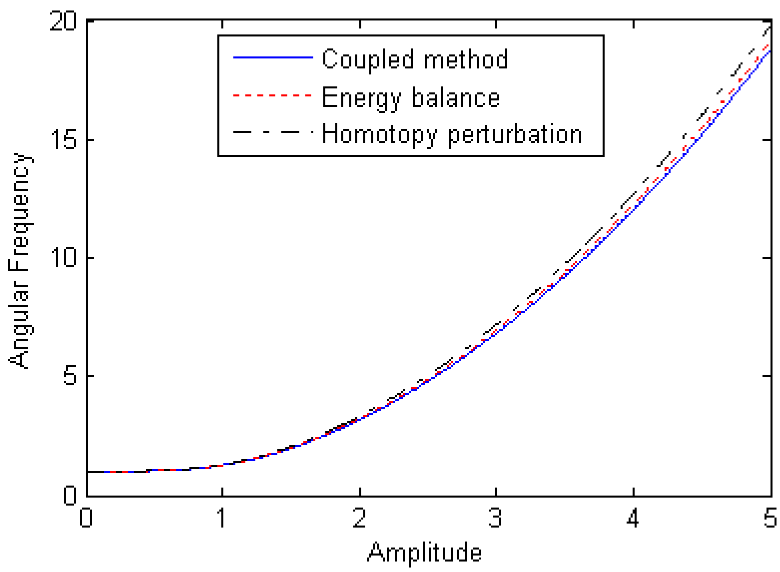

Figure 1. Comparison of the coupled method solution with the Homotopy perturbation solution and energy balance solution $\varepsilon_1 = \varepsilon_2 = 1$.

3.2- Example 2

We consider the cubic-quintic nonlinear oscillator [9]:

$$u'' + u + \varepsilon_1 u^3 + \varepsilon_2 u^5 = 0 \tag{25}$$

With the initial condition of: $u(0) = A$, $u'(0) = 0$

Suppose that the frequency of Eq. (25) is ω

We construct following homotopy:

$$u'' + \omega^2 u + p[\varepsilon_1 u^3 + \varepsilon_2 u^5 + (1-\omega^2)u] = 0\ ,\ p \in [0\ ,1] \tag{26}$$

We assume that the periodic solution to equation Eq. (26) may be written as a power series in p:

$$u = u_0 + pu_1 + p^2 u_2 + \dots , \tag{27}$$

Substituting Eq. (27) into Eq. (26), collecting terms of the same power of p, gives:

$$u_0'' + \omega^2 u_0 + = 0\ ,\ u_0(0) = A\ ,\ u_0'(0) = 0 \tag{28}$$

And:

$$u_1'' + \omega^2 u_1 + \varepsilon_1 u_0^3 + \varepsilon_2 u_0^5 + (1-\omega^2)u_0 = 0\ ,\ u_1(0) = 0\ ,\ u_1'(0) = 0 \tag{29}$$

The solution of Eq. (28) is $u_0 = A\cos\omega t$, where ω will be identified from the variational formulation for u_1, which reads:

$$J(u_1) = \int_0^T \{-\frac{1}{2}u_1'^2 + \frac{1}{2}\omega^2 u_1^2 + (1-\omega^2)\ u_0 u_1\ + \varepsilon_1 u_0^3 u_1 + \varepsilon_2 u_0^5 u_1\}\ dt\ \ ,\ T = \frac{2\pi}{\omega} \tag{30}$$

To better illustrate the procedure, we choose the simplest trail function:

$$u_1 = B(\cos\omega t - \frac{1}{5}\cos 3\omega t) \tag{31}$$

Substituting u_1 into the functional Eq. (30) results in:

$$J(B,\omega) = \left\{ \frac{1}{1200} \frac{(1200A\pi - 192\omega^2 B\pi - 1200A\omega^2\pi + 840\varepsilon_1 A^3\pi + 675\varepsilon_2 A^5\pi)B}{\omega} \right\} \tag{32}$$

Setting:

$$\frac{\partial J}{\partial B} = 0\ , \text{and} \frac{\partial J}{\partial \omega} = 0 \tag{33}$$

We obtain:

$$1 - \omega^2 + \frac{7}{10}\varepsilon_1 A^2 + \frac{9}{16}\varepsilon_2 A^4 = 0,\ \text{and ,B=0} \tag{34}$$

The first order approximate solution is obtained, which reads:

$$\omega = \sqrt{1 + \frac{7}{10}\varepsilon_1 A^2 + \frac{9}{16}\varepsilon_2 A^4} \tag{35}$$

In order to compare with Modified Lindsted-Poincare solution: Double series Expansion, we write J. H. He's result [9]:

$$\omega = \sqrt{1 + \frac{3}{4}\varepsilon_1 A^2 + \frac{5}{8}\varepsilon_2 A^4} \tag{36}$$

Table 2.Comparison of coupled method frequency with parameter expanding frequency and energy balance frequency $\varepsilon_1 = \varepsilon_2 = 1$.

A	Coupled method frequency	Modified Lindsted-Poincare frequency
0.1	1.0035	1.0038
0.2	1.0143	1.0154
0.3	1.0332	1.0356
0.4	1.0613	1.0658
0.5	1.1001	1.1075
1	1.5042	1.5411
5	19.2370	20.2577

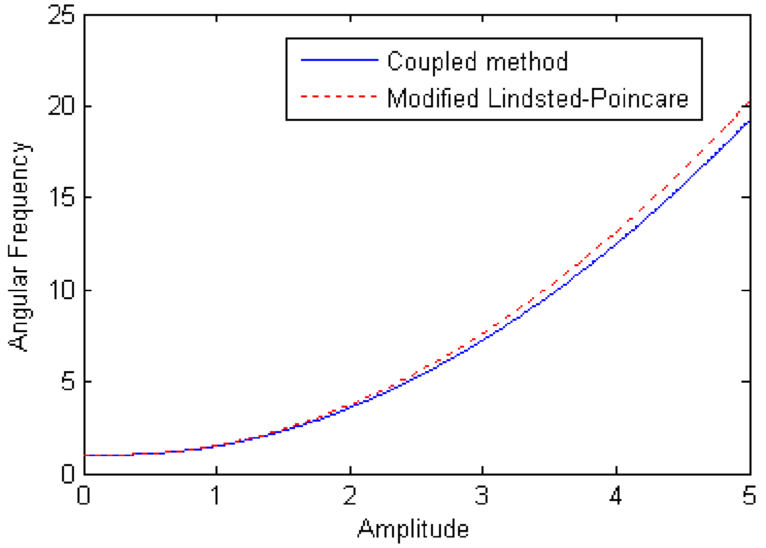

Figure2. Comparison of the coupled method solution with the Modified Lindsted-Poincare solution and energy balance solution.

4- Conclusions

This paper has proposed a new method for solving accurate analytical approximations to strong nonlinear oscillations.
The solution procedure of Coupled Method of Homotopy Perturbation Method and Variational Method is of deceptive simplicity and the insightful solutions obtained are of high accuracy even for the one-order approximation. Unlike the classical perturbation method involving expansion over a small parameter, these

approximate analytical frequencies are valid for small as well as large amplitudes of oscillation as it is not restricted to the presence of a small parameter.
The most important advantages of this method as compared to the previous methods are its simplicity and flexibility in application. The method can also be extended to wide range of problems such as (singular) nonlinear boundary value problems, delay differential equations, autonomous systems and other problems of mathematical physics. We think that the method have great potential which still needs further development.

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