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Coupled transverse and torsional vibrations in a mechanical system with two identical beams

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The paper aims to study a plane system with bars, with certain symmetries. Such problems can be encountered frequently in industry and civil engineering. Considerations related to the economy of the design process, constructive simplicity, cost and logistics make the use of identical parts a frequent procedure. The paper aims to determine the properties of the eigenvalues and eigenmodes for transverse and torsional vibrations of a mechanical system where two of the three component bars are identical. The determination of these properties allows the calculus effort and the computation time and thus increases the accuracy of the results in such matters. [http://dx.doi.org/10.1063/1.4985271]

I. INTRODUCTION

Symmetries appearing in engineering systems can lead, in some cases, to the simplification of the dynamic analysis made for these structures. The consequence would be shortening the design time and lowering the costs. The information provided by a repetitive structure can help the computational effort. The static analysis for such a system is presented in strength of the material. In dynamic case, in which the elastic elements lead to vibrations, some properties have been observed for a long time[1] (Meirovitch) but a systematic study of the problem has not yet been made. Particular cases have been studied in Refs. 2–6. In the following we will study a mechanical system composed of 3 bars of which two are identical, situated in a plane. It will study the transverse out of plane and torsional vibrations of such a system that is strongly coupled.

II. DESCRIPTION OF THE SYSTEM

The mechanical system considered (Fig. 1), is made of two identical cantilever beams AC and BC which are rigidly connected, perpendicular (egg, by welding in a system engineering) with a third cantilever beam CD (with constant cross section too). The three beams which are found in a plane can have transversal vibrations in a perpendicular direction on the plane ABD and torsional vibrations. The endpoints A and B are free: it results that the bending moment, the torque and the shear force are null in these two points (6 boundary conditions). The endpoint D of the beam CD is clamped; it results that the deflection, the tangent slope and the torsion angle are null (three conditions).

For the intermediary point C the deflections of the C point of the three beams are equal (two conditions). The torsion angle of the beam DC in C is equal with the tangent slope of the beam AC and BC in C (two conditions). The torsion angle of the beams AC and BC in C are equal with the tangent slope of the beam DC in C (two boundary conditions). If we consider the balance of an infinitesimal element surrounding the point C, the shearing forces in the two identical beams, together, must be equal with the shearing force in the beam CD. A similar relationship is obtained for the equilibrium

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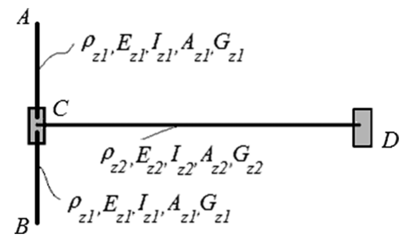

FIG. 1. The sketch of the mechanical system.

of the moments. The bending moments appearing in the beams AC and BC in the point C must be equal with the torque of the beam CD in C and the torques appearing in the beams AC and BC in C must be equal with the bending moment in C for the beam CD. These considerations will define the last three boundary conditions. Finally we have 18 boundary conditions that offer us 18 integration constants.

III. MOTION EQUATIONS OF THE TRANSVERSE AND TORSIONAL VIBRATIONS

For a continuous beam with constant cross section, the vibrations of the system are described, in the absence of distributed loads along the bar, by the classical equations (i.e. References 7–9)

$$\frac{\partial^4 v}{\partial x^4} + \frac{\rho A}{E I_z} \frac{\partial^2 v}{\partial t^2} = 0. \tag{1}$$

In relation (1) we use the following notations: v is the beam deflection, x is the ordinate of the point with the deflection v, ρ represents the density of the beam material, A - the cross section, E is Young's modulus, I_z is the second moment of the area with respect to the z axis.

According to the classical procedure for solving differential equations[10,11] we are looking for a solution in the form:

$$v(x,t) = \Phi(x) \sin(pt + \theta). \tag{2}$$

Putting relation (2) to verify (1) at any moment, we get:

$$\frac{\partial^4 \Phi}{\partial x^2} - p^2 \frac{\rho A}{E I_z} \Phi = 0. \tag{3}$$

If we denote with:

$$\lambda^4 = \frac{\rho A}{E I_z}, \tag{4}$$

(1) becomes:

$$\frac{\partial^4 \Phi}{\partial x^4} - p^2 \lambda^4 \Phi = 0, \tag{5}$$

where Φ represents the function that offers the deformed beam (eigenmode) that will vibrate with the eigenvalue p. Using the rel. (5) for the cantilever beams AC, BC, CD we obtain:

$$\text{For the beam AC}: \quad \frac{\partial^4 \Phi_{AC}}{\partial x^4} - \frac{\rho_1 A_1}{E_1 I_{z1}} p^2 \Phi_{AC} = 0. \tag{6}$$

$$\text{For the beam BC}: \quad \frac{\partial^4 \Phi_{BC}}{\partial x^4} - \frac{\rho_1 A_1}{E_1 I_{z1}} p^2 \Phi_{BC} = 0. \tag{7}$$

$$\text{For the beam CD}: \quad \frac{\partial^4 \Phi_{CD}}{\partial x^4} - \frac{\rho_2 A_2}{E_2 I_{z2}} p^2 \Phi_{CD} = 0. \tag{8}$$

The index 1 refers to the beams AC and BC and the index 2 to the beam CD. We denote:

$$\frac{\rho_1 A_1}{E_1 I_{z1}} = \lambda_1^4; \quad \frac{\rho_2 A_2}{E_2 I_{z2}} = \lambda_2^4. \tag{9}$$

The solutions of the differential equations (6), (7) and (8) will be:

$$\Phi_{AC}(x) = C_1^{AC} sin\lambda_1\sqrt{p}x + C_2^{AC} cos\lambda_1\sqrt{p}x + C_3^{AC} sh\lambda_1\sqrt{p}x + C_4^{AC} ch\lambda_1\sqrt{p}x; \tag{10}$$

$$\Phi_{BC}(x) = C_1^{BC} sin\lambda_1\sqrt{p}x + C_2^{BC} cos\lambda_1\sqrt{p}x + C_3^{BC} sh\lambda_1\sqrt{p}x + C_4^{BC} ch\lambda_1\sqrt{p}x; \tag{11}$$

$$\Phi_{CD}(x) = C_1^{CD} sin\lambda_2\sqrt{p}x + C_2^{CD} cos\lambda_2\sqrt{p}x + C_3^{CD} sh\lambda_2\sqrt{p}x + C_4^{CD} ch\lambda_2\sqrt{p}x. \tag{12}$$

For the torsional vibrations the equation describing the vibration of the x section is:

$$\frac{\partial^2\varphi}{\partial x^2} - \frac{J}{GI_p}\frac{\partial^2\varphi}{\partial t^2} = 0, \tag{13}$$

where: φ – angle of torsion of the section, x is the ordinate of the point with the torsion φ, $J = \rho I_p$, ρ represents the density of the beam material, I_p - the polar second moment of the area, G is shear's modulus.

We consider the solution φ under the form:

$$\varphi(x,t) = \psi(x)\sin(pt + \theta). \tag{14}$$

Introducing in (13) we obtain:

$$\frac{\partial^2\psi}{\partial x^2} + p^2\delta^2\psi = 0, \tag{15}$$

where: $\delta^2 = \frac{J}{GI_p}$.

In our case we are dealing with three bars, so we get three differential equations, written for the three parts, AC, BC, and CD. We have:
For the beam AC:

$$\frac{\partial^2\psi_{AC}}{\partial x^2} + p^2\delta_1^2\psi_{AC} = 0. \tag{16}$$

For the beam BC:

$$\frac{\partial^2\psi_{BC}}{\partial x^2} + p^2\delta_1^2\psi_{BC} = 0. \tag{17}$$

For the beam CD:

$$\frac{\partial^2\psi_{CD}}{\partial x^2} + p^2\delta_2^2\psi_{CD} = 0. \tag{18}$$

The solutions of the differential equations (16), (17) and (18) are:

$$\psi_{AC}(x) = D_1^{AC} sin\delta_1 px + D_2^{AC} cos\delta_1 px; \tag{19}$$

$$\psi_{BC}(x) = D_1^{BC} sin\delta_1 px + D_2^{BC} cos\delta_1 px; \tag{20}$$

$$\psi_{CD}(x) = D_1^{CD} sin\delta_2 px + D_2^{CD} cos\delta_2 px. \tag{21}$$

The free vibrations of the mechanical system are obtained using the equations (10), (11), (12), (19), (20), (21). We have 18 unknowns, the integration constants.

If we denote by M^b the bending moment of a beam in the section x, T the shear force and M^t the torque, the boundary conditions can be written:

a) For the beam AC the endpoint A is free, it results: $M_{AC}^b(0,t) = 0$; $T_{AC}(0,t) = 0$; $M_{AC}^t(0,t) = 0$;
b) For the beam BC the endpoint B is free, it results: $M_{BC}^b(0,t) = 0$; $T_{BC}(0,t) = 0$; $M_{AC}^t(0,t) = 0$;
c) For the beam CD the endpoint D is clamped, it results: $v_{CD}(l_2,t) = 0$; $v'_{CD}(l_2,t) = 0$; $\varphi(l_2,t) = 0$.

Totally we have 9 boundary conditions. In the following we will express in an analytical form these relations. The bending moment, the shear force and the torque can be expressed using the relations:[7,12–14]

$$M^b(x) = -EI_z \frac{\partial^2 v(x)}{\partial x^2}; \quad T(x) = EI_z \frac{\partial^3 v(x)}{\partial x^3}; \quad M^t(x) = GI_p \frac{\partial \varphi(x)}{\partial x}. \tag{22}$$

By differentiation it obtains (for the beam AC):

$$\begin{aligned}\frac{\partial^2 v(x)}{\partial x^2} &= \Phi''_{AC}(x)\sin(pt+\theta) = \\ &= (\lambda_1\sqrt{p})^2\left[-C_1^{AC} sin\lambda_1\sqrt{p}x - C_2^{AC} cos\lambda_1\sqrt{p}x + C_3^{AC} sh\lambda_1\sqrt{p}x + C_4^{AC} ch\lambda_1\sqrt{p}x\right]\sin(pt+\theta);\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{\partial^3 v(x)}{\partial x^3} &= \Phi'''_{AC}(x)\sin(pt+\theta) = \\ &= (\lambda_1\sqrt{p})^3\left[-C_1^{AC} cos\lambda_1\sqrt{p}x + C_2^{AC} sin\lambda_1\sqrt{p}x + C_3^{AC} ch\lambda_1\sqrt{p}x + C_4^{AC} sh\lambda_1\sqrt{p}x\right]\sin(pt+\theta);\end{aligned} \tag{24}$$

$$\frac{\partial \varphi(x)}{\partial x} = \psi'_{AC}\sin(pt+\theta) = \delta_1 p\left[D_1^{AC} cos\delta_1 px - D_2^{AC} sin\delta_1 px\right]\sin(pt+\theta), \tag{25}$$

for the beam AC and similar relations for the beams BC and CD.

If we use the boundary conditions (23), (24) and (25) in (22) and, taking into account the 9 conditions a), b) and c) previously written, we obtain the relations:

$$-C_2^{AC} + C_4^{AC} = 0, \tag{26}$$

$$-C_1^{AC} + C_3^{AC} = 0, \tag{27}$$

$$D_1^{AC} = 0, \tag{28}$$

$$-C_2^{BC} + C_4^{BC} = 0, \tag{29}$$

$$-C_1^{BC} + C_3^{BC} = 0, \tag{30}$$

$$D_1^{BC} = 0, \tag{31}$$

$$C_1^{CD} sin\lambda_2\sqrt{p}l_2 + C_2^{CD} cos\lambda_2\sqrt{p}l_2 + C_3^{CD} sh\lambda_2\sqrt{p}l_2 + C_4^{CD} ch\lambda_2\sqrt{p}l_2 = 0, \tag{32}$$

$$C_1^{CD} cos\lambda_2\sqrt{p}l_2 - C_2^{CD} sin\lambda_2\sqrt{p}l_2 + C_3^{CD} ch\lambda_2\sqrt{p}l_2 + C_4^{CD} sh\lambda_2\sqrt{p}l_2 = 0, \tag{33}$$

$$D_1^{CD} sin\delta_2 pl_2 + D_2^{CD} cos\delta_2 pl_2 = 0. \tag{34}$$

The continuity of the system in the point C leads to the following conditions: the deflections of the three bars in the point C are equal: $v_{AC}(l_1,t) = v_{BC}(l_1,t) = v_{CD}(0,t)$ (**two conditions**), the tangent slope in C of the bars AC and BC is equal with the torsion angle of the bar CD in C: $v'_{AC}(l_1,t) = v'_{BC}(l_1,t) = \varphi_{CD}(0,t)$ (**two conditions**), the torsion angles of the bars AC and BC in C are equal with the tangent slope of the bar CD in C: $\varphi_{AC}(l_1,t) = \varphi_{BC}(l_1,t) = v'_{CD}(0,t)$ (**two conditions**). These six conditions lead to the relations:

$$C_1^{AC} sin\lambda_1\sqrt{p}l_1 + C_2^{AC} cos\lambda_1\sqrt{p}l_1 + C_3^{AC} sh\lambda_1\sqrt{p}l_1 + C_4^{AC} ch\lambda_1\sqrt{p}l_1 = C_2^{CD} + C_4^{CD}, \tag{35}$$

$$C_1^{BC} sin\lambda_1\sqrt{p}l_1 + C_2^{BC} cos\lambda_1\sqrt{p}l_1 + C_3^{BC} sh\lambda_1\sqrt{p}l_1 + C_4^{BC} ch\lambda_1\sqrt{p}l_1 = C_2^{CD} + C_4^{CD}, \tag{36}$$

$$\lambda_1\sqrt{p}\left(C_1^{AC}sin\lambda_1\sqrt{p}l_1 - C_2^{AC}cos\lambda_1\sqrt{p}l_1 + C_3^{AC}sh\lambda_1\sqrt{p}l_1 + C_4^{AC}ch\lambda_1\sqrt{p}l_1\right) = D_2^{AC}, \tag{37}$$

$$\lambda_1\sqrt{p}\left(C_1^{BC}sin\lambda_1\sqrt{p}l_1 - C_2^{BC}cos\lambda_1\sqrt{p}l_1 + C_3^{BC}sh\lambda_1\sqrt{p}l_1 + C_4^{BC}ch\lambda_1\sqrt{p}l_1\right) = D_2^{AC}, \tag{38}$$

$$D_1^{AC}sin\delta_1 pl_1 + D_2^{AC}cos\delta_1 pl_1 = \lambda_2\sqrt{p}\left(C_1^{CD} + C_3^{CD}\right), \tag{39}$$

$$D_1^{BC}sin\delta_1 pl_1 + D_2^{BC}cos\delta_1 pl_1 = \lambda_2\sqrt{p}\left(C_1^{CD} + C_3^{CD}\right). \tag{40}$$

To determine all 18 constants we need three more conditions. These are obtained by considering the balance of an infinitesimal mass element around the point C.

We have: The sum of the shearing forces occurring in C for the beams AC and BC must be equal to the shearing force in C corresponding to the beam CD:

$$T_1 + T_2 = T. \tag{41}$$

The shear force in the section x, the bending moment and the torque are defined by rel. (22) and (24). Replacing in (41) it is possible to obtain the condition for the balance of shear forces:

$$\begin{aligned} -\left(C_1^{AC} + C_1^{BC}\right)cos\lambda_1\sqrt{p}l_1 + \left(C_2^{AC} + C_2^{BC}\right)sin\lambda_1\sqrt{p}l_1 + \left(C_3^{AC} + C_3^{BC}\right)ch\lambda_1\sqrt{p}l_1 + \\ + \left(C_4^{AC} + C_4^{BC}\right)sh\lambda_1\sqrt{p}l_1 = \frac{E_2I_{p2}}{E_1I_{z1}}\left(\frac{\lambda_2}{\lambda_1}\right)^3\left(C_1^{CD} + C_3^{CD}\right). \end{aligned} \tag{42}$$

With the notation:

$$a_3 = \frac{E_2I_{p2}}{E_1I_{z1}}\left(\frac{\lambda_2}{\lambda_1}\right)^3,$$

it is possible to write:

$$\begin{aligned} -\left(C_1^{AC} + C_1^{BC}\right)cos\lambda_1\sqrt{p}l_1 + \left(C_2^{AC} + C_2^{BC}\right)sin\lambda_1\sqrt{p}l_1 + \left(C_3^{AC} + C_3^{BC}\right)ch\lambda_1\sqrt{p}l_1 + \\ + \left(C_4^{AC} + C_4^{BC}\right)sh\lambda_1\sqrt{p}l_1 = a_3\left(C_1^{CD} + C_3^{CD}\right). \end{aligned} \tag{43}$$

Similarly, it is possible to write the balance of the moments:

$$M^{b1} + M^{b2} = M^t; \tag{44}$$

$$M^{t1} + M^{t2} = M^b. \tag{45}$$

The bending moment and the torque in the section x are obtained by considering relations (22), (23) and (25). Replacing (22) in (44) and (45) it is possible to obtain the balance of the moments:

$$E_1I_{z1}\frac{\partial^2\Phi_{AC}(l_1)}{\partial x^2} + E_1I_{z1}\frac{\partial^2\Phi_{BC}(l_1)}{\partial x^2} = G_2I_{p2}\frac{\partial\varphi(x)}{\partial x}; \tag{46}$$

$$G_1I_{p1}\frac{\partial\varphi(x)}{\partial x} + G_1I_{p1}\frac{\partial\varphi(x)}{\partial x} = E_2I_{z2}\frac{\partial^2\Phi_{CD}(0)}{\partial x^2}. \tag{47}$$

Using rel. (23) and (25) the balance of the moment's equation (46) and (47) leads to:

$$\begin{aligned} -\left(C_1^{AC} + C_1^{BC}\right)sin\lambda_1\sqrt{p}l_1 + \left(C_2^{AC} + C_2^{BC}\right)cos\lambda_1\sqrt{p}l_1 + \left(C_3^{AC} + C_3^{BC}\right)sh\lambda_1\sqrt{p}l_1 + \\ + \left(C_4^{AC} + C_4^{BC}\right)ch\lambda_1\sqrt{p}l_1 = \frac{G_2I_{p2}}{E_1I_{z1}}\left(\frac{1}{\lambda_1}\right)^2\delta_1\left(-D_1^{AC}\right); \end{aligned} \tag{48}$$

$$D_1^{AC} + D_1^{BC} = \frac{(\lambda_2)^2}{\delta_1}\left(-C_2^{CD} + C_4^{CD}\right). \tag{49}$$

If we denote: $a_2 = \frac{G_2I_{p2}}{E_1I_{z1}}\left(\frac{1}{\lambda_1}\right)^2\delta_1$ and $a_1 = \frac{\lambda_2^2}{\delta_1}$ rel. (48) and (49) become:

$$\begin{aligned} -\left(C_1^{AC} + C_1^{BC}\right)sin\lambda_1\sqrt{p}l_1 + \left(C_2^{AC} + C_2^{BC}\right)cos\lambda_1\sqrt{p}l_1 + \left(C_3^{AC} + C_3^{BC}\right)sh\lambda_1\sqrt{p}l_1 + \\ + \left(C_4^{AC} + C_4^{BC}\right)ch\lambda_1\sqrt{p}l_1 = a_2\left(-D_1^{AC}\right); \end{aligned} \tag{50}$$

$$D_1^{AC}+D_1^{BC}=a_1\left(-C_2^{CD}+C_4^{CD}\right). \tag{51}$$

To determine the integration constants that obey the boundary conditions it is necessary to solve the linear homogeneous system (26)–(40), (43), (48), (49) in order to determine the constants C_1^{AB}, C_2^{AB}, C_3^{AB}, C_4^{AB}, C_1^{AC}, C_2^{AC}, C_3^{AC}, C_4^{AC} , C_1^{CD}, C_2^{CD}, C_3^{CD}, C_4^{CD}, D_1^{AB}, D_2^{AB}, D_1^{BC}, D_2^{BC}, D_1^{CD}, D_2^{CD}.

We denote:

$$\{C\}=\left\{\begin{matrix}C^{AC}\\ C^{BC}\\ C^{CD}\end{matrix}\right\}=$$

$$=\left[\,C_1^{AB}\ C_2^{AB}\ C_3^{AB}\ C_4^{AB}\ D_1^{AB}\ D_2^{AB}\ C_1^{AC}\ C_2^{AC}\ C_3^{AC}\ C_4^{AC}\ D_1^{BC}\ D_2^{BC}\ C_1^{CD}\ C_2^{CD}\ C_3^{CD}\ C_4^{CD}\ D_1^{CD}\ D_1^{CD}\,\right]^T.$$

To have a no null solution the determinant of the system must be null. This condition offers us first the eigenfrequencies and, using these, it is possible to obtain the constants defining the eigenmodes of vibration. If we denote:

$$A_{11}=\begin{bmatrix}0 & -1 & 0 & 1 & 0 & 0\\ -1 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ sin\lambda_1\sqrt{p}l_1 & cos\lambda_1\sqrt{p}l_1 & sh\lambda_1\sqrt{p}l_1 & ch\lambda_1\sqrt{p}l_1 & 0 & 0\\ cos\lambda_1\sqrt{p}l_1 & -sin\lambda_1\sqrt{p}l_1 & ch\lambda_1\sqrt{p}l_1 & sh\lambda_1\sqrt{p}l_1 & 0 & 0\\ 0 & 0 & 0 & 0 & sin\delta_1 pl_1 & cos\delta_1 pl_1\end{bmatrix}; \tag{52}$$

$$A_{13}=\begin{bmatrix}0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 0 & -1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\lambda_1\sqrt{p}}\\ \lambda_2\sqrt{p} & 0 & \lambda_2\sqrt{p} & 0 & 0 & 0\end{bmatrix}; \tag{53}$$

$$A_{31}=\begin{bmatrix}-cos\lambda_1\sqrt{p}l_1 & sin\lambda_1\sqrt{p}l_1 & ch\lambda_1\sqrt{p}l_1 & sh\lambda_1\sqrt{p}l_1 & 0 & 0\\ -sin\lambda_1\sqrt{p}l_1 & cos\lambda_1\sqrt{p}l_1 & sh\lambda_1\sqrt{p}l_1 & ch\lambda_1\sqrt{p}l_1 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 0\end{bmatrix}; \tag{54}$$

$$A_{33}=\begin{bmatrix}-a_3 & 0 & -a_3 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & a_z & 0\\ 0 & a_1 & 0 & -a_1 & 0 & 0\\ sin\lambda_2\sqrt{p}l_2 & cos\lambda_2\sqrt{p}l_2 & sh\lambda_2\sqrt{p}l_2 & ch\lambda_2\sqrt{p}l_2 & 0 & 0\\ cos\lambda_2\sqrt{p}l_2 & -sin\lambda_2\sqrt{p}l_2 & ch\lambda_2\sqrt{p}l_2 & sh\lambda_2\sqrt{p}l_2 & 0 & 0\\ 0 & 0 & 0 & 0 & sin\delta_2 pl_2 & cos\delta_2 pl_2\end{bmatrix}. \tag{55}$$

The matrix of the whole system is:

$$S_{18\times 18}=\begin{bmatrix}A_{11} & 0 & A_{13}\\ 0 & A_{11} & A_{13}\\ A_{31} & A_{31} & A_{33}\end{bmatrix}, \tag{56}$$

and (26)–(40), (43), (48), (49) becomes:

$$[S]\{C\}=\{0\}. \tag{57}$$

The condition:

$$\det(S) = 0 \tag{58}$$

offers the eigenvalues of the differential system (26)–(40), (43), (48), (49).

IV. EIGENVALUES

In the following we will present a property of the eigenvalues of such a system. Consider only one of the beams AC (or BC), free at the end A (or B) and clamped in the end C. The transverse vibrations of this are described by relation (1), written under the form:

$$\frac{\partial^4 v}{\partial x^4} + \frac{\rho_1 A_1}{E_1 I_{z1}}\frac{\partial^2 v}{\partial t^2} = 0 \tag{59}$$

If we choose for v the function:

$$v(x,t) = \Phi(x)\sin(pt+\theta) \tag{60}$$

then by introducing it in (59) we obtain:

$$\frac{\partial^4 \Phi}{\partial x^4} - p^2\frac{\rho_1 A_1}{E_1 I_{z1}}\Phi = 0\,; \quad \lambda_1^4 = \frac{\rho_1 A_1}{E_1 I_{z1}}, \tag{61}$$

with the solution:

$$\Phi_{AC}(x) = C_1^{AC} sin\lambda_1\sqrt{p}x + C_2^{AC} cos\lambda_1\sqrt{p}x + C_3^{AC} sh\lambda_1\sqrt{p}x + C_4^{AC} ch\lambda_1\sqrt{p}x \tag{62}$$

Considering the torsional vibrations, the equation describing the vibration of the x section of the bar is (13):

$$\frac{\partial^2 \varphi}{\partial x^2} - \frac{J}{GI_p}\frac{\partial^2 \varphi}{\partial t^2} = 0 \tag{63}$$

Choosing the solution φ under the form:

$$\varphi(x,t) = \varphi_o \psi(x)\sin(pt+\theta) \tag{64}$$

and introducing it in (63) it obtains:

$$\frac{\partial^2 \psi}{\partial x^2} + p^2\delta^2\Phi = 0 \tag{65}$$

The solution of the differential equation (65) will be:

$$\psi_{AC}(x) = D_1^{AC} sin\delta_1 px + D_2^{AC} cos\delta_1 px \tag{66}$$

The end point A is free while the end C is clamped. From these conditions it results: $M_{AC}^b(0,t) = 0$ $T_{AC}(0,t) = 0$ $M_{AC}^t(0,t) = 0$; $v_{AC}(l1,t) = 0$; $v'_{AC}(l1,t) = 0$; $\varphi(l_1,t) = 0$.

Using these boundary conditions in (62) and (66) the integration constants C_1, C_2, C_3, C_4, D_1 and D_2 are determined from the homogenous linear system:

$$[A_{11}]\{C\} = 0 \tag{67}$$

The condition det (A_{11}) = 0 determine the eigenvalues of the AC (or BC). After some calculus it results that:

$$cos\lambda_1\sqrt{p}l_1\, ch\lambda_1\sqrt{p}l_1 = 1 \quad \text{and} \quad tg\lambda_1\sqrt{p}l_1 = 1 \tag{68}$$

from where it is possible to obtain the eigenvalues of a single bar AC or BC, clamped in C and free in A (or B).

We will prove the following:

T1. The eigenvalues for the beam AC, clamped at the end C and free in A are eigenvalues for the whole system too (represented in Fig. 1)

Proof: We must prove that det (A_{11})=0 implies det (S)=0. In paper[15] the property is proved for a more general case. It results that the property is valid for our case too.

Hence, the eigenvalues for a single beam, clamped at one end and free in the other end are eigenvalues for the whole system of beams, clamped in D and free in A and B.

V. EIGENVECTORS

If we calculated the eigenvalues of the matrix (56) then the eigenmodes can be obtained using rel. (57) written in the form:

$$[s]\{\Phi\} = \{0\}$$

We denoted with:

$$\{\Phi\} = \begin{Bmatrix} \Phi_s \\ \Phi_d \\ \Phi_m \end{Bmatrix}$$

the vector of integration constants obtained from the condition (24'), which provides finally the eigenmode. The components Φ_s, Φ_d and Φ_m correspond to the beams AC, BC and CD. We can prove the following two results:

T2. For the eigenvalues that are the same for the beam AC (or BC) (Fig. 2) and for the whole system (Fig. 1) the eigenmodes have the form:

$$\boldsymbol{\Phi} = \begin{Bmatrix} \boldsymbol{\Phi_1} \\ -\boldsymbol{\Phi_1} \\ \mathbf{0} \end{Bmatrix} \tag{69}$$

(The existence of these eigenvalues is ensured by the theorem T1).

Proof: For the eigenvalues offered by relation (68) the following homogeneous system must be solved:

$$\begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{11} & A_{13} \\ A_{31} & A_{31} & A_{33} \end{bmatrix} \begin{Bmatrix} \Phi_s \\ \Phi_d \\ \Phi_m \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{70}$$

with

$$\det A_{11} = 0 \tag{71}$$

Condition (71) implies that a vector Φ_s can be found such that:

$$A_{11}\Phi_s = 0 \tag{72}$$

Rel. (70) becomes:

$$A_{13}\Phi_m = 0 \tag{73}$$

$$A_{11}\Phi_d + A_{13}\Phi_m = 0 \tag{74}$$

$$A_{31}\left(\Phi_s + \Phi_d\right) + A_{33}\Phi_m = 0 \tag{75}$$

From (73), because $\det A_{13} \neq 0$ it results that:

$$\Phi_m = 0 \tag{76}$$

and introducing it in (75) we will get $\Phi_s = -\Phi_d$. This relation verifies (74) too, using (72). If we denote $\Phi_s = \Phi_1$ than it results (69).

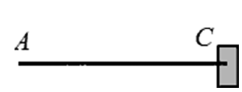

FIG. 2. One single beam.

T3. For the other eigenvalues, not covered by theorem T2, the form of the eigenmodes is:

$$\boldsymbol{\Phi} = \begin{Bmatrix} \boldsymbol{\Phi_1} \\ \boldsymbol{\Phi_1} \\ \boldsymbol{\Phi_3} \end{Bmatrix} \tag{77}$$

Proof: For the other eigenvalues, introducing in rel. (70), with det $A \neq 0$ we obtain:

$$A\Phi_s + B\Phi_m = 0 \tag{78}$$

$$A\Phi_d + B\Phi_m = 0 \tag{79}$$

$$C\,(\Phi_s + \Phi_s) + D\Phi_m = 0 \tag{80}$$

Subtracting (78) from (79) we obtain:

$$A(\Phi_s - \Phi_m) = 0 \tag{81}$$

If det $A \neq 0$, it results that $\Phi_s - \Phi_m = 0$ and therefore $\Phi_s = \Phi_m = \Phi_1$.

For the eigenvalues of the system which coincide with those of a single beam clamped at one end and free at the other, the component of the eigenmodes corresponding to the two identical bars vibrates in opposite phase and the third bar is not moving. For other eigenvalues the components of the eigenmodes are identical.

VI. CONCLUSIONS

In many engineering applications the structural symmetry which exists in such mechanical systems can be used to facilitate the calculation of the eigenvalues and eigenvectors of these systems. It is a step necessary in order to solve the system of differential equations that offers the dynamical response of a linear system. The paper outlines the state of a mechanical structure consisting of two identical cantilever beams rigidly connected by a third, with two free end and a clamped ends. For such kind of structure we have demonstrated properties of the eigenvalues and eigenvectors that allow ease and simplify the calculation of real structures. This allows shortening the time and cost calculations.

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