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COUPLED VIBRATIONS OF A STRUCTURE SUBMERGED IN A COMPRESSIBLE FLUID

by

O. C. ZIENKIEWICZ *

and

R. E. NEWTON **

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* Dean of Applied Science, Professor of Civil Engineering, University of Wales, Swansea, U.K.

** Professor of Mechanical Engineering. U.S.Naval Postgraduate School, Monterey, U.S.A. (currently visiting Professor, University of Wales, Swansea)

1. Introduction

In this Symposium concerned with ship structures many papers on general application of finite element methods are presented. To avoid duplication of matter and discussion of well known computational systems attention will be focussed here on a special problem which has not been dealt with previously in the literature and which in some aspects is relevant to shipbuilding problems. The range of application of the methodology presented is obviously much wider and has repercussions ranging from oscillations in rocket fuel systems through earthquake response of water retaining structures to electro magnetic vibration situations.

The problem of incompressible motion of fluid masses and of coupling such effects to the structural vibration response has been dealt with in the Finite Element text(1) and in some earlier publication(2)

An extension to dealing with oscillations of compressible fluids has been achieved more recently(3)(4).

In this paper a review of this earlier work is included in the context of solving the complete fluid-structure interaction problem.

For incompressible fluid problems the concept of an influence matrix was introduced by Zienkiewicz and Nath(5)(6) using an electric analogue solution.

Alternative formulation of influence coefficients from exact source solution[7] has been employed recently by Yugan et al.[8] coupling this with a finite element structural matrix. Such concepts are difficult to utilise in compressible solutions and here a direct finite element treatment of both the fluid and structural continuum will be outlined.

A recent attempt to deal with the oscillation of bodies submerged in compressible fluid uses complex response functions for the fluid phase[9] but assumes mode shape invariance.

2. The structure-discretisation

The discretisation of the structural problem by the finite element process into the (assembled) stiffness equation system

$$[K]\{\delta\} + [C]\{\dot{\delta}\} + [M]\{\ddot{\delta}\} = \{R\} \quad * \qquad (1)$$

is well known and described in texts[1]. In above $[K]$ is the structural stiffness matrix, $\{\delta\}$ are the nodal displacements and $\{R\}$ the (generalised) nodal loads. Matrices $[C]$ and $[M]$ are the corresponding damping and mass matrices of the structure calculated in the proper, consistent, manner.

At this stage we need concern ourselves only with the interaction forces due to fluid pressure

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*Dots indicate time differentiation

$$\frac{\partial \delta}{\partial t} = \dot{\delta} \text{ etc.}$$

p at the interfaces between the structure and the fluid.

If the total generalised nodal forces are divided into two parts

$$\{R\} = \{F\} + \{P\} \qquad (2)$$

where the first are due to external forces and the latter due to the fluid pressure on interface, we can write consistently for a node 'i'

$$P_i = \int_S N_i' p \, dS \qquad (3)$$

where N_i' is the appropriate shape function defining the displacement pattern in direction normal to the boundary, p the pressure on interface S, and the integration covers the whole interface using the shape function appropriate to the subregion.

3. The Fluid

3.1 Basic theory

If the fluid pressure p is considered as the excess over stated gravity pressure then the Navier-Stokes equation of fluid motion in Cartesian co-ordinates can be written in the x direction as

$$-\frac{\partial u}{\partial t} = \frac{1}{\rho}\frac{\partial p}{\partial x} + \left[u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right] - \frac{\mu}{\rho}\nabla^2 u - \frac{\mu}{3\rho}\frac{\partial}{\partial x}\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] \qquad (4)$$

with similar expressions for the other co-ordinate direction. In above u, v and w stand for Cartesian velocity components and μ and ρ for viscosity and density of the fluid respectively.

In addition to equation 4 continuity relationship

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + \frac{1}{K}\frac{\partial p}{\partial t} = 0 \qquad (5)$$

has to be satisfied. The last term, with k being the bulk modulus of the fluid gives the storage due to compressibility.

On differentiation of (5) with respect to time and substitution of equation (4) with the small convective acceleration term in square brackets omitted we have

$$- \nabla^2 p + \frac{\rho}{K}\frac{\partial^2 p}{\partial t^2} + \frac{4}{3}u\nabla^2\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

and on using equation (5) again and noting that

$$c = \sqrt{\frac{K}{\rho}} \quad \text{(the sonic velocity)} \qquad (6)$$

we obtain the final governing equation

$$\nabla^2 p + \frac{4}{3}\frac{\mu}{K}\nabla^2\dot{p} - \frac{1}{c^2}\ddot{p} = 0 \qquad (7)$$

This equation, together with the necessary boundary conditions defines the fluid phase problem.

Neglecting the viscous term we have on the boundary for (4)

$$\frac{\partial p}{\partial x} = -\rho\frac{\partial u}{\partial t}$$

$$\frac{\partial p}{\partial y} = -\rho\frac{\partial v}{\partial t}$$

$$\frac{\partial p}{\partial z} = -\rho\frac{\partial w}{\partial t}$$

or quite generally i taking the normal, n,

$$\frac{\partial p}{\partial n} = - \rho \frac{\partial V_n}{\partial t} = - \rho \dot{V}_n \qquad (8)$$

when V_n is the velocity in the normal direction of the boundary.

On free surface in the absence of surface waves

$$p = 0 \qquad (9)$$

becomes a suitable condition.

This is generally adequate but if gravity waves are generated and it is important to take these into consideration a more elaborate condition can be included. This is derived in ref. 5. If z is the vertical direction and y the gravity acceleration on z = constant free surface

$$\frac{1}{g} \ddot{p} + \frac{\partial p}{\partial z} = 0$$

supercedes the simpler condition (9) (upward direction of z is implied).

3.2 Finite element discretisation of the fluid problem by the Galerkin process

The spatial discretisation of equation (7) and the appropriate boundary conditions can be accomplished by a variational process as described in ref.1. Alternatively a direct approach via the use of the Galerkin weighted residual process can be used.

The viscous terms will be neglected at this stage due to their relative unimportance.

If quite generally we write that at any instant

$$p = \sum_{i=1}^{m} N_i p_i , \qquad N_i(x,y,z) \qquad (11)$$

in which p_i is a set of nodal pressure values which are time dependant we have for the i^{th} weighted residual equation (7) (Galerkin)

$$\int_R N_i \left[\nabla^2 \Sigma N_j p_j - \frac{\mu}{k} - \frac{1}{c^2} \Sigma N_j \ddot{p}_j \right] dR = 0 \qquad (12)$$

where R is the region under consideration.

Using Green's theorem these can be transformed to

$$- \int\int_R \left(\frac{\partial N_i}{\partial x} \sum \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \sum \frac{\partial N_j}{\partial y} + \frac{\partial N_i}{\partial z} \sum \frac{\partial N_j}{\partial z} \right) dR \; p_j$$

$$- \frac{1}{c^2} \int_R N_i \Sigma N_j \; dR \; \ddot{p}_j \qquad (13)$$

$$+ \int_S N_i \sum \frac{\partial N_j}{\partial n} \; dS \; p_j = 0$$

in which j = 1 to m(the total node number)

The last term can be written as

$$\int_S N_i \frac{\partial p}{\partial n} dS \qquad (14)$$

and through this the boundary condition on the normal pressure gradient (8) incorporated.

The whole system of equations (13) can be written in a matrix form

$$[H]\{p\} + [Q]\{\ddot{p}\} = \{B\} \qquad (15)$$

i.e. in the form familiar in finite element analysis in which

$$H_{ij} = \Sigma h_{ij}$$
$$Q_{ij} = \Sigma q_{ij} \qquad (16)$$
$$B_i = \Sigma b_i$$

in which the summation is carried over all the subregions or elements and the lower case letters show the contributions of each element to the terms of integral(13)

i.e.

$$h_{ij} = \int_{Re}\left(\frac{\partial N_i}{\partial x}\cdot\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\cdot\frac{\partial N_j}{\partial y} + \frac{\partial N_i}{\partial z}\cdot\frac{\partial N_j}{\partial z}\right)dR$$

$$q_{ij} = \frac{1}{c^2}\int_{Re} N_i N_j \, dR \qquad (17)$$

$$b_i = \int_{Se} N_i \frac{\partial p}{\partial n} \, dS$$

where Re and Se denote region and external boundary of an element.

(These expressions are achieved in an alternative way in Chapter 10 of reference 1).

4. Surface wave generation

While prescribed boundary acceleration or indeed prescribed pressures lead to a vector {B} on a free surface the condition given by equation (10) leads to another form of contribution.

Now

$$\frac{\partial p}{\partial n} = - \frac{1}{g} \ddot{p} \qquad (10)$$

and substituting into the expression (14) we find that a contribution of the form

$$\int_S N_i \left(- \frac{1}{g} \ddot{p}\right) dS = - \frac{1}{g} \int_S N_i \Sigma N_j dS \; \ddot{p}_j \qquad (18)$$

arises leading to a term

$$[Q_o]\{\ddot{p}\} \text{ augmenting the } Q \qquad (19)$$

matrix in which

$$q_{o_{ij}} = - \frac{1}{g} \int_{Se} N_i N_j dS$$

where Se is the free surface of an element.

In general this additional term is of minor importance.

5. The coupling of structure and fluid

The 'force' terms {P} of equation (1) and {B} of equation (15) determine the coupling between the fluid and structure parts of the problem. This coupling occurs via the interface and attention to the displacements and pressures there has now to be given.

The nodal forces P_i due to pressures are by equation (3)

$$P_i = \int_S N_i' \, p dS = \int_S N_i' \Sigma N_j p_j dS$$

or

$$\{P\} = \lfloor L \rfloor \{p\} \qquad (20)$$

with $L_{ij} = \Sigma \ell_{ij}$ $\qquad \ell_{ij} = \int_{Se} N_i' N_j dS$

Similary, noting that on the interface

$$\frac{\partial p}{\partial n} = - \rho \dot{V}_n = - \rho \Sigma N_j' \ddot{\delta} j$$

we have for the forcing term of equation 15

$$B_i = - \int N_i \rho \Sigma N_j' \ddot{\delta} j \qquad (21)$$

or writing $\{B\} = \lceil S \rceil \{\ddot{\delta}\}$

with $S_{ij} = \Sigma s_{ij}$

and $s_{ij} = - \rho \int_{Se} N_i N j' dS$

Thus quite an important observation can be made that

$$\rho \lfloor L \rfloor = - \lceil S \rceil^T \qquad (22)$$

All the integrationsin above matrices are confined to the interface surfaces and it should be observed that in general N_i' is in fact a two or three component vector, depending on the relative directions of the normal and global co-ordinates of the structure.

6. Radiation damping - infinite boundary.

While the viscous damping term was deliberately excluded as its effect in compression oscillations is known to be small another form of damping occurs in the fluid phase if the extent of this is large. Waves originating at the hull of a vibrating ship, for instance, travel far and are finally absorbed without their reflection having any effect on the response of the structure.

Thus in problems not enclosed by fully reflecting boundaries an energy loss always occurs and it is inappropriate to have free undamped oscillations under such conditions.

In a numerical representation an infinite boundary has always to be truncated at some 'sufficiently large' distance Fig. 1.

At such a boundary a suitable condition has to be imposed ensuring that no waves are reflected.

Considering a direction n normal to the boundary the wave equation governing the problem (7) lead for plane waves to a form

$$p = F_1(n - ct) + F_2(n + ct) \qquad (23)$$

In this F_1 stands for a wave advancing with a velocity c towards the boundary and F_2 for the returning wave which by our requirements should not exist.

Differentiating with respect to n and t and taking F_2 as zero we have

$$\frac{\partial p}{\partial n} = F_1' \text{ and } \frac{\partial p}{\partial t} = - c\ F_1'$$

which gives

$$\frac{\partial p}{\partial n} = - \frac{1}{c} \frac{\partial p}{\partial t} \qquad (24)$$

as the 'non reflecting' boundary condition.* The corresponding boundary integral (14) is now

$$- \frac{1}{c}\int_S N_i \frac{\partial p}{\partial t}\ dS = - \frac{1}{c}\int_S N_i \Sigma N_j\ dS\ \dot{p}_j \qquad (25)$$

This leads to an additional term in the matrix equation (15)

$$[D]\{\dot{p}\} \qquad (26)$$

with

$D_{ij} = \Sigma d_{ij}$ and element contribution of

$$d_{ij} = - \frac{1}{c}\int_{Se} N_i N_j\ dS$$

7. The assembled problem

The complete problem of fluid structure dynamic interaction has now been formulated and can be summarised.

Equation (1) is rewritten using (20) as

$$[K]\{\delta\} + [C]\{\dot{\delta}\} + [M]\{\ddot{\delta}\} = [L]\{p\} + \{F\} \qquad (27)$$

stating the structure behaviour.

Similarly equation (15) shows the fluid behaviour and by using (21),(22) and (26) can be written as

*Footnote

This derivation is exactly valid only for plane wave situations. In a real situation a test must be made to determine if the 'infinite' boundary has been placed 'far enough'. This is the case if the position of the boundary is far enough not to affect the results at the focus of interest and, generally, two or more trial solutions should be attempted.

$$[H]\{p\} + [D]\{\dot{p}\} + [\bar{Q}]\{\ddot{p}\} = -\rho[L]^T\{\ddot{\delta}\} \qquad (28)$$

in which $\bar{Q}$ combines Q and Q_c terms.

These relations which are easily obtained by standard operations once the element idealisation has been decided govern the full response of the system which can now be discussed in more detail.

7.1 Structures in an incompressible fluid with no surface waves

In this case equation (28) becomes simply

$$[H]\{p\} = -\rho[L]^T\{\ddot{\delta}\} \qquad (29)$$

and we have on inversion

$$\{p\} = -[H]^{-1}\rho[L]^T\{\ddot{\delta}\} \qquad (30)$$

Substitution into (27) gives

$$[K]\{\delta\} + [C]\{\dot{\delta}\} + ([M] + [L][H]^{-1}[L]^T\rho)\{\ddot{\delta}\} = \{F\} \quad (31)$$

i.e. a standard structures problem amended only by the addition of an additional mass matrix (added mass).

Treatment of such problems is standard and has been described in 24 in ref. (1).

It should be noted that in fact complete inversion of $[H]$ is not necessary as the coupling occurs only via interface nodes. Thus partitioning of $\{\delta\}$ and $\{p\}$ should be adopted.

By omitting $[C]$ and $\{F\}$ terms the natural frequencies of the structure-fluid complex can be found.

7.2 Frequency response of the total system

If the excitation force is written in the complex form

$$\{F\} = \{F_o\}e^{i\omega t}$$

then in steady state response

$$\{p\} = \{p_o\}e^{i\omega t} \text{ and } \{\delta\} = \{\delta_o\}e^{i\omega t}$$

where in general all quantities are complex.

Substitution into equation (28) permits once again the determination of $\{\delta_o\}$ to be achieved from equation (27) now operating in real and complex parts of the various quantities (ref. 1. p.179).

Thus numerically response to any frequency input can be obtained and full characteristics obtained.

7.3 Free vibrations of the system

Natural frequencies of the whole system are obviously important in the analysis of coupled problems. Omitting thus the forcing and damping terms from (27) and (28) we can write a combined equation for simple sinusoidal response of frequency ω, as

$$\begin{bmatrix} K & -L \\ 0 & H \end{bmatrix}\begin{Bmatrix} \delta \\ p \end{Bmatrix} - \omega^2 \begin{bmatrix} M & 0 \\ \rho L^T & \bar{Q} \end{bmatrix}\begin{Bmatrix} \delta \\ p \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (33)$$

which leads to an eigenvalue problem in principle allowing the determination of the modes and natural frequencies. Unfortunately the above equation leads to a non standard, unsymmetric eigenvalue problem, for which specialised approaches would be necessary (this despite the inherent symmetry of the $[K]$ $[H]$ $[L]$ and $[\bar{Q}]$ matrices).

A simple modification suggested by B. M. Irons allows a symmetric form to be retained. From the second equation we have

$$p = H^{-1}\ \omega^2\ (\rho L^T\delta + \bar{Q}p)$$

which is substituted into the first row. Multiplying the above by $\bar{Q}$ gives the new second row and the modified symmetric system becomes

$$\begin{bmatrix} \rho K & 0 \\ 0 & \bar{Q} \end{bmatrix} \begin{Bmatrix} \delta \\ p \end{Bmatrix} - \omega^2 \begin{bmatrix} \rho M + \rho^2 L H^{-1} L^T & \rho L H^{-1} \bar{Q} \\ \bar{Q} H^{-1} \rho L^T & \bar{Q} H^{-1} \bar{Q} \end{bmatrix} \begin{Bmatrix} \delta \\ p \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (34)$$

In this form, now standard, eigenvalues can be found.

8. Some applications

A few illustrative problems are attached here to show the scope of the process. Only rather simple examples are given but obviously problems of any complexity can be treated.

A simple rectangular element of linear type with corner nodes is used generally though in areas of particular interest an additional node is added on center-line of one side.

8.1 Example 1. Vertical motion of a ship in a rectangular channel

The added mass contributed by the water to a rigid ship undergoing vertical harmonic oscillation can be found by application of Eq. 15. The two-dimensional system considered is illustrated in Fig. 2. As shown, the water is represented by fifteen rectangular elements with a finer mesh adjacent to the hull. Two elements, Nos. 12 and 15, have a mid-side node on the face in contact with the hull. The remaining elements have only corner nodes.

Results are given in Fig. 3 for two channel sizes. In Case A the channel is 240 feet deep and 480 feet wide and in Case B it is 480 feet deep and 960 feet wide. The separate frequency scales for the two cases are chosen so that fluid resonances, in the absence of a ship, occur at the same abscissae. Approximate locations of vertical asymptotes are indicated.

A study of the curves of Fig. 3 discloses that, at frequencies of practical concern for a ship of this size, the channel dimensions and fluid compressibility have little effect on the added mass. It is of interest to observe that, for the larger channel of Case B, the effects of resonance are confined to very narrow frequency bands.

8.2 Example 2. Fluid pressures generated by structural deformation

The rigid ship hull of Example 1 may be replaced by a flexible hull which is allowed to deflect laterally and the resulting fluid pressures at the hull can be deduced from the fluid discretisation used above. Hull displacements are represented by the nodal coordinates shown in Fig. 4 with parabolic interpolation used to derive values between nodes. As before, the governing equation is Eq. 15.

Table 1 presents results for nodal pressures resulting from unit nodal accelerations, i.e. gives the effective mass 'influence' matrix[1]. In consideration of the conclusions drawn from Example 1, data are reported only for $\omega = 0$ (incompressible case, $\lfloor Q \rceil = 0$). The channel dimensions are those of Case A. Because of geometric symmetry, no data are given for accelerations at nodes 6 to 9.

Table 1. Pressures on ship hull for unit nodal accelerations.

	$\ddot{u}_1$	$\ddot{u}_2$	$\ddot{u}_3$	$\ddot{v}_3$	$\ddot{v}_4$	$\ddot{v}_5$
p_1	0	0	0	0	0	0
p_2	2.2	24.9	5.9	3.5	9.6	3.2
p_3	-0.4	15.5	9.0	9.0	23.8	7.5
p_4	-0.2	9.8	5.5	7.8	44.6	18.4
p_5	-0.1	6.9	3.8	3.4	34.1	29.5
p_6	0	0	0	0	0	0
p_7	0.0	1.3	0.7	0.7	4.8	3.2
p_8	0.0	3.0	1.6	1.7	11.3	7.5
p_9	-0.1	5.0	2.7	2.7	21.0	18.4

Notes: Pressures in lb./ft^2., accelerations in ft./sec.^{2}
Geometric arrangement as in Fig. 2, Case A.
Node numbering according to Fig. 4.

8.3 Example 3. Principal modes of a coupled fluid - structure system.

The principal modes of an idealized coupled system, Fig. 5, are found from Eq. 33. The system illustrated represents, in two-dimensional idealization, a dry dock 160 feet wide with tapered concrete walls 80 feet high. For analysis the wall is represented by two tapered Euler-Bernoulli beam elements. The fluid is discretised as four square elements with corner nodes. The two elements, Nos. 3 and 4, in contact with the wall each have a mid-side node on the fluid-structure interface.

The lowest frequencies for the symmetric modes of this system are summarised in Table 2.

Table 2. Frequencies* of symmetric modes, Example 3.

Mode no.	Coupled modes (compressible)	Coupled modes (incompressible)	Uncoupled modes
1	18.1	18.2	29.7+
2	62.9	65.9	83.3+
3	110.2	154.8	92.8x
4	159.0	368.2	181.8+

*Frequencies in radians/second

+Wall modes, fluid absent

xFluid mode, rigid walls

Examination of these data clearly shows the marked effect of the fluid in reducing the frequencies of the bare structure, especially in the fundamental mode. At these frequencies the compressibility of the fluid has little effect. This has been confirmed by analyzing the system with an incompressible fluid (placing $[Q] = 0$ in Eq. 33).

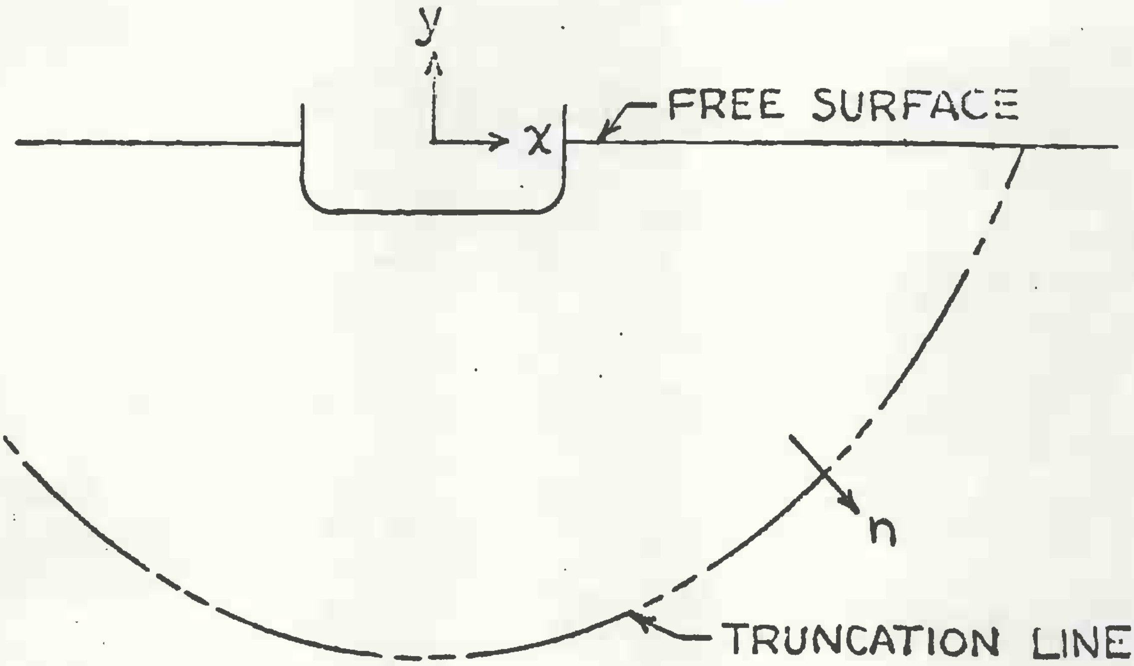


FIG. 1. TRUNCATION OF FLUID REGION.

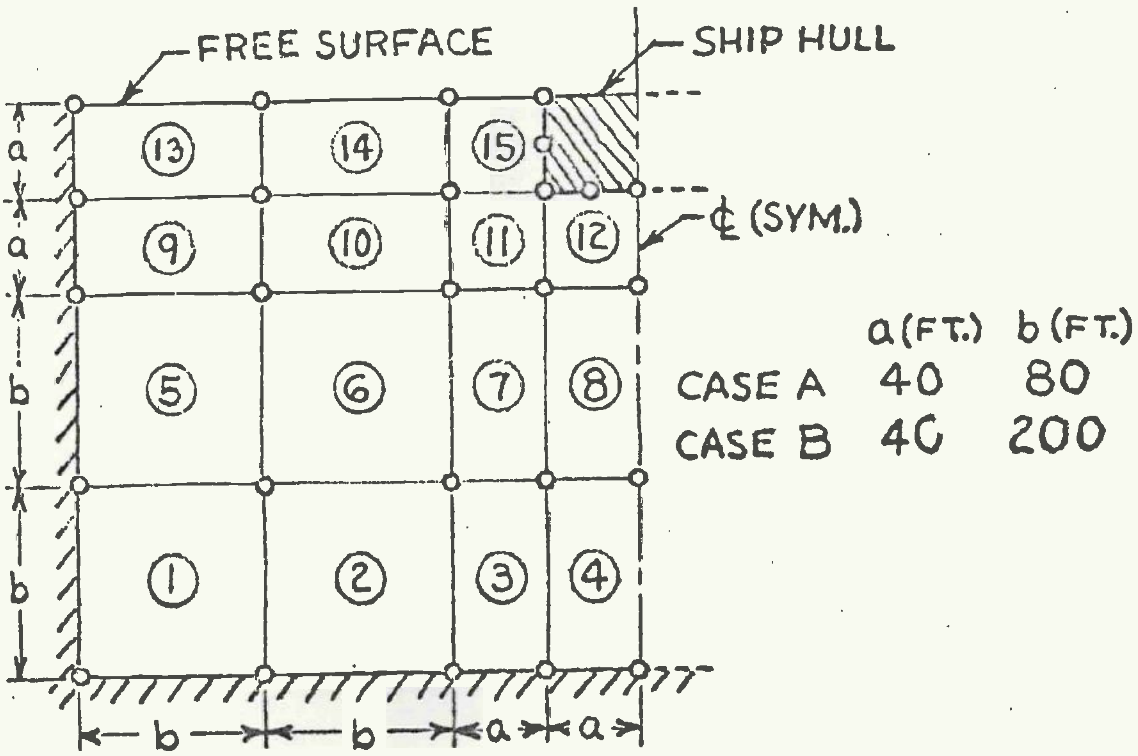


FIG. 2. SYSTEM ANALYZED, EXAMPLES 1 & 2.

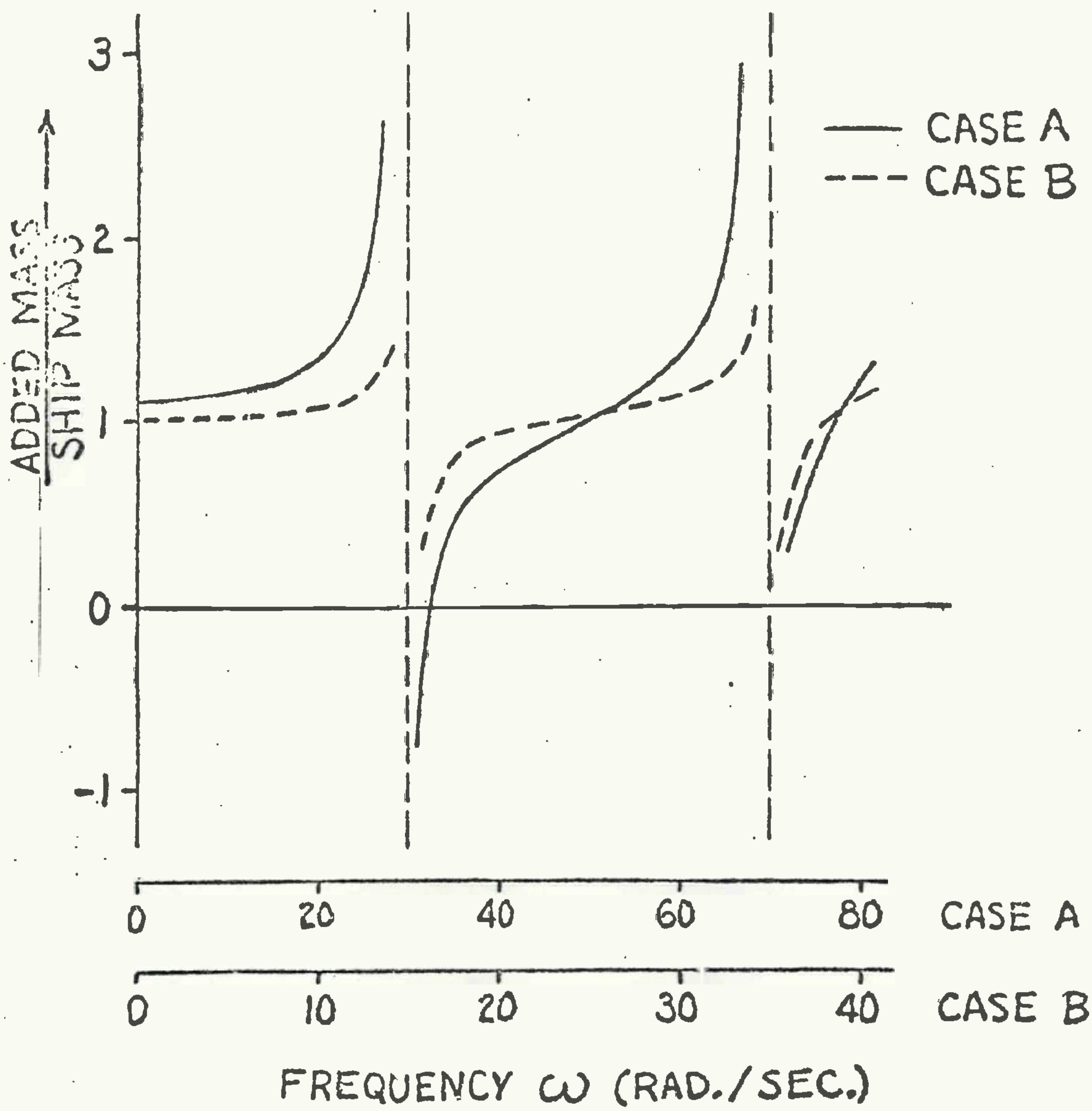


FIG. 3. ADDED MASS VS. FREQUENCY FOR VERTICAL MOTION OF A RIGID SHIP.

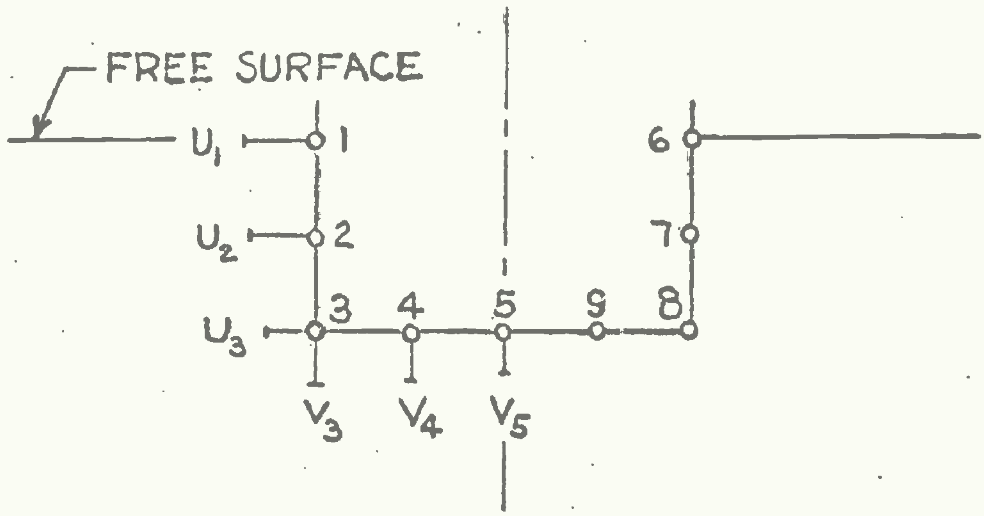
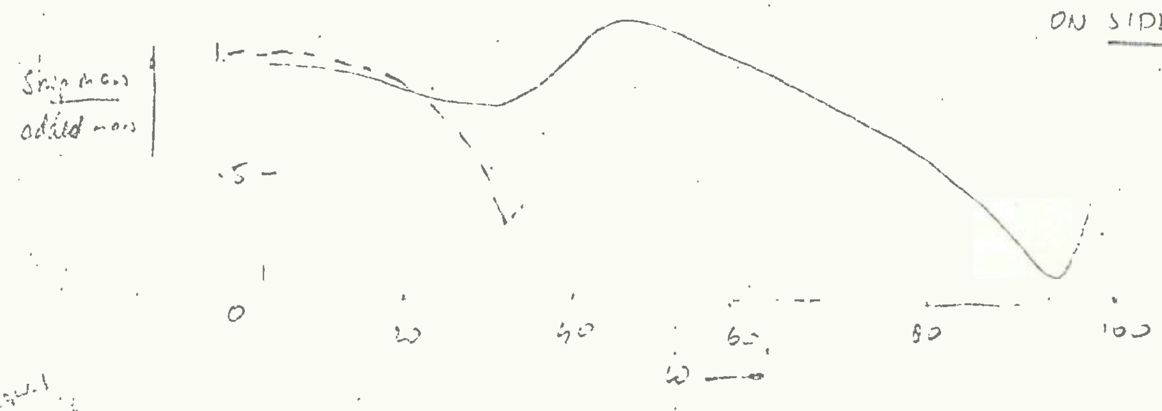


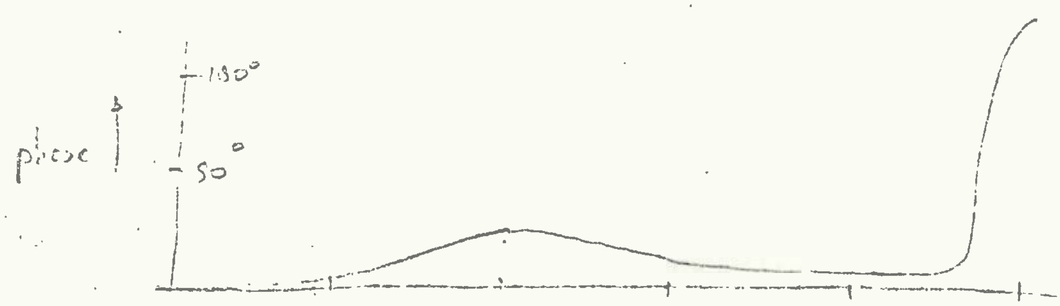
FIG. 4. HULL NODAL DISPLACEMENTS AND PRESSURE NODES.

TER CONFERENTIE GETOOND
OOK VOOR BOTTOM + SIDES BY ELKAAR

— 240' deep, US
--- 240 " 950'

RADIATION DAMPING
ON SIDES only





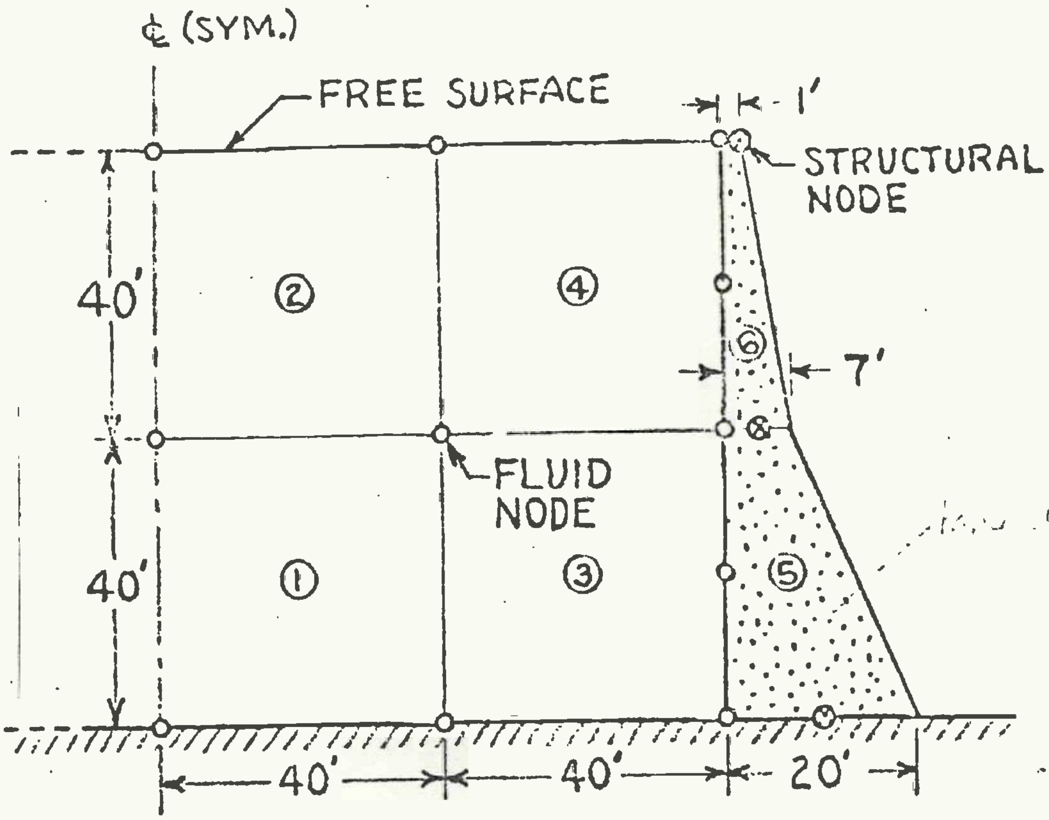


FIG. 5. COUPLED SYSTEM, EXAMPLE 3.

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