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COUPLER-POINT CURVE SYNTHESIS USING HOMOTOPY METHODS

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Coupler-Point Curve Synthesis Using
Homotopy Methods

By

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ABSTRACT

Title of Thesis : Coupler-Point Curve Synthesis Using Homotopy Methods

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A new numerical method called "Homotopy" method (Continuation method) is applied to the problem of four-bar coupler-point-curve synthesis. We have shown that, for five precision points, the link lengths of a four-bar linkage can be found by the "General Homotopy" method. For nine precision points, the "Cheater's Homotopy" can be applied to find some four-bar linkages that will guide a coupler point through the nine prescribed positions. The nine-coupler-points synthesis problem is highly non-linear and highly singular. We have also shown that Newton-Raphson's method and Powell's method, in general, tend to converge to the singular condition or do not converge at all, while the Cheater's homotopy always works. The powerfulness of Cheater's homotopy opens a new frontier for dimensional synthesis of mechanisms.

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

NOMENCLATURE

M = coupler-point

$\underline{Z}_i$ = a vector describing the length and orientation of a link in space

$\underline{R}_j$ = the position vector of a coupler-point

$\underline{\delta}_j$ = path-increment vector

θ_j = rotation angle of the coupler

ϕ_j = rotation angle of the input crank

ψ_j = rotation angle of the output crank

t = Homotopy parameter

S = Homotopy path

B_i = random constants used in the General Homotopy

C_i = coefficients of a polynomial system to be solved

B^*_i = random constants used in the Cheater's Homotopy

C^*_i = coefficients of starting polynomials used in the Cheater's Homotopy

D = Bezout number (total degree)

D_0 = number of paths used in the Cheater's Homotopy

CHAPTER 1. INTRODUCTION

§ 1-1. Introduction

It is often desired to have a mechanism to guide a point along a specified path. The path traced by a point on the coupler link of a four-bar linkage is known as the coupler curve and the generating point is the coupler-point. The design of such a mechanism to generate a prescribed coupler curve is called coupler-point-curve synthesis.

Various methods of coupler-point-curve synthesis have been studied extensively. In the early development, graphical methods using trial and error and intuition predominated. Several graphical methods for the design of a four-bar linkage to guide a coupler-point through two, three, and four specified positions, can be found in the literature [3,7,12,19,25]. Hrones and Nelson [8] created a catalog of four-bar linkages that contain over 7000 coupler curves. The catalog can be used by a designer in selecting a proper linkage with the desired coupler curve.

The graphical methods have their own merit and are used in simple problems where high accuracy is not needed. The increasing need for solving more difficult problems and for higher accuracy has led to the development of analytical methods. However, the mathematics of some synthesis problems becomes formidable even for the case of four-bar linkage, and it becomes more so as the number of links is increased. Recent development of numerical techniques has been a great help in this respect.

Several mathematical techniques for modeling the motion of bar linkages have been developed. These include complex-number method, Freudenstein's method, and loop-closure-equations technique. Closed-form solutions have been derived for the case of five prescribed coupler points and four corresponding crank angles [1,4,5,14,22,23,26]. However, for nine precision points, the problem has not been previously solved in a satisfactory manner. Alt [1] presented the problem without solving it. Roth [20] and Sieker [24] presented some solution techniques which do not work at all the times.

In this thesis, we used "General Homotopy" and "Cheater's Homotopy" to solve the problems of five and nine precision points, respectively. In solving the five-points-problem, we used the General Homotopy to solve the system of design equations instead of reducing the problem to a single equation in one unknown. Thus, we can obtain all the solutions, real or complex, systematically. For nine-positions, we applied the Cheater's Homotopy method to find some solutions, real or complex. The Cheater's Homotopy always find some solutions, although sometimes the solutions may be complex.

§ 1-2. The Design Equations

The design equations for the coupler-point-curve synthesis may be derived by vectors method. Consider the four-bar linkage, designated by the four pivots O_A, O_B, A, B, and the coupler point

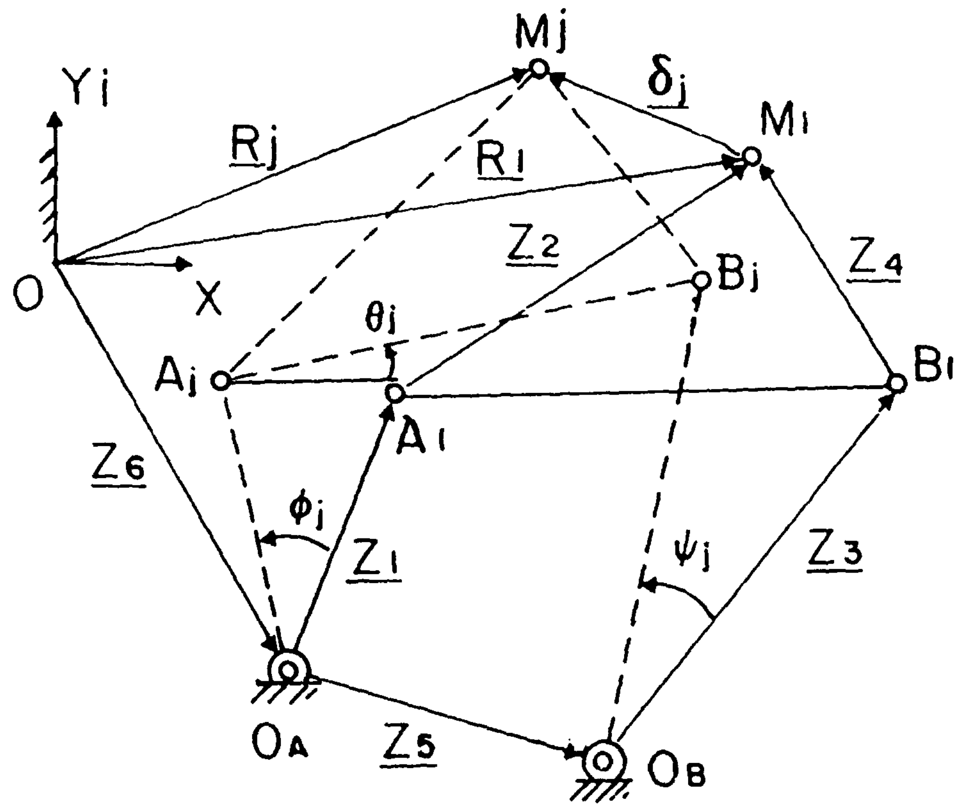

Figure 1-1 Plane Vector $\underline{Z}_1$ to $\underline{Z}_6$ Defining the jth Position of a Four-Bar with Respect to the Reference Position

M, as shown in Fig. 1-1. The first link, O_AO_B, is fixed to the ground. At every position, we can write two independent vector-loop equations with respect to a Cartesian coordinate system. At position 1, we have :

$$\begin{cases} \underline{Z}_6 + \underline{Z}_1 + \underline{Z}_2 = \underline{R}_1 \\ \underline{Z}_6 + \underline{Z}_5 + \underline{Z}_3 + \underline{Z}_4 = \underline{R}_1 \end{cases} \qquad (1\text{-}1)$$

We consider the plane vectors $\underline{Z}_1$ to $\underline{Z}_6$ as describing the mechanism in its reference position and define the jth position in terms of these vectors as follows :

$$\begin{cases} \underline{Z}_6 + \underline{Z}_1\, e^{i\phi j} + \underline{Z}_2\, e^{i\theta j} = \underline{R}_j \\ \underline{Z}_6 + \underline{Z}_5 + \underline{Z}_3\, e^{i\psi j} + \underline{Z}_4\, e^{i\theta j} = \underline{R}_j\ , \quad j = 2,3,\dots \end{cases} \qquad (1\text{-}2)$$

where, ϕj, θj, and ψj are the angular displacements of link O_AA, AB and O_BB from their respective reference positions. By subtracting the reference position, Eqs. (1-1), from those describing the jth position, Eqs. (1-2), we can obtain the path-increment equations :

$$\begin{cases} \underline{Z}_1\,(e^{i\phi j} - 1) + \underline{Z}_2\,(e^{i\theta j} - 1) = \underline{\delta}_j \\ \underline{Z}_3\,(e^{i\psi j} - 1) + \underline{Z}_4\,(e^{i\theta j} - 1) = \underline{\delta}_j \end{cases} \qquad (1\text{-}3)$$

where the path-increment vector $\underline{\delta}_j$ is given by $\underline{\delta}_j = \underline{R}_j - \underline{R}_1$. Rearranging Eqs. (1-3), we obtain :

$$\begin{cases} \underline{Z}_1 \, e^{i\phi j} = \underline{\delta}_j - \underline{Z}_2 \left(e^{i\theta j} - 1 \right) + \underline{Z}_1 \\ \underline{Z}_3 \, e^{i\psi j} = \underline{\delta}_j - \underline{Z}_4 \left(e^{i\theta j} - 1 \right) + \underline{Z}_3 \end{cases} \qquad (1\text{-}4)$$

The complex conjugates of Eqs. (1-4) are :

$$\begin{cases} \bar{\underline{Z}}_1 \, e^{-i\phi j} = \bar{\underline{\delta}}_j - \bar{\underline{Z}}_2 \left(e^{-i\theta j} - 1 \right) + \bar{\underline{Z}}_1 \\ \bar{\underline{Z}}_3 \, e^{-i\psi j} = \bar{\underline{\delta}}_j - \bar{\underline{Z}}_4 \left(e^{-i\theta j} - 1 \right) + \bar{\underline{Z}}_3 \end{cases} \qquad (1\text{-}5)$$

where $(\bar{\ })$ denotes the complex conjugate of ( ). Multiplying each equation of (1-4) by its corresponding complex conjugate shown in (1-5), then multiplying the resulting equations by $e^{i\theta j}$, we obtain:

$$\begin{cases} \underline{A}_{0j} \, X_j^2 + \underline{A}_{1j} \, X_j + \underline{A}_{2j} = 0 \\ \underline{B}_{0j} \, X_j^2 + \underline{B}_{1j} \, X_j + \underline{B}_{2j} = 0 \end{cases} \qquad (1\text{-}6)$$

where,

$$X_j = e^{i\theta j} - 1$$

$$\underline{A}_{0j} = -\underline{Z}_2 \, \bar{\underline{Z}}_2 - \underline{Z}_2 \, (\bar{\underline{\delta}}_j + \bar{\underline{Z}}_1)$$

$$\underline{A}_{1j} = \underline{A}_{2j} + (\underline{\delta}_j \, \bar{\underline{Z}}_2 - \bar{\underline{\delta}}_j \, \underline{Z}_2) + (\underline{Z}_1 \, \bar{\underline{Z}}_2 - \bar{\underline{Z}}_1 \, \underline{Z}_2)$$

$$\underline{A}_{2j} = \underline{\delta}_j \, \bar{\underline{\delta}}_j + \underline{\delta}_j \, \bar{\underline{Z}}_1 + \bar{\underline{\delta}}_j \, \underline{Z}_1$$

$$\underline{B}_{0j} = - \underline{Z}_4 \, \bar{\underline{Z}}_4 - \underline{Z}_4 \, (\bar{\underline{\delta}}_j + \bar{\underline{Z}}_3)$$

$$\underline{B}_{1j} = \underline{B}_{2j} + (\underline{\delta}_j \, \bar{\underline{Z}}_4 - \bar{\underline{\delta}}_j \, \underline{Z}_4) + (\underline{Z}_3 \, \bar{\underline{Z}}_4 - \bar{\underline{Z}}_3 \, \underline{Z}_4)$$

and

$$\underline{B}_{2j} = \underline{\delta}_j\,\bar{\underline{\delta}}_j + \underline{\delta}_j\,\bar{\underline{Z}}_3 + \bar{\underline{\delta}}_j\,\underline{Z}_3$$

Considering X_j as the variable, we form the Sylvester's dyalitic eliminant of Eq. (1-6) :

$$\begin{bmatrix} \underline{A}_{0j} & \underline{A}_{1j} & \underline{A}_{2j} & 0 \\ 0 & \underline{A}_{0j} & \underline{A}_{1j} & \underline{A}_{2j} \\ \underline{B}_{0j} & \underline{B}_{1j} & \underline{B}_{2j} & 0 \\ 0 & \underline{B}_{0j} & \underline{B}_{1j} & \underline{B}_{2j} \end{bmatrix} = 0 \qquad (1\text{-}7)$$

Equation (1-7) can be decomposed into Eq.(1-8) :

$$\underline{A}_{0j} \begin{bmatrix} \underline{A}_{0j} & \underline{A}_{1j} & \underline{A}_{2j} \\ \underline{B}_{1j} & \underline{B}_{2j} & 0 \\ \underline{B}_{0j} & \underline{B}_{1j} & \underline{B}_{2j} \end{bmatrix} + \underline{B}_{0j} \begin{bmatrix} \underline{A}_{1j} & \underline{A}_{2j} & 0 \\ \underline{A}_{0j} & \underline{A}_{1j} & \underline{A}_{2j} \\ \underline{B}_{0j} & \underline{B}_{1j} & \underline{B}_{2j} \end{bmatrix} = 0 \qquad (1\text{-}8)$$

Let :

$$\underline{Z}_j = Z_{jx} + i\,Z_{jy}, \qquad j = 1,\ldots,4 \qquad (1\text{-}9)$$

Substituting Eq. (1-9) into Eq. (1-8), we obtain the real part as follows :

$$\left[A_{0jx} B_{2jx} - A_{2jx} B_{0jx} \right]^2 + \left[A_{0jy} B_{2jx} - A_{2jx} B_{0jy} \right]^2$$
$$+ 4 \left[(A_{0jy} B_{2jx} - A_{2jx} B_{0jy}) \cdot (A_{0jx} B_{0jy} - A_{0jy} B_{0jx}) \right] = 0, \qquad j = 2,3,\ldots \qquad (1\text{-}10)$$

where the subscript x represents the X component, and y represents the Y component of the vector. For example, A_{0jx} = X component of $\underline{A}_{0j}$, A_{0jy} = Y component of $\underline{A}_{0j}$, etc. In a design problem, we are interested in finding four-bar linkages that can guide their coupler points through a set of prescribed positions. We may consider Eq. (1-10) as the design equations. Then the unknowns involved are $\underline{Z}_1$, $\underline{Z}_2$, $\underline{Z}_3$, and $\underline{Z}_4$. The relationship between the number of precision points and the number of solutions is listed in Table 1-1 :

Table 1-1 Relationship Between the Number of Precision Points and the Number of the Solutions

Number of Precision Points	Number of Scalar Equations	Number of Scalar Unknowns	Number of Solutions	Number of free choices
3	2	8	∞^6	6
4	3	8	∞^5	5
5	4	8	∞^4	4
6	5	8	∞^3	3
7	6	8	∞^2	2
8	7	8	∞^1	1
9	8	8	Finite	0

Equation (1-10) is a seventh-degree polynomial in which both coefficients of the constant and first-degree terms are zero identically. Hence, $\underline{Z}_1 = \underline{Z}_2 = \underline{Z}_3 = \underline{Z}_4 = 0$ is a multiple root to the system of equation defined by (1-10). It can be shown that any solution satisfying the following equation is also a multiple root.

$$\frac{A_{ojx}}{B_{0jx}} = \frac{A_{0jy}}{B_{0jy}} = \frac{A_{2jx}}{B_{2jx}} \qquad (1-11)$$

It is obvious that Eq.(1-11) can be satisfied when $\underline{Z}_1 = \underline{Z}_3$ and $\underline{Z}_2 = \underline{Z}_4$. Furthermore, if $\underline{Z}_2 = \underline{Z}_4 = 0$, then Eq.(1-10) is equal to zero identically. These solutions are known as singular solutions, since the Jacobian of the function is equal to zero at these points. Iterative techniques such as Newton-Raphson's method tend to converge to these singular conditions. Since the nine-position synthesis problem is highly non-linear and contains many singular points, it would be difficult to find any solution if a general iterative technique is used. In what follows, we shall show that the Homotopy method can always find some solutions.

CHAPTER 2 HOMOTOPY METHODS (CONTINUATION METHODS)

The problem of solving a system of N polynomial equations in N unknowns is common in many fields of science and engineering. The number of isolated solutions of a system is bounded by the total degree of the system.[1] Most systems of N polynomial equations in N unknowns arising in applications are deficient[2], in the sense that they have fewer solutions than that predicted by the total degree of the system. A number of numerical methods for solving all the solutions of an N-polynomial system can be found in the literature [9,10,11,15,16,27]. The General Homotopy method developed by Morgan [15,16,27] can be applied to solve all the solutions of a system of polynomial equations. The limitation of this method is that the system must be "small", i.e. its total degree can not be too large. Li, et al. [9,10] developed more efficient algorithms by following fewer "Homotopy paths". However, it is sometimes difficult to arrange the highest order terms in product forms. Sometimes, the highest order terms may be singular. Thus, a Cheater's Homotopy [11] was developed to overcome these difficulties. In what follows, both homotopy methods will be reviewed.

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1. The total degree is the product of the degree of the individual equation, also called the Bezout Number, See [9,10].
2. It means there are some zeros at infinity, see [9,10].

§ 2-1. General Homotopy Method

The General Homotopy [15,16,27] method is a numerical method used to find all solutions to a system of N polynomial equations with complex coefficients in N unknowns. The procedure is described as follows.

Let the system of polynomial equations be

$$\begin{cases} P_1(X_1,\ldots,X_n) = 0 \\ \quad\ldots \\ P_n(X_1,\ldots,X_n) = 0 \end{cases} \tag{2-1}$$

where the degree of each equation is D_i, $i = 1,\ldots,n$. Bezout's theorem predicts the maximum number of isolated solutions to be $D = D_1 \cdots D_n$, the product of the degree of the polynomials. This Bezout number is the total paths the General Homotopy has to follow. Morgan defined the General Homotopy function as :

$$\begin{aligned} H_j(X_1,\ldots,X_n,t) &= t\ P_j(X_1,\ldots,X_n) + (1-t)\ Q_j(X_1,\ldots,X_n) \\ &= t\ P_j(X_1,\ldots,X_n) + (1-t)\ (K_j \cdot X_j^{d_j} - b_j), \\ & \qquad j = 1,\ldots,n \end{aligned} \tag{2-2}$$

where t is a real variable called the homotopy parameter and, K_j and b_j are random complex numbers. Note that when $t = 0$, Eq. (2-2) reduces to :

$$H_j(X_1,\ldots,X_n,t) = K_j\ (X_j^{d_j} - B_j^{d_j}), \qquad j = 1,\ldots,n \tag{2-3}$$

which can be easily solved.

When $t = 1$, Eq.(2-2) reduces to

$$H_j(X_1,\ldots,X_n,t) = P_j(X_1,\ldots,X_n), \quad j = 1,\ldots,n \qquad (2\text{-}4)$$

which is the equation, Eq. (2-1), we want to solve. Thus, as t changes from 0 to 1, the solution of $H(\underline{X},t) = 0$ change from solutions of $Q(\underline{X}) = 0$ to solutions of $P(\underline{X}) = 0$.

We start at a solution to $H_j(X_1,\ldots,X_n,0) = 0$, $j = 1,2,\ldots,n$ and increment t by a small positive number Δt. We then solve $H_j(X_1,\ldots,X_n,\Delta t) = 0$, $j = 1,2,\ldots,n$, using the solution to $H_j(X_1,\ldots,X_n,0)$ as the initial guess. Increment t by Δt again and solve $H_j(X_1,\ldots,X_n,2\Delta t) = 0$, using the solution to $H_j(X_1,\ldots,X_n,\Delta t)$ as the initial guess. The process is continued until $t = 1$. Such a sequence of solutions is called a "Homotopy path". The idea is that each solution to Eq.(2-3) yields a solution to Eq.(2-4) by tracing out such a Homotopy path.

Now, we need a method of extrapolating from the solutions of $H_j(X_1,\ldots,X_n,t) = 0$ to that of $H_j(X_1,\ldots,X_n,t+\Delta t) = 0$. There are several possible methods. In this paper, we apply the following two methods used by Morgan :

1. Newton-Raphson's method
2. Define a differential equation whose solutions are the Homotopy paths. Solve differential equation using numerical integration method.

Method 2 is faster than method 1 and less prone to failure. Nevertheless, we can use Newton-Raphson's method at the end of the homotopy path to refine the final solution.

GENERAL HOMOTOPY PROCEDURE :

Step 1). Choose complex number K_j, b_j at random, and use the solutions of Eq.(2-3) as the starting points.

Step 2). Follow all the paths, $D = D_1 \cdots D_n$, defined by $H_j(X_1,\ldots,X_n,t) = 0$ to reach all solutions of $P_j(X_1,\ldots,X_n) = 0$.

§ 2-2. Cheater's Homotopy Method

The Cheater's Homotopy was recently introduced by Li, et al. [11]. Let a system of polynomial equations be :

$$\left\{ \begin{array}{l} P_1\ (C_1,C_2,\ldots C_m,X_1,X_2,\ldots,X_n) = 0 \\ \qquad\qquad \cdots \\ P_n\ (C_1,C_2,\ldots C_m,X_1,X_2,\ldots,X_n) = 0 \end{array} \right. \qquad (2\text{-}5)$$

where C_j are the coefficients and X_j are the variables of the system. Suppose we know the solutions, $X_j^* = (\ X_1,\ldots,X_n\ )$, to the following polynomial system :

$$\left\{ \begin{array}{l} Q_1(X_1,\ldots,X_n) = P_1(C_1^*,\ldots,C_m^*,X_1,\ldots,X_n) + B_1^* = 0 \\ \qquad\qquad \cdots \\ Q_n(X_1,\ldots,X_n) = P_n(C_1^*,\ldots,C_m^*,X_1,\ldots,X_n) + B_n^* = 0 \end{array} \right. \qquad (2\text{-}6)$$

where ($C_1^*, \ldots, C_m^*$) are the coefficients and ($B_1^*, \ldots, B_n^*$) a set of random complex numbers. Then, we can solve the solutions of Eq. (2-5) by defining the Cheater's Homotopy function as follows :

$$H_j(X_1, \ldots X_n, t)$$

$$= P_j\left[(1-t)C_1^* + tC_1, \ldots, (1-t)C_m^* + tC_m, X_1, \ldots, X_n \right] + (1-t)\ B_j^*$$

$$= (1-t)\ Q_j(X_1, \ldots, X_n) + t\ P_j(X_1, \ldots, X_n),$$

$$j = 1,2,\ldots,n \qquad (2\text{-}7)$$

It follows that every solution of $P_j(X_1, \ldots, X_n) = 0$ is reached by a path beginning at a point of X_j^*. Following the same process depicted in last section, we see that at $t = 0$, Eq. (2-7) reduces to :

$$H_j(X_1, \ldots, X_n, t) = P_j(C_1^*, \ldots, C_m^*, X_1, \ldots, X_n) + B_j^*$$
$$= Q_j(X_1, \ldots, X_n),$$

$$j = 1,2,\ldots,n \qquad (2\text{-}8)$$

for which the solutions are known. At $t = 1$, Eq. (2-7) reduces to

$$H_j(X_1, \ldots, X_n, t) = P_j(C_1, \ldots, C_m, X_1, \ldots, X_n),$$

$$j = 1,2,\ldots,n \qquad (2\text{-}9)$$

which is the system of equations we want to solve. If we increase t from 0 to 1 incrementally, and solve Eq. (2-7) for every t. We generate a homotopy path from the solution of equation $Q_j(X_1, \ldots, X_n) = 0$ to that of $P_j(X_1, \ldots, X_n) = 0$. When t arrives at 1, we get the solution we want. It is important to note that

C_i^* and B_j^* in Eq. (2-7) must be random complex numbers. Otherwise, it might not have the required property of smoothness and accessibility [9,10,11].

CHEATER'S HOMOTOPY PROCEDURE :

Step 1). Choose complex numbers $(B_1^*,\ldots,B_n^*)$ and $(C_1^*,\ldots,C_m^*)$ at random, and use the method described in Sec. 2-1 to solve Eq. (2-6). The number of solutions D_0, is bounded by the total degree D, i.e. $D_0 \le D$.

Step 2). For each new choice of coefficients $(C_1,\ldots,C_n)$, follow the D_0 paths defined by $H_j(X_1,\ldots,X_n,t) = 0$ in Eq. (2-7) to reach all solutions of $P_j(C_1,\ldots,C_n,X_1,\ldots,X_n) = 0$.

In this procedure, the first step is to solve the system with a set of random coefficients B_j^* and C_i^*, one time only, following all $D = D_1\cdots D_n$ paths. In subsequent runs, these D_0 solutions are used as "seeds" to initialize paths for various values of C_i. This is the "cheating" part. Instead of starting over from scratch each time, we use the seeds determinated in step 1 as starting points. If the system is deficient, this method results in fewer paths , D_0, to be followed in subsequent runs.

CHAPTER 3 : ALLPICATION OF THE GENERAL HOMOTOPY METHOD TO THE FIVE-POSITION SYNTHESIS PROBLEM

There are several methods [1,4,5,14,22,23,26] to solve five-coupler-points synthesis problem. Traditional method is to reduce the system of design equations to one equation in one unknown. In this thesis, we use the General Homotopy to solve the system of equations instead of making the reduction.

§ 3-1. Traditional Approach

For five prescribed coupler points, the first equation of (1-3) can be used as the design equation :

$$\underline{Z}_1 (e^{i\phi_j} - 1) + \underline{Z}_2 (e^{i\theta_j} - 1) = \underline{\delta}_j, \qquad j = 2,3,4,5 \tag{3-1}$$

where $\underline{\delta}_j = \underline{R}_j - \underline{R}_1$ are known from the prescribed positions and θ_j, $\underline{Z}_1$, $\underline{Z}_2$, and ϕ_j are unknowns. We can choose θ_j arbitrarily and solve Eq.(3-1) for the remaining unknowns. Considering $\underline{Z}_1$ and $\underline{Z}_2$ as two unknowns, the augmented matrix M of Eq.(3-1) can be written as :

$$M = \begin{bmatrix} e^{i\phi_2}-1 & e^{i\theta_2}-1 & \underline{\delta}_2 \\ e^{i\phi_3}-1 & e^{i\theta_3}-1 & \underline{\delta}_3 \\ e^{i\phi_4}-1 & e^{i\theta_4}-1 & \underline{\delta}_4 \\ e^{i\phi_5}-1 & e^{i\theta_5}-1 & \underline{\delta}_5 \end{bmatrix} \tag{3-2}$$

For the system of equations, (3-1), to have simultaneous solutions for the dyad vectors $\underline{Z}_1$ and $\underline{Z}_2$, matrix M must be of rank two. Thus there are two compatibility equations simultaneously fitted to the problem of five precision points :

$$\det \begin{bmatrix} e^{i\phi_2}-1 & e^{i\theta_2}-1 & \underline{\delta}_2 \\ e^{i\phi_3}-1 & e^{i\theta_3}-1 & \underline{\delta}_3 \\ e^{i\phi_4}-1 & e^{i\theta_4}-1 & \underline{\delta}_4 \end{bmatrix} = 0 \qquad (3\text{-}3)$$

and,

$$\det \begin{bmatrix} e^{i\phi_2}-1 & e^{i\theta_2}-1 & \underline{\delta}_2 \\ e^{i\phi_3}-1 & e^{i\theta_3}-1 & \underline{\delta}_3 \\ e^{i\phi_5}-1 & e^{i\theta_5}-1 & \underline{\delta}_5 \end{bmatrix} = 0 \qquad (3\text{-}4)$$

Expanding Eqs.(3-3) and (3-4), we obtain :

$$\begin{cases} \underline{\Delta}_2\ e^{i\phi_2} + \underline{\Delta}_3\ e^{i\phi_3} + \underline{\Delta}_4 e^{i\phi_4} - \underline{\Delta}_1 = 0 \\ \underline{\Delta}_2'\ e^{i\phi_2} + \underline{\Delta}_3'\ e^{i\phi_3} + \underline{\Delta}_4 e^{i\phi_5} - \underline{\Delta}_1' = 0 \end{cases} \qquad (3\text{-}5)$$

where:

$$\underline{\Delta}_2 = \begin{bmatrix} e^{i\theta_3}-1 & \underline{\delta}_3 \\ e^{i\theta_4}-1 & \underline{\delta}_4 \end{bmatrix}$$

$$\underline{\Delta}_3 = - \begin{bmatrix} e^{i\theta_2}-1 & \underline{\delta}_2 \\ e^{i\theta_4}-1 & \underline{\delta}_4 \end{bmatrix}$$

$$\underline{\Delta}_4 = \begin{bmatrix} e^{i\theta_2}-1 & \underline{\delta}_2 \\ e^{i\theta_3}-1 & \underline{\delta}_3 \end{bmatrix}$$

$$\underline{\Delta}_2' = \begin{bmatrix} e^{i\theta_3}-1 & \underline{\delta}_3 \\ e^{i\theta_5}-1 & \underline{\delta}_5 \end{bmatrix}$$

$$\underline{\Delta}_3' = -\begin{bmatrix} e^{i\theta_2}-1 & \underline{\delta}_2 \\ e^{i\theta_5}-1 & \underline{\delta}_5 \end{bmatrix}$$

$$\underline{\Delta}_1 = \underline{\Delta}_2 + \underline{\Delta}_3 + \underline{\Delta}_4$$

$$\underline{\Delta}_1' = \underline{\Delta}_2' + \underline{\Delta}_3' + \underline{\Delta}_4$$

To eliminate ϕ_4 and ϕ_5, Eq.(3-5) is rearranged as :

$$\begin{cases} \underline{\Delta}_2 \; e^{i\phi_2} + \underline{\Delta}_3 \, e^{i\phi_3} - \underline{\Delta}_1 = - \underline{\Delta}_4 \, e^{i\phi_4} \\ \underline{\Delta}_2' \; e^{i\phi_2} + \underline{\Delta}_3' \, e^{i\phi_3} - \underline{\Delta}_1' = - \underline{\Delta}_4 \, e^{i\phi_5} \end{cases} \tag{3-5a}$$

The complex conjugates of Eq.(3-5a) also hold true :

$$\begin{cases} \overline{\underline{\Delta}}_2 \, e^{-i\phi_2} + \overline{\underline{\Delta}}_3 \, e^{-i\phi_3} - \overline{\underline{\Delta}}_1 = - \overline{\underline{\Delta}}_4 \, e^{-i\phi_4} \\ \overline{\underline{\Delta}}_2' \, e^{-i\phi_2} + \overline{\underline{\Delta}}_3' \, e^{-i\phi_3} - \overline{\underline{\Delta}}_1' = - \overline{\underline{\Delta}}_4 \, e^{-i\phi_5} \end{cases} \tag{3-6}$$

Multipling each equation in (3-5a) by the corresponding conjugate equations shown in (3-6), and after some simplification, we obtain :

$$\begin{cases} \underline{C}_1 \, e^{i\phi_3} + \underline{D}_1 + \overline{\underline{C}}_1 \, e^{-i\phi_3} = 0 \\ \underline{C}_2 \, e^{i\phi_3} + \underline{D}_2 + \overline{\underline{C}}_2 \, e^{-i\phi_3} = 0 \end{cases} \tag{3-7}$$

where :

$$\underline{C}_1 = \underline{\Delta}_3 \left(-\bar{\underline{\Delta}}_1 + \bar{\underline{\Delta}}_2 \, e^{-i\phi_2} \right)$$

$$\underline{D}_1 = -\bar{\underline{\Delta}}_1 \underline{\Delta}_2 \, e^{i\phi_2} - \underline{\Delta}_1 \bar{\underline{\Delta}}_2 \, e^{-i\phi_2} - \underline{\Delta}_4 \bar{\underline{\Delta}}_4 + \underline{\Delta}_1 \bar{\underline{\Delta}}_1 + \underline{\Delta}_2 \bar{\underline{\Delta}}_2 + \underline{\Delta}_3 \bar{\underline{\Delta}}_3$$

$$\underline{C}_2 = \underline{\Delta}_3' \left(-\bar{\underline{\Delta}}_1' + \bar{\underline{\Delta}}_2' \, e^{-i\phi_2} \right)$$

$$\underline{D}_2 = -\bar{\underline{\Delta}}_1' \underline{\Delta}_2' \, e^{i\phi_2} - \underline{\Delta}_1' \bar{\underline{\Delta}}_2' \, e^{-i\phi_2} - \underline{\Delta}_4 \bar{\underline{\Delta}}_4 + \underline{\Delta}_1' \bar{\underline{\Delta}}_1' + \underline{\Delta}_2' \bar{\underline{\Delta}}_2' + \underline{\Delta}_3' \bar{\underline{\Delta}}_3'$$

Multipling Eq. (3-7) by $e^{i\phi_3}$, we obtain :

$$\begin{cases} \underline{C}_1 \, e^{2i\phi_3} + \underline{D}_1 \, e^{i\phi_3} + \bar{\underline{C}}_1 = 0 \\ \underline{C}_2 \, e^{2i\phi_3} + \underline{D}_2 \, e^{i\phi_3} + \bar{\underline{C}}_2 = 0 \end{cases} \qquad (3\text{-}8)$$

We can eliminate powers of $e^{i\phi_3}$, from Eqs. (3-7) and (3-8) by using Sylvester's dyalitic eliminant :

$$E = \det \begin{bmatrix} 0 & \underline{C}_1 & \underline{D}_1 & \bar{\underline{C}}_1 \\ 0 & \underline{C}_2 & \underline{D}_2 & \bar{\underline{C}}_2 \\ \underline{C}_1 & \underline{D}_1 & \bar{\underline{C}}_1 & 0 \\ \underline{C}_2 & \underline{D}_2 & \bar{\underline{C}}_2 & 0 \end{bmatrix} = 0 \qquad (3\text{-}9)$$

After expanding and simplifying the determinant, a polynomial in $e^{i\phi_2}$ is obtained :

$$\sum_m A_m \, e^{im\phi_2} = 0 \qquad (3\text{-}10)$$

where m = -3, -2, -1, 0, 1, 2, 3, and all the coefficients A_m are deterministic function of the quantities $\underline{\delta}_j$ and θ_j (j =

2,3,4,5). Note that A_{-i} and A_i (i = 1,2,3) are each other's complex conjugates, and A_0 is a real number. Thus Eq.(3-10) is real, i.e., its imaginary part vanishes identically. The real part of Eq.(3-10) have the form :

$$\sum_m \left[P_m \cos(m\phi_2) + Q_m \sin(m\phi_2) \right] = 0, \qquad m = 1,2,3 \qquad (3\text{-}11)$$

where P_m and Q_m are known real numbers. Let $t = \tan(\phi_2/2)$. Then, Eq.(3-11) can be written as :

$$\sum_{n=0}^{6} A_n t^n = 0 \qquad (3\text{-}12)$$

There are two trivial solutions : $\phi_2 = 0$ and $\phi_2 = \theta_2$. So, dividing Eq.(3-12) by the factor $t \cdot [t - \tan(\theta_2/2)]$, it can be reduced to a fourth-degree polynomial :

$$t^4 + \lambda_3 t^3 + \lambda_2 t^2 + \lambda_1 t^1 + \lambda_0 = 0 \qquad (3\text{-}13)$$

Equation (3-13) can yield zero, two or four real roots. Each real root gives a value of ϕ_2, which can be back substituted into Eq. (3-7) to solve for ϕ_3, and then Eq. (3-5) to obtain ϕ_4 and ϕ_5. Any two equations of Eq. (3-1) can used to solve for $\underline{Z}_1$ and $\underline{Z}_2$, which are known as the Burmester point pairs. By combining two different Burmester point pairs, we can construct up to six different four-bar linkages.

§ 3-2. General Homotopy Approach

In this section, we describe the application of General Homotopy to solve the five coupler-points problem. First, we present the design equations for the General Homotopy. The first equation of (1-6) can be simplified as :

$$2\cdot(A_{0jx} \cos\theta_j - A_{0jy} \sin\theta_j) + A_{1jx} = 0 \qquad j = 2,\ldots,5 \qquad (3\text{-}14)$$

where

$$A_{0jx} = (-Z_{2x}^2 - Z_{2y}^2 - Z_{2x}\,\delta_{jx} - Z_{2y}\,\delta_{jy} - Z_{1x}\,Z_{2x} - Z_{1y}\,Z_{2y})$$

$$A_{0jy} = (Z_{1y}\,Z_{2x} - Z_{1x}\,Z_{2y} + Z_{2x}\,\delta_{jy} - Z_{2y}\,\delta_{jx})$$

$$A_{1jx} = \delta_{jx}^2 + \delta_{jy}^2 + 2\,Z_{1x}\,\delta_{jx} + 2\,Z_{1y}\,\delta_{jy} + 2\,Z_{2x}^2 + 2\,Z_{2y}^2 + 2\,Z_{2x}\,\delta_{jx} + 2\,Z_{2y}\,\delta_{jy} + 2\,Z_{1x}\,Z_{2x} + 2\,Z_{1y}\,Z_{2y}$$

where $\underline{\delta}_j = \underline{R}_j - \underline{R}_1$ are known from the five given precision coupler-points, and the angular displacement of the coupler, θ_j, with $j = 2,\ldots,5$ can be chosen arbitrarily. The four scaler unknowns are : Z_{1x}, Z_{1y}, Z_{2x}, and Z_{2y}. To facilitate further discussion, we will define the X and Y components of $\underline{Z}_1$ as X_1 and X_2, whereas the components of $\underline{Z}_2$ will be defined as X_3 and X_4 (i.e., $X_1 = Z_{1x}$, $X_2 = Z_{1y}$, $X_3 = Z_{2x}$, and $X_4 = Z_{2y}$). The degree of Eq.(3-14) is two. Therefore, the total degree number is $2^4 = 16$. Since it is a small number, we can apply the General Homotopy method to solve the problem. There are sixteen paths to be followed from the starting points to the final solutions. If we consider Eq. (3-14) as the design equation, $P_j(X_1,X_2,X_3,X_4) = 0$, the General Homotopy function can be defined as follows :

$$H_j(X_1,\ldots,X_4,t) = t\, P_j(X_1,\ldots,X_4) + (1-t)\cdot K_j\cdot(X_j^2 - B_j^2),$$

$$j = 2,\ldots,5 \quad (3\text{-}15)$$

where K_j and B_j are randomly chosen complex numbers. When t = 0, Eq.(3-15) reduces to

$$H_j(X_1,\ldots,X_4,t) = K_j\ (X_j^2 - B_j^2),$$

$$j = 2,\ldots,5 \quad (3\text{-}16)$$

Solving Eq.(3-16) for X_j, we obtain :

$$X_j = \pm B_j, \qquad j = 2,\ldots,5 \quad (3\text{-}17)$$

There are sixteen solutions. These solutions are used as the starting points for the homotopy paths. As t is incremented from 0 to 1, we get all the solutions we want. However, some of the solutions may diverge to infinity [15].

Since the degree of Eq. (3-13) is four, there are at most four solutions. Consequently, twelve of the sixteen homotopy paths will diverge to infinity. If some of them are complex solutions, they must exist in conjugate pairs. Thus, corresponding to each choice of θ_j, $j = 2,\ldots,5$, the number of real solutions can be zero, two, or four. Each real solution corresponds to a dyad. Any combination of two different dyads can constitutes a four-bar linkage[1] that will guide its coupler point through the five prescribed precision points.

1.Because the two dyad have same θ_j.

Once $\underline{Z}_1, \ldots, \underline{Z}_4$ are found, we can find the remaining unknowns in Eqs.(1-1) and (1-2) by back substitution. First, we solve Eqs.(1-5) for ϕ_j and ψ_j. Then we solve Eq.(1-1) for $\underline{Z}_5$ and $\underline{Z}_6$. From these data, we can calculate the coordinates of points O_A, O_B, A, B, and M in every precision point and the length of every link.

§ 3-3. Application of Roberts-Chebyshev Theorem

When we consider a curve traced by the coupler-point of a planar four-bar linkage, some other planar four-bar linkages tracing the identical coupler-point curve may be found by applying the Roberts-Chebyshev theorem. This theorem states that three different planar four-bar linkages will trace identical coupler curves. These linkages do not look alike. Their only shared property is that they can trace identical coupler curves. This kind of linkages are known as cognate linkages [13,22]. The velocity and acceleration characteristics of the cognate linkages are not necessarily identical. This means that cognate linkages are not always equivalent linkages.[1]

Consider the given four-bar designated by the four pivots O_A, O_B, A, B, and the coupler point M shown in Fig. 3-1, We can draw two parallelograms, O_AAMA_1 and O_BBMB_2, and two triangles $\Delta\ A_1MC_1$ and $\Delta\ MB_2C_2$ similar to the triangle Δ ABM. Finally, a third parallelogram $O_CC_1MC_2$ can be depicted by taking into account C_1O_C

1.The equivalent linkage have same instantaneous velocities and acceleration. They can be also called instantaneous linkage.

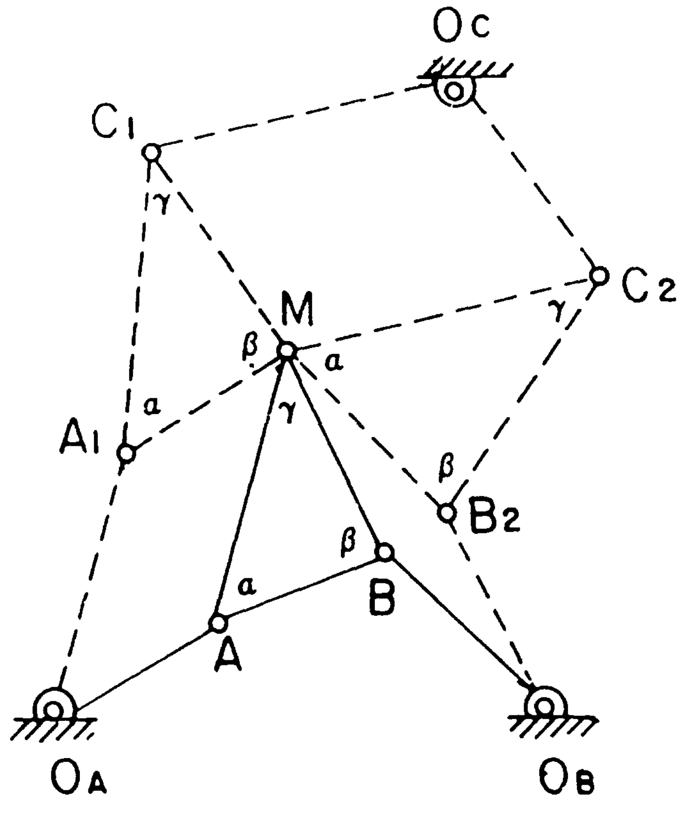

Figure 3-1 Cognate Four-Bar Linkages Obtained by

Roberts-Chebyshev Theorem

and C_2O_C. As a result, we can obtain the three cognate four-bar linkages :

1). Given linkage : O_AABO_B, with ground link O_AO_B and coupler point M.

2). Cognate linkage 1 : $O_AA_1C_1O_C$, with ground link O_AO_C and coupler point M.

3). Cognate linkage 2 : $O_BB_2C_2O_C$, with ground link O_BO_C and coupler point M.

Therefore, we can construct two four-bar linkages from each of the four-bar obtained in the previous section.

§ 3-4. Numerical Examples

In this section, we use two examples to illustrate the theory. The computations were performed on VAX 750 VMS computer, in accordance with programs written in the Fortran programming language. The numerical accuracy are up to 10^{-16} using double precision.

Example 1.

Table 3-1 lists the five desired precision points. Choosing θ_2 = 10.00, θ_3 = 15.00, θ_4 = 20.00 and θ_5 = 25.00, and then applying the General Homotopy method, we found four sets of real solutions as listed in Table 3-2. By combining any two of these solutions, we can form six different four-bar linkages. Then, we apply Roberts-Chebyshev Theorem to find two cognate linkages for each of these six four-bar linkages. Hence, a total of eighteen four-bar linkages have been identified. Table 3-3 shows three cognate four-bar linkages obtained from the combination of Sol. 1 and Sol. 4 listed in Table 3-2. The cognate four-bar linkages formed from the combination of Sol. 2 and Sol. 4 listed in Table 3-2 are shown in Table 3-4. Note that many more four-bar linkages can be found by choosing different θj's.

Table 3-1 Five Precision Points Used for Example 1

Coupler Point M_i	X - Coordinate	Y - Coordinate
M_1	0.896186660	-0.098029166
M_2	1.515143000	-0.854496080
M_3	1.713869000	-0.300992320
M_4	1.664202900	0.332410880
M_5	1.301183400	0.921538060

Table 3-2 Solutions of Example 1

	Solution 1	Solution 2
X_1	-1.573127954 + 0.000000000 i	3.238956530 + 0.000000000 i
X_2	0.536864225 + 0.000000000 i	0.313100908 + 0.000000000 i
X_3	7.930062456 + 0.000000000 i	-3.704433040 + 0.000000000 i
X_4	1.676584252 + 0.000000000 i	-3.173755471 + 0.000000000 i

	Solution 3	Solution 4
X_1	47.654623041 + 0.000000000 i	-0.188274472 + 0.000000000 i
X_2	-14.122356345 + 0.000000000 i	0.662781720 + 0.000000000 i
X_3	21.610554685 + 0.000000000 i	2.540201974 + 0.000000000 i
X_4	-15.065604572 + 0.000000000 i	-0.225088167 + 0.000000000 i

Table 3-3 Cognate Four-Bar Linkages Obtained from

Sol.1 and Sol.4 in Table 3-2

	Original Four-Bar	Chebyshev 1	Chebyshev 2
X_1	-1.573127954	7.930062456	2.540201974
X_2	0.536864225	1.676584252	-0.225088167
X_3	7.930062456	-1.573127954	-0.188274472
X_4	1.676584252	0.536864225	0.662781720
X_5	-0.188274472	-0.142093297	0.539387260
X_6	0.662781720	0.966714693	-0.509025037
X_7	2.540201974	0.539387260	-0.142093297
X_8	-0.225088167	-0.509025037	0.966714693

Table 3-4 Cognate Four-Bar Linkages Obtained from

Sol.2 and Sol.4 in Table 3-2

	Original Four-Bar	Chebyshev 1	Chebyshev 2
X_1	3.238956530	-3.704433040	2.540201974
X_2	0.313100908	-3.173755471	-0.225088167
X_3	-3.704433040	3.238956530	-0.188274472
X_4	-3.173755471	0.313100908	0.662781720
X_5	-0.188274472	-0.251902070	1.090665209
X_6	0.662781720	0.416432467	-0.504388179
X_7	2.540201974	1.090665209	-0.251902070
X_8	-0.225088167	-0.504388179	0.416432467

Example 2.

Table 3-5 lists another set of five precision points. Again, we choose θ_2 = 10.00, θ_3 = 15.00, θ_4 = 20.00, and θ_5 = 25.00. We found four real solutions as listed in Table 3-6. Table 3-7 shows the cognate four-bar linkages obtained from the combination of Sol. 2 and Sol. 3 listed in Table 3-6.

Table 3-5 Five Precision Points Used for Example 2

Coupler Point M_i	X - Coordinate	Y - Coordinate
M_1	1.000000000	0.000000000
M_2	1.514419000	-0.856816990
M_3	1.709746300	-0.323059980
M_4	1.711962400	0.311115900
M_5	1.394774300	0.973082000

Table 3-6 Solutions of Example 2

	Solution 1	Solution 2
X_1	–1.738452449 + 0.000000000 i	1.808703160 + 0.000000000 i
X_2	0.715947433 + 0.000000000 i	0.627956529 + 0.000000000 i
X_3	8.607234884 + 0.000000000 i	-0.339727801 + 0.000000000 i
X_4	0.829597213 + 0.000000000 i	-2.386883834 + 0.000000000 i

	Solution 3	Solution 4
X_1	6.460652740 + 0.000000000 i	0.065461578 + 0.000000000 i
X_2	-0.804343772 + 0.000000000 i	0.756770007 + 0.000000000 i
x_3	11.121121872 + 0.000000000 i	2.550095826 + 0.000000000 i
X_4	-10.499644935 + 0.000000000 i	-0.553093852 + 0.000000000 i

Table 3-7 Cognate Four-Bar Linkages Obtained from

Sol.2 and Sol.3 in Table 3-6

	Original Four-Bar	Chebyshev 1	Chebyshev 2
X_1	1.808703160	-0.339727801	11.121121872
X_2	0.627956529	-2.386883834	-10.499644935
X_3	-0.339727801	1.808703160	6.460652470
X_4	-2.386883834	0.627956529	-0.804343772
X_5	6.460652470	-0.384089825	2.046524463
X_6	-0.804343772	1.049794766	0.401000302
X_7	11.121121872	2.046524463	-0.384089825
X_8	-10.499644935	0.401000302	1.049794766

CHAPTER 4. APPLICATION OF THE CHEATER'S HOMOTOPY METHOD TO THE NINE-POSITION SYNTHESIS PROBLEM

In this chapter, we present a new method to solve the nine-coupler-points synthesis problem. To date, there are at least three published formulations of this problem [1,20,24]. Roth's work [20] is concerned with the synthesis of geared five-bar mechanisms[1], while Alt [1] and Sieker [24] treated the problem associated with four-bar linkages. Roth suggested a numerical method which is similar to homotopy method [20]. However, he did not use complex algorithm and the random constant technique. In order to avoid singular conditions he proposed the so-called "Bootstrap" method which includes "position interchange" and "quality index". However, he did not suggest how to find the starting mechanisms and the method does not necessarily results in a solution. Alt [1] made no attempt to solve the problem. Finally, Seiker's [24] suggestion of the selection method seems incomplete. The Cheater's homotopy, we use in this thesis is a proven method that always works.

1.When the gear ratio is plus one, the geared five-bar mechanism is equivalent to a four-bar linkage.

§ 4-1. Cheater's Homotopy Approach

According to Sec. 2-2, once we have found all the solutions to a polynomial system, then we can obtain all the solutions of a similar polynomial system with different coefficients by the Cheater's Homotopy . For the nine-points synthesis problem, we can use Eq. (1-10) as the design equation :

$$P_j(C_1, \ldots, C_m, X_1, \ldots, X_8)$$

$$= \left[A_{0jx}\, B_{2jx} - A_{2jx}\, B_{0jx} \right]^2 + \left[A_{j0y}\, B_{2jx} - A_{2jx}\, B_{0jy} \right]^2$$

$$+ 4 \cdot \left[(A_{0jy}\, B_{2jx} - A_{2jx}\, B_{0jy}) \cdot (A_{0jx}\, B_{0jy} - A_{0jy}\, B_{0jx}) \right],$$

$$j = 2, \ldots, 9 \qquad (4\text{-}1)$$

where :

$$A_{0jx} = - X_3^2 - X_4^2 - X_3\, \delta_{jx} - X_1 X_3 - X_4\, \delta_{jy} - X_2 X_4$$

$$A_{0jy} = - X_4\, \delta_{jx} - X_1 X_4 + X_3\, \delta_{jy} + X_2 X_3$$

$$A_{2jx} = \delta_{jx}^2 + \delta_{jy}^2 + 2\, \delta_{jx} X_1 + 2\, \delta_{jy} X_2$$

$$B_{0jx} = - X_7^2 - X_8^2 - X_7\, \delta_{jx} - X_5 X_7 - X_8\, \delta_{jy} - X_6 X_8$$

$$B_{0jy} = - X_8\, \delta_{jx} - X_5 X_8 + X_7\, \delta_{jy} + X_6 X_7$$

$$B_{2jx} = \delta_{jx}^2 + \delta_{jy}^2 + 2\, \delta_{jx} X_5 + 2\, \delta_{jy} X_6$$

Where, $X_1 = Z_{1x}$, $X_2 = Z_{1y}$, $X_3 = Z_{2x}$, $X_4 = Z_{2y}$, $X_5 = Z_{3x}$, $X_6 = Z_{3y}$, $X_7 = Z_{4x}$, and $X_8 = Z_{4y}$.

Each of the equations in (4-1) is a seventh-degree polynomial. Thus the total degree of the polynomial system is 7^8= 5,764,801. Obviously, this number is too large to apply the

General Homotopy to find all the solutions. Hence, we chose to apply the Cheater's Homotopy to find some solutions. If we have already had some solutions to a polynomial system, then we can use it as the starting points to find some solutions to an similar polynomial system with different coefficients by the Cheater's Homotopy. For the nine-coupler-points synthesis problem, first, we find some four-bars which satisfy five of the nine precision points, using General Homotopy method discussed in the previous chapter. Then, these four-bar linkages are used as the starting mechanisms. The coupler point of such a starting mechanism will pass through five of the nine desired precision points. However, it will not pass through the remaining four precision points. For the reason of preserving the characteristics of the polynomial system and reducing computer time, we choose four additional coupler points from the starting mechanism to be as close to the remaining four precision points as possible. The starting mechanism along with its nine coupler points constitute the starting functions, $Q_j(X_1,...,X_8)$, in the Cheater's Homotopy. Let Q_j function be :

$$Q_j(C_1',...,C_m',X_1,...,X_8)$$

$$= P_j(C^*_1,...,C^*_m,X_1,...,X_8) + B^*_j$$

$$= \left[A_{0jx} B_{2jx} - A_{2jx} B_{0jx} \right]^2 + \left[A_{j0y} B_{2jx} - A_{2jx} B_{0jy} \right]^2$$

$$+ 4 \cdot \left[(A_{0jy} B_{2jx} - A_{2jx} B_{0jy}) \cdot (A_{0jx} B_{0jy} - A_{0jy} B_{0jx}) \right] + B^*_j ,$$

$$j = 2,..,9 \qquad (4\text{-}2)$$

where B^*_j are random complex numbers and C^*_i are the coefficients of the polynomial $P_j(C^*_1, \ldots, C^*_m, X_1, \ldots, X_8) = 0$. The coefficients C^*_i are function of $\underline{\delta}_j$, where $\underline{\delta}_j = \underline{R}_j - \underline{R}_1$. Thus, C^*_i are function of the nine coupler points of the starting mechanism. This indicates that once the coupler points of the starting mechanism are determined, the coefficients C_i' of the starting function Q_j are uniquely defined. Let the Cheater's Homotopy function be :

$$H_j(X_1, \ldots, X_8, t)$$

$$= t \cdot P_j(C_1, \ldots, C_m, X_1, \ldots, X_8) + (1-t)\, Q_j(C_1', \ldots, C_m', X_1, \ldots, X_8)$$

$$= t \cdot P_j(C_1, \ldots, C_m, X_1, \ldots, X_8) + (1-t)\left[P_j(C_1^*, \ldots, C_m^*, X_1, \ldots, X_8) + B_j^*\right], \quad j = 2, \ldots, 9 \qquad (4\text{-}3)$$

At $t = 0$, the solution to the above equation is the starting mechanism. At $t = 1$, Eq.(4-3) reduces to the system of design equations we want to solve. Following the Homotopy path by incrementing t from 0 to 1, we can find the solution. We summarize the procedure as follows :

<u>CHEATER'S HOMOTOPY PROCEDURE :</u>

Step 1). Choose five alternate points from the given nine precision points.

Step 2). Use the General Homotopy to solve the five-precision-points problem, and use the four-bar linkages found as the starting mechanisms. The coupler-point of the starting mechanism passes through five of the nine given points.

Step 3). Choose four additional coupler points to be as close to the remaining four precision points as possible.

Step 4). Determine the complex random number $B^{*}j$.(See section 4-2).

Step 5). Use Cheater's Homotopy to solve the problem.

§ 4-2. Generation of the Random Numbers

In the Cheater's Homotopy method, we need to determine some random complex numbers $B^{*}j$. The other important criterion is that the coefficient $C^{*}i$ must be complex. They are used to ensure the smoothness and accessibility properties of Homotopy. After we have constructed a four-bar linkage, we use a small trick to generate these random complex numbers $B^{*}j$ and $C^{*}i$. The coefficients $C^{*}i$ defined by the starting mechanism and its nine coupler points are real. If we add a small random complex number to the coefficient $C^{*}i$, the functional value of $Pj(C_1^{*},\dots,C_m^{*},X_1,\dots,X_8)$ will no longer be zero. We take this small residual value as $-B^{*}j$ so that $Qj(C',\dots,C'_m,X_1,\dots,X_8) = 0$. The value of these complex numbers relative to $C^{*}i$ should be in some range. If it is too small (e.g., $< 10^{-7}$), the effect of these numbers is insignificant. Then it defeats the purpose of adding these numbers. In this case, it is easy to converge to the singular solution as t arrives at 1 or to cause the Homotopy path to fail (i.e., the Homotopy path does not meet the Smoothness and

Accessibility criteria). On the other hand, if the values are too large, the characteristics of the polynomials will deviate greatly from the original system. Thus, it might cause the Homotopy paths S to converge to complex solutions which is not desirable. There is no theory to predict the size of these complex numbers. According to our experience, the range of these complex numbers can be chosen in the order of 10^{-2} to 10^{-4}.

§ 4-3. Numerical Examples

Given nine precision points, we can choose positions 1, 3, 5, 7, and 9 and apply the General Homotopy to find some starting four-bar linkages. Then, we apply the Cheater's Homotopy to synthesize four-bar linkages for the nine precision points.

Example 3.

Table 4-1 lists the desired nine precision points. Note that the odd number points have been taken from the five positions listed in Table 3-1, to avoid repeating the same process again. Hence, the four-bar linkages found in Chapter 3 can be used directly as the starting mechanisms. There are eighteen starting mechanisms obtained from example 1. Applying the Cheater's Homotopy, we found two of these starting mechanisms converge to real solutions while the other sixteen converges to complex solutions.

Table 4-2(a) lists the link lengths of the Chebyshev 1 linkage obtained from example 1, Table 3-3 and used as the starting mechanism. The coupler-point-curve of this starting four-bar linkage is shown in Figure 4-1. Table 4-2(b) shows the perturbed coupler points, and Table 4-2(c) shows the complex number B^{*}_{j} calculated from the residual value of the function P_j as a result of the perturbation. This starting mechanism converge to a real solution as listed in Table 4-3. Figure 4-2 shows the coupler-point curve of the resulting four-bar. Note that the four-bar found is very different from the starting four-bar. The coupler-link rotation angles, θ_j, $j = 2,3,\ldots,9$ are also totally different from the initial θ_j's. Also note that the nine-precision-coupler-points have been chosen such that point 1 doesn't fall on the smooth curve formed by the remaining points. This makes the synthesis problem a very difficult one.

By using the original four-bar listed in Table 3-4 as the starting mechanism, we found another four-bar linkage satisfying this nine-position problem. Table 4-4(a) to 4-4(c) list the starting four-bar, perturbed coupler points and the random numbers B^{*}_{j} . Table 4-5 shows the four-bar found. Figures 4-3 and 4-4 show the coupler curves of the starting and resultant four-bar, respectively. We note that the two four-bar linkages found are cognate four-bar linkages.

Table 4-6(a) to 4-6(c) lists the starting mechanism, perturbed coupler points and the random numbers B_j^{*} that were generated by the Chebyshev 2 linkage listed in Table 3-3. This starting mechanism converge to a complex solution as listed in Table 4-7.

Table 4-1 Nine Precision Points Used for Example 3

Coupler-Point M_i	X - Coordinate	Y - Coordinate
M_1	0.896186660	-0.098029166
M_2	1.215653500	-1.187491000
M_3	1.515143000	-0.854496080
M_4	1.675477500	-0.487680580
M_5	1.713869000	-0.300992320
M_6	1.721523600	0.032699525
M_7	1.664202900	0.332410880
M_8	1.498417100	0.744355760
M_9	1.301183400	0.921538060

Table 4-2(a) Starting Four-Bar - Example 3

	Starting Four-bar linkage X_i
X_1	7.930062456 + 0.000000000 i
X_2	1.676584252 + 0.000000000 i
X_3	-1.573127954 + 0.000000000 i
X_4	0.536864225 + 0.000000000 i
X_5	-0.142093297 + 0.000000000 i
X_6	0.966714693 + 0.000000000 i
X_7	0.539387260 + 0.000000000 i
X_8	-0.509025037 + 0.000000000 i

Table 4-2(b) Perturbed Coupler points - Example 3

Coupler Point	X - Coordinate	Y - Coordinate
M_1	0.896188558 + 0.000189768 i	-0.098029374 - 0.000020758 i
M_2	1.235854846 + 0.000052709 i	-1.197241812 - 0.000051062 i
M_3	1.515145770 + 0.000277044 i	-0.854497642 - 0.000156245 i
M_4	1.671085400 + 0.000062916 i	-0.502154668 - 0.000018906 i
M_5	1.713872375 + 0.000337461 i	-0.300992913 - 0.000059265 i
M_6	1.724348150 + 0.000081131 i	0.004104246 + 0.000000193 i
M_7	1.664206842 + 0.000394250 i	0.332411667 + 0.000078748 i
M_8	1.301183699 + 0.000029862 i	0.921538271 + 0.000021149 i
M_9	1.301185738 + 0.000233823 i	0.921539716 + 0.000165600 i

Table 4-2(c) Random Number B^*_j

	Random Number B^*_j
B^*_1	-0.000175398 - 0.017620614 i
B^*_2	0.000089363 + 0.008476548 i
B^*_3	0.000368945 + 0.034488178 i
B^*_4	-0.000737279 - 0.077496516 i
B^*_5	0.000959904 + 0.093519389 i
B^*_6	-0.002624841 - 0.278921348 i
B^*_7	0.001169117 + 0.112837206 i
B^*_8	-0.002471391 - 0.257027083 i

Table 4-3 Four-Bar Linkage Found Using the Starting Mechanism

Listed in Table 4-2

	Four-Bar Linkage Vector X_i	θ_j	ϕ_j	ψ_j
X_1	5.053231840 + 0.000000000 i	173.05	5.62	342.49
X_2	0.911854117 + 0.000000000 i	200.36	7.56	354.64
X_3	-0.264524071 + 0.000000000 i	222.54	9.02	6.32
X_4	0.776972270 + 0.000000000 i	232.62	9.67	12.08
X_5	0.973133191 + 0.000000000 i	249.72	10.78	22.52
X_6	-0.429958241 + 0.000000000 i	264.42	11.87	32.57
X_7	-0.271767001 + 0.000000000 i	276.55	14.89	51.31
X_8	0.393275674 + 0.000000000 i	252.40	21.38	76.22

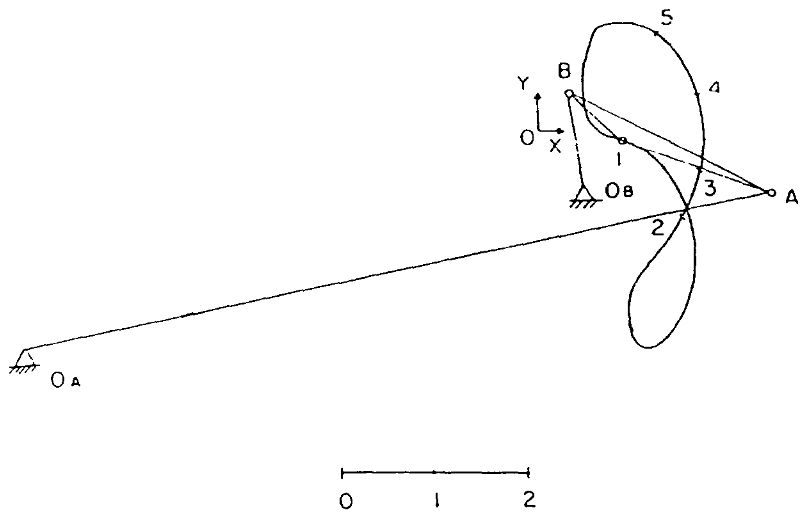

Figure 4-1 Starting Four-Bar Given in Table 4-2(a) and it's

Coupler Curve - Example 3

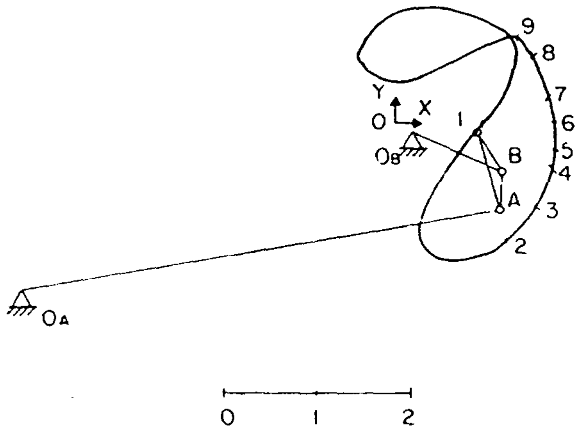

Figure 4-2 Four-Bar Found in Table 4-3 and it's

Coupler Curve - Example 3

Table 4-4(a) Starting Four-Bar - Example 3

	Starting Four-Bar Linkage X_i
X_1	3.238956530 + 0.000000000 i
X_2	0.313100908 + 0.000000000 i
X_3	-3.704433040 + 0.000000000 i
X_4	-3.173755471 + 0.000000000 i
X_5	-0.188274472 + 0.000000000 i
X_6	0.662781720 + 0.000000000 i
X_7	2.540201974 + 0.000000000 i
X_8	-0.225088167 + 0.000000000 i

Table 4-4(b) Perturbed Coupler Points - Example 3

Coupler Point	X - Coordinate	Y - Coordinate
M_1	0.896187609 + 0.000094884 i	-0.098029270 - 0.000010379 i
M_2	1.515143323 + 0.000032310 i	-0.854496262 - 0.000018222 i
M_3	1.515144385 + 0.000138522 i	-0.854496861 - 0.000078122 i
M_4	1.678315089 + 0.000031594 i	-0.476592260 - 0.000008972 i
M_5	1.713870687 + 0.000168730 i	-0.300992616 - 0.000029632 i
M_6	1.722618400 + 0.000040525 i	0.027344926 + 0.000000643 i
M_7	1.664204871 + 0.000197125 i	0.332411274 + 0.000039374 i
M_8	1.476371953 + 0.000016941 i	0.735537770 + 0.000008440 i
M_9	1.301184569 + 0.000116911 i	0.921538888 + 0.000082800 i

Table 4-4(c) Random Number B^*_j - Example 3

	Random Number B^*_j
B^*_1	0.004910767 + 0.486387086 i
B^*_2	-0.007116488 - 0.723861148 i
B^*_3	0.004775361 + 0.472042473 i
B^*_4	-0.004797831 - 0.488148437 i
B^*_5	0.002218529 + 0.218577888 i
B^*_6	-0.002338933 - 0.240090693 i
B^*_7	0.000180692 + 0.016537634 i
B^*_8	0.000145380 + 0.013994693 i

Table 4-5 Four-Bar Linkage Found Using the Starting Mechanism Listed in Table 4-4

	Four-Bar Linkage Vector X_i	θ_j	ϕ_j	ψ_j
X_1	2.269940196 + 0.000000000 i	5.62	342.49	173.05
X_2	-0.156914237 + 0.000000000 i	7.56	354.64	200.36
X_3	-4.446743976 + 0.000000000 i	9.02	6.32	222.54
X_4	-4.597693014 + 0.000000000 i	9.67	12.08	232.62
X_5	-0.264524071 + 0.000000000 i	10.78	22.52	249.72
X_6	0.776972270 + 0.000000000 i	11.87	32.57	264.43
X_7	5.053231840 + 0.000000000 i	14.89	51.31	276.55
X_8	0.911854117 + 0.000000000 i	21.38	76.22	252.40

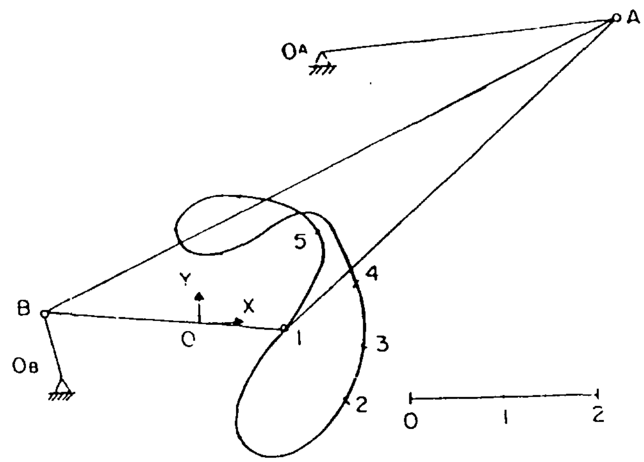

Figure 4-3 Starting Four-Bar Given in Table 4-4(a) and it's

Coupler Curve - Example 3

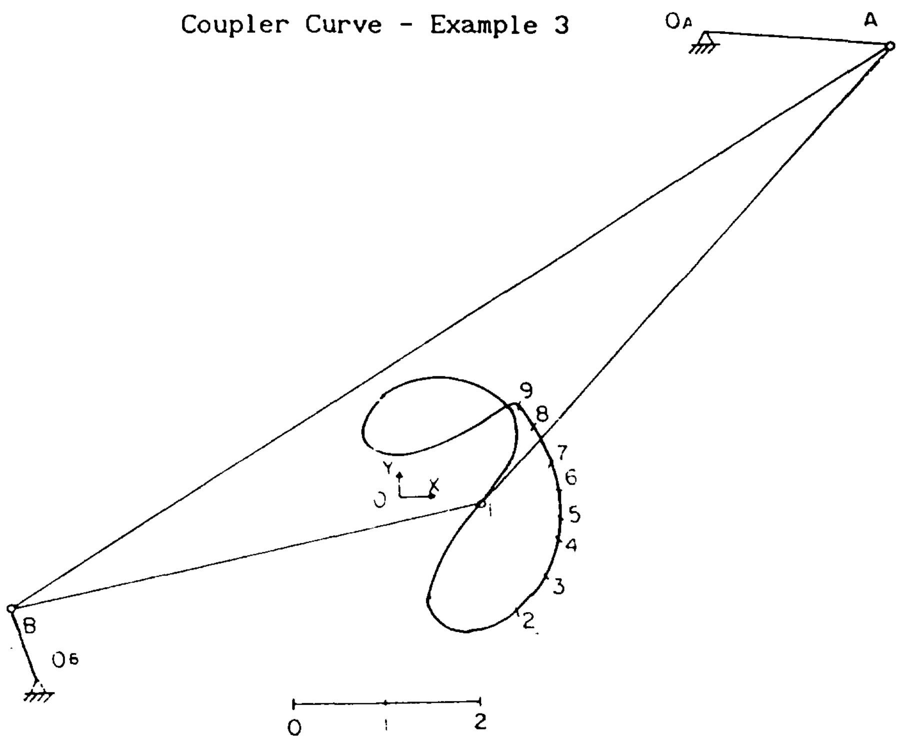

Figure 4-4 Four-Bar Found in Table 4-5 and it's

Coupler Curve - Example 3

Table 4-6(a) Starting Four-Bar - Example 3

	Starting Four-Bar Linkage X_i
X_1	2.540201974 + 0.000000000 i
X_2	-0.225088167 + 0.000000000 i
X_3	-0.188274472 + 0.000000000 i
X_4	0.662781720 + 0.000000000 i
X_5	0.539387260 + 0.000000000 i
X_6	-0.509025037 + 0.000000000 i
X_7	-0.142093297 + 0.000000000 i
X_8	0.966714693 + 0.000000000 i

Table 4-6(b) Perturbed Coupler Points - Example 3

Coupler Point	X - Coordinate	Y - Coordinate
M_1	0.896188558 + 0.000189768 i	-0.098029374 - 0.000020758 i
M_2	1.235854846 + 0.000052709 i	-1.197241812 - 0.000051062 i
M_3	1.515145770 + 0.000277044 i	-0.854497642 - 0.000156245 i
M_4	1.671085400 + 0.000062916 i	-0.502154668 - 0.000018906 i
M_5	1.713872375 + 0.000337461 i	-0.300992913 - 0.000059265 i
M_6	1.724348150 + 0.000081131 i	0.004104246 + 0.000000193 i
M_7	1.664206842 + 0.000394250 i	0.332411667 + 0.000078748 i
M_8	1.301183699 + 0.000029862 i	0.921538271 + 0.000021149 i
M_9	1.301185738 + 0.000233823 i	0.921539716 + 0.000165600 i

Table 4-6(c) Random Number B^*_j - Example 3

	Random Number B^*_j
B^*_1	-0.000002983 - 0.000299695 i
B^*_2	0.000001520 + 0.000144171 i
B^*_3	0.000006275 + 0.000586582 i
B^*_4	-0.000012540 - 0.001318078 i
B^*_5	0.000016326 + 0.001590598 i
B^*_6	-0.000044644 - 0.004743955 i
B^*_7	0.000019885 + 0.001919160 i
B^*_8	-0.000042034 - 0.004371573 i

Table 4-7 Complex Solution of Example 3 by Using Table 4-6 as Starting Points

	Solution X_i
X_1	-0.130274805 + 0.122718869 i
X_2	-0.131199982 - 0.016539961 i
X_3	0.080214750 - 0.260901692 i
X_4	-0.545170718 + 0.009173337 i
X_5	-0.437602670 + 0.350193498 i
X_6	0.207290033 + 0.235174620 i
X_7	0.619843501 - 0.051105023 i
X_8	-0.340171305 - 0.581427794 i

Example 4.

As another example, Table 4-8 lists another set of nine precision points. Again, the odd number points (i.e., M_1, M_3, M_5, M_7, and M_9) are taken from that of example 2. Thus, we can solve this problem by using the solutions found in example 2 as starting points. Table 4-9(a) to 4-9(c) list the starting mechanism, perturbed coupler points, and the random numbers generated from the Chebyshev 1 listed in Table 3-7. Table 4-10 lists the solutions found by the Cheater's Homotopy method. Figs. 4-5 and 4-6 show the coupler curves of the starting and resulting four-bar linkages, respectively. Again, the resultant four-bar is very different from the starting four-bar. The coupler-point curves of the two mechanisms are totally different, although both curves pass through the five specified positions. This demonstrates the power of Cheater's homotopy as a tool for solving a set of highly non-linear, singular polynomial system.

Table 4-8 Nine Precision Points Used for Example 4

Coupler Point Mi	X - Coordinate	Y - Coordinate
M1	1.000000000	0.000000000
M2	1.210153700	-1.193562100
M3	1.514419000	-0.856816990
M4	1.672618000	-0.490052250
M5	1.709746300	-0.323059980
M6	1.735739500	0.017302200
M7	1.711962400	0.311115900
M8	1.565230700	0.760035300
M9	1.394774300	0.973082000

Table 4-9(a) Starting Four-Bar - Example 4

	Starting Four-Bar Linkage X_i
X_1	-0.339727801 + 0.000000000 i
X_2	-2.386883834 + 0.000000000 i
X_3	1.808703160 + 0.000000000 i
X_4	0.627956529 + 0.000000000 i
X_5	-0.384089825 + 0.000000000 i
X_6	1.049794766 + 0.000000000 i
X_7	2.046524463 + 0.000000000 i
X_8	0.401000302 + 0.000000000 i

Table 4-9(b) Perturbed Coupler Points - Example 4

Coupler Point	X - Coordinate	Y - Coordinate
M_1	1.000001059 + 0.000105875 i	0.000000000 + 0.000000887 i
M_2	1.251135552 + 0.000026680 i	-0.851158265 + 0.000000179 i
M_3	1.514420385 + 0.000138456 i	-0.856817773 + 0.000000766 i
M_4	1.667901141 + 0.000031398 i	-0.495741050 + 0.000000158 i
M_5	1.709747983 + 0.000168325 i	-0.323060298 + 0.000000825 i
M_6	1.740915738 + 0.000040955 i	0.016629245 + 0.000000391 i
M_7	1.711964428 + 0.000202782 i	0.311116269 + 0.000036852 i
M_8	1.554278088 + 0.000017835 i	0.746558590 + 0.000008567 i
M_9	1.394775553 + 0.000125320 i	0.973082874 + 0.000087431 i

Table 4-9(c) Random Number B^*_j - Example 4

	Random Number B^*_j
B^*_1	-0.000142850 - 0.014148129 i
B^*_2	-0.000160540 - 0.006719557 i
B^*_3	0.000298245 + 0.030265201 i
B^*_4	-0.000328937 - 0.030386400 i
B^*_5	0.000397444 + 0.039260288 i
B^*_6	-0.000669947 - 0.068153203 i
B^*_7	0.000517895 + 0.051401249 i
B^*_8	-0.000490246 - 0.049235418 i

Table 4-10 Four-Bar Linkage Found Using the Starting Mechanism

Listed in Table 4-9

	Four-Bar Linkage Vector X_i	θ_j	ϕ_j	ψ_j
X_1	6.609200130 + 0.000000000 i	40.29	355.79	334.22
X_2	7.117464581 + 0.000000000 i	170.00	7.09	77.62
X_3	-0.792492794 + 0.000000000 i	190.85	7.88	77.56
X_4	0.745932329 + 0.000000000 i	199.25	8.16	76.80
X_5	0.826204995 + 0.000000000 i	215.72	8.58	74.09
X_6	0.779116826 + 0.000000000 i	318.98	359.78	344.90
X_7	-1.008253769 + 0.000000000 i	256.72	8.58	60.87
X_8	0.441683619 + 0.000000000 i	278.23	7.77	50.73

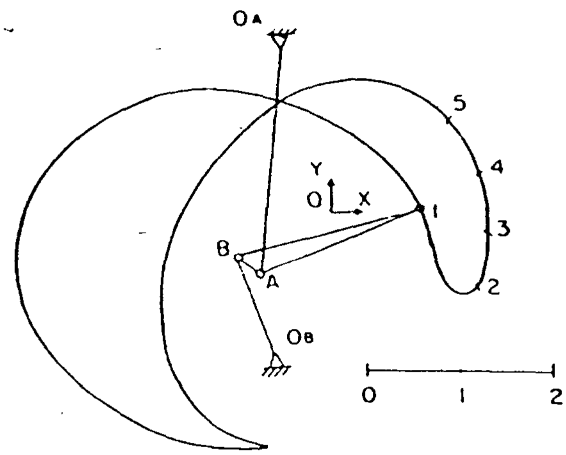

Figure 4-5 Starting Four-Bar Given in Table 4-9(a) and it's Coupler Curve - Example 4

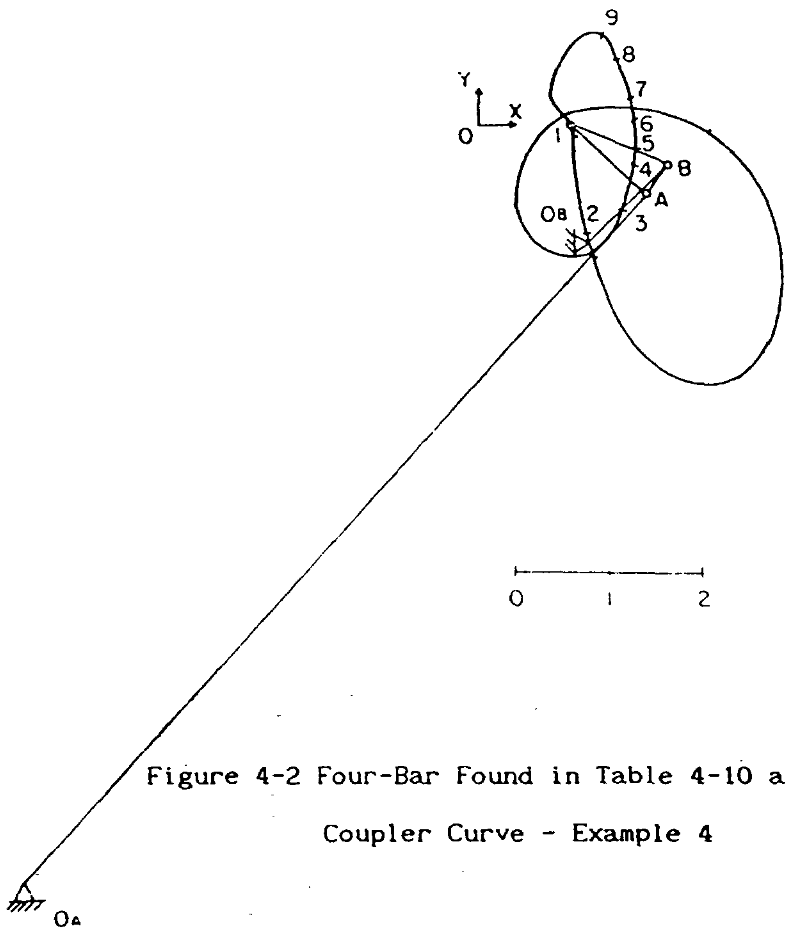

Figure 4-2 Four-Bar Found in Table 4-10 and it's Coupler Curve - Example 4

§ 4-4. Comparison with Newton-Raphson's Method and Powell's Method

There are other numerical methods that can be used to solve a system of non-linear equations. For this nine precision-points problem, we also tried both Newton-Raphson's and Powell's methods. Our experience is that both methods are difficult to yield any solutions (real or complex) at all, because of the high non-linearity of the polynomial system and because of the existence of singular solutions as discussed in Chapter 1. In general, Newton-Raphson's method will converge to the singular solutions $\underline{Z}_1 = \underline{Z}_3$ and $\underline{Z}_2 = \underline{Z}_4$, and Powell's method will converge to either $\underline{Z}_2 = \underline{Z}_4 = 0$, or $\underline{Z}_1 = \underline{Z}_3$ and $\underline{Z}_2 = \underline{Z}_4$.

For examples 3 and 4, we tried Newton-Raphson's Method and Powell's Method using the same starting points used in the Cheater's Homotopy. Table 4-11 and Table 4-12 list the numbers of solutions obtained by various different methods. For example 3, Cheater's Homotopy method found all the eighteen solutions (two reals and sixteen complex); Newton-Raphson's method found one real solution and seventeen singular; and Powell's method found one real solution, six singular and others did not converge. For example 4, Cheater's Homotopy found all eighteen solutions (one real and seventeen complex); Newton-Raphson's method found four complex and fourteen singular solutions; And Powell's method found seventeen singular solutions and one did not converge. The powerfulness of Cheater's homotopy in avoiding the singular condition and non-convergence has been demonstrated by this highly non-linear and highly singular polynomial system.

Table 4-11 Comparison of Homotopy Method, Newton-Raphson's Method, and Powell's Method - Example 3

Numbers of	Homotopy Method	Newton-Raphson's Method	Powell's Method
Real Solution	2	1	1
Complex Solution	16	0	0
Singular Solution	0	17	6

Table 4-12 Comparison of Homotopy Method, Newton-Raphson's Method, and Powell's method - Example 4

Numbers of	Homotopy Method	Newton-Raphson's Method	Powell's Method
Real Solution	1	0	0
Complex Solution	17	4	0
Singular Solution	0	14	17

CHAPTER 5. DISCUSSION AND CONCLUSION

Roth [20] presented a "Bootstrap" method for the nine-coupler-points synthesis problem. The method has some flavor of continuation. However, the author did not mention how to choose the starting mechanisms. In addition, he did not use the complex algorithm and the starting points did not include small random complex constants. For this reason, according to Homotopy criteria stated in [9,10,11], the Homotopy path is easy to fail due to the lack of smoothness and accessibility properties. Although Roth introduced the "position interchange" and "quality-index control" techniques to overcome the difficulty, there is still no guarantee for convergence.

Newton-Raphson's method tends to converge to the singular solutions. Powell's method either converges to singular solutions or does not converge at all.

The homotopy methods presented in this work always finds some solutions, although sometimes the solutions may be complex. The number of solutions found is always equal to the number of starting mechanisms. The General Homotopy method can be used to solve the five-coupler-points problem, while the Cheater's Homotopy method can be used to find some solutions to the nine-coupler-points synthesis problem. The powerfulness of the homotopy methods opens a new frontier for dimensional synthesis of mechanisms, and other problems involving a system of polynomial equations.

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