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Coupling Constant Dependence of the Kaluza-Klein Spectrum in Five-Dimensional SQCD on S^1

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We investigate the Kaluza-Klein (KK) spectrum of $\mathcal{N}=1$ supersymmetric gauge theory compactified on a circle. We concentrate on a model with the gauge group $SU(2)$ and four massless matter fields in the fundamental representation. We derive the exact mass formula of KK modes using Seiberg-Witten theory. From the mass formula and the D3-brane probe realization, we determine the spectrum of the KK modes of the matter fields and gauge fields. We find that the lightest KK state of the gauge fields is stable in the entire vacuum moduli space, while the lightest KK state of the matter fields decays more readily than other KK states in one region of the moduli space. This region becomes small as we decrease the five-dimensional gauge coupling constant g_5, and vanishes in the limit $g_5 \to 0$. This result continuously connects the known KK spectrum in the weak coupling limit and that in the strong coupling limit.

§1. Introduction

Recently, various four-dimensional models embedded in higher-dimensional spacetime have been thoroughly investigated.[1)] Their spectra generally include Kaluza-Klein (KK) states, which carry nonzero momentum in extra dimensions. As an example, let us consider a circle compactification of five-dimensional gauge theory with a massless matter field. The effective four-dimensional theory has a tower of KK modes of the gauge field, say $A_\mu^{(n)}$, and of the matter field, say $\psi^{(n)}$, where n runs over the integers and μ runs over $0, 1, 2, 3$. The states $A_\mu^{(n)}$ and $\psi^{(n)}$ possess fifth-dimensional momentum, n/R, where R is the compactification radius. Therefore $A_\mu^{(n)}$ and $\psi^{(n)}$ with nonzero n are KK states. In this paper, we investigate the stability of KK states.

A state A is kinematically unstable and decays into states B and C when all the charges are conserved in the decay process and their masses satisfy the inequality $M(A) \geq M(B) + M(C)$. In the case of the five-dimensional model mentioned above, the masses of $A_\mu^{(n)}$ and $\psi^{(n)}$ are classically $|n|/R$. Thus, for $n > 1$, $A_\mu^{(n)}$ can decay into $A_\mu^{(n-1)}$ and $A_\mu^{(1)}$, and $\psi^{(n)}$ can decay into $A_\mu^{(n-1)}$ and $\psi^{(1)}$. Similar decay processes occur for $n < 1$. From these considerations, we see that the stable KK states are $A_\mu^{(\pm 1)}$ and $\psi^{(\pm 1)}$. This result is supported by the perturbative analysis given in Ref. 2), where the above inequality among masses in various models is evaluated to one-loop order. Nonperturbative behavior of the KK spectrum was found in the strong coupling limit of a supersymmetric extension of the five-dimensional model.[3)] In Ref. 3), the KK spectrum of a circle compactification of the five-dimensional $\mathcal{N}=1$ supersymmetric model with the gauge group $SU(2)$ and $N_f = 5, 6, 7$ massless fundamental matter fields is studied in the strong coupling limit. The four-dimensional effective theory possesses $\mathcal{N}=2$ supersymmetry, and then the exact mass formula can be derived by using Seiberg-Witten theory.[4] In addition, the theory has D3-brane probe realization,[5] where the inequality is diagrammatically evaluated using string junctions.[6] Using these techniques, it was shown that $A_\mu^{(n)}$ can decay into $A_\mu^{(n-1)}$ and $A_\mu^{(1)}$, similarly to the perturbative result, while $\psi^{(n-1)}$ decays more readily than $\psi^{(n)}$ in a certain region of the vacuum moduli space.

Now we know that the perturbative KK spectrum is different from the spectrum in the strong coupling limit. How does the spectrum change as the five-dimensional coupling constant g_5 varies from 0 to ∞? To answer this question, we shall generalize the analysis in Ref. 3) to the case with finite g_5. In §2, we derive the exact mass formula for finite g_5, using Seiberg-Witten theory. From the mass formula and the D3-brane probe realization, we determine the stability of KK modes in §3. In this way, we show that the nonperturbative behavior also appears in the case of finite coupling constant. As we decrease g_5, the region in which the nonperturbative behavior appears shrinks, and in the limit $g_5 \to 0$, it vanishes, and the perturbative spectrum is reproduced.

§2. Seiberg-Witten solution

2.1. *Seiberg-Witten curve*

Five-dimensional $\mathcal{N}=1$ supersymmetric gauge theory compactified on a circle is effectively described by four-dimensional $\mathcal{N}=2$ supersymmetric gauge theory. It includes an adjoint complex Higgs scalar field ϕ as the superpartner of the gauge field, and it has a vacuum moduli space parametrized by the vacuum expectation value of ϕ. The low energy effective Lagrangian and the mass formula are derived from the Seiberg-Witten curve.[4]

For the theory with the gauge group $SU(2)$ and N_f matter fields in the fundamental representation, which we refer to as "quarks", the Seiberg-Witten curve is written as[7]

$$y^2 = x^3 + f(u)x + g(u), \tag{1}$$

$$f(u) = \sum_{i=0}^{4} a_i u^i, \qquad g(u) = \sum_{i=0}^{6} b_i u^i, \tag{2}$$

where $u = \langle \mathrm{Tr}\exp(2\pi R\phi)\rangle - 2$,[8] a gauge invariant moduli parameter, which takes values in CP^1. The constants a_i and b_i depend on the parameters appearing in the theory such as the five-dimensional coupling constant g_5, the compactification radius R, and the masses of the matter fields m_i $(i=1,\cdots,N_f)$.*) In this subsection, we

*) We should introduce a cutoff parameter to define five-dimensional gauge theories, because they are not renormalizable. Here we formally turn off the parameter by taking a low energy limit. Divergent terms could appear in this limit, but we discard them.

determine these constants. For simplicity, let us assume $N_f = 4$ and $m_i = 0$ for all i. In this case, the moduli space becomes simple, because the four-dimensional limit of the theory is conformal and the dynamically generated scale Λ does not appear.

The zero points of the discriminant of (1),

$$\Delta(u) = 4f(u)^3 + 27g(u)^2, \tag{3}$$

are determined from the global symmetry.[9)] The correspondence between the global symmetry and the zeros of the $\Delta(u)$ is presented in Refs. 10) and 11). Because our model possesses the flavor symmetry $SO(8)$, we find that a_0 and b_0 should vanish and $\Delta(u)$ is proportional to u^6.[10)] In addition, the symmetry is extended to broken affine $SO(8)$.[12)] This is because the states possess not only $SO(8)$ charges but also the KK charge n. In other words, an $SO(8)$ multiplet has copies labeled by integers n. This is precisely the structure of affine $SO(8)$ multiplets. The affine symmetry is broken at the scale $1/R$, because the copies have different masses. The broken affine symmetry requires two additional zero points.[11)] From these restrictions, we conclude that the curve of our model is that defined by

$$\begin{aligned} y^2 = x^3 &+ \left\{ a_2 + u\left(a_3 - \frac{3u}{L^4}\right)\right\} u^2 x \\ &+ \frac{L^6}{216}\left(a_3 - \frac{6u}{L^4}\right)\left\{a_3^2 + \frac{24a_3u}{L^4} + \frac{36}{L^4}\left(a_2 - \frac{2u^2}{L^4}\right)\right\}u^3. \end{aligned} \tag{4}$$

To simplify this curve, we scale and shift the variables as

$$y \to \frac{1}{24\sqrt{3}}y, \qquad x \to \frac{1}{12}\left\{x - \frac{L^2}{3}\left(a_3 - \frac{u}{L^4}\right)u\right\}, \qquad u \to \frac{1}{6}u \tag{5}$$

and set $a_3 = b/L^2$ and $a_2 = (3c^2 - b^2)/12$. In this way we obtain the simple form

$$y^2 = x^3 + \left(\frac{u}{L^2} - b\right)ux^2 + c^2u^2x. \tag{6}$$

Then the discriminant is given by

$$\Delta(u) = c^4u^6\left\{u - (b+2c)L^2\right\}\left\{u - (b-2c)L^2\right\}. \tag{7}$$

Next we consider the constants b, c and L. They are functions of g_5 and R. In the following, we derive explicit forms of the functions by matching the curve (6) with these existing in two limits. We choose R/g_5^2 and R as independent variables and complexify R/g_5^2 to $\tau = 4\pi i R/g_5^2 + \theta/2\pi$, the bare coupling constant of the four-dimensional effective theory. The parameter θ comes from a background Abelian gauge field.[13)]

Firstly, we consider the limit $R \to 0$ with fixed τ. In this limit, the theory reduces to the four-dimensional $\mathcal{N} = 2$ $SU(2)$ gauge theory with $N_f = 4$ massless quarks and the coupling constant τ. Its Seiberg-Witten curve is given by[4)]

$$y^2 = x^3 - \frac{1}{4}g_2(\tau)u^2x - \frac{1}{4}g_3(\tau)u^3, \tag{8}$$

where $g_2(\tau)$ and $g_4(\tau)$ are the following Eisenstein series:

$$g_2(\tau) = \frac{60}{\pi^4} \sum_{(m,n)\in Z^2_{\neq 0}} \frac{1}{(m+n\tau)^4}, \qquad g_4(\tau) = \frac{140}{\pi^6} \sum_{(m,n)\in Z^2_{\neq 0}} \frac{1}{(m+n\tau)^6}. \tag{9}$$

The mass dimensions of y, x and u in (8) are 3, 2 and 2, respectively. Setting the same mass dimensions in (6), we find that the mass dimensions of b and c are 0 and that of L is 1. Hence b and c depend only on τ, and therefore L can be written as $1/R$ times a function of τ, say $f(\tau)$. The function $f(\tau)$ is removed from the curve by scaling parameters as $u \to f(\tau)u$, $b \to b/f(\tau)$ and $c \to c/f(\tau)$. Altogether, rearranging the curve (6) yields the following equation:

$$y^2 = x^3 + \left(R^2 u - b\right) ux^2 + c^2 u^2 x. \tag{10}$$

Now we take the limit $R \to 0$ in (10) and compare with (8). Shifting x as $x \to x - bu/3$ in (8), we find

$$b = -3e_i(\tau), \tag{11}$$

$$c^2 = 3e_i^2(\tau) + e_1(\tau)e_2(\tau) + e_2(\tau)e_3(\tau) + e_3(\tau)e_1(\tau). \tag{12}$$

Here $e_i(\tau)$ $(i=1,2,3)$ are the solutions of the equation $x^3 - g_2(\tau)x/4 - g_3(\tau)/4 = 0$. They are written as

$$\begin{aligned} e_1(\tau) &= \frac{1}{3}(-\theta_1^4(\tau) + 2\theta_3^4(\tau)), \\ e_2(\tau) &= \frac{1}{3}(-\theta_1^4(\tau) - \theta_3^4(\tau)), \\ e_3(\tau) &= \frac{1}{3}(2\theta_1^4(\tau) - \theta_3^4(\tau)), \end{aligned} \tag{13}$$

where θ_1 and θ_3 are

$$\theta_1(\tau) = \sum_{n\in Z} e^{i\pi\tau(n+1/2)^2}, \qquad \theta_3(\tau) = \sum_{n\in Z} e^{i\pi\tau n^2}. \tag{14}$$

Secondly, we consider the limit $g_5 \to \infty$ with $\theta = 0$ and fixed R. In this limit, the flavor symmetry $SO(8)$ is enhanced to E_5,[13] and the curve should be[7),12]

$$y^2 = x^3 + (R^2 u - 4)ux^2 + 4u^2 x. \tag{15}$$

Now, we take the limit of the curve (10). From (13), (14), and the relations $e_1(\tau) = e_2(-1/\tau)/\tau^2$, $e_2(\tau) = e_1(-1/\tau)/\tau^2$ and $e_3(\tau) = e_3(-1/\tau)/\tau^2$, we see that $e_1(\tau) \sim -1/3\tau^2$, $e_2(\tau) \sim 2/3\tau^2$ and $e_3(\tau) \sim -1/3\tau^2$ in this limit. Thus $b \sim 1/\tau^2$ and $c^2 \sim 0$ when i in (11) and (12) is 1 or 3, while $b \sim 2/\tau^2$ and $c^2 \sim 1/\tau^4$ for $i=2$. In the former case, we cannot make (10) coincide with (15). In the latter case, we can make it coincide by scaling the parameters as $x \to x/4\tau^4$, $y \to y/8\tau^6$ and $u \to u/2\tau^2$. Then we choose $i=2$ in (11) and (12). Thus we have

$$b = \theta_1^4(\tau) + \theta_3^4(\tau), \tag{16}$$

$$c = \theta_1^2(\tau)\theta_3^2(\tau). \tag{17}$$

In summary, the Seiberg-Witten curve of our model is defined by (10), where b and c are given by (16) and (17). Its discriminant is (7) with $L=1/R$. The zero points of $\Delta(u)$ are located at $u=0$ and $(b\pm 2c)/R^2$. It is known that extra massless states appear at each zero point.[4)] For simplicity, we set $R=1$ in the following.

2.2. *Mass formula*

From the Seiberg-Witten curve (10), we derive the mass formula of stable states called BPS states. For this purpose, we derive the periods of the curve,

$$\Pi(u) = \begin{pmatrix} \omega_D(u) \\ \omega(u) \end{pmatrix} = \begin{pmatrix} \oint_\beta \frac{dx}{y} \\ \oint_\alpha \frac{dx}{y} \end{pmatrix}, \tag{18}$$

where α and β are the homology cycles on the torus given by (10) with fixed u.

The periods $\Pi(u)$ are determined from the Picard-Fuchs equation,

$$\left\{ \frac{d^2}{du^2} + \frac{3u^2 - 4bu + b^2 - 4c^2}{u(u-b-2c)(u-b+2c)} \frac{d}{du} + \frac{4u^2 - 2bu - b^2 + 4c^2}{4u^2(u-b-2c)(u-b+2c)} \right\} \Pi(u) = 0. \tag{19}$$

Setting $\Pi(u) = u^{-1/2} k(w)$ with $w = -\{u - (b+2c)\}/4c$, this equation becomes

$$\frac{d^2k}{dw^2} + \frac{1-2w}{w(1-w)} \frac{dk}{dw} - \frac{1/4}{w(1-w)} k = 0. \tag{20}$$

This is the standard hypergeometric equation with $\alpha = \beta = 1/2$ and $\gamma = 1$. It has two independent solutions,

$$K(w) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \left\{ \Gamma(-s)\Gamma\left(\frac{1}{2}+s\right) \right\}^2 (1-w)^s, \tag{21}$$

$$K'(w) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i} \left\{ \Gamma(-s)\Gamma\left(\frac{1}{2}+s\right) \right\}^2 w^s. \tag{22}$$

Series expansions of $K(w)$ and $K'(w)$ for $|w| < 1$, $|1-w| < 1$ and $|1/w| < 1$ are easily derived. For instance, the expansions for $|w|<1$ are

$$K(w) = \pi \sum_{n=0}^{\infty} \left\{ \frac{\Gamma(\frac{1}{2}+n)}{n!} \right\}^2 w^n, \tag{23}$$

$$K'(w) = -\sum_{n=0}^{\infty} \left\{ \frac{\Gamma(\frac{1}{2}+n)}{n!} \right\}^2 w^n \left\{ \log w + 4\sum_{r=0}^{n-1} \left(\frac{1}{2r+1} - \frac{1}{2r+2} \right) - 2\log 4 \right\}. \tag{24}$$

The periods are given by linear combinations of the functions

$$\Pi(u) = u^{-\frac{1}{2}} \begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} \begin{pmatrix} K(w) \\ K'(w) \end{pmatrix}. \tag{25}$$

The coefficients $c_1, \cdots, c_4$ are determined by direct calculation of the elliptic integrals (18) for $|w| < 1$ and comparing the result with the expansions (23) and (24). We obtain

$$\begin{pmatrix} c_1 & c_2 \\ c_3 & c_4 \end{pmatrix} = \frac{1}{\sqrt{c}\,\pi^2} \begin{pmatrix} 2\pi & 0 \\ -\pi - 2i\log 4 & -i\pi \end{pmatrix}. \tag{26}$$

The periods undergo monodromy around the zeros of $\Delta(u)$. The monodromy matrices acting on $\Pi(u)$ around $u=0$, $b-2c$ and $b+2c$ are

$$M_0 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad M_- = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}, \qquad M_+ = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \tag{27}$$

respectively. From these matrices, we can determine the kinds of states that become massless at the zeros; when the monodromy matrix around a first-order zero is

$$M_{(p,q)} = \begin{pmatrix} 1-pq & p^2 \\ -q^2 & 1+pq \end{pmatrix}, \tag{28}$$

a state (p,q) whose electric and magnetic charges of unbroken $U(1)$ gauge symmetry are p and q, respectively, becomes massless. Thus we see that $(2,-1)$ becomes massless at $u=b-2c$ and $(0,1)$ at $u=b+2c$. Moreover, six states become massless simultaneously at $u=0$, the sixth order zero. Because M_0 is expressed as $M_{(1,0)}^4 M_{(2,-1)} M_{(0,1)}$, the massless states are $(2,-1)$, $(0,1)$ and four $(1,0)$.

Now we can write down the mass formula. It is described by the integration of the periods over u.[4)] Each bound of the integrals is chosen so as to reproduce the extra massless states at the zero points correctly. Then the mass formula of a state with electric and magnetic charges (p,q) and the charges of broken affine symmetry related to the singularities at $u=b\pm 2c$, say n_1 and n_2, is given by

$$M_{(p,q,n_1,n_2)}(u) = \left| Z_{(p,q,n_1,n_2)}(u) \right|, \tag{29}$$

$$Z_{(p,q,n_1,n_2)}(u) = pa(u) - qa_D(u) + n_1 s_1 + n_2 s_2, \tag{30}$$

where

$$a_D(u) = \int_0^u \omega_D(u')du', \tag{31}$$

$$a(u) = \int_0^u \omega(u')du', \tag{32}$$

$$s_1 = \int_0^{b+2c} \omega_D(u)du = 8\pi \arcsin\sqrt{\frac{b+2c}{4c}}, \tag{33}$$

$$s_2 = -\int_0^{b-2c} \{2\omega(u) + \omega_D(u)\}\, du = 8\pi \arcsin\sqrt{\frac{b-2c}{4c}}. \tag{34}$$

In the following, we report the results of evaluations of the values of a and a_D using numerical integration with *Mathematica*. For simplicity, we assume $\theta = 0$. Then b and c are real, $c > 0$ and $b > 2c$. Hence the zeros of $\Delta(u)$, $u = 0$, $b-2c$ and $b+2c$, are aligned from the left on the real axis of the u-plane. We set the branch cuts of a_D and a from these zeros to ∞ along the real axis as depicted in Fig. 1.

When we cross each cut from the lower half u-plane, ${}^t(a_D(u), a(u), s_1, s_2)$ is changed by the matrices and we have

$$\widetilde{M}_0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{M}_- = \begin{pmatrix} -1 & -4 & -2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{M}_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{35}$$

for the cut from $u=0$, $b-2c$ and $b+2c$, respectively. To conserve (30), we should change the charges ${}^t(p, q, n_1, n_2)$ simultaneously. The matrices acting on ${}^t(p, q, n_1, n_2)$ are given by

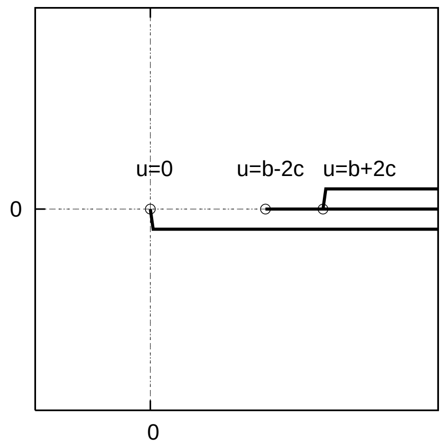

Fig. 1. Three brunch cuts.

$$K_0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad K_- = \begin{pmatrix} -1 & -4 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ -1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad K_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}. \tag{36}$$

§3. Stability of KK modes

Four-dimensional theory described by Seiberg-Witten curve (1) appears in type IIB string theory as the world volume theory of a D3-brane probe in a 7-brane background.[5] In our model, we employ the background constructed from six 7-branes $[1,0]^4[2,-1][1,0]$ at $z=0$, a 7-brane $[2,-1]$ at $z=b-2c$, and a 7-brane $[0,1]$ at $z=b+2c$. Here z is a complex coordinate of the space transverse to the 7-branes, $[1,0]$ denotes a D7-brane, and $[p,q]$ denotes an $SL(2,Z)$-dual 7-brane.[14] The metric on the z-plane is given by[15]

$$ds^2 = \mathrm{Im}\,\widetilde{\tau}(z) \left| \frac{da(z)}{dz} dz \right|^2, \tag{37}$$

where $\widetilde{\tau}(z) = da_D(z)/da(z)$. In this background, the world volume theory of a D3-brane probe located at $z=u$ is our model with the moduli parameter u. States of the model correspond to strings ending on the D3-brane. Therefore, the spectrum of states corresponds to the spectrum of strings which can end on the D3-brane probe. Then to find the spectrum of KK modes, we study the spectrum of corresponding strings.

In IIB string theory, there appear a fundamental string, $(1,0)$, and its $SL(2,Z)$-dual strings, (p,q). The ends of a string (p,q) are located on a D3-brane or a 7-brane with the same charges, $[p,q]$. In addition, strings can merge with each other and form a string junction.[14] Thus there are three kinds of strings ending on the D3-brane probe: a string connecting a 7-brane and the D3-brane, a string with both ends on the D3-brane, and a string emanating from a string junction and ending on the D3-brane. Some strings are related to string junctions transitionally, due to the string creation at 7-branes.[16] In any case, a string (p, q) ending on the D3-brane is detected as a state with electric charge p and magnetic charge q. In particular, a string $(1, 0)$ connecting $[1, 0]$ at $z = 0$ and the D3-brane corresponds to a quark, and a string $(1, 0)$ with both ends on the D3-brane corresponds to a gauge field. The winding number of a string around the 7-branes is equivalent to the KK charge n.[17]

To be stable, a string stretches along a geodesic that minimizes the string mass. The mass of a string (p, q) along a curve C is given by

$$\int_C T_{(p,q)} ds = \int \left| dZ_{(p,q,0,0)} \right| , \tag{38}$$

where $T_{(p,q)} = |p - q\tilde{\tau}| / \sqrt{\operatorname{Im} \tilde{\tau}}$, the tension of (p, q).[14] In order to minimize this quantity, points on a geodesic emanating from $z = z_0$ must satisfy the equation

$$\operatorname{Arg} \left\{ Z_{(p,q,0,0)}(z) - Z_{(p,q,0,0)}(z_0) \right\} = \phi. \tag{39}$$

Here ϕ is a constant between 0 and 2π. It takes the same value for geodesics of the strings in a stable junction.[6] Therefore, the mass of a string junction constructed from $n_1(2, -1)$ from $z = b - 2c$, $n_2(0, 1)$ from $z = b + 2c$, $(p - 2n_1, q + n_1 - n_2)$ from $z = 0$ and an outgoing string (p, q) coincides with (29) when the outgoing string ends on the D3-brane at $z = u$. Thus the junction corresponds to a state with the charges (p, q, n_1, n_2). Note that the geodesic equation of the outgoing string (p, q) is rewritten as

$$\operatorname{Arg}\{ Z_{(p,q,n_1,n_2)}(z) \} = \phi. \tag{40}$$

Hereafter, we refer to the string satisfying (40) as (p, q, n_1, n_2). When a string (p, q, n_1, n_2) crosses branch cuts, it undergoes monodromy described by the matrices (36).

We note that ϕ parameterizes the direction of geodesics. As ϕ varies, the geodesic (39) moves around the point $z = z_0$ and sweeps some region in the z-plane. When this region includes a point $z = u$, the string can end on the D3-brane probe at $z = u$ and is detected as a stable state. Therefore, the region in the z-plane through which the geodesic of a string passes corresponds to the region in the moduli u-plane where the corresponding state is stable.[6] With this in mind, in the following, we seek the regions of KK modes of quarks $\psi^{(n)}$, unbroken $U(1)$ gauge fields $A_\mu^{(n)}$ and W-bosons $W_\mu^{(n)}$ by evaluating (39) and (40) for corresponding strings. In the following, we assume $b - 2c = 4$ and $b + 2c = 6$, unless explicitly stated otherwise.

3.1. *Quarks*

We now consider the cases of quarks. First, we consider $\psi^{(0)}$. It corresponds to a string $(1, 0)$ emanating from $z = 0$, that is, $(1, 0, 0, 0)$. The geodesics for various ϕ are evaluated as depicted in Fig. 2. From that figure, we see that the string sweeps

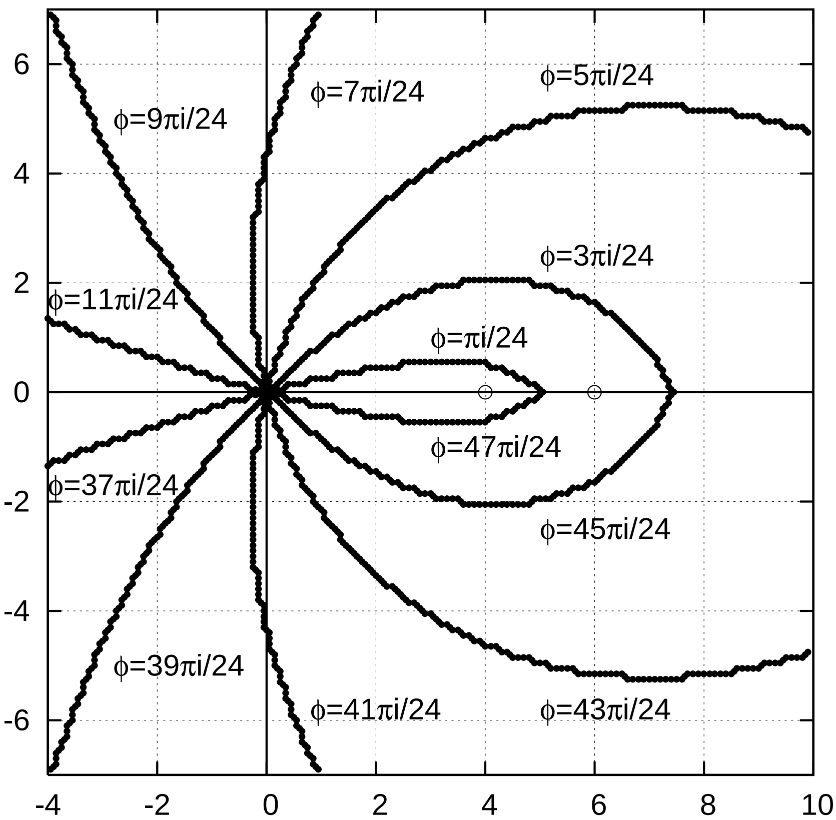

Fig. 2. The string that corresponds to a quark KK zero mode $\psi^{(0)}$ sweeps out the entire of the z-plane. This result implies that $\psi^{(0)}$ is stable for any value of the moduli parameter u.

the entire z-plane. Thus, we can conclude that $\psi^{(0)}$ is stable in the entire moduli u-plane.

Secondly, we consider $\psi^{(1)}$. It corresponds to a string $(1, 0, 0, 0)$ going around the 7-branes once in the counterclockwise direction. The string crosses the branch cuts in the region satisfying $\mathrm{Re}\, z > b + 2c$. Then the charges are changed to $K_+K_-K_0{}^t(1, 0, 0, 0) = {}^t(1, 0, 1, 1)$, as depicted in Fig. 3(A). Thus, $\psi^{(1)}$ corresponds to a string $(1, 0, 1, 1)$. The string sweeps a region in the first quadrant of the z-plane as we increase ϕ. Moreover, as depicted in Figs. 3(B)–(C), the string hits the 7-brane $[0, 1]$ at $z = b + 2c$ and becomes a string junction. The detailed configuration near the merging point of the strings is shown in Fig. 4. The junction is constructed from three strings: $(1, -1, 1, 0)$, which comes from $(1, 0, 0, 0)$ crossing the cuts between $z = 0$ and $z = b - 2c$, $(0, 1, 0, 1)$ emanating from $[0, 1]$ at $z = b + 2c$, and the outgoing string $(1, 0, 1, 1)$. As we increase ϕ further, the merging point draws a curve C_1, as shown in Fig. 4, and $(1, 0, 1, 1)$ sweeps outside the region S_1 surrounded by C_1 and the real axis. Thus, $\psi^{(1)}$ is stable outside S_1 in the moduli u-plane and disappears inside it. In addition, when the D3-brane is located on C_1, $(1, 0, 1, 1)$ can decay into

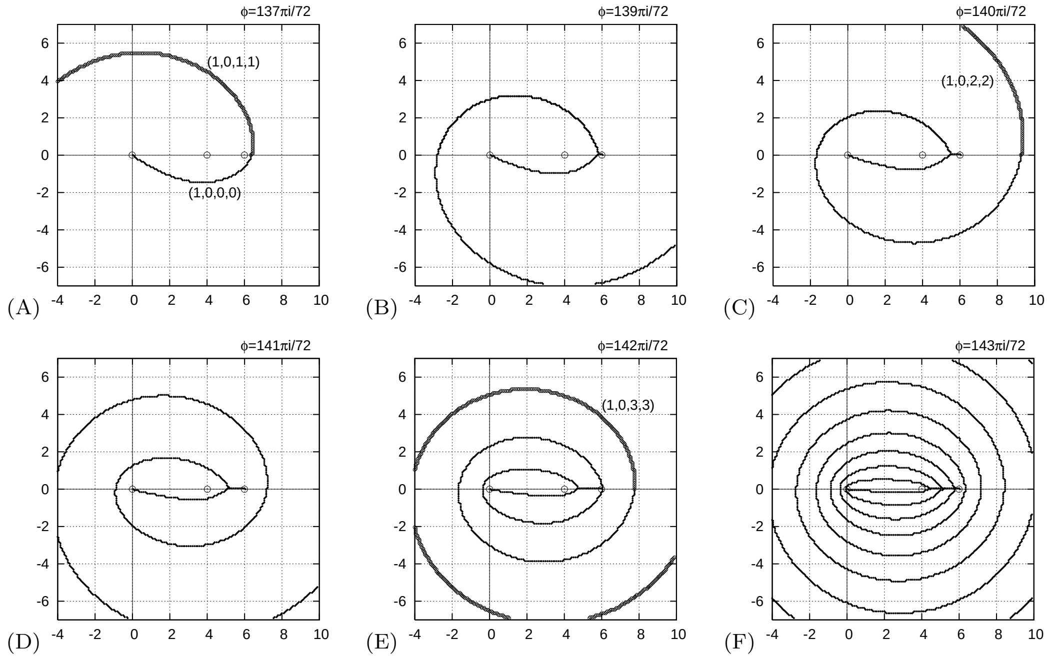

Fig. 3. The string geodesics corresponding to $\psi^{(n)}$. The string $(1,0,n,n)$ corresponds to the quark $\psi^{(n)}$. Here, we show the strings up to $n=8$. In Figs. 3(A)–(F), some string junctions are constructed. (For details, see Fig. 4.)

$(0,1,0,1)$ and $(1,-1,1,0)$. Thus $\psi^{(1)}$ becomes marginally stable when the moduli parameter u is on C_1.

Thirdly, we consider $\psi^{(2)}$. It corresponds to $(1,0,2,2)$, which comes from a string $(1,0,1,1)$ going around the 7-branes once more (see Figs. 3(C)–(D)). As depicted in Fig. 3(E), the string hits the 7-brane at $z=b+2c$ again and becomes a complicated string junction. As we increase ϕ, the merging point of the strings draws a curve C_2, and $(1,0,2,2)$ sweeps outside the region S_2 surrounded by C_2 and the real axis. Thus, $\psi^{(2)}$ is stable outside the region and disappears inside it. Note that the merging point is apparently below C_1. Consequently, S_2 is inside S_1, as shown in Fig. 5.

On the basis of these observations, we next consider $\psi^{(n)}$ $(n \geq 3)$. In general, $\psi^{(n)}$ corresponds to a string $(1,0,n,n)$, which appears as $(1,0,n-1,n-1)$ crosses the branch cuts in the region satisfying $\mathrm{Re}\, z > b+2c$. The string hits $[0,1]$ at $z=b+2c$, and an additional string $(0,1,0,1)$ is created. The merging point of the strings draws a curve C_n, which connects $z=b\pm 2c$ in the upper half u-plane, as we increase ϕ. Simultaneously, $(1,0,n,n)$ sweeps outside the region S_n surrounded by C_n and the real axis. Thus, $\psi^{(n)}$ appears outside S_n and disappears inside it. The region S_n is inside S_{n-1}, as shown in Fig. 5. Therefore, the quarks $\psi^{(n)}$ disappear in increasing order of n as we change the moduli parameter from a value above C_1 to the segment $[b-2c, b+2c]$. Similar analysis can be done for $\psi^{(n)}$ with negative n. It corresponds to a string $(1,0,0,0)$ going around the 7-branes $|n|$ times in the clockwise direction. We can derive the curve of marginal stability for $\psi^{(n)}$ and obtain the mirror image of

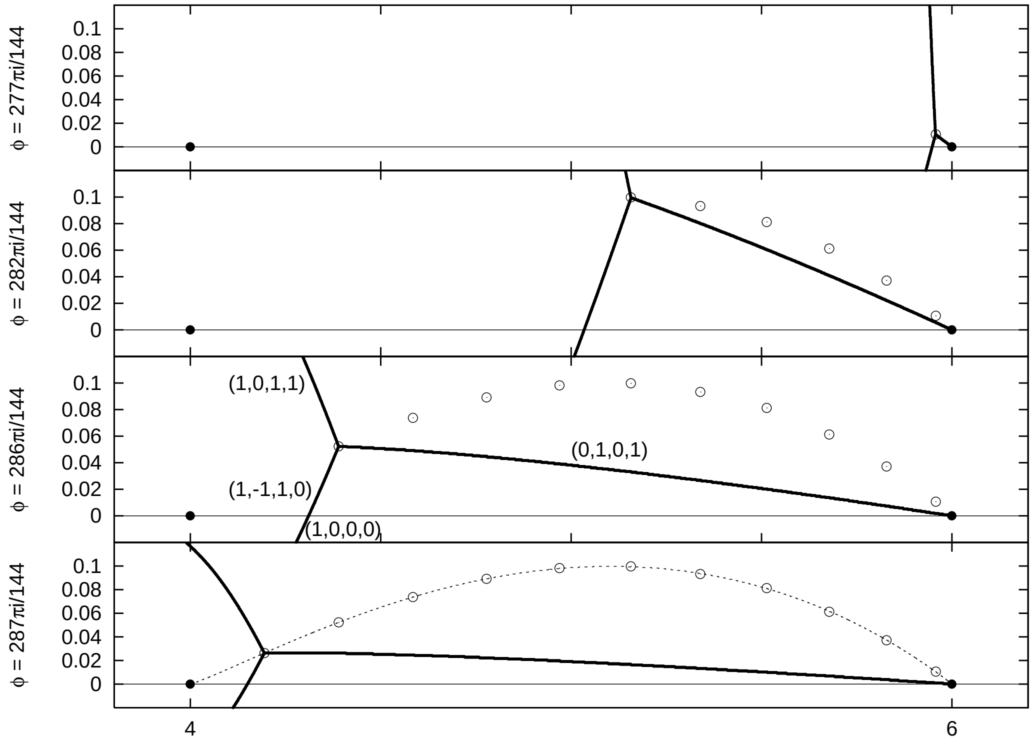

Fig. 4. The enlarged illustrations of a string junction with the outgoing string $(1,0,1,1)$. We illcustrate the point where the strings merge by a blob $\circ$. Connecting these blobs, we get a curve stretched from the 7-brane at $z=b+2c\,(=6)$ to the one at $z=b-2c\,(=4)$. We call this curve C_1, the curve of marginal stability for $\psi^{(1)}$.

that for $\psi^{(-n)}$ with respect to the real axis in the u-plane. These results are similar to the quark KK spectrum in the strong coupling limit, $g_5 \to \infty$.[3)]

Now we determine the g_5 dependence of the quark KK spectrum. From (16) and (17), we see that the positions of the singularities, $z=b\pm 2c$, depend on g_5, as shown in Fig. 6. The distance between the singularities becomes small as we decrease g_5. Then the curves of marginal stability, which connect the two singularities, also become small. As an example, we plot the curve of $\psi^{(1)}$ for $g_5=3,4,5,6$ in Fig. 7. This curve shrinks to a point in the limit $g_5 \to 0$, where the two singularities at $z=b\pm 2c$ collide. Therefore, the nonperturbative jumps of the quark KK spectrum disappear in this limit, as suggested by the perturbative analysis. Contrastingly, in the limit $g_5 \to \infty$, $z=b-2c$ coincides with $z=0$. Therefore the quark KK spectrum in the strong coupling limit is reproduced.

3.2. *W-bosons*

Next we consider $W_\mu^{(0)}$. It corresponds to the string $(1,0)$ emanating from the D3-brane going around the 7-branes at $z=0$, and coming back to the D3-brane. The string crosses the branch cut from $z=0$ and becomes a string $(-1,0)$ because of the monodromy. Both the geodesic of $(1,0)$ and that of $(-1,0)$ are the same as that of

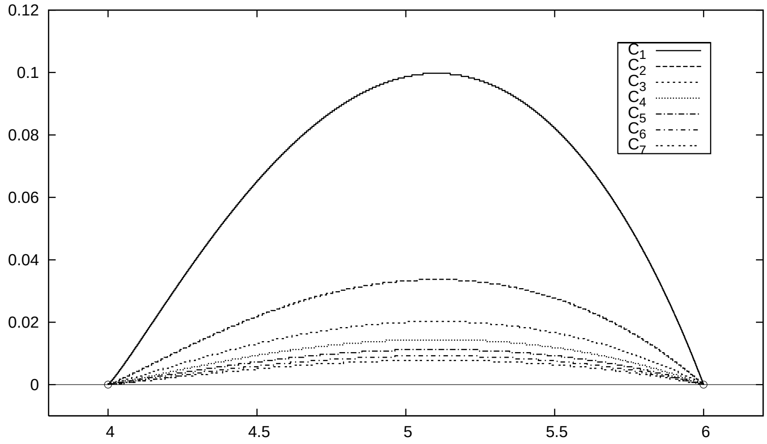

Fig. 5. The marginal stability curves for $\psi^{(n)}$ $(1 \leq n \leq 7)$. Here we also set $z \pm 2c = 6, 4$.

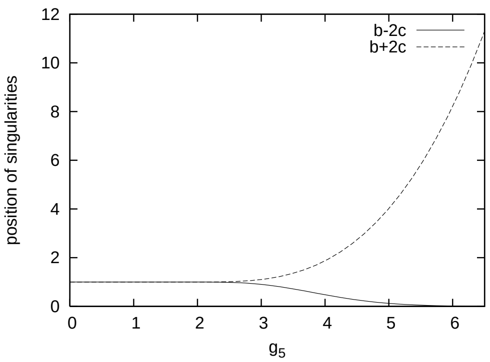

Fig. 6. The positions of the two singularities at $z = b \pm 2c$ depend on g_5, and they collide in the limit $g_5 \to 0$. The singularity $z = b - 2c$ collides with that at $z = 0$ in the strong coupling limit, $g_5 \to \infty$.

$(1, 0, 0, 0)$, the string corresponding to $\psi^{(0)}$. Therefore, the region where the geodesic for $W_\mu^{(0)}$ sweeps is the same as that for $\psi^{(0)}$. Hence the stability of $W_\mu^{(0)}$ is the same as that of $\psi^{(0)}$; it is stable for the entire moduli u-plane. In addition, the KK state $W_\mu^{(n)}$ corresponds to the string corresponding to $W_\mu^{(0)}$ which goes around the 7-branes n times in the counterclockwise direction. The string has the same geodesic for $\psi^{(n)}$. Thus we conclude that $W_\mu^{(n)}$ is stable outside S_n and disappears inside it.

3.3. *Unbroken $U(1)$ gauge bosons*

Before concluding this paper, we consider unbroken $U(1)$ gauge bosons. We start with $A_\mu^{(0)}$. It corresponds to the string $(1, 0)$ localized on the D3-brane probe at $z = u$. Because this string appears for any value of u, $A_\mu^{(0)}$ is stable over the entire u-plane.

Next we consider $A_\mu^{(1)}$. It corresponds to the string $(1, 0)$ emanating from the D3-brane probe, going around all the 7-branes once in the counterclockwise direction, and coming back to the D3-brane. The geodesic equation is given by

$$\mathrm{Arg}\{a(z) - a(x)\} = 0, \tag{41}$$

where x is a point through which the string passes. The geodesics for various x are loops, as shown in Fig. 8. As we decrease x from ∞ to $b + 2c$, the loop becomes small. When $x = b + 2c$, the loop hits the 7-brane at $z = b + 2c$ and becomes a

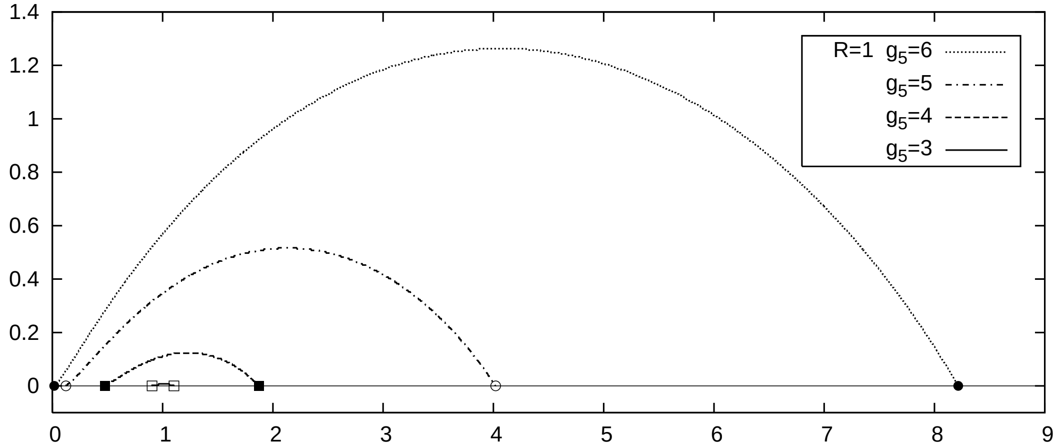

Fig. 7. The marginal stability curve C_1 of $\psi^{(1)}$ for $g_5 = 3, 4, 5, 6$. The weaker g_5 becomes, the shorter the curve C_1 becomes.

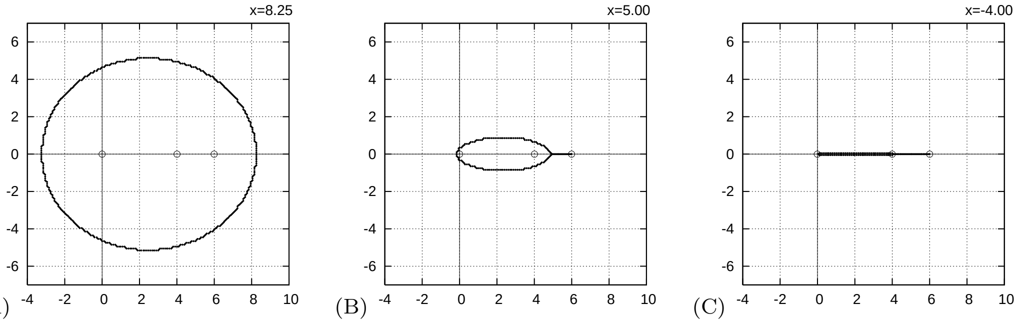

Fig. 8. The string geodesics corresponding to the lightest gauge KK state $A_\mu^{(1)}$.

string junction, as depicted in Fig. 8(B). As we decrease x from $b + 2c$ to $b - 2c$, the loop part of the junction collapses to a straight line, as depicted in Fig. 8(C). In any case, when the D3-brane is located on the loop, the junction cannot decay, and is observed as a stable $A_\mu^{(1)}$. Because the loop sweeps the entire z-plane, except the segment between $z=0$ and $z=b+2c$, $A_\mu^{(1)}$ is stable in the entire u-plane except on that segment. If the D3-brane is located on this segment, the junction can decay into two parts. Thus this segment is the marginal stability curve of $A_\mu^{(1)}$.

Finally, we consider $A_\mu^{(n)}$ $(n>1)$. This corresponds to a string $(1, 0)$ going around the 7-branes n times. This string corresponds to n loop strings or loop string junctions derived for $A_\mu^{(1)}$. Thus, $A_\mu^{(n)}$ can decay into n $A_\mu^{(1)}$ for any value of u. Similarly, it can be shown that $A_\mu^{(-1)}$ is stable in the entire u-plane, except on the segment between $z=0$ and $z=b+2c$, and $A_\mu^{(-n)}$ $(n > 1)$ can decay into n $A_\mu^{(-1)}$ for any value of u.

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