

NASA Contractor Report 181606

ICASE REPORT NO. 87-70

ICASE

**COUPLING FINITE ELEMENT AND
SPECTRAL METHODS: FIRST RESULTS**

**(NASA-CR-181606) COUPLING FINITE ELEMENT
AND SPECTRAL METHODS: FIRST RESULTS Final
Report (NASA) 46 p CSCL 12A**

N88-17379

**G3/64 Unclas
0124772**

**Christine Bernardi
Naima Debit
Yvon Maday**

**Contract No. NAS1-18107
November 1987**

**INSTITUTE FOR COMPUTER APPLICATIONS IN SCIENCE AND ENGINEERING
NASA Langley Research Center, Hampton, Virginia 23665**

Operated by the Universities Space Research Association

NASA

**National Aeronautics and
Space Administration**

**Langley Research Center
Hampton, Virginia 23665**

**COUPLING FINITE ELEMENT AND SPECTRAL METHODS:
FIRST RESULTS**

Christine Bernardi*, Naima Debit*, and Yvon Maday**

ABSTRACT

A Poisson equation on a rectangular domain is solved by coupling two methods: the domain is divided in two squares; a finite element approximation is used on the first square, and a spectral discretization is used on the second one. Two kinds of matching conditions on the interface are presented and compared; in both cases, error estimates are proved.

This research was supported by the National Aeronautics and Space Administration under NASA Contract No. NAS1-18107 while the third author was in residence at the Institute for Computer Applications in Science and Engineering (ICASE), NASA Langley Research Center, Hampton, VA 23665.

*Analyse Numérique, Tour 55-65, Université Pierre et Marie Curie, 4 place Jussieu, F-75252 Paris Cedex 05, France.

**Université Paris XII, and Analyse Numérique, Tour 55-65, Université Pierre et Marie Curie, 4 place Jussieu, F-75252 Paris Cedex 05, France.

1. INTRODUCTION

To approximate the solutions of partial differential equations, a number of methods can be successfully applied: among them, spectral type methods, in which the discrete solution is a polynomial of high degree, are known to be very accurate when the solution approximated is very smooth (see [GO][CHQZ] for a general description of these methods). Their main drawback lies in the difficulty to take into account the singularities of the function approximated, as well as the difficulty in handling domains with a complicated boundary. This last problem is usually solved by decomposition into subdomains and/or transformation of coordinates. On the other hand, the finite element method, where the discrete solution, restricted to very small domains called "elements," is a polynomial of low degree, is well suited to problems with complex geometries, but its accuracy is limited by the degree of the polynomials (general properties of finite elements are analyzed in [C]). Several attempts have been made to set the two methods in a unified framework and obtain the advantages of each one. The spectral element method [P], which consists of using a spectral algorithm on a fixed number of subdomains, is presently developed for a growing number of problems (see, for instance, [F], [KP], and [MP]); on the opposite side, the so-called p-version of the finite element method, where the discrete functions are polynomials of fixed high degree on each element, is studied by several authors ([BSK], [SV], [V] for instance).

The idea of this paper is very different: as previously presented by K. Z. Korszak and A. T. Patera [KP], it consists of dividing the domain where the problem must be solved in two parts; then, the problem will be approximated by a finite element method on the first part and by a spectral method on the second. Consequently, the discrete space will consist of functions which are piecewise polynomial on one part and a restriction of a high degree poly-

nomial on the other with a matching condition on the interface. Here, we present and compare two kinds of matching conditions: the first kind is a punctual one, i.e., we bias the functions to be continuous at the nodes of the finite elements on the interface; the second kind is an integral one, since we require the trace of the finite element function on the interface to be the L^2 -projection of the trace of the polynomial onto the finite element space. Of course, both algorithms will be nonconforming in the general case, since it is impossible to match a high-degree polynomial and a piecewise polynomial function on the interface in a continuous way. However, in a finite element context, nonconforming methods have proved themselves to be as efficient as the conforming ones (see for instance [CR] or [RT]). Moreover, numerical experiments¹ [KP] already show the interest of the coupling technique, which turns out to be easy to implement and very flexible to fit both the problem and the domain.

In this paper, we analyze the coupling method on a test problem and in a model domain. The domain is simply the rectangle $\Omega = (-1,1) \times (0,1)$, which we divide in two parts, $\Omega^- = (-1,0) \times (0,1)$ and $\Omega^+ = (0,1) \times (0,1)$; we denote by γ the interface $\{0\} \times (0,1)$, and by \mathbf{n} the unit vector which is orthogonal to γ and directed from Ω^- to Ω^+ . For a given function f on Ω , the Poisson problem we want to approximate is the following one: Find a function u on Ω such that

$$(1.1) \quad \begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

¹N. Debit, Thesis in preparation.

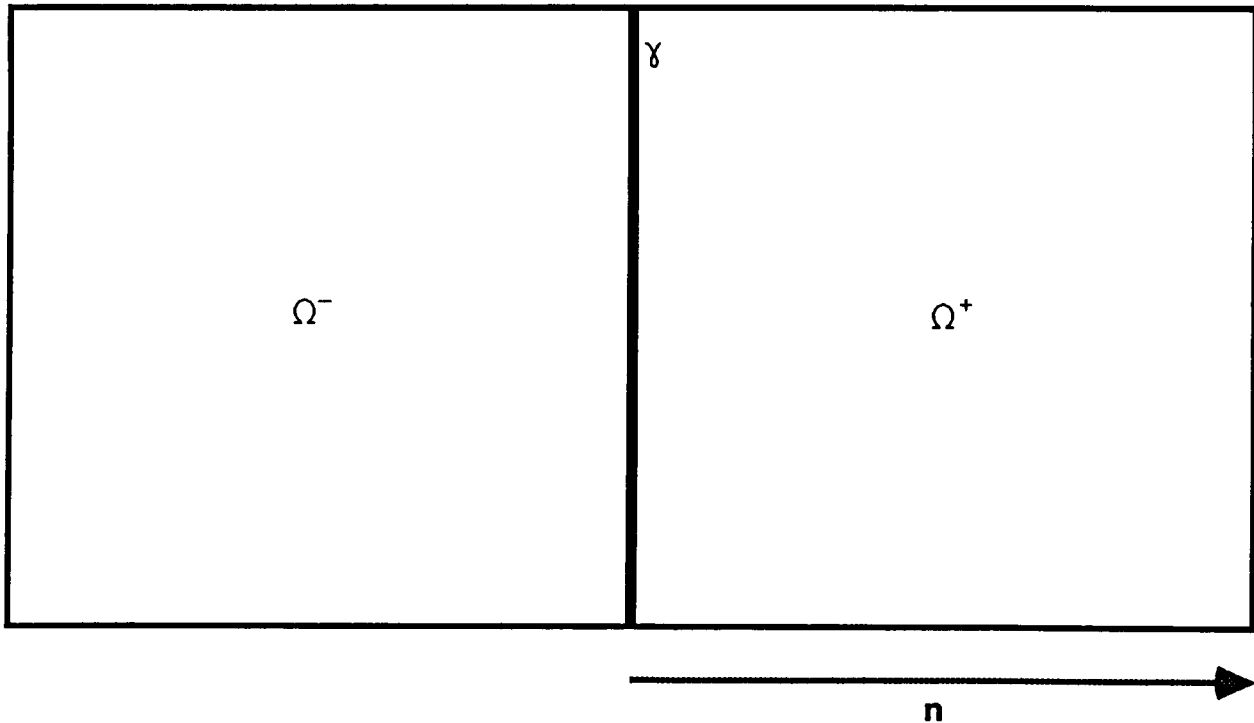


Figure 1.1. Decomposition of the domain.

An outline of the paper is as follows. In Section II, we introduce the discrete spaces and state the discrete problems. Sections III and IV are devoted respectively to the analysis of the consistency error and of the approximation error. The final error estimates, together with concluding remarks, are given in Section V.

The main results of this paper were first presented in [BDM].

Notation: Let Δ denote any open interval of the real line or any domain in \mathbb{R}^2 with a polygonal boundary. For any real number s , we consider the classical Hilbert Sobolev spaces $H^s(\Delta)$, provided with the usual norm $\|\cdot\|_{s,\Delta}$, and also, when s is an integer, with the semi-norm $|\cdot|_{s,\Delta}$. For any real number $s \geq 0$ and any p , $1 \leq p \leq +\infty$, we also use the Sobolev spaces $W^{s,p}(\Delta)$, provided with the norm $\|\cdot\|_{s,p,\Delta}$. Finally, for any real

number $s \geq 0$, $H_0^s(\Delta)$ stands for the closure in $H^s(\Delta)$ of the space of indefinitely differentiable functions with a compact support in Δ .

Throughout this paper, with any function v defined on Ω , we associate the pair $v^* = (v^-, v^+)$, where v^- (resp. v^+) denotes the restriction of v to Ω^- (resp. Ω^+). The following scalar product on $L^2(\Omega^-) \times L^2(\Omega^+)$

$$(1.2) \quad (u^*, v^*) = \int_{\Omega^-} u^-(\mathbf{x})v^-(\mathbf{x})d\mathbf{x} + \int_{\Omega^+} u^+(\mathbf{x})v^+(\mathbf{x})d\mathbf{x}$$

coincides with the usual one on $L^2(\Omega)$. We also provide the product $H^1(\Omega^-) \times H^1(\Omega^+)$ with the norm

$$(1.3) \quad \|v^*\| = [(v^*, v^*) + (\nabla v^*, \nabla v^*)]^{1/2};$$

the space of pairs v^* in $H^1(\Omega^-) \times H^1(\Omega^+)$ with v continuous through γ , is isomorphic to $H^1(\Omega)$. Finally, we define on $H^1(\Omega^-) \times H^1(\Omega^+)$ the bilinear form

$$(1.4) \quad \forall (u^*, v^*) \in [H^1(\Omega^-) \times H^1(\Omega^+)]^2, \quad a(u^*, v^*) = (\nabla u^*, \nabla v^*).$$

Clearly, for any function f in $L^2(\Omega)$, problem (1.1) is equivalent to the following one: Find u in $H_0^1(\Omega)$ such that

$$(1.5) \quad \forall v \in H_0^1(\Omega), \quad a(u^*, v^*) = (f^*, v^*).$$

This variational form is precisely the one which will be used in order to define the discrete problems.

In all that follows, c, c', c'', \dots , are generic positive constants, independent of the discretization parameters.

2. THE DISCRETE SPACES AND PROBLEMS

2.1. Definition of the discrete spaces.

We have to define a discrete space on each subdomain Ω^- and Ω^+ , and then we must match conditions on the interface.

Let h be a real parameter, $0 < h \leq 1$, which will tend to 0. With each value of h , we associate a triangulation T_h of the domain Ω^- , i.e., a finite set of triangles such that the intersection of two triangles is either empty or a vertex or an edge and such that

$$(2.1) \quad \bar{\Omega}^- = \bigcup_{K \in T_h} K;$$

h is the upper bound of the diameters of the triangles of T_h . We denote by h_K the diameter of any triangle K in T_h , and by ρ_K the diameter of the inscribed circle in K . Next, we assume that the family $(T_h)_h$ is regular in the following sense (cf. [C, Section 3.1] or [B, Def. 3.1]): there exists a constant $\tau > 0$ such that, for any h , and for any K in T_h , the following inequality holds

$$(2.2) \quad \rho_K \geq \tau h_K.$$

Let k be a fixed integer ≥ 1 . For any closed subset A of \mathbb{R} (resp. \mathbb{R}^2), we denote by $P_k(A)$ the set of the restriction to A of polynomials of one variable (resp. two variables) with total degree $\leq k$. With any triangulation T_h , we have the associated finite dimensional X_h defined by

$$(2.3) \quad X_h = \{v_h \in C^0(\bar{\Omega}^-); \forall K \in T_h, v_h|_K \in P_k(K) \text{ and } v_h = 0 \text{ on } \partial\bar{\Omega} \setminus \gamma\}.$$

We also need the finite dimensional trace space

$$(2.4) \quad x_h = \{v_h|_\gamma, v_h \in X_h\}.$$

In order to build an appropriate basis of X_h and x_h , we consider each triangle K as the support of a Lagrange finite element $(K, P_k(K), E_K)$, where E_K is the set of all points in K with barycentric coordinates $i/k, j/k$, and $(k-i-j)/k$, $0 \leq i, j \leq k, i+j \leq k$; it is well-known [C, Thm. 2.2.1] that this set of points is $P_k(K)$ -unisolvant. Next, we set

$$(2.5) \quad E_h = \bigcup_{K \in T_h} E_K,$$

and also

$$(2.6) \quad \xi_h = \{a \in E_h \cap \gamma\}.$$

To each point a in $E_h \cap (\bar{\Omega}^- \cup \gamma)$, we associate its Lagrange interpolant, i.e., the unique q_a of X_h which is equal to 1 in a and vanishes at any

other point of Ξ_h . Then, the set $\{q_{\mathbf{a}, \mathbf{a}} \in \Xi_h \cap (\Omega^- \cup \gamma)\}$ is a basis of the space X_h and the set $\{q_{\mathbf{a}}|_{\gamma}, \mathbf{a} \in \Xi_h\}$ is a basis of the space X_h .

Next, let N be an integer ≥ 1 , which will tend to $+\infty$. For any integer $n \geq 0$, we denote by $\mathcal{Q}_n(\bar{\Omega}^+)$ the set of the restrictions to $\bar{\Omega}^+$ of polynomials of two variables with degree $\leq n$ with respect to each variable. For each integer N , we consider the finite dimensional space X_N defined by

$$(2.7) \quad X_N = \{v_N \in \mathcal{Q}_N(\bar{\Omega}^+); v_N = 0 \text{ on } \partial\Omega^+ \setminus \gamma\}.$$

Let $(L_n)_{n \in \mathbb{N}}$ be the family of Legendre polynomials on $[0,1]$, i.e., of orthogonal polynomials on $[0,1]$, such that $L_n, n \in \mathbb{N}$ is of degree n and satisfies $L_n(0) = 1$. We recall that the set $\{L_m \otimes L_n, 0 \leq m, n \leq N\}$ is a basis of $\mathcal{Q}_N(\bar{\Omega}^+)$. However, we shall also characterize the polynomials of Ξ_N by ponctual values: let $\zeta_j, 0 \leq j \leq N$, be the roots of the polynomial $\zeta(1 - \zeta)L_N(\zeta)$, with $0 = \zeta_0 < \zeta_1 < \dots < \zeta_N = 1$; we set

$$(2.8) \quad \Xi_N = \{(\zeta_i, \zeta_j), 0 \leq i, j \leq N\},$$

and

$$(2.9) \quad \xi_N = \Xi_N \cap \gamma = \{(0, \zeta_j), 1 \leq j \leq N-1\}.$$

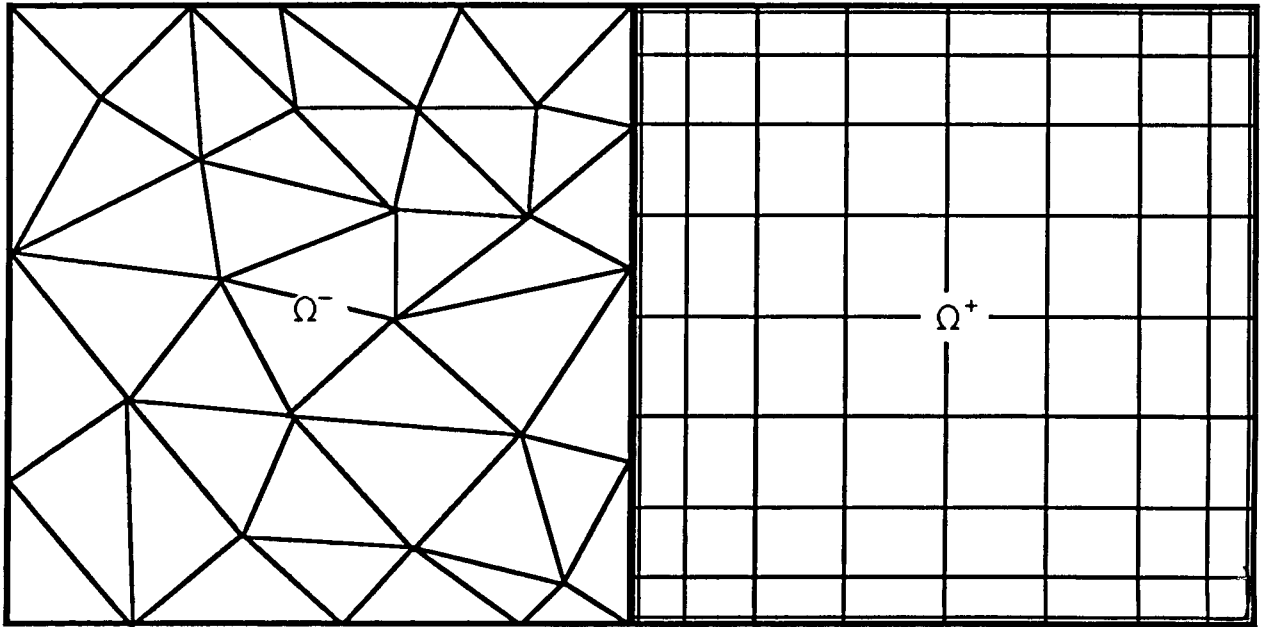


Figure 2.2. The triangulation T_h and the set Ξ_N .

Finally, with each value of h and N , we associate the discretization parameter $\delta = (h, N^{-1})$. The pair u^* , where u is the solution of problem (1.1), will be approximated in a subspace of $X_h \times X_N$, made of all pairs which satisfy a matching condition on the interface γ . More precisely, we are going to consider two kinds of matching conditions, with which we associate two kinds of discrete spaces, both denoted by V_δ :

1) punctual matching condition: the space V_δ^P is defined by

$$(2.10) \quad V_\delta^P = \{v_\delta^* = (v_h, v_N) \in X_h \times X_N; \forall a \in \xi_h, v_h(a) = v_N(a)\};$$

2) integral matching condition: the space V_δ^1 is defined by

$$(2.11) \quad V_\delta^1 = \{v_\delta^* = (v_h, v_N) \in X_h \times X_N; \forall q_h \in X_h, \int_Y (v_h - v_N)(0, y) q_h(y) dy = 0\}.$$

The space V_δ^P has already been used in [KP]. Here, we compare the two kinds of spaces, and the integral matching condition will turn out to be better.

Remark 2.1: We immediately notice that both methods are nonconforming since the functions v_δ , associated with a pair v_δ^* of V_δ , are generally discontinuous through γ and consequently do not belong to $H_0^1(\Omega)$. Indeed, for $N \geq k$, a function v_δ , with the pair v_δ^* in $X_h \times X_N$, belongs to $H_0^1(\Omega)$ if and only if it is a polynomial of $P_k(\bar{\Omega}) \cap H_0^1(\Omega)$.

Remark 2.2: From a numerical point of view, to enforce the punctual matching condition, one has to interpolate polynomials of X_N at every point of ξ_h and hence must store the values $L_n(a)$, $0 \leq n \leq N$, $a \in \xi_h$. On the other hand, to enforce the integral matching condition, one needs to store the integrals $\int_Y L_n(y) q_a(0, y) dy$, $0 \leq n \leq N$, $a \in \xi_h$. Consequently, the cost of the two methods is of the same order.

However, when k is equal to 1, for a given value of N , it is possible to choose the triangulation T_h such that the sets ξ_h and ξ_N coincide. Then, since the polynomials of X_N are characterized by their values at the points of ξ_N , enforcing the punctual matching condition would be less expensive. But this would require very strong restrictions on the triangulation T_h ; in particular, the parameters h and N would be linked

by a relation of the type $h \geq cN^{-1}$. Moreover, the triangulation could not be uniformly regular since it is well-known (see [S, Thm. 6.21.3]) that the ζ_j , $1 \leq j \leq N-1$, satisfy: $\zeta_j = \sin^2 \theta_j$, with $(2j - 1)\pi/4N < \theta_j < (j + 1)\pi/2(N + 1)$; hence the points of ξ_N are not all equally distributed (they cluster to ± 1). That is why we would not recommend such a choice.

2.2. Definition of the discrete problems.

We are now in a position to define the discrete problems. We recall (see [DR, Section 2.7] or [H, Chapter 25]) that there exist positive weights ρ_j , $0 \leq j \leq N$, such that the Gauss-Lobatto quadrature formula

$$(2.12) \quad \int_0^1 \phi(\zeta) d\zeta \approx \sum_{j=0}^N \phi(\zeta_j) \rho_j$$

is exact on all polynomials with degree $\leq 2N - 1$.

With each point $\mathbf{a} = (\zeta_i, \zeta_j)$ in ξ_N , we associate the weight $\rho_{\mathbf{a}} = \rho_i \rho_j$. We now introduce the following discrete bilinear form on $L^2(\Omega^-) \times C^0(\Omega^+)$

$$(2.13) \quad (u^*, v^*)_{\delta} = \sum_{K \in \mathcal{T}_h} \int_K u^-(x) v^-(x) dx + \sum_{\mathbf{a} \in \xi_N} u^+(\mathbf{a}) v^+(\mathbf{a}) \rho_{\mathbf{a}},$$

which coincides with the usual scalar product on $L^2(\Omega^-) \times C_{N-1}(\Omega^+)$. Finally, we define on $H^1(\Omega^-) \times C^1(\Omega^+)$ the bilinear form

$$(2.14) \quad \forall (u^*, v^*) \in [H^1(\Omega^-) \times C^1(\Omega^+)]^2, a_{\delta}(u^*, v^*) = (\nabla u^*, \nabla v^*)_{\delta}.$$

Then, for any pair f^* given in $L^2(\Omega^-) \times C^0(\overline{\Omega^+})$, for each kind of matching condition, the discrete problem is the following: Find u_δ , with u_δ^* in V_δ , such that

$$(2.15) \quad \forall v_\delta^* \in V_\delta, \quad a_\delta(u_\delta^*, v_\delta^*) = (f^*, v_\delta^*)_\delta.$$

Remark 2.3: Of course, in definition (2.13), one could, by using a quadrature formula, replace each integral $\int_K u^-(\mathbf{x})v^-(\mathbf{x})d\mathbf{x}$ by its approximation. The resulting algorithm will be thoroughly analyzed.²

We recall [CQ, Lemma 3.2] the property

$$(2.16) \quad \sum_{j=0}^N L_N(\zeta_j)^2 \rho_j = (2 + N^{-1}) \int_0^1 L_N(\zeta^2) d\zeta.$$

Since the quadrature formula (2.12) is exact on all polynomials of degree $\leq 2N-1$, the discrete scalar product $(\cdot, \cdot)_\delta$ is uniformly equivalent to (\cdot, \cdot) on $L^2(\Omega^-) \times \mathbf{Q}_N(\overline{\Omega^+})$. Consequently, the form a_δ satisfies the following properties of continuity

$$(2.17) \quad \forall (u^*, v^*) \in [H^1(\Omega^-) \times \mathbf{Q}_N(\overline{\Omega^+})]^2, \quad |a_\delta(u^*, v^*)| \leq c \|u^*\| \|v^*\|,$$

and of ellipticity

²N. Debit, Thesis in preparation.

$$(2.18) \quad \forall u^* \in H^1(\Omega^-) \times \mathbf{Q}_N(\Omega^+), \quad a_\delta(u^*, u^*) \geq (\nabla u^*, \nabla u^*).$$

Now, since both $\partial\Omega^- \cap \partial\Omega$ and $\partial\Omega^+ \cap \partial\Omega$ have a positive measure, it follows from the Poincaré-Friedrichs inequality that the semi-norm:

$v^* \rightarrow (\nabla v^*, \nabla v^*)^{1/2}$ is a norm equivalent to $\|\cdot\|$ on the space $\{v^* \in H^1(\Omega^-) \times H^1(\Omega^+); v = 0 \text{ on } \partial\Omega\}$, which yields in our particular case

$$(2.19) \quad \forall u^* \in H^1(\Omega^-) \times \mathbf{Q}_N(\Omega^+), \quad a_\delta(u^*, u^*) \geq c \|u^*\|^2.$$

Thus we have proved the following result.

Proposition 2.4: In both cases of ponctual and integral matching conditions, problem (2.15) has a unique solution u_δ with u_δ^* in V_δ .

The purpose of what follows is to give an error estimate between the solutions u and u_δ of problems (1.1) and (2.15) respectively. We begin with a classical bound (see [C, Thm. 4.2.2]).

Proposition 2.5: In both cases of ponctual or integral matching conditions, the solutions u and u_δ of problems (1.1) and (2.15) satisfy

$$(2.20) \quad \begin{aligned} \|u^* - u_\delta^*\| &\leq c \inf_{v_\delta^* \in V_\delta} \{\|u^* - v_\delta^*\| \\ &+ \sup_{w_\delta^* \in V_\delta} [a(v_\delta^*, w_\delta^*) - a_\delta(v_\delta^*, w_\delta^*)] / \|w_\delta^*\| \} \\ &+ \sup_{w_\delta^* \in V_\delta} [(f^*, w_\delta^*) - (f^*, w_\delta^*)_\delta] / \|w_\delta^*\| \end{aligned}$$

$$+ \sup_{w_\delta^* = (w_h, w_N) \in V_\delta} \int_\gamma (\partial u / \partial n)(0, y)(w_N - w_h)(0, y) dy / \|w_\delta^*\|.$$

Proof: Let v_δ^* be any element in V_δ . Using (2.19), we have

$$\begin{aligned} c \|u_\delta^* - v_\delta^*\|^2 &\leq a_\delta(u_\delta^* - v_\delta^*, u_\delta^* - v_\delta^*) \\ &= -a(v_\delta^*, u_\delta^* - v_\delta^*) + a(v_\delta^*, u_\delta^* - v_\delta^*) - a_\delta(v_\delta^*, u_\delta^* - v_\delta^*) + (f^*, u_\delta^* - v_\delta^*)_\delta. \end{aligned}$$

Next, it follows from (2.1) that, for any w^* in $H^1(\Omega^-) \times H^1(\Omega^+)$ such that w vanishes on $\partial\Omega$,

$$\begin{aligned} (f^*, w^*) &= \int_\Omega f(\mathbf{x})w(\mathbf{x})d\mathbf{x} = -\int_{\Omega^-} (\Delta u)(\mathbf{x})w^-(\mathbf{x})d\mathbf{x} - \int_{\Omega^+} (\Delta u)(\mathbf{x})w^+(\mathbf{x})d\mathbf{x} \\ &= a(u^*, w^*) + \int_\gamma (\partial u / \partial n)(0, y)(w^+ - w^-)(0, y)dy. \end{aligned}$$

Setting $w^* = u_\delta^* - v_\delta^* = (w_h, w_N)$ and combining the result with the previous inequality, we obtain

$$\begin{aligned} c \|u_\delta^* - v_\delta^*\|^2 &\leq a(u^* - v_\delta^*, u_\delta^* - v_\delta^*) + a(v_\delta^*, u_\delta^* - v_\delta^*) - a_\delta(v_\delta^*, u_\delta^* - v_\delta^*) \\ &\quad - (f^*, u_\delta^* - v_\delta^*) + (f^*, u_\delta^* - v_\delta^*)_\delta + \int_\gamma (\partial u / \partial n)(0, y)(w_N - w_h)(0, y)dy, \end{aligned}$$

and (2.20) follows.

We are now interested in deriving a bound for:

1) the consistency error term

$$\sup_{w_\delta^* = (w_h, w_N) \in V_\delta} \int_\gamma (\partial u / \partial n)(0, y)(w_N - w_h)(0, y) dy / \|w_\delta^*\|, \quad \text{and}$$

2) the approximation error term

$$\inf_{v_\delta^* \in V_\delta} \|u^* - v_\delta^*\|$$

since estimating the two other terms of (2.20) is a standard result in spectral methods.

3. ANALYSIS OF THE CONSISTENCY ERROR

The aim of this section is to study the term

$\int_\gamma (\partial u / \partial n)(0, y)(w_N - w_h)(0, y) dy$, for any pair $w_\delta^* = (w_h, w_N)$ in V_δ , where u is a given function on Ω which we shall assume to be sufficiently smooth. This analysis involves only one-dimensional approximation operators.

3.1 The case of ponctual matching condition.

We recall that there exists an interpolation operator i_h from $\{v \in C^0(\bar{\gamma}); v(0) = v(1) = 0\}$ into \mathfrak{X}_h such that, for any function v continuous on $\bar{\gamma}$ and vanishing at 0 and 1, $i_h v$ is the only element of \mathfrak{X}_h which satisfies

$$(3.1) \quad \forall a \in \xi_h, (i_h v)(a) = v(a).$$

Moreover, for any real number ℓ , $1 \leq \ell \leq k+1$, there exists a constant c such that, if the function v belongs to $W^{\ell,p}(\gamma)$ for a number p , $1 \leq p \leq +\infty$, the following interpolation error holds [C, Thm. 3.1.5]

$$(3.2) \quad \|v - \hat{v}_h\|_{m,p,\gamma} \leq ch^{\ell-m} \|v\|_{\ell,p,\gamma}, \quad m = 0 \text{ or } 1.$$

First, we prove suitable inverse inequalities in the space $Q_N(\Omega^+)$, which complete those given in [Q, Section 3.1] and could be useful in other applications.

Lemma 3.1: Any polynomial q_N on $[0,1]$ with degree $\leq N$ satisfies

$$(3.3) \quad \|q_N\|_{1,1,(0,1)} \leq 2N \|q_N\|_{0,\infty,(0,1)}.$$

Proof: First, we have

$$\|q_N\|_{0,1,(0,1)} \leq \|q_N\|_{0,\infty,(0,1)}.$$

Next, let $\alpha_1, \dots, \alpha_{K-1}$ be the zeros of \hat{q}_N (i.e., the extrema of q_N) belonging to $(0,1)$, in increasing order. Of course, K is $\leq N$. Then, it suffices to compute

$$\begin{aligned} & \int_0^1 |\hat{q}_N(\zeta)| d\zeta \\ &= |q_N(\alpha_1) - q_N(0)| + \sum_{k=1}^{K-2} |q_N(\alpha_{k+1}) - q_N(\alpha_k)| + |q_N(1) - q_N(\alpha_{K-1})| \\ &\leq 2K \|q_N\|_{0,\infty,(0,1)}. \end{aligned}$$

Remark 3.2: The inequality (3.3) is optimal. Indeed, choosing q_N equal to the Chebyshev polynomial with degree N , i.e., $T_N(\zeta) = \cos(N \operatorname{Arccos}(1 - 2\zeta))$, we see that

$$\int_0^1 |T_N(\zeta)| d\zeta = 2N.$$

Corollary 3.3: There exists a constant c such that, for any p and p^* , $1 \leq p \leq 2$ and $1/p + 1/p^* = 1$, any polynomial q_N on $[0,1]$ with degree $\leq N$ satisfies

$$(3.4) \quad \|q_N\|_{1,p,(0,1)} \leq cN^{3-2/p} \|q_N\|_{0,p^*,(0,1)}.$$

Proof: We recall the well-known inverse inequality [CQ1, Lemma 2.1].

$$\|q_N\|_{1,(0,1)} \leq cN^2 \|q_N\|_{0,(0,1)}.$$

Interpolating between this inequality and (3.3) gives (3.4), with a constant independent of p (see [BL, Thm. 1.1.1] or [LM, Chap. 1, Th. 5.1]).

Corollary 3.4: There exists a constant c such that, for any real number p , $1 \leq p \leq 2$, any polynomial q_N on $[0,1]$ with degree $\leq N$ satisfies

$$(3.5) \quad \|q_N\|_{1,p,(0,1)} \leq cN^{3/2-1/p} \|q_N\|_{1/2,(0,1)}.$$

Proof: For $p > 1$, setting $1/p^* = 1 - 1/p$, we know from [BL, Thm. 6.4.5] that $H^{1/2}(\gamma)$ is the interpolation space with index $1/2$ between $W^{1,p}(\gamma)$ and $L^{p^*}(\gamma)$; hence we derive (3.5) for $p > 1$ from (3.4). That implies

$$\|q_N\|_{1,1,(0,1)} \leq cN^{3/2-1/p} \|q_N\|_{1/2,(0,1)}$$

for p tending to 1 , which proves (3.5) with $p = 1$.

We need a precise version of the Sobolev imbedding (further details can be found in [T] for instance).

Lemma 3.5: There exists a constant c such that, for any real number p , $2 < p < +\infty$, any function v in $H^{1/2}(0,1)$ satisfies

$$(3.6) \quad \|v\|_{0,p,(0,1)} \leq c\sqrt{p} \|v\|_{1/2,(0,1)}$$

Proof: Since $H^{1/2}(0,1)$ coincides with the space of the restrictions of the functions of $H^{1/2}(\mathbb{R})$, it suffices to prove (3.6) with $(0,1)$ replaced by \mathbb{R} . First, let ϵ be a real number > 0 . For any function v in $H^{1/2+\epsilon}(\mathbb{R})$, denoting by \hat{v} the Fourier transform of v , we have

$$v(\zeta) = (2\pi)^{-1/2} \int_{\mathbb{R}} \hat{v}(\xi) e^{-i\zeta\xi} d\xi, \quad \text{so that}$$

$$\begin{aligned} \|v\|_{0,\infty,\mathbb{R}} &\leq c \int_{\mathbb{R}} |\hat{v}(\xi)| d\xi \\ &\leq c \left(\int_{\mathbb{R}} |\hat{v}(\xi)|^2 (1 + |\xi|^2)^{1/2+\epsilon} d\xi \right)^{1/2} \left(\int_{\mathbb{R}} (1 + |\xi|^2)^{-1/2-\epsilon} d\xi \right)^{1/2}. \end{aligned}$$

Since the map: $v \rightarrow \left(\int_{\mathbb{R}} |\hat{v}(\xi)|^2 (1 + |\xi|^2)^{1/2+\epsilon} d\xi \right)^{1/2}$ is a norm on $H^{1/2+\epsilon}(\mathbb{R})$ equivalent to the one which is obtained by interpolation, we must estimate the constant $\int_{\mathbb{R}} (1 + |\xi|^2)^{-1/2-\epsilon} d\xi$. The change of variables $\xi = \cotan \omega$ gives

$$\int_{\mathbb{R}} (1 + |\xi|^2)^{-1/2-\varepsilon} d\xi = 2 \int_0^{\pi/2} (\sin\omega)^{-1+2\varepsilon} d\omega \leq c \int_0^{\pi/2} \omega^{-1+2\varepsilon} d\omega = c/\varepsilon.$$

We deduce

$$\|v\|_{0,\infty,\mathbb{R}} \leq c/\sqrt{\varepsilon} \|v\|_{1/2+\varepsilon,\mathbb{R}}.$$

Finally, for a real number $p > 2$, we choose $\varepsilon = 1/(p-2)$ and we interpolate this last inequality with the identity $\|v\|_{0,\mathbb{R}} = \|v\|_{0,\mathbb{R}}$. We obtain

$$\|v\|_{0,p,\mathbb{R}} \leq (c/\sqrt{\varepsilon})^{1/(1+2\varepsilon)} \|v\|_{1/2,\mathbb{R}} \leq c\sqrt{p} \|v\|_{1/2,\mathbb{R}}.$$

We are now in a position to prove the following result.

Proposition 3.6: For any function u in $H_0^1(\Omega) \cap H^2(\Omega)$, the following estimate holds for any $w_\delta^* = (w_h, w_N)$ in V_δ^p

$$(3.7) \quad \int_{\gamma} (\partial u / \partial n)(0, y)(w_N - w_h)(0, y) dy \leq chN^{1/2} \|u\|_{2, \Omega^-} \|w_\delta^*\|.$$

Proof: Let p be any real number, $2 < p < +\infty$, and p^* be such that $1/p + 1/p^* = 1$. Since u belongs to $H^2(\Omega)$, the trace $\partial u / \partial n$ belongs to $H^{1/2}(\gamma)$; hence Lemma 3.5 gives

$$\|\partial u / \partial n\|_{0,p,\gamma} \leq c\sqrt{p} \|u\|_{2, \Omega^-}.$$

Next, let us compute

$$\int_{\gamma} (\partial u / \partial n)(0, y)(w_N - w_h)(0, y) dy \leq \|\partial u / \partial n\|_{0,p,\gamma} \|w_h - w_N\|_{0,p^*,\gamma}$$

$$\leq c\sqrt{p} \|u\|_{2,\Omega} \|w_h - w_N\|_{0,p^*,\gamma}.$$

But, due to the definition (2.10) of v_δ^p , w_h is equal to $i_h w_N$ on γ , so that, by (3.2)

$$\int_\gamma (\partial u / \partial n)(0,y) (w_N - w_h)(0,y) dy \leq c\sqrt{p} h \|u\|_{2,\Omega} \|w_N\|_{1,p^*,\gamma}.$$

Applying Corollary 3.4 gives

$$\begin{aligned} \int_\gamma (\partial u / \partial n)(0,y) (w_N - w_h)(0,y) dy &\leq c \sqrt{p} h N^{3/2-1/p^*} \|u\|_{2,\Omega} \|w_N\|_{1/2,\gamma} \\ &\leq c \sqrt{p} h N^{1/2} N^{1/p} \|u\|_{2,\Omega} \|w_N\|_{1/2,\gamma}. \end{aligned}$$

Choosing $p = \log N$, we obtain the desired result.

Remark 3.7. Of course, the estimate (3.7) is not what we would wish, since the convergence is obtained only if the discretization parameters are linked by the following condition:

$$(3.8) \quad \lim[hN^{1/2} \sqrt{\log N}] = 0,$$

(in fact, in (3.7) and in this condition, h can be replaced by \tilde{h} which is the greatest of the lengths of the edges of triangles K in T_h contained in $\bar{\gamma}$).

Remark 3.8: The estimate (3.7) is independent of k ; indeed, we do not know how to improve it for large values of k .

3.2 The case of integral matching condition.

This case turns out to be simpler. We denote by π_h the Hilbert projection operator from $L^2(\gamma)$ onto \mathfrak{X}_h . We have for any v in $H_0^1(\gamma)$

$$\|v - \pi_h v\|_{0,\gamma} \leq \|v - \hat{i}_h v\|_{0,\gamma},$$

so that, for any v in $H_0^1(\gamma) \cap H^\ell(\gamma)$, $1 \leq \ell \leq k+1$,

$$(3.9) \quad \|v - \pi_h v\|_{0,\gamma} \leq ch^\ell \|v\|_{\ell,\gamma}.$$

By interpolation, this inequality also holds for any v in $H_0^\ell(\gamma)$, $1/2 < \ell < 1$. Finally, recalling that the interpolation space with index $1/2$ between $H_0^1(\gamma)$ and $L^2(\gamma)$ is $H_{00}^{1/2}(\gamma)$ (see [LM, Chap. 1, Th. 11.7]) and denoting by $\|\cdot\|_{1/2^*,\gamma}$ the norm of $H_{00}^{1/2}(\gamma)$, we also obtain for any v in $H_{00}^{1/2}(\gamma)$

$$(3.10) \quad \|v - \pi_h v\|_{0,\gamma} \leq ch^{1/2} \|v\|_{1/2^*,\gamma}.$$

Now, we prove the following.

Proposition 3.9: For any function u in $H_0^1(\Omega) \cap H^2(\Omega)$ such that the function u^- belongs to $H^\ell(\Omega^-)$, where ℓ is a real number, $2 \leq \ell \leq k + 5/2$, the following estimate holds for any $w_\delta^* = (w_h, w_N)$ in V_δ^1

$$(3.11) \quad \int_\gamma (\partial u / \partial n)(0,y) (w_N - w_h)(0,y) dy \leq ch^{\ell-1} \|u^-\|_{\ell,\Omega^-} \|w_\delta^*\|.$$

Proof: Let $w_\delta^* = (w_h, w_N)$ be any element in V_δ^1 ; due to the definition (2.11) of V_δ^1 , w_h coincides with $\pi_h w_N$ on γ . We compute

$$\begin{aligned} \int_{\gamma} (\partial u / \partial n)(0, y)(w_N - w_h)(0, y) dy &= \int_{\gamma} (\partial u / \partial n)(0, y)(w_N - \pi_h w_N)(0, y) dy \\ &= \int_{\gamma} [(\partial u / \partial n) - \pi_h (\partial u / \partial n)(0, y)](w_N - \pi_h w_N)(0, y) dy, \end{aligned}$$

so that

$$\int_{\gamma} (\partial u / \partial n)(0, y)(w_N - w_h)(0, y) dy \leq \|(\partial u / \partial n) - \pi_h (\partial u / \partial n)\|_{0, \gamma} \|w_N - \pi_h w_N\|_{0, \gamma}.$$

If $\ell - 1/2$ is not an integer, we note that $\partial u / \partial n$ belongs to $H_0^1(\gamma) \cap H^{\ell-3/2}(\gamma)$ and that, since w_N vanishes on $\partial\Omega^+ \setminus \gamma$, $w_N|_{\gamma}$ belongs to $H_{00}^{1/2}(\gamma)$. Applying (3.9) or (3.10) to bound the first term and (3.10) to bound the second one, we obtain

$$\int_{\gamma} (\partial u / \partial n)(0, y)(w_N - w_h)(0, y) dy \leq ch^{\ell-1} \|\partial u / \partial n\|_{\ell-3/2, \gamma} \|w_N\|_{1/2^*, \gamma},$$

(with $\|\partial u / \partial n\|_{1/2, \gamma}$ replaced by $\|\partial u / \partial n\|_{1/2^*, \gamma}$ in the case $\ell = 2$), and the result follows. The case where $\ell - 1/2$ is an integer follows by an interpolation argument.

Remark 3.10: Clearly, the estimate (3.11) is much better than (3.6), since it is independent of N . In fact, the term $\int_{\gamma} (\partial u / \partial n)(0, y)(w_N - w_h)(0, y) dy$ goes to 0 whenever the discretization parameter h decreases to 0.

4. ANALYSIS OF THE APPROXIMATION ERROR

We begin by recalling some properties of the approximation by finite element functions and by polynomials in the two-dimensional case.

First, since for each K in T_h the set E_K is $P_k(K)$ -unisolvent, there exists an interpolation operator T_h from $\{v \in C^0(\bar{\Omega}^-); v = 0 \text{ on } \partial\Omega^- \setminus \gamma\}$ into X_h such that, for any function v continuous on $\bar{\Omega}^-$ and vanishing on $\partial\Omega^- \setminus \gamma$, $T_h v$ is the only element of X_h which satisfies

$$(4.1) \quad \forall a \in E_h \cap (\Omega^- \cup \gamma), (T_h v)(a) = v(a).$$

Moreover, if the function v belongs to $H^\ell(\Omega^-)$ for a real number ℓ , $2 \leq \ell \leq k + 1$, the following interpolation error holds [C, Thm. 3.1.5]

$$(4.2) \quad \|v - T_h v\|_{m, \Omega^-} \leq ch^{\ell-m} \|v\|_{\ell, \Omega^-}, \quad m = 0 \text{ or } 1.$$

Next, we state the following result which can be derived in the same way as in [M2, Thm. 3.2].

Lemma 4.1: Let ρ be a real number ≥ 1 such that $\rho - 1/2$ is not an integer. There exists a projection operator Π_N^ρ from the space $\{v \in H^\rho(\Omega^+); v = 0 \text{ on } \partial\Omega^+ \setminus \gamma\}$ onto $\{v_N \in Q_N(\bar{\Omega}^+); v_N = 0 \text{ on } \partial\Omega^+ \setminus \gamma\}$ such that, if a function v vanishing on $\partial\Omega^+ \setminus \gamma$ belongs to $H^\sigma(\Omega^+)$ for a real number $\sigma \geq \rho$, the following error estimate holds

$$(4.3) \quad \|v - \Pi_N^\rho v\|_{\mu, \Omega^+} \leq cN^{\mu-\sigma} \|v\|_{\sigma, \Omega^+}, \quad 0 \leq \mu \leq \rho.$$

Now, we are going to approximate a function u of $H_0^1(\Omega)$ by a function v_δ , with v_δ^* in V_δ . In fact, we shall set

$$(4.4) \quad v_\delta^* = (T_h u^- + q_h, \Pi_N^\rho u^+),$$

where q_h will be chosen in X_h so that v_δ satisfies the matching condition. Of course, the choice of q_h depends on this condition.

4.1 The case of punctual matching condition.

We immediately prove the following result.

Proposition 4.2: For any function u in $H_0^1(\Omega)$ such that the pair u^* belongs to $H^\ell(\Omega^-) \times H^\sigma(\Omega^+)$, where ℓ and σ are real numbers, $2 \leq \ell \leq k+1$ and $\sigma \geq 2$, there exists a pair v_δ^* in V_δ^p such that

$$(4.5) \quad \|u^* - v_\delta^*\| \leq c \{ h^{\ell-1} \|u^-\|_{\ell, \Omega^-} + (h^{\sigma-1} + N^{1-\sigma}) \|u^+\|_{\sigma, \Omega^+} \}.$$

Proof: For any function z defined on Ω^+ , let us denote by \bar{z} the function defined on Ω^- by

$$\forall (x,y) \in \Omega^-, \quad \bar{z}(x,y) = z(-x,y).$$

Next, we take $2 \leq \rho \leq \inf\{k+1, \sigma\}$, $\rho - 1/2 \notin \mathbb{N}$, and choose v_δ^* as in (4.4) with q_h equal to $-T_h(\bar{u}^+ - \bar{\Pi}_N^\rho u^+)$. Clearly, we have at any point a of ξ_h

$$(\mathcal{T}_h u^-)(a) + q_h(a) = u^-(a) - u^+(a) + (\overline{\Pi}_N^\rho u^+)(a) = (\Pi_N^\rho u^+)(a),$$

so that v_δ^* belongs to V_δ^P . Moreover, we write

$$\begin{aligned} \|u^* - v_\delta^*\| &\leq \|u^- - \mathcal{T}_h u^-\|_{1,\Omega^-} + \|q_h\|_{1,\Omega^-} + \|u^+ - \Pi_N^\rho u^+\|_{1,\Omega^+} \\ &\leq \|u^- - \mathcal{T}_h u^-\|_{1,\Omega^-} + \|(\text{id} - \mathcal{T}_h)(\overline{u^+} - \overline{\Pi}_N^\rho u^+)\|_{1,\Omega^-} + 2\|u^+ - \Pi_N^\rho u^+\|_{1,\Omega^+}. \end{aligned}$$

Finally, applying (4.2) and Lemma 4.1, we deduce

$$\begin{aligned} \|u^* - v_\delta^*\| &\leq c\{h^{\ell-1} \|u^-\|_{\ell,\Omega^-} + h^{\rho-1} \|\overline{u^+} - \overline{\Pi}_N^\rho u^+\|_{\rho,\Omega^-} + N^{1-\sigma} \|u^+\|_{\sigma,\Omega^+}\} \\ &\leq c\{h^{\ell-1} \|u^-\|_{\ell,\Omega^-} + h^{\rho-1} N^{\rho-\sigma} \|u^+\|_{\sigma,\Omega^+} + N^{1-\sigma} \|u^+\|_{\sigma,\Omega^+}\}. \end{aligned}$$

Applying the convexity inequality $\alpha\beta \leq \alpha^p/p + \beta^q/q$ ($\alpha \geq 0, \beta \geq 0, 1/p + 1/q = 1$), gives the proposition.

4.2. The case of integral matching condition.

As far as the approximation error is concerned, this case is less simple. First, we introduce the following lifting operator R_h from \mathbf{x}_h , into \mathbf{X}_h : for any v_h in \mathbf{x}_h , $R_h v_h$ is equal to v_h on γ and vanishes in any point a of $E_h \setminus \gamma$. This amounts to stating that

$$(4.6) \quad \forall v_h = \sum_{a \in \xi_h} v_h(a) q_a|_\gamma \in \mathbf{x}_h, \quad R_h v_h = \sum_{a \in \xi_h} v_h(a) q_a.$$

We recall that π_h stands for the Hilbert projection operator from $L^2(\gamma)$ onto \mathbf{x}_h . Our purpose is to prove a stability result for the operator $R_h \pi_h$.

We use now a technique due to M. Crouzeix and V. Thomee [CT]. We first introduce the subset ξ_h^0 of the points of ξ_h which are a vertex of a triangle K in T_h , and the subspace of \mathbf{x}_h

$$(4.7) \quad \mathbf{x}_h^2 = \{v_h \in \mathbf{x}_h; \forall a \in \xi_h^0, v_h(a) = 0\};$$

we denote by \mathbf{x}_h^1 the orthogonal subspace to \mathbf{x}_h^2 in \mathbf{x}_h with respect to the scalar product of $L^2(\gamma)$ (when k is equal to 1, the space \mathbf{x}_h^2 is simply $\{0\}$, and the space \mathbf{x}_h^1 coincides with \mathbf{x}_h). Let π_h^i , $i = 1$ or 2 , be the orthogonal projection from $L^2(\gamma)$ onto \mathbf{x}_h^i ; we have of course

$$(4.8) \quad \pi_h = \pi_h^1 + \pi_h^2.$$

Hence, we are going to prove a stability result successively for $R_h \pi_h^2$ and $R_h \pi_h^1$.

First, we recall some classical notation. Let \hat{K} be the "reference" triangle with vertices $(1,0)$, $(0,1)$, and $(0,0)$. For any triangle K in T_h , there exists an affine mapping F_K which maps \hat{K} onto K ; let B_K be the Jacobian matrix of F_K , and h_K be the diameter of K . Moreover, if the triangle K has an edge E contained in $\bar{\gamma}$, we assume that F_K maps the edge \hat{E} with vertices $(0,0)$ and $(1,0)$ onto E , and we denote by B_E the Jacobian matrix of the restriction of F_K to \hat{E} . For any function q on K , we set: $\hat{q} = q \circ F_K$.

We denote by \mathcal{T}_h the set of triangles K in T_h which have an edge contained in $\bar{\gamma}$. For any K in \mathcal{T}_h , we set

$$(4.9) \quad Y_K = \text{Span}\{q_a, a \in \xi_h \cap \partial K\}.$$

Lemma 4.3: Let K be a triangle of \mathcal{T}_h , and let E be the edge of K contained in $\bar{\gamma}$. Any polynomial q in Y_K satisfies

$$(4.10) \quad \|q\|_{1,K} \|q\|_{0,1,E} / \|q\|_{0,E}^2 \leq c.$$

Proof: Due to assumption (2.2), it is well-known (see [C, Thm. 3.1.2 and 3.1.3] or [B, Lemma 2.3]) that, for any function q in $H^1(K)$,

$$\|q\|_{1,K} \leq c \|B_K^{-1}\| |(\det B_K)|^{1/2} \|\hat{q}\|_{1,\hat{K}} \leq c' \|\hat{q}\|_{1,\hat{K}},$$

$$\|q\|_{0,1,E} = |(\det B_E)| \|\hat{q}\|_{0,1,\hat{E}} \leq c^{-h_K} \|\hat{q}\|_{0,1,\hat{E}},$$

and

$$\|q\|_{0,E} = |(\det B_E)|^{1/2} \|\hat{q}\|_{0,1,\hat{E}} \geq c^{-h_K^{1/2}} \|\hat{q}\|_{0,\hat{E}},$$

whence

$$\|q\|_{1,K} \|q\|_{0,1,E} / \|q\|_{0,E}^2 \leq c \|\hat{q}\|_{1,\hat{K}} \|\hat{q}\|_{0,1,\hat{E}} / \|\hat{q}\|_{0,\hat{E}}^2.$$

Finally, if q belongs to Y_K , \hat{q} belongs to the space \hat{Y} spanned by the Lagrange polynomials of the points $(j/k, 0)$, $0 \leq j \leq k$. Clearly, on this last finite-dimensional space, the semi-norm: $\hat{q} \rightarrow \|\hat{q}\|_{0,1,\hat{E}}$ and $\hat{q} \rightarrow \|\hat{q}\|_{0,\hat{E}}$ are norms and the three norms $\|\cdot\|_{1,\hat{K}}$, $\|\cdot\|_{0,\hat{E}}$ and $\|\cdot\|_{0,1,\hat{E}}$ are equivalent, completing the proof of the Lemma.

We need the following Gagliardo-Nirenberg inequality.

Lemma 4.4: Let s be a real number, $0 \leq s < 1/2$. Let E be the edge of a triangle of T_h , contained in $\bar{\gamma}$. Any function w in $H^1(E)$ satisfies

$$(4.11) \quad \|w\|_{0,\infty,E} \leq c \|w\|_{s,E}^{1/2} \|w\|_{1-s,E}^{1/2}$$

Proof: Using the previous notation, we set $\hat{w} = w \circ F_K$, and applying the classical Gagliardo-Nirenberg inequality, we obtain

$$\|\hat{w}\|_{0,\infty,\hat{E}} \leq c \|\hat{w}\|_{s,\hat{E}}^{1/2} \|\hat{w}\|_{1-s,\hat{E}}^{1/2}$$

Next, applying [C, Thm. 3.1.2 and 3.1.3] or [B, Lemma 2.3] gives

$$\|\hat{w}\|_{0,\hat{E}} \leq c |(\det B_E)|^{-1/2} \|w\|_{0,E} \leq c^{-1} h_K^{-1/2} \|w\|_{0,E},$$

$$\|\hat{w}\|_{1,\hat{E}} \leq c \|B_E\| |(\det B_E)|^{-1/2} \|w\|_{1,E} \leq c^{-1} h_K^{1/2} \|w\|_{1,E}.$$

Interpolating between these two inequalities, we have

$$\|\hat{w}\|_{s,\hat{E}} \leq c h_K^{s-1/2} \|w\|_{s,E} \quad \text{and} \quad \|\hat{w}\|_{1-s,\hat{E}} \leq c^{-1} h_K^{1/2-s} \|w\|_{1-s,E},$$

so that

$$\|w\|_{0,\infty,E} = \|\hat{w}\|_{0,\infty,\hat{E}} \leq c h_K^{s/2-1/4} h_K^{1/4-s/2} \|w\|_{s,E}^{1/2} \|w\|_{1-s,E}^{1/2}.$$

We are now in a position to prove the following lemma.

Lemma 4.5: Let s be a real number, $0 \leq s < 1/2$. The operator $R_h \pi_h^2$ satisfies the following stability property: for any function w in $H^{1-s}(\gamma)$,

$$(4.12) \quad \|R_h \pi_h^2 w\|_{1,\Omega} \leq c \|w\|_{s,\gamma}^{1/2} \|w\|_{1-s,\gamma}^{1/2}.$$

Proof: Let w be any function in $H^{1-s}(\gamma)$. If $E \subset \bar{\gamma}$ is the edge of a triangle K of \mathcal{T}_h , there exists a polynomial in \mathbf{x}_h which is equal to $\pi_h^2 w$ on E and vanishes on $\bar{\gamma} \setminus E$; using this polynomial and the definition of π_h^2 , we have

$$\|\pi_h^2 w\|_{0,E}^2 = \int_E w(y) (\pi_h^2 w)(y) dy \leq \|w\|_{0,\infty,E} \|\pi_h^2 w\|_{0,1,E}.$$

Now, applying Lemma 4.3, we obtain

$$\|R_h \pi_h^2 w\|_{1,K} \leq \|\pi_h^2 w\|_{0,E} / \|\pi_h^2 w\|_{0,1,E} \leq c \|w\|_{0,\infty,\gamma}.$$

Using Lemma 4.4, we derive

$$\|R_h \pi_h^2 w\|_{1,k} \leq c \|w\|_{s,E}^{1/2} \|w\|_{1-s,E}^{1/2}.$$

Adding up this inequality on all triangles K of \mathcal{T}_h and using the Cauchy-Schwarz inequality, we obtain (4.12).

Next, we consider the operator π_h^1 . Following [CT] and using the theory of orthogonal polynomials, we can easily find a polynomial ψ in $P_k([0,1])$ such that

$$(4.13) \quad \left[\begin{array}{l} \forall \phi \in P_k([0,1]) / \phi(0) = \phi(1) = 0, \int_0^1 \psi(y)\phi(y)dy = 0, \\ \psi(0) = 0 \quad \text{and} \quad \psi(1) = 1. \end{array} \right.$$

We denote by $(0, y_i)$, $1 \leq i \leq M$, the points of ξ_h^0 , with $0 < y_1 < \dots < y_m < 1$. Setting $y_0 = 0$ and $y_{M+1} = 1$, we denote by $E_{i+1/2}$, $0 \leq i \leq M$, the edge with vertices $(0, y_i)$ and $(0, y_{i+1})$. Finally, we define the function ψ_i , $1 \leq i \leq M$, by

$$(4.14) \quad \psi_i(y) = \begin{cases} \psi((y - y_{i-1})/(y_i - y_{i-1})) & \text{if } y \in E_{i-1/2}, \\ \psi((y - y_{i+1})/(y_i - y_{i+1})) & \text{if } y \in E_{i+1/2}, \\ 0 & \text{elsewhere.} \end{cases}$$

It is an easy matter to see that the set $\{\psi_i, 1 \leq i \leq M\}$ is a basis of x_h^1 .

We next establish the following lemma.

Lemma 4.6: The following stability property holds for any i , $1 \leq i \leq M$,

$$(4.15) \quad \|R_h \psi_i\|_{1, \Omega} \leq c.$$

Proof: As a result of (2.2), the support of ψ_i is made of a finite number of triangles K of T_h , bounded independently of h (see [B, Rem. 3.2]). Let K be such a triangle, and assume that F_K maps the point $(0,1)$ onto $(0, y_i)$.

1) If K meets γ only in $(0, y_1)$, $R_h \psi_1|_K$ coincides with the Lagrange polynomial q associated with this point, hence

$$\|R_h \psi_1\|_{1,K} \leq c \|B_K^{-1}\| |(\det B_K)|^{1/2} \|q\|_{1,\hat{K}} \leq c'.$$

2) If K has an edge E contained in $\bar{\gamma}$, we introduce the polynomial ψ^* of $P_k(\hat{K})$ which coincides with ψ on $[0,1] \times \{0\}$ and vanishes at any point of \hat{E}_K which does not belong to \hat{E} , i.e., at any point with coordinates $(j/k, \ell/k)$, $0 \leq j \leq k$, $1 \leq \ell \leq k - j$; then, $R_h \psi_1|_K$ coincides with $\psi^* \circ F_K^{-1}$ and

$$\|R_h \psi_1\|_{1,K} \leq c \|B_K^{-1}\| |(\det B_K)|^{1/2} \|\psi^*\|_{1,\hat{K}} \leq c'.$$

completing the proof.

Let w be any function in $L^2(\gamma)$. Setting $\pi_h^1 w = \sum_{i=1}^M \lambda_i \psi_i$, we note that the vector of coefficients $\lambda = (\lambda_i)_{1 \leq i \leq M}$ is given by

$$(4.16) \quad G\lambda = \mu,$$

where $G = (g_{ij})_{1 \leq i, j \leq M}$ is the square matrix of order M defined by

$$(4.17) \quad \forall (i, j), 1 \leq i, j \leq M, g_{ij} = \int_0^1 \psi_i(y) \psi_j(y) dy,$$

and where $\mu = (\mu_i)_{1 \leq i \leq M}$ is the vector of \mathbb{R}^M given by

$$(4.18) \quad \forall i, 1 \leq i \leq M, \mu_i = \int_0^1 w(y) \psi_i(y) dy.$$

It is easy to see that the matrix G is symmetric and tridiagonal. Its coefficients have been computed in [CT, Lemma 2]:

$$(4.19) \quad \begin{aligned} g_{ii} &= (y_{i+1} - y_{i-1})/k(k+2) \quad \text{and} \\ g_{i,i+1} &= (-1)^{k-1} (y_{i+1} - y_i)/k(k+1)(k+2). \end{aligned}$$

Using the arguments of [CT], we prove the following lemma.

Lemma 4.7: With the notation (4.16) to (4.18), the following estimate holds

$$(4.20) \quad \sum_{i=1}^M \lambda_1^2 \leq c \sum_{i=1}^M \mu_i^2 / (y_{i+1} - y_{i-1})^2.$$

Proof: Let D be the diagonal matrix with coefficients $d_i = g_{ii}$, $1 \leq i \leq M$. We have $G = D(\text{Id} + K)$, where K is a tridiagonal matrix with a zero diagonal; moreover, all the coefficients of K are bounded by $1/(k+1)$. If $\|\cdot\|_2$ denotes the matrix norm associated with the euclidean norm on \mathbb{R}^M , it has been computed in [CT] that, for any integer $\ell \geq 1$,

$$\|K^\ell\|_2 \leq (2\ell + 1)^{1/2} (k+1)^{-\ell}.$$

Hence, we deduce

$$\|(\text{Id} + K)^{-1}\|_2 \leq 1 + \sum_{\ell \in \mathbb{N}} \|K^\ell\|_2 \leq 1 + \sum_{\ell \in \mathbb{N}} (2\ell + 1)^{1/2} (k+1)^{-\ell},$$

so that $\|(\text{Id} + K)^{-1}\|_2$ is bounded by a constant depending only on k . This proves the lemma, since

$$\lambda = (\text{Id} + K)^{-1} D^{-1} \mu.$$

We are finally in a position to prove the following lemma.

Lemma 4.8: Let s be a real number, $0 \leq s < 1/2$. The operator $R_h \pi_h^1$ satisfies the following stability property: for any function w in $H^{1-s}(\gamma)$,

$$(4.21) \quad \|R_h \pi_h^1 w\|_{1, \Omega^-} \leq c \|w\|_{s, \gamma}^{1/2} \|w\|_{1-s, \gamma}^{1/2}.$$

Proof: Let w be any function in $H^{1-s}(\gamma)$. Setting $\pi_h^1 w = \sum_{i=1}^M \lambda_i \psi_i$, we have

$$R_h \pi_h^1 w = \sum_{i=1}^M \lambda_i R_h \psi_i;$$

we note that each ψ_i , $1 \leq i \leq M$, is orthogonal in $H^1(\Omega^-)$ to any ψ_j , $1 \leq j \leq M$, $i \neq j$, but at most two, whence

$$\|R_h \pi_h^1 w\|_{1, \Omega^-}^2 \leq 3 \sum_{i=1}^M \lambda_i^2 \|R_h \psi_i\|_{1, \Omega^-}^2.$$

Lemma 4.6 then yields

$$\|R_h \pi_h^1 w\|_{1, \Omega^-}^2 \leq c \sum_{i=1}^M \lambda_i^2.$$

Using Lemma 4.7 with the notation (4.18) we obtain

$$\|R_h \pi_h^1 w\|_{1, \Omega^-}^2 \leq c \sum_{i=1}^M \mu_i^2 / (y_{i+1} - y_i)^2.$$

Noting that the support of ψ_i is $E_{i-1/2} \cup E_{i+1/2}$, we compute

$$\mu_i \leq \|w\|_{0,\infty,(y_{i-1}, y_{i+1})} \|\psi_i\|_{0,1,\gamma} = (y_{i+1} - y_{i-1}) \|w\|_{0,\infty,(y_{i-1}, y_{i+1})} \|\psi\|_{0,1,(0,1)},$$

so that

$$\|R_h \pi_h^1 w\|_{1,\Omega^-}^2 \leq c \sum_{i=1}^M \|w\|_{0,\infty,(y_{i-1}, y_{i+1})}^2 \leq c \sum_{i=1}^M \|w\|_{0,\infty,E_{i-1/2}}^2.$$

Thanks to Lemma 4.4, we derive

$$\|R_h \pi_h^1 w\|_{1,\Omega^-}^2 \leq c \sum_{i=1}^M \|w\|_{s,E_{i-1/2}} \|w\|_{s,E_{i-1/2}},$$

and using the Cauchy-Schwarz inequality gives the result

Proposition 4.9: For any function u in $H_0^1(\Omega)$ such that the pair u^* belongs to $H^\ell(\Omega^-) \times H^\sigma(\Omega^+)$, where ℓ and σ are real numbers, $2 \leq \ell \leq k+1$ and $\sigma \geq 2$, there exists a pair v_δ^* in V_δ^1 such that

$$(4.22) \quad \|u^* - v_\delta^*\| \leq c \{h^{\ell-1} \|u^-\|_{\ell,\Omega^-} + N^{1-\sigma} \|u^+\|_{\sigma,\Omega^+}\}.$$

Proof: Let s be a real number, $0 < s < 1/2$; we take $\rho = 3/2-s$. We choose v_δ^* as in (4.4) with q_h equal to $R_h \pi_h(\Pi_h^\rho u^+ - T_h u^-)$. Then, we have

$$\pi_h(T_h u^- + q_h) - \pi_h \Pi_N^\rho u^+ = 0,$$

so that v_δ^* belongs to V_δ^1 . Moreover, we compute

$$\begin{aligned}
 \|u^* - v_\delta^*\| &\leq \|u^- - T_h u^-\|_{1,\Omega^-} + \|R_h \pi_h (\Pi_N^\rho u^+ - T_h u^-)\|_{1,\Omega^-} + \|u^+ - \Pi_N^\rho u^+\|_{1,\Omega^+} \\
 &\leq \|u^- - T_h u^-\|_{1,\Omega^-} + \|R_h \pi_h (u^- - T_h u^-)\|_{1,\Omega^-} + \|R_h \pi_h (u^+ - \Pi_N^\rho u^+)\|_{1,\Omega^-} \\
 &\quad + \|u^+ - \Pi_N^\rho u^+\|_{1,\Omega^+}.
 \end{aligned}$$

Lemmas 4.5 and 4.8, together with (4.8), imply

$$\begin{aligned}
 \|u^* - v_\delta^*\| &\leq \|u^- - T_h u^-\|_{1,\Omega^-} + \|u^- - T_h u^-\|_{s,\gamma}^{1/2} \|u^- - T_h u^-\|_{1-s,\gamma}^{1/2} \\
 &\quad + \|u^+ - \Pi_N^\rho u^+\|_{s,\gamma}^{1/2} \|u^+ - \Pi_N^\rho u^+\|_{1-s,\gamma}^{1/2} + \|u^+ - \Pi_N^\rho u^+\|_{1,\Omega^+}.
 \end{aligned}$$

Using the trace theorem, we obtain

$$\begin{aligned}
 \|u^* - v_\delta^*\| &\leq \|u^- - T_h u^-\|_{1,\Omega^-} + \|u^- - \mathcal{I}_h u^-\|_{s+1/2,\Omega^-}^{1/2} \|u^- - \mathcal{I}_h u^-\|_{3/2-s,\Omega^-}^{1/2} \\
 &\quad + \|u^+ - \Pi_N^\rho u^+\|_{s+1/2,\Omega^+}^{1/2} \|u^+ - \Pi_N^\rho u^+\|_{3/2-s,\Omega^+}^{1/2} + \|u^+ - \Pi_N^\rho u^+\|_{1,\Omega^+}.
 \end{aligned}$$

Finally, estimates (3.2) and (4.2), together with an interpolation argument, the trace theorem and Lemma 4.1 prove the result.

Remark 4.10: Here also, the error is better for the integral matching condition than for the punctual one. Indeed, in (4.22), the two discretization parameters are involved in a completely independent way.

5. FINAL ESTIMATES AND CONCLUSION

First, we recall an estimate which follows at once from a standard result in spectral methods [CQ2, Lemma 3.2] [MQ, Formula (3.22)].

Lemma 5.1: For any function f in $L^2(\Omega)$ such that the function f^+ belongs to $H^\rho(\Omega^+)$, where ρ is a real number > 1 , the following estimate holds for any $w^* = (w_h, w_N)$ in V_δ

$$(5.1) \quad (f^*, w_\delta^*) - (f^*, w_\delta^*)_\delta \leq cN^{1-\rho} \|f^+\|_{\rho, \Omega^+} \|w_N\|_{0, \Omega^+}.$$

Our main results are stated in the two following theorems.

Theorem 5.2: Assume that the solution u of problem (1.1) is such that the pair u^* belongs to $H^2(\Omega^-) \times H^\sigma(\Omega^+)$, where σ is a real number ≥ 2 . Assume moreover that the function f of $L^2(\Omega)$ is such that the function f^+ belongs to $H^\rho(\Omega^+)$, where ρ is a real number > 1 . Then, in the case of the punctual matching condition, the solutions u and u_δ of problems (1.1) and (2.15) satisfy

$$(5.2) \quad \|u^* - u_\delta^*\| \leq c \{ hN^{1/2} \sqrt{\log N} \|u^-\|_{2, \Omega^-} + (h^{\sigma-1} + N^{1-\sigma}) \|u^+\|_{\sigma, \Omega^+} + N^{1-\rho} \|f^+\|_{\rho, \Omega^+} \}.$$

Theorem 5.3: Assume that the solution u of problem (1.1) is such that the pair u^* belongs to $H^\ell(\Omega^-) \times H^\sigma(\Omega^+)$, where ℓ and σ are real numbers, $2 \leq \ell \leq k+1$ and $\sigma \geq 2$. Assume moreover that the function f of $L^2(\Omega)$ is such that the function f^+ belongs to $H^\rho(\Omega^+)$, where ρ is a real

number > 1 . Then, in the case of the integral matching condition, the solutions u and u_δ of problems (1.1) and (2.15) satisfy

$$(5.3) \quad \|u^* - u_\delta^*\| \leq c \{ h^{\ell-1} \|u^-\|_{\ell, \Omega^-} + N^{1-\sigma} \|u^+\|_{\sigma, \Omega^+} + N^{1-\rho} \|f^+\|_{\rho, \Omega^+} \}.$$

Proof: We set $\tilde{\delta} = (h, (N-1)^{-1})$. Of course, we apply Proposition 2.5 and, in (2.20), we choose $v_\delta^* = (v_h, v_N)$ equal to the pair defined in Propositions 4.2 and 4.9 respectively, but with δ replaced by $\tilde{\delta}$. Since v_N belongs to $Q_{N-1}(\bar{\Omega}^+)$ and the quadrature formula (2.12) is exact on all polynomials of degree $\leq 2N-1$, this implies that, for any w_δ^* in V_δ ,

$$a(v_\delta^*, w_\delta^*) = a_\delta(v_\delta^*, w_\delta^*).$$

Then the estimates (5.2) and (5.3) follow from (2.20), Propositions 3.6 and 3.9 respectively, Propositions 4.2 and 4.9 respectively, and Lemma 5.1.

By a classical duality method, it is possible to derive an improved estimate for $\|u - u_\delta\|_{0, \Omega}$ in the case of the integral matching condition.

Proposition 5.4: Under the assumptions of Theorem 5.3, in the case of the integral matching condition, the solutions u and u_δ of problems (1.1) and (2.15) satisfy

$$(5.4) \quad \|u - u_\delta\|_{0, \Omega} \leq c \{ h^{\ell-1} (h + N^{-1}) \|u^-\|_{\ell, \Omega^-} + N^{1-\sigma} (h + N^{-1}) \|u^+\|_{\sigma, \Omega^+} + N^{1-\rho} \|f^+\|_{\rho, \Omega^+} \}.$$

Proof: We have

$$\|u - u_\delta\|_{0,\Omega} = \sup_{g \in L^2(\Omega)} \int_{\Omega} (u - u_\delta)(\mathbf{x})g(\mathbf{x})d\mathbf{x} / \|g\|_{0,\Omega}.$$

Let g be any function in $L^2(\Omega)$. The unique solution w in $H_0^1(\Omega)$ of the problem

$$-\Delta w = g \text{ in } \Omega,$$

(5.5)

$$w = 0 \text{ on } \partial\Omega,$$

satisfies

$$(5.6) \quad \|w\|_{2,\Omega} \leq c\|g\|_{0,\Omega}.$$

Setting $u_\delta^* = (u_h, u_N)$, we compute

$$\int_{\Omega} (u - u_\delta)(\mathbf{x})g(\mathbf{x})d\mathbf{x} = a(u^* - u_\delta^*, w^*) + \int_{\Gamma} (\partial w / \partial n)(0, y)(u_N - u_h)(0, y)dy.$$

Hence, for any w_δ^* in $V_{\tilde{\delta}}$, with $\tilde{\delta} = (h, (N-1)^{-1})$, we have

$$\begin{aligned} \int_{\Omega} (u - u_\delta)(\mathbf{x})g(\mathbf{x})d\mathbf{x} &= a(u^* - u_\delta^*, w^* - w_\delta^*) + (f^*, w_\delta^*) - (f^*, w_\delta^*)_\delta \\ &\quad + \int_{\Gamma} (\partial w / \partial n)(0, y)(u_N - u_h)(0, y)dy. \end{aligned}$$

Choosing w_δ^* as defined in Proposition 4.9 and using Lemma 5.1, we obtain

$$(5.7) \quad \int_{\Omega} (u - u_{\delta})(\mathbf{x})g(\mathbf{x})d\mathbf{x} \leq c\{(h + N^{-1})\|u^* - u_{\delta}^*\| + N^{1-\rho}\|f^+\|_{\rho, \Omega^+}\}\|w\|_{2, \Omega}$$

$$+ \int_{\Gamma} (\partial w / \partial n)(0, y)(u_N - u_h)(0, y)dy.$$

It remains to estimate this last term. But we note that $u_h|_{\Gamma}$ is equal to $\pi_h u_N$, so that

$$\int_{\Gamma} (\partial w / \partial n)(0, y)(u_N - u_h)(0, y)dy$$

$$= \int_{\Gamma} [(\partial w / \partial n) - \pi_h(\partial w / \partial n)](0, y)(u_N - \pi_h u_N)(0, y)dy$$

$$= \int_{\Gamma} [(\partial w / \partial n) - \pi_h(\partial w / \partial n)](0, y)[(u - \pi_h u) - (id - \pi_h)(u - u_N)](0, y)dy$$

$$\leq \|(\partial w / \partial n) - \pi_h(\partial w / \partial n)\|_{0, \Gamma} (\|u - \pi_h u\|_{0, \Gamma} + \|(id - \pi_h)(u - u_N)\|_{0, \Gamma}).$$

Using (3.9) and (3.10) yields

$$\int_{\Gamma} (\partial w / \partial n)(0, y)(u_N - u_h)(0, y)dy \leq c h^{1/2} \|\partial w / \partial n\|_{1/2^*, \Gamma}$$

$$(h^{\ell-1/2} \|u^-\|_{\ell, \Omega^-} + h^{1/2} \|u^+ - u_N\|_{1, \Omega^+}),$$

which together with (5.6) and (5.7), gives (5.4).

The detailed analysis we have performed allows us to compare the two algorithms, corresponding to different matching conditions. Indeed, whatever the regularity of the exact solution is, we obtain better convergence results in the case of the integral matching condition. Since we have already noted

that the computational cost of the two methods is of the same order, we think that this last algorithm has to be preferred. Numerical tests³ [KP] which are currently being implemented should confirm the theoretical results.

As already stated in this paper, we are only concerned with a model problem on a model domain. However, in this very simple example, it turns out that the order of accuracy in the finite element domain is simultaneously restricted by the degree of polynomials and by the regularity of the solution, while in the spectral domain it is only limited by the regularity of the solution. That is why we believe that, in more general problems, the finite element domain must be chosen in such a way that it contains a neighborhood of both the singularities of the solution (in the case of hyperbolic equations with shock waves for instance) and the singularities of the boundary of the domain (for instance, corners of polygons which induce singularities of the solution even if the righthand member is very smooth). Then, local refinements of the mesh can be applied to improve the convergence, in a much simpler way than for the p -version of finite elements. These techniques are presently being developed by the second author.³

ACKNOWLEDGEMENT

We are very grateful to Professor J. L. Lions who accepted to present a note about our work at the Académie des Sciences of Paris. We are also greatly indebted to Vivette Girault for her kind and helpful advice.

³N. Debit, Thesis in preparation.

REFERENCES

- [BSK] I. Babuska, B. Szabo, and I. N. Katz - The p-Version of the Finite Element Method, SIAM J. Numer. Anal., Vol. 18 (1981), pp. 515-545.
- [BL] J. Berg and J. Löfström - Interpolation Spaces: An Introduction, Springer-Verlag (1976).
- [B] C. Bernardi - Optimal Finite Element Interpolation on Curved Domains, submitted to SIAM J. Numer. Anal.
- [BDM] C. Bernardi, N. Debit, and Y. Maday - Couplage de méthodes spectrale et d'éléments finis: premiers résultats d'approximation, Note aux C.R.A.S. 305, Paris, Série I, no. 8 (1987), pp. 353-356.
- [CHQZ] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang - Spectral Methods with Applications to Fluid Dynamics, Springer-Verlag (1987).
- [CQ1] C. Canuto and A. Quarteroni - Approximation Results for Orthogonal Polynomials in Sobolev Spaces, Math. Comp., Vol. 38 (1982), pp. 67-86.
- [CQ2] C. Canuto and A. Quarteroni - Spectral and Pseudo-Spectral Methods for Parabolic Problems with Nonperiodic Boundary Conditions, Calcolo, Vol. 18 (1981), pp. 197-218.

- [C] P. G. Ciarlet - The Finite Element Method for Elliptic Problems, North-Holland (1978).
- [CR] M. Crouzeix and P.-A. Raviart - Conforming and Nonconforming Finite Elements Methods for Solving the Stationary Stokes Equations I, R.A.I.R.O., Vol. 7, R3 (1973), pp. 33-75.
- [CT] M. Crouzeix and V. Thomée - The Stability in L_p and W_p^1 of the L_2 -Projection onto Finite Element Function Spaces, Math. Comp., Vol. 48 (1987), pp. 521-532.
- [DR] P. J. Davis and P. Rabinowitz - Methods of Numerical Integration, Academic Press (1985).
- [F] D. Funaro - A Multidomain Spectral Approximation of Elliptic Equations, Numer. Meth. for P.D.E., Vol. 2 (1986), pp. 187-205.
- [GO] D. Gottlieb and S. A. Orszag - Numerical Analysis of Spectral Methods, SIAM CBMS, Philadelphia (1977).
- [H] Handbook of Mathematical Functions, Dover Publications, Inc. (1970).
- [KP] K. Z. Korczak and A. T. Patera - An Isoparametric Spectral Element Method and its Application to Incompressible Two-Dimensional Flow in Complex Geometries, in Proc. 6th GAMM Conference on Numerical Methods in Fluid Mechanics, Vieweg (1986).

- [LM] J.-L. Lions and E. Magenes - Problèmes aux limites non homogènes et applications, Vol. I and II, Dunod (1968).
- [M1] Y. Maday - Analysis of Spectral Projectors in One-Dimensional Domains, submitted to Math. Comp.
- [M2] Y. Maday - Analysis of Spectral Projectors in Multi-Dimensional Domains, submitted to Math. Comp.
- [MP] Y. Maday and A. T. Patera - Spectral Element Methods for the Incompressible Navier-Stokes Equations, to appear in State of the Art Surveys in Computational Mechanics, A. Noor (ed.), A.S.M.E., (1987).
- [MQ] Y. Maday and A. Quarteroni - Legendre and Chebyshev Spectral Approximations of Burgers' Equation, Numer. Math., Vol. 37 (1981), pp. 321-332.
- [P] A. T. Patera - A Spectral Element Method for Fluid Dynamics: Laminar Flow in a Channel Expansion, J. Comp. Phys., Vol. 54 (1984), pp. 468-488.
- [Q] A. Quarteroni - Some Results of Bernstein and Jackson Type for Polynomial Approximation in L^p -Spaces, Japan J. Appl. Math., Vol. 1 (1984), pp. 173-181.
- [RT] P.-A. Raviart and J.-M. Thomas - Primal Hybrid Finite Element Methods for Second Order Elliptic Equations, Math. Comp., Vol. 31 (1977), pp. 391-413.

[SV] L. R. Scott and M. Vogelius - Norm Estimates for a Maximal Right Inverse of the Divergence Operator in Spaces of Piecewise Polynomials, Mod. Math. et Anal. Num., Vol. 19 (1985), pp. 111-143.

[S] G. Szegö - Orthogonal Polynomials, A.M.S. Publications (1978).

[T] G. Talenti - Best Constant in Sobolev Inequality, Ann. Math. Pura ed Appl., Vol. 110, Serie IV (1976), pp. 253-272.

[V] M. Vogelius - A Right-Inverse for the Divergence Operator in Spaces of Piecewise Polynomials. Application to the p-Version of the Finite Element Method, Numer. Math., Vol. 41 (1983), pp. 19-37.