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COUPLING MESHBASED AND MESHFREE METHODS BY A TRANSFER OPERATOR APPROACH

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Abstract: In contrast to the well known meshbased methods like the finite element method, meshfree methods do not rely on a mesh. The advantage of meshfree methods lies in the fact, that they need no mesh generation and can thus better cope with geometric changes and high dimensional problems. However besides their great applicability, meshfree methods are rather time consuming. Thus, it seems favorable to combine both methods, by using meshfree methods only in a small part of the domain, where a mesh is disadvantageous, and a meshbased method for the rest of the domain. We motivate, that this coupling between the two simulation techniques can be considered as saddle point problem and show the stability of this coupling. Thereby a novel transfer operator is introduced, which interacts in the transition zone, where both methods coexist.

AMS Subject Classification: 65L60, 65Pxx
Key Words: meshfree, coupling, multiscale, *inf-sup* condition

1. Introduction

The usage of computer simulations in engineering and science has increasingly became a more important and almost indispensable tool. For solving complex problems in solid mechanics related to partial differential equations, meshbased methods like the finite element method are a robust and powerful tool. In a meshbased method, the continuum domain is partitioned into subdomains by

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a mesh. This mesh serves as a topological map, which gives the neighbor relation between the different nodes. This process allows to convert a differential equation into a set of algebraic equations. They have reached great success for solving problems in solid mechanics and other related problems in engineering. However, the performance of each method depends on the problem under consideration, thus it is not surprising that the applicability of meshbased methods to engineering problems is limited. For instance, crack propagation problems meshbased methods suffer from difficulties, since after each crack propagation step a remeshing has to be done.

In contrast to meshbased methods, the comparably new class of meshfree methods only rely on a set of scattered data (particles) without any a priori knowledge of the neighbor relations. Consequently meshfree methods do not suffer from the problems related to mesh generation and mesh refinement. Due to their great flexibility, meshfree methods became very popular in the engineering [14, 27, 12] as well as in the mathematical community [2, 17] and succeeds in capturing the interest of a broader community of researchers. In the literature a large number of names for individual methods can be found. For an overview of existing meshfree methods, their theory and application we refer to [6, 28]. However besides its great and rich area of applications there are some aspects in meshfree methods which could benefit from improvements. The flexibility of mesh free methods compared to the finite elements is typically paid with a certain amount of computational burden. Moreover the meshfree lack of the Kronecker Delta property, i.e. for particles p and q and the associated basis function $\varphi_p(q)$ we have $\varphi_p(q) \neq \delta_{pq}$, which makes it hard to impose essential boundary conditions. A strategy to benefit from each method is to combine them in order to profit from either one. As an example for such a strategy, we consider a dynamic crack problem. In contrast to the finite element method, in a meshfree method no mesh needs to conform internal boundaries (cracks), moreover no remeshing for the propagation of the crack in necessary. On the other hand the computational burden of meshfree methods is harder, thus the domain with a crack is divided into two parts. In Ω^{II} is a quiet small area around the crack tip where a meshfree method is applied, since no remeshing for the propagation of the crack is necessary. Moreover there is no mesh which has to conform with the internal boundaries. For the rest of the domain, Ω^{I}, a finite element discretization can be used efficiently, cp. Figure 1.

In the literature several approaches for coupling mesh free and mesh based methods have been proposed. The possibly first approach can be found in [7] based on a ramp function approach. Thereby the authors introduce interface elements between the meshfree and meshbased method, in order to enforce the

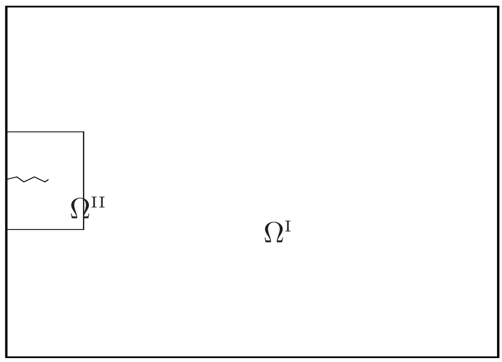

Figure 1: A $2d$ solid with a crack. The domain Ω is decomposed into a small part Ω^{II} close to the crack, where a high resolution is needed and a larger domain where a less accurate simulation suffices (Ω^{I}).

continuity condition of the displacements between both methods. In [18] a coupling with reproducing condition is employed. Analogue to the approach of [7] an interface region with a mixed approximation of a meshfree and meshbased method with the constraint, that the finite element shape functions are not modified, is used. The advantage of this approach is, that a consistency of any desired order can be achieved. For more details we refer to [20]. In the approach of [24] the total displacement field is decomposed hierarchically into a part that can be captured by finite elements and a part that can not be represented by finite elements but by a meshfree method. The disadvantage is that meshfree basis function have to be evaluated for the whole domain. This is of course oppositional to a strategy to diminish computational effort by using a meshfree method only on small parts of the domain.

This paper is structured as follows. In the second section we introduce a saddle point formulation in a continuous setting. In the next section a meshfree method on the one and a mesh based method on the other part of the domain is introduced such that a discrete saddle point formulation can be written. In the fourth section the transfer operator, which connect the meshfree part with the meshbased part over the interface is given. In the last section, we show that this choice of a transfer operator ensures a stable and well defined coupling.

2. The Continuous Setting

In the following we consider a domain $\Omega \subset \mathbb{R}^d$ which is separated into

$$\Omega = \Omega^{\mathrm{I}} \cup \Omega^{\mathrm{II}} \qquad \Xi = \Omega^{\mathrm{I}} \cap \Omega^{\mathrm{II}},$$

where $\mathrm{meas}_d(\Xi) > 0$, i.e. we consider an overlapping decomposition.

We denote by $L^2(\Omega)$ the space of 2-Lebesgue-integrable functions on Ω. Moreover, the standard scalar product is given by $(\cdot,\cdot)_{L^2(\Omega)}$ and the norm

$$\|u\|_{L^2(\Omega)} := \left(\int_\Omega |u|^2\, d\Omega\right)^{1/2}.$$

Let α be a multi index with $\|\alpha\|_1 := \sum_{i=1}^d |\alpha_i|$, Then, for a bounded domain Ω and for $m \in \mathbb{N}$ we denote by $H^m(\Omega)$ the Sobolev space, given by

$$H^m := H^{m,2}(\Omega) = \{u \in L^p(\Omega) \,|\, \text{th. ex.} \partial^\alpha u \in L^2(\Omega) \;\; \forall \|\alpha\| \leq m\}$$

and equipped with the norm

$$\|u\|_{H^m(\Omega)} := \left(\sum_{\|\alpha\|_1 \leq m} \int_\Omega |\partial^\alpha u|^2 d\Omega\right)^{1/2}$$

where the weak derivative (if it exists) is defined by

$$\int_\Omega \partial^\alpha u \varphi\, d\Omega = (-1)^{\|\alpha\|_1} \int_\Omega u \partial^\alpha \varphi\, d\Omega \qquad \forall \varphi \in C_0^\infty(\Omega).$$

Here, $C_0^\infty(\Omega)$ is the set of infinitely differentiable functions with a compact support in Ω.

We assume, that we have a static problem, moreover the energies on the respective domains are given by the bilinear forms

$$a_{\mathrm{I}}(\cdot,\cdot) \quad : H^1(\Omega^{\mathrm{I}}) \times H^1(\Omega^{\mathrm{I}}) \to \mathbb{R} \tag{1}$$
$$a_{\mathrm{II}}(\cdot,\cdot) \quad : H^1(\Omega^{\mathrm{II}}) \times H^1(\Omega^{\mathrm{II}}) \to \mathbb{R} \tag{2}$$

respectively and the external forces are given by the linear forms

$$f_{\mathrm{I}}(\cdot) \quad : H^1(\Omega^{\mathrm{I}}) \to \mathbb{R} \tag{3}$$
$$f_{\mathrm{II}}(\cdot) \quad : H^1(\Omega^{\mathrm{II}}) \to \mathbb{R}. \tag{4}$$

We furthermore define the Lagrange multiplier space by $M := (H^1(\Xi))'$, where we denote by $(H^1(\Xi))'$ the dual of $H^1(\Xi)$. Let us denote by $\mathbf{H}_{\mathrm{I,II}}$ the product space of $H^1(\Omega^{\mathrm{I}})$ and $H^1(\Omega^{\mathrm{II}})$, i.e.

$$\mathbf{H}_{\mathrm{I,II}} := H^1(\Omega^{\mathrm{I}}) \times H^1(\Omega^{\mathrm{II}}),$$

which is a Sobolev space with the product norm [1]

$$|||v||| := (\|v_I\|^2_{H^1(\Omega^{\mathrm{I}})} + \|v_{II}\|^2_{H^1(\Omega^{\mathrm{II}})})^{1/2}. \tag{5}$$

Furthermore we define

$$\mathbb{H}_{\mathrm{I,II}} := \mathbf{H}_{\mathrm{I,II}} \backslash \{0\}\,.$$

such that on this space the bilinear from

$$a(\cdot,\cdot) := a_{\mathrm{I}}(\cdot,\cdot) + a_{\mathrm{II}}(\cdot,\cdot)$$

and

$$f(\cdot) = f_{\mathrm{I}}(\cdot) + f_{\mathrm{II}}(\cdot),$$

can be given. Now, the saddle point formulation reads as follows:

Find $(u, \lambda) \in \mathbf{H}_{\mathrm{I,II}} \times M$

$$a(u,v) + b(\lambda, \begin{bmatrix} u_{\mathrm{I}} \\ u_{\mathrm{II}} \end{bmatrix}) \quad = f(u) \quad \forall v \in \mathbf{H}_{\mathrm{I,II}} \tag{6}$$

$$b(\mu, \begin{bmatrix} u_{\mathrm{I}} \\ u_{\mathrm{II}} \end{bmatrix}) \quad = 0 \quad \forall v \in \mathbf{H}_{\mathrm{I,II}} \tag{7}$$

where

$$b(\lambda, \begin{bmatrix} u_{\mathrm{I}} \\ u_{\mathrm{II}} \end{bmatrix}) := (\lambda, u_{\mathrm{I}} - u_{\mathrm{I}})_{L^2(\Xi)}.$$

2.1. The inf-sup Condition for the Continuous Saddle Point Problem

From the saddle point theory it is well known that the choice of the space M of the Lagrange multipliers is essential for the well posedness of the saddle point formulation. More precisely the space $\mathbb{H}_{\mathrm{I,II}}$ and the multiplier space have to satisfy the *inf-sup* condition [3, 10, 11], which is given by

$$\exists \beta : \forall \lambda \in M, \sup_{u \in \mathbb{H}_{I,II}} \frac{b(\lambda, u)}{|||u|||} \geq \beta \|\lambda\|_{(H^1(\Xi))'}. \tag{8}$$

Other choices of the multiplier space can result in non optimal estimates for the discretized problem. Here, we show that our choice $M = (H^1(\Xi))'$ fulfills these demands. To do so let us recall the following facts.

For the relation between an element and its dual, we need the Riesz Representation Theorem: Let $\mathcal{V}$ be a Hilbert space and let $\mathcal{V}'$ be its dual. Let $l \in \mathcal{V}'$. Then there exists a unique $u \in \mathcal{V}$ for which

$$l(v) = (v, u) \qquad \forall v \in \mathcal{V}. \tag{9}$$

In addition we have

$$\|l\|_{\mathcal{V}'} = \|u\|_{\mathcal{V}}.$$

Furthermore, let $\mathcal{V} \subset U$ be Hilbert spaces and let us assume, that the embedding $\mathcal{V} \hookrightarrow U$ is continuous and dense. We identify U' with its Riesz representation of U. Then we have the Gelfand triple

$$\mathcal{V} \subset U \subset \mathcal{V}'. \tag{10}$$

With these tools we are now prepared for the

Theorem 1. *Let us assume that there exists a bounded extension operator onto $H^1(\Omega^I)$ (cf. [29][Theorem 5, p. 181]) such that for $u \in H^1(\Xi)$ we have*

$$u_I = \mathcal{E}(u) \in H^1(\Omega^I) \tag{11}$$

with

$$C\|\mathcal{E}(u)\|_{H^1(\Omega^I)} \leq \|u\|_{H^1(\Xi)}, \tag{12}$$

then the inf-sup condition

$$\exists \beta : \forall \lambda \in M, \sup_{u \in \mathbb{H}_{I,II}} \frac{b(\lambda, u)}{|||u|||} \geq \beta \|\lambda\|_{(H^1(\Xi))'} \tag{13}$$

holds.

Proof. Since

$$\sup_{u \in \mathbb{H}_{\mathrm{I,II}}} \frac{b(\lambda, u)}{|||u|||} \geq \sup_{u_{\mathrm{I}} \in H^1(\Omega^{\mathrm{I}}) \setminus \{0\}} \frac{(\lambda, u_{\mathrm{I}})_{L^2(\Xi)}}{\|u_{\mathrm{I}}\|_{H^1(\Omega^{\mathrm{I}})}}.$$

it is adequate to show

$$\exists \beta : \forall \lambda \in M, \quad \sup_{u_{\mathrm{I}} \in H^1(\Omega^{\mathrm{I}}) \setminus \{0\}} \frac{(\lambda, u_{\mathrm{I}})_{L^2(\Xi)}}{\|u_{\mathrm{I}}\|_{H^1(\Omega^{\mathrm{I}})}} \geq \beta \|\lambda\|_{(H^1(\Xi))'}, \tag{14}$$

then (13) follows.

Due to the Gelfand triple (10), with $\mathcal{V} = L^2(\Xi)$ and $U = H^1(\Xi)$, we can write formally [30]

$$(\lambda, u)_{L^2(\Xi)} = \langle \lambda, u \rangle_{(H^1(\Xi))' \times H^1(\Xi)},$$

where $\langle \cdot, \cdot \rangle$ is the duality between $(H^1(\Xi))' \times H^1(\Xi)$. Then, by applying the Riesz Theorem (9) there exists $u_\lambda \in H^1(\Xi)$ such that

$$\langle \lambda, u \rangle_{(H^1(\Xi))' \times H^1(\Xi)} = (u_\lambda, u)_{H^1(\Xi)}, \qquad \forall u \in H^1(\Xi),$$

i.e. u_λ is the representing element of λ. Thus we write

$$(\lambda, u)_{L^2(\Xi)} = (u_\lambda, u)_{H^1(\Xi)}. \tag{15}$$

Inserting (15) into (14) yields

$$\sup_{u_{\mathrm{I}} \in H^1(\Omega^{\mathrm{I}}) \setminus \{0\}} \frac{(\lambda, u_{\mathrm{I}})_{L^2(\Xi)}}{\|u_{\mathrm{I}}\|_{H^1(\Omega^{\mathrm{I}})}} = \sup_{u_{\mathrm{I}} \in H^1(\Omega^{\mathrm{I}}) \setminus \{0\}} \frac{(u_\lambda, u_{\mathrm{I}})_{H^1(\Xi)}}{\|u_{\mathrm{I}}\|_{H^1(\Omega^{\mathrm{I}})}}$$

By choosing $u_{\mathrm{I}} = \mathcal{E}(u_\lambda)$ we obtain

$$\sup_{u_{\mathrm{I}} \in H^1(\Omega^{\mathrm{I}}) \setminus \{0\}} \frac{(u_\lambda, u_{\mathrm{I}})_{H^1(\Xi)}}{\|u_{\mathrm{I}}\|_{H^1(\Omega^{\mathrm{I}})}} \geq \frac{(u_\lambda, u_\lambda)_{H^1(\Xi)}}{\|\mathcal{E}(u_\lambda)\|_{H^1(\Omega^{\mathrm{I}})}} \tag{16}$$

$$\geq \frac{1}{C} \frac{\|u_\lambda\|^2_{H^1(\Xi)}}{\|u_\lambda\|_{H^1(\Xi)}} \tag{17}$$

$$\geq \frac{1}{C} \frac{\|u_\lambda\|_{H^1(\Xi)} \|\lambda\|_{(H^1(\Xi))'}}{\|u_\lambda\|_{H^1(\Xi)}} \tag{18}$$

$$\geq \beta \|\lambda\|_{(H^1(\Xi))'}. \tag{19}$$

Thus we have

$$\sup_{u \in \mathbb{H}_{I,II}} \frac{b(\lambda, u)}{|||u|||} \geq \beta \|\lambda\|_{(H^1(\Xi))'},$$

with $\beta = 1/C$. □

Thus the wellposedness of the saddle point problem given by (6) and (7) is ensured.

3. The Discrete Counterpart

In this section we consider the discrete counterpart of the continuous saddle point formulation. In contrast to most existing works, we do not consider the discretization of two finite element meshes with different mesh size. Here, we act on a suggestion given in the first section, by discretizing Ω_{I} with a meshbased and Ω_{II} with a meshfree method. Analogue to the continuous *inf-sup* condition (8) we have to proof an *inf-sup* condition for the discrete saddle point problem. To do so we choose for a meshfree method the moving least squares method and for the meshbased method the finite element method.

3.1. Meshbased Methods

In order to approximate a continuous displacement field u, we employ a finite element discretization of lower order. Let $\mathcal{T}^h$ denote a mesh with mesh size parameter $h > 0$, such that the family $\{\mathcal{T}^h\}_h$ is shape regular.

Here, we use Lagrangian conforming finite elements of first order (P_1) for the displacements u and denote the set of all nodes of $\mathcal{T}^h$ by $\mathcal{N}_h$. The finite element space $\mathcal{V}_h \subset H^1(\Omega)$ is spanned by the nodal basis

$$\mathcal{V}_h = \mathrm{span}_{p\in\mathcal{N}_h}\{\psi_p^h\}.$$

The Lagrangian basis functions $\psi_p^h \in \mathcal{V}_h$ are uniquely characterized by the Kronecker-delta property

$$\psi_p^h(q) = \delta_{pq}\,, \qquad p, q \in \mathcal{N}_h\,, \tag{20}$$

where δ_{pq} is the Kronecker-delta. Any function $u_h \in \mathcal{V}_h(\Omega)$ can uniquely be written as

$$u_h = \sum_{p\in\mathcal{N}_h} \bar{u}_p \psi_p^h\,, \tag{21}$$

where $(\bar{u}_p)_{p\in\mathcal{N}_h} \in \mathbb{R}^{d\cdot|\mathcal{N}_h|}$, $\bar{u}_p \in \mathbb{R}^d$, is the coefficient vector. We can identify each element of $\mathcal{V}_h$ with its coefficient vector $(u_p)_{p\in\mathcal{N}_h}$. In the forthcoming, we omit the superscript h whenever possible.

3.2. Meshfree Methods

As aforementioned, meshfree methods do not use mesh which makes them more applicable for problems with highly localized behavior. The Smoothed Particle

Hydrodynamics [25, 26] - initially developed for solving astrophysical hydrodynamical phenomena without boundaries - can be considered as the first meshfree method. Since then much effort in the further developments has been made: Reproducing Kernel Particle Method [23], Element-Free Galerkin Method [5], Moving Least Squares Method (MLS) [22], Partition of Unity Method [2] and HP Clouds [15]. We mark, that this list is far from exhaustive.

Most of the approaches for the construction of a meshfree approximation lead back to basically three methods. These are the kernel methods, the moving least squares methods and the Partition of Unity Methods.

However, even though that these methods differ in their form, i.e. kernel method use an integral representation whereas MLS are based on sums, they share commonalities. Even more: any discrete Kernel method that is consistent is identical to a related MLS approximation and any MLS approximation can serve as a Partition of Unity Method. Here we consider the Moving Least Squares Method, introduced by Shepard and further developed by Lancaster and Salkauskas.

Let us assume that we have the given values $u_i = u(x_i)$, $i = 1, ..., N$ of a scattered data set $\chi = \{x_1, ..., x_N\}$ where u is some smooth function.

Our aim is to find a function $u^\delta : \overline{\Omega} \to \mathbb{R}$, such that

$$u^\delta(x_i) \approx u_i \text{ for all } i = 1, ..., N. \tag{22}$$

In order to construct a MLS fit, we consider the approximation space being the space $\mathbb{P}_m$ of polynomials with the basis $\{P_i\}_{i=1}^n$ of degree $n := \binom{m+d}{d}$ in d variables and a set of non-negative weight functions

$$w_i : \mathbb{R}^d \to \mathbb{R}_0^+$$

and the dilatation parameter ρ_i of $w_i(x) = w\left(\frac{x-x_i}{\rho_i}\right)$. We furthermore claim that

$$\bigcup_{i=1}^N \text{supp } w_i \supset \Omega\,.$$

Now, we minimize for each fixed x the quadratic functional

$$J(\wp)(x) = \sum_{i=1}^N w_i(x)(u_i - \wp(x_i))^2 \tag{23}$$

over all $\wp \in \mathbb{P}_m$.

In order to minimize (23), we set the derivative of (23) equal to zero and obtain the system of equations

$$\sum_{i=1}^{N} w_i(x)u_i p_k(x_i) = \sum_{i=1}^{N} w_i(x) \sum_{i=1}^{n} p_l(x_i)p_k(x_i)c(x) \qquad k = 1, ..., n. \tag{24}$$

In order to rewrite (24) in matrix from, we define the vectors
$p(x) := [p_1(x)\ p_2(x)\ \cdots\ p_n(x)]^T$, $w(x) := [w_1(x)\ w_2(x)\ \cdots\ w_N(x)]^T$,
$b := [u_1\ u_2\ \cdots\ u_N]^T$ and $a(x) := [a_1(x)\ a_2(x)\ ...\ a_n(x)]^T$.
With these definitions (24) can be written as

$$A(x)a(x) = B(x)b, \tag{25}$$

where the matrices $B := (B_{ik})_{\substack{i=1,...,N \\ k=1,...,n}}$ and $A(x) := (A_{lk})_{l,k=1,...,n}$, are given by

$$B_{ik} = w_i(x)p_k(x) \text{ and } A_{lk} = \sum_{i=1}^{N} p_l(X_i)w_i(x)p_l(x_i)$$

respectively.

The above matrix $A(x)$ is also known as Gram's matrix. The minimizer $u^\delta(x)$ of (23) is given by the linear combination

$$u^\delta(x) = \sum_{i=1}^{N} u_i \varphi_i(x), \tag{26}$$

where the shape functions φ_i are

$$\varphi_i(x) = p^T(x_i)[A(x)]^{-1} w_i(x)p(x_i). \tag{27}$$

Thus, with the local approximation spaces given by $\mathcal{V}_i \subset H^1(\Omega \cap \text{supp}(w_i))$ we obtain the global approximation space

$$\mathcal{V}_\delta = \sum_i \varphi_i \mathcal{V}_i = \{\sum_i \varphi_i v_i | v_i \in \mathcal{V}_i\} \subset H^1(\Omega).$$

Remarks. 1. Since the approximation space in the MLS is the space of polynomials the Gram matrix $A(x)$ is called moment matrix for the weights. Note, that this matrix has to be inverted for the evaluation of a MLS shape function. In order to guarantee that the matrix $A(x)$ is invertible, we have to claim that the sets $\chi_N(\Omega) \cap \text{supp}\ (w_i)$ are $\mathbb{P}_m$ unisolvent in the sense that zero is the only function in $\mathbb{P}_m$ that vanishes on $\chi_N(\Omega) \cap \text{supp}\ (w_i)$.

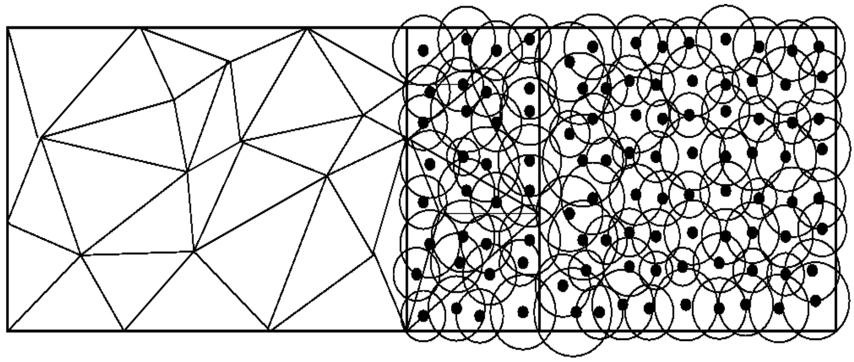

Figure 2: The domain Ω with two different discretizations. On Ω^{I} a meshbased method is employed and on Ω^{II} a meshfree method is used. In the intersection Ξ both methods coexist.

2. The choice of the dilatation parameter ρ is closely related to the accuracy and stability of the approximation.

With

$$\mathcal{V}_{h,\delta} := \mathcal{V}_h \times \mathcal{V}_\delta$$

equipped with the norm $|||\cdot|||$, we are now in a position to give the discrete saddle point formulation:

Find $(u_{h,\delta}, \lambda_h) \in \mathcal{V}_{h,\delta} \times \mathcal{M}_h$

$$a(u_{h,\delta}, v_{h,\delta}) + b(\lambda_h, \begin{bmatrix} u_h \\ u_\delta \end{bmatrix}) = f(u_{h,\delta}) \quad \forall v_{h,\delta} \in \mathcal{V}_{h,\delta} \tag{28}$$

$$b(\mu_h, \begin{bmatrix} u_h \\ u_\delta \end{bmatrix}) = 0 \quad \forall \mu_h \in \mathcal{M}_h\,. \tag{29}$$

The multiplier space $\mathcal{M}_h$ is equipped with the norm $\|\cdot\|_{(H^1(\Xi))'}$.

4. Construction and Analysis of the Transfer Operator

In this section we introduce the transformation operator π which allows for a transfer from the meshfree discretization to the meshbased discretization. Going along with this construction we show the discrete *inf-sup* condition, where we employ techniques from domain decomposition methods.

In order to impose the constraints, we define the following L^2 Projection $\pi_h : L^2(\overline{\Xi}) \to \mathcal{V}_h$ by

$$\pi_h(w) \in \mathcal{V}_h \ : \ (\pi_h(w), \mu)_{L^2(\Omega)} = (w, \mu)_{L^2(\Omega)} \qquad \forall \mu \in \mathcal{M}_h\,, \tag{30}$$

where, the multiplier space $\mathcal{M}_h$ is defined by

$$\mathcal{M}_h = \operatorname{span}\{\mu_s \,|\, s \in \mathcal{N}_h\}\,. \tag{31}$$

Here, the basis functions μ_s, $s \in \mathcal{N}_h$ are assumed to have the local support $\operatorname{supp}\mu_s \subseteq \operatorname{supp}\lambda_s|_{\overline{\Xi}}$. As is the case in the mortar setting, there are several possible choices for the basis functions μ_s of $\mathcal{M}_h$. We follow the standard approach, see, e.g. [8, 4] by setting

$$\mu_s = \psi_s|_{\overline{\Xi}}\,, \quad s \in \mathcal{N}_h\,. \tag{32}$$

Remark. Of course, for the construction of the L^2 projection the domain and in particular the boundary of the domain is crucial. Since here, the main concern is the general construction of the transfer operator from the coarse to the fine scale, we neglect the domain aspect.

The algebraic representation of (30) is given by

$$\pi(w) = M^{-1}Rw,$$

where we have identified $\pi(w)$ and w with their respective coefficients. For the first matrix R, we need to evaluate integrals of the form $R_{p\alpha} = \int \mu_p\varphi_\alpha$. The matrix M with entries $M_{qp} = \int \psi_q\mu_q$ has the character of a finite element mass matrix. For a fast evaluation of matrices, especially for R we refer to [16].

4.1. The Discrete inf-sup Condition

In this section we elaborate the technical details for the coupling of the two discretizations, thereby we examine the stability of the L^2 projection between the mesh free partition of unity method and the mesh based finite element method.

Here, we want to ensure the stability of the discrete counterpart of the saddle point problem. More precisely, we want to show, that the discrete *inf-sup* condition holds, i.e.

$$\sup_{u_{h,\delta} \in \mathcal{V}_{h,\delta}\setminus\{0\}} \frac{b(\lambda_h, \begin{bmatrix} u_h \\ u_\delta \end{bmatrix})}{|||u_{h,\delta}|||} \geq \beta \|\lambda_h\|_{\mathcal{M}_h}, \tag{33}$$

where β is independent of h and δ.

For the proof of the *inf-sup* condition, we follow the approach of Fortin [19, 11], by succeeding the following steps:

Step 1 Defining an operator $\hat{\pi} : \mathbf{H}_{I,II} \to \mathcal{V}_{h,\delta}$ with

$$b(v - \hat{\pi} v, \mu_h) = 0 \qquad \forall \mu_h \in \mathcal{M}_h.$$

Step 2 Showing, that π_h is $H^1(\Xi)$-stable.

Step 3 Showing, that $|||\hat{\pi}||| \le c$, with $c > 0$ and c is independent of h.

Step 1 We define for $u = (u_{\mathrm{I}}, u_{\mathrm{II}})^T$

$$\hat{\pi}(u) = \begin{pmatrix} \pi_h(u_{\mathrm{I}} - u_{\mathrm{II}}) \\ 0 \end{pmatrix},$$

where π_h is given in (30). Thus we have, that

$$b(\mu_h, u - \hat{\pi} u) = b(\mu_h, \begin{bmatrix} u_{\mathrm{I}} \\ u_{\mathrm{II}} \end{bmatrix} - \hat{\pi} u) = (\mu_h, u_{\mathrm{I}} - u_{\mathrm{II}} - \pi_h(u_{\mathrm{I}} - u_{\mathrm{II}}))_{L^2(\Xi)} = 0.$$

Step 2 To show the H^1 stability for a wider class of multiplier spaces we follow [21]. This class of multiplier spaces is characterized by the following assumptions:

M1 The discrete multiplier space $\mathcal{M}_h$ contains constant functions.

M2 We have that $\dim(\mathcal{V}_h|_\Xi) = \dim(\mathcal{M}_h)$.

M3 There exists a constant C independent of h, such that

$$\|u_h\|_{L^2(\Xi)} \le C \sup_{\lambda_h \in \mathcal{M}_h} \frac{(u_h, \lambda_h)_{L^2(\Xi)}}{\|\lambda_h\|_{L^2(\Xi)}} \qquad \forall u_h \in \mathcal{V}_h$$

Corollary 2. *Let the triangulation $\mathcal{T}^h$ be globally quasi uniform, that is $h_t \ge c\bar{h}$ for all $t \in \mathcal{T}^h$. Moreover let the domain Ξ be polygonal, and M1-M3 hold. Then*

$$\|\pi_h u\|_{H^1(\Xi)} \le C \|u\|_{H^1(\Xi)}, \qquad u \in H^1(\Xi)$$

where C does not depend on the mesh size.

Proof. By our assumption, that the mesh is quasi uniform, there exists an operator $Q : L^2(\Xi) \to \mathcal{V}_h$ (Clément interpolation), such that [13]

$$\|Qu\|^2_{H^1(\Xi)} + \sum_{t\in\mathcal{T}^h} h_t^{-2}\|(I-Q)u\|^2_{L^2(t)} \le C\|u\|^2_{H^1(\Xi)}, \tag{34}$$

where h_t denotes the diameter of t. Let us for fixed $u \in H^1(\Xi)$ show, that the operator Q full fills

$$\|(\pi_h - Q)u\|_{H^1(\Xi)} \le C\|u\|_{H^1(\Xi)}. \tag{35}$$

By the inverse inequality, we have

$$\|(\pi_h - Q)u\|^2_{H^1(\Xi)} \le h^{-2}\|(\pi_h - Q)u\|^2_{L^2(\Xi)}. \tag{36}$$

Together with M3 and by exploiting that π_h is a L^2 projection, we obtain

$$\begin{aligned}\|(\pi - Q)u\|_{L^2(\Xi)} &\le C\sup_{\theta_h\in\mathcal{M}_h}\tfrac{((\pi_h-Q)u,\theta_h)_{L^2(\Xi)}}{\|\theta_h\|_{L^2(\Xi)}}\\ &= C\sup_{\theta_h\in\mathcal{M}_h}\tfrac{((I-Q)u,\theta_h)_{L^2(\Xi)}}{\|\theta_h\|_{L^2(\Xi)}}.\end{aligned}$$

We then furthermore have that

$$\begin{aligned}&C\sup_{\theta_h\in\mathcal{M}_h}\frac{((I-Q)u,\theta_h)_{L^2(\Xi)}}{\|\theta_h\|_{L^2(\Xi)}}\\ &\le C\sup_{\theta_h\in\mathcal{M}_h}\frac{\sum_{t\in\mathcal{T}} h_t^{-2}\|(I-Q)u\|_{L^2(t)}h_t^2\|\theta_h\|_{L^2(t)}}{\|\theta_h\|_{L^2(\Xi)}}\\ &\le C\sup_{\theta_h\in\mathcal{M}_h}\frac{\left(\sum_{t\in\mathcal{T}} h_t^{-2}\|(I-Q)u\|^2_{L^2(t)}\right)^{1/2}\left(\sum_{t\in\mathcal{T}} h_t^2\|\theta_h\|^2_{L^2(t)}\right)^{1/2}}{\|\theta_h\|_{L^2(\Xi)}}\\ &\le Ch\|u\|_{H^1(\Xi)}.\end{aligned}$$

In the last step we used (34), in particular we only need, that $\sum_{t\in\mathcal{T}} h_t^{-2}\|(I-Q)u\|^2_{L^2(t)} \le C\|u\|^2_{H^1(\Xi)}$.

Thus we obtain (35). By the triangle inequality and (34) we have

$$\begin{aligned}\|\pi_h u\|^2_{H^1(\Xi)} &\le \|(\pi_h - Q)u\|^2_{H^1(\Xi)} + \|Qu\|^2_{H^1(\Xi)}\\ &\le C\|u\|^2_{H^1(\Xi)}\end{aligned}$$

□

Step 3 We can show, that

$$|||\hat{\pi}||| \leq c \qquad c \neq c(h), c > 0.$$

To do so, we consider

$$\begin{aligned}
|||\hat{\pi}||| &= \sup_{v \in \mathbb{H}_{\mathrm{I,II}}} \frac{|||\hat{\pi} v|||}{|||v|||} \\
&= \sup_{v \in \mathbb{H}_{\mathrm{I,II}}} \frac{\|\pi_h(v_{\mathrm{I}} - v_{\mathrm{II}})\|_{H^1(\Xi)}}{(\|v_{\mathrm{I}}\|^2_{H^1(\Omega^{\mathrm{I}})} + \|v_{\mathrm{II}}\|_{H^1(\Omega^{\mathrm{II}})})^{1/2}} \\
&\leq \sup_{v \in \mathbb{H}_{\mathrm{I,II}}} \frac{\|\pi_h v_{\mathrm{I}}\|_{H^1(\Omega^{\mathrm{I}})} + \|\pi_h v_{\mathrm{II}}\|_{H^1(\Omega^{\mathrm{II}})}}{(\|v_{\mathrm{I}}\|^2_{H^1(\Omega^{\mathrm{I}})} + \|v_{\mathrm{II}}\|_{H^1(\Omega^{\mathrm{II}})})^{1/2}} \qquad (37) \\
&\leq C \frac{\|v_{\mathrm{I}}\|_{H^1(\Omega^{\mathrm{I}})} + \|v_{\mathrm{II}}\|_{H^1(\Omega^{\mathrm{II}})}}{(\|v_{\mathrm{I}}\|^2_{H^1(\Omega^{\mathrm{I}})} + \|v_{\mathrm{II}}\|_{H^1(\Omega^{\mathrm{II}})})^{1/2}} \qquad (38) \\
&\leq c.
\end{aligned}$$

Since $a + b \leq 2(a^2 + b^2)$, $a, b > 0$ by Young's inequality. For the step from (37) to (38) we exploited the $H^1(\Xi)$ stability of π_h. We mark, that in [9] the H^1 stability of the projection π_h has been shown, if the multiplier space is chosen as $\mathcal{M}_h = \mathcal{V}_h$.

Finally, we can now prove

Theorem 3. *Under the above assumptions, we have that for the discrete saddle point problem the inf-sup condition holds.*

Proof. Analogously to the Fortin operator [19], we have

$$\begin{aligned}
\beta \|\lambda_h\|_{(H^1(\Xi))'} &\leq \sup_{u \in \mathbb{H}_{\mathrm{I,II}}} \frac{b(\lambda_h, u)}{|||u|||} \qquad (39) \\
&= \sup_{u \in \mathbb{H}_{\mathrm{I,II}}} \frac{b(\lambda, \hat{\pi} u)}{|||u|||} \qquad (40) \\
&\leq c \sup_{u \in \mathbb{H}_{\mathrm{I,II}}} \frac{(\lambda_h, \pi_h(u_{\mathrm{I}} - u_{\mathrm{II}}))_{L^2(\Xi)}}{|||\hat{\pi} u|||} \qquad (41) \\
&= c \sup_{u_{\delta,h} \in \mathcal{V}_{h,\delta} \setminus \{0\}} \frac{b(\lambda_h, u_{\delta,h})}{|||u_{\delta,h}|||} \qquad (42)
\end{aligned}$$

□

Summing up, we have developed a new transfer operator based on a weak coupling approach. The key idea is to construct the transfer operator on the basis of weighted local averaging instead of using point wise taken values, for the coupling between a meshfree and a meshbased method. Moreover, we have shown for the static case, that our weak coupling operator for the coupling of a meshfree and a mesh based method is H^1 stable.

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