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Coupling of Element Free Galerkin and Hybrid Boundary Element methods using modified variational formulation

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Abstract

A novel method is proposed by coupling the Element Free Galerkin (EFG) and the Hybrid Boundary Element (HBE) methods to achieve solution efficiency and accuracy for stress analysis in solids. A modified variational formulation is derived for the present coupled EFG/HBE method so that the continuity and compatibility can be preserved on the interface between the domains of EFG and HBE. The coupled EFG/HBE method has been coded in FORTRAN. The validity and efficiency of the proposed method are demonstrated through a number of example problems. It is found that the present method can take advantages of both EFG and HBE methods. The present method is very easy to implement, and very flexible for obtaining displacements and stresses of desired accuracy in solids, as the efforts for meshing the problem domain have been significantly reduced due to the use of Boundary Element Method (BEM).

KEYWORDS: Meshless Method; Element Free Galerkin Method; Boundary Element Method; Stress Analysis; Numerical Analysis

1. Introduction

Meshless methods have become recently attractive alternatives for problems in computational mechanics, as it does not require a mesh to discretize the problem domain, and the approximate solution is constructed entirely in terms of a set of scattered nodes. The principle attraction of the meshless methods is the possibility of simplifying adaptivity and problems with moving boundaries and discontinuities, such as phase changes and crack propagation.

Some meshless methods are proposed and achieved remarkable progress, such as Reproducing Kernel Particle (RKP) method by Liu et al. (1995), Meshless Local Petrov-Galerkin (MLPG) method by Atluri and Zhu (1998) and Element Free Galerkin (EFG) method by Belytschko et al. (1994). The EFG method is a very promising method for the treatment of partial different equations. It has been successfully applied in a large variety of problems. However, there exists some inconvenience or disadvantages in using EFG. First, it is difficult to implement essential boundary conditions in EFG, because the shape function, which constructed by Moving Least Squares (MLS) approximation, lacks the delta function property. Second, the EFG is computationally expensive for some problems, as the MLS approximation has to be performed for each Gauss point of integration over the background integration mesh for the entire problem domain. The numerical integration can be computationally very expensive especially for problems with infinite or semi-infinite domains.

Some strategies (Zhu et al., 1998; Liu, 1999; Liu and Gu, 1999; Liu and Yang, 1998) have been developed for the alleviation of the above-mentioned problems. Coupling the EFG with other established numerical methods can also be a possible solution. For certain problems it is desirable and beneficial to combine a few methods together in order to exploit their advantages while evading their disadvantages (Liu et al., 1992). In coupling EFG with other methods, EFG is used only in the sub-domains where their unique advantages are beneficial, such as in the areas of crack growth, and Finite Element (FE) or Boundary Element (BE) method is employed in the remaining part of the domain. Some research work has been done in the coupled EFG/FE method (Belytschko and Organ, 1995; Hegen, 1996). The major difficulty of the coupling is how to satisfy the displacement compatibility condition on the interface between the domains of the two methods. Interface element methods and methods based on extension of weak forms have been so far employed in the coupled EFG/FE.

For some specific problems, the Boundary Element Method (BEM) is undoubtedly superior to the 'domain' type techniques such as FE and EFG. Therefore, the idea of combining BE with other numerical techniques is naturally of great interest in many practical problems. A coupled EFG/BE method has been recently presented by Gu and Liu (1999). An interface element is formulated and used along the interface between EFG and BE domains. The shape function used within interface element is continuous from the EFG domain across to the interface element. However the derivative of shape functions is discontinuous across the boundary. In addition, the symmetrization of the BE stiffness matrix has to be done in the coupled EFG/BE method. All these can lead to a loss of accuracy and efficiency. In this paper, the attention is focused on finding an effective approach to avoid the above-mentioned disadvantages in coupling EFG with BE methods.

In the late eighties, alternative BE formulations have been developed based on generalized variational principles. Dumont (1988) has proposed a hybrid stress BE formulation based on the Hellinger-Reissner principle. DeFigueiredo and Brebbia (1989), DeFigueiredo (1991) and Jin et al. (1996) presented a Hybrid displacement Boundary Element (HBE) formulations. The HBE formulation led to a symmetric stiffness matrix. This property of symmetry can be an added advantage in coupling the HBE with other methods.

A novel coupled EFG/HBE method for continuum mechanics problems is presented in this paper. The compatibility condition on the interface boundary is introduced into the variational formulations of EFG and HBE using Lagrange multipliers. Coupled system equations have been derived based on the variational formulation. A program of the coupled method has been developed in FORTRAN, and several numerical examples are presented to demonstrate the convergence, validity and efficiency of the coupled method.

Compared to the EFG/BE approach developed earlier by the authors (Gu and Liu, 1999), the present EFG/HBE advances mainly in the following:

a) The coupled system equations are formulated in a different but more general manner.
b) System matrices obtained by EFG/HBE are symmetric, and there is no need for operation of symmetrization.
c) The order of continuity of the shape functions obtained near the interface is higher, as a modified variational formulation is used.
d) There is no need for interface elements, and therefore mesh generation becomes much simpler, and there is no special treatment needed on the interface.

2. EFG formulation

2.1 Moving Least Squares interpolant

In this section a briefing of MLS approximation is given. More details can be found in a paper by Lancaster and Salkauskas(1981).

Consider a problem domain Ω. To approximate a function $u(\mathbf{x})$ in Ω, a finite set of $\mathbf{p}(\mathbf{x})$ called basis functions is considered in the space coordinates $\mathbf{x}^{\mathrm{T}}=[x, y]$. The basis functions in two-dimension is given by

$$\mathbf{p}^{\mathrm{T}}(\mathbf{x})=[1, x, y, x^2, xy, y^2 \ldots] \tag{1}$$

The MLS interpolant $u^h(x)$ is defined in the domain Ω by

$$u^h(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x}) a_j(\mathbf{x}) = \mathbf{p}^{\mathrm{T}}(\mathbf{x})\mathbf{a}(\mathbf{x}) \tag{2}$$

where m is the number of basis functions, the coefficient $a_j(x)$ in equation (2) is also functions of $\mathbf{x}$; $\mathbf{a}(\mathbf{x})$ is obtained at any point $\mathbf{x}$ by minimizing a weighted discrete $\mathbf{L}_2$ norm of:

$$J = \sum_{i=1}^{n} w(\mathbf{x}-\mathbf{x}_i)[\mathbf{p}^{\mathrm{T}}(\mathbf{x}_i)\mathbf{a}(\mathbf{x}) - u_i]^2 \tag{3}$$

where n is the number of points in the neighborhood of $\mathbf{x}$ for which the weight function $w(\mathbf{x}\text{-}\mathbf{x}_i)\neq 0$, and u_i is the nodal value of u at $\mathbf{x}=\mathbf{x}_i$.

The stationarity of J with respect to $\mathbf{a}(\mathbf{x})$ leads to the following linear relation between $\mathbf{a}(\mathbf{x})$ and u_i:

$$\mathbf{A}(\mathbf{x})\mathbf{a}(\mathbf{x})=\mathbf{B}(\mathbf{x})\mathbf{u} \tag{4}$$

Solving $\mathbf{a}(\mathbf{x})$ from equation (4) and substituting it into equation (2), we have

$$u^h(\mathbf{x}) = \sum_{i=1}^{n} \phi_i(\mathbf{x}) u_i \tag{5}$$

where the MLS shape function $\phi_i(\mathbf{x})$ is defined by

$$\phi_i(\mathbf{x}) = \sum_{j=1}^{m} p_j(\mathbf{x})(\mathbf{A}^{-1}(\mathbf{x})\mathbf{B}(\mathbf{x}))_{ji} \tag{6}$$

where $\mathbf{A}(\mathbf{x})$ and $\mathbf{B}(\mathbf{x})$ are the matrices defined by

$$\mathbf{A}(\mathbf{x}) = \sum_{i=1}^{n} w_i(\mathbf{x})\mathbf{p}^{\mathrm{T}}(\mathbf{x}_i)\mathbf{p}(\mathbf{x}_i) \quad w_i(\mathbf{x})=w(\mathbf{x}\text{-}\mathbf{x}_i) \tag{7}$$

$$\mathbf{B}(\mathbf{x})=[w_1(\mathbf{x})\mathbf{p}(\mathbf{x}_1), w_2(\mathbf{x})\mathbf{p}(\mathbf{x}_2),\ldots,w_n(\mathbf{x})\mathbf{p}(\mathbf{x}_n)] \tag{8}$$

2.2 Discrete equations of EFG

Consider the following two-dimensional problem of solid mechanics in domain Ω bounded by Γ:

$$\nabla\boldsymbol{\sigma}+\mathbf{b}=0 \qquad \text{in } \Omega \tag{9}$$

where σ is the stress tensor, which corresponds to the displacement field $\mathbf{u}=\{u, v\}^{\mathrm{T}}$, $\mathbf{b}$ is the body force vector, and ∇ is the divergence operator. The boundary condition are given as follows:

$$\mathbf{u}=\bar{\mathbf{u}} \qquad \text{on } \Gamma_{\mathrm{u}} \tag{10}$$

$$\sigma\cdot\mathbf{n}=\bar{\mathbf{t}} \qquad \text{on } \Gamma_{\mathrm{t}} \tag{11}$$

in which the superposed bar denotes prescribed boundary values and $\mathbf{n}$ is the unit outward normal to domain Ω.

The principle of minimum potential energy can be stated as follows: The solution of a problem in the small displacement theory of elasticity is the vector function $\mathbf{u}$ which minimizes the total potential energy Π given by

$$\Pi=\int_{\Omega}\frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}}\cdot\boldsymbol{\sigma}\,\mathrm{d}\Omega-\int_{\Omega}\mathbf{u}^{\mathrm{T}}\cdot\mathbf{b}\mathrm{d}\Omega-\int_{\Gamma_t}\mathbf{u}^{\mathrm{T}}\cdot\bar{\mathbf{t}}\mathrm{d}\Gamma \tag{12}$$

with the boundary condition (10), where $\boldsymbol{\varepsilon}$ is the strain.

Because the MLS interpolant function lacks the delta function property, the accurate and efficient imposition of essential boundary condition often presents difficulties. Strategies have been developed for alleviating this problem, such as using Lagrange multipliers (Belytschko et al., 1994), using FE (Krongauz and Belytschko, 1996), penalty method (Liu and Yang, 1998), and so on. In the coupled EFG/HBE method, it is desirable to include the essential boundary into the HBE domain. The essential boundary conditions can then be easily imposed as in the HBE method. For some problems, it maybe difficult or not efficient to include the essential boundary into the HBE domain. The method of Lagrange multipliers is employed here to enforce the essential boundary conditions in the EFG domain. In this case, the variational form of equation (12) should be posed as follows.

$$\Pi = \int_{\Omega} \frac{1}{2} \boldsymbol{\varepsilon}^{\mathrm{T}} \cdot \boldsymbol{\sigma}\, \mathrm{d}\Omega - \int_{\Omega} \mathbf{u}^{\mathrm{T}} \cdot \mathbf{b}\mathrm{d}\Omega - \int_{\Gamma_t} \mathbf{u}^{\mathrm{T}} \cdot \bar{\mathbf{t}}\mathrm{d}\Gamma - \int_{\Gamma_u} \boldsymbol{\lambda}^{\mathrm{T}} \cdot (\mathbf{u} - \bar{\mathbf{u}})\mathrm{d}\Gamma \qquad (13)$$

where the $\boldsymbol{\lambda}$ is given by the following approximation

$$\boldsymbol{\lambda} = \mathbf{N}^{\mathrm{T}} \boldsymbol{\lambda}^{\mathrm{e}} \qquad (14)$$

where $\mathbf{N}$ is interpolation function, $\boldsymbol{\lambda}^{\mathrm{e}}$ is the unknown parameter. Substituting the expression of $\mathbf{u}$ and $\boldsymbol{\lambda}$ given in equations (5) and (14) into equation (13), and using the stationary condition yields

$$\begin{bmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{G}^{\mathrm{T}} & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} + \mathbf{d} \\ \mathbf{q} \end{Bmatrix} \qquad (15)$$

where

$$\mathbf{K}_{ij} = \int_{\Omega} \mathbf{B}_i^{\mathrm{T}} \mathbf{D} \mathbf{B}_j \mathrm{d}\Omega \qquad (16a)$$

$$\mathbf{G}_{ij} = -\int_{\Gamma_u} \phi_i \cdot N_j \mathrm{d}\Gamma \qquad (16b)$$

$$\mathbf{f}_i = \int_{\Gamma} \phi_i \mathbf{t} \mathrm{d}\Gamma \qquad (16c)$$

$$\mathbf{d}_i = \int_{\Omega} \phi_i \mathbf{b} \mathrm{d}\Omega \qquad (16d)$$

$$\mathbf{q}_i = \int_{\Omega} \phi_i \bar{\mathbf{u}} \mathrm{d}\Omega \qquad (16e)$$

$$\mathbf{B}_i = \begin{bmatrix} \phi_{i,x} & 0 \\ 0 & \phi_{i,y} \\ \phi_{i,y} & \phi_{i,x} \end{bmatrix} \qquad (16f)$$

$$\mathbf{D} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ 0 & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \text{ for plane stress} \tag{16g}$$

in which **f** is the equivalent nodal force, and **d** is the vector due to the distributed sources or body forces. A comma in equation (16f) designates a partial derivative with respect to the indicated spatial variable.

3 Hybrid displacement BE formulation

Equation (12) should satisfy the boundary condition (10) and the compatibility condition

$$\tilde{\mathbf{u}} = \mathbf{u} \qquad \text{on } \Gamma \tag{17}$$

where $\tilde{\mathbf{u}}$ is the displacement field on the boundary, and **u** is the displacement in the domain. Now subsidiary condition (17) is introduced into the variational expression (12) by introducing a set of Lagrange multipliers λ. Thus the modified variational principle can be written as

$$\Pi = \int_\Omega \frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}} \cdot \boldsymbol{\sigma}\, \mathrm{d}\Omega - \int_\Omega \mathbf{u}^{\mathrm{T}} \cdot \mathbf{b}\mathrm{d}\Omega - \int_{\Gamma_t} \tilde{\mathbf{u}}^{\mathrm{T}} \cdot \bar{\mathbf{t}}\mathrm{d}\Gamma + \int_\Gamma \boldsymbol{\lambda}^{\mathrm{T}} \cdot (\tilde{\mathbf{u}} - \mathbf{u})\mathrm{d}\Gamma \tag{18}$$

The Euler equations for the above equation are obtained when the first variation is set equal to zero. As the Lagrange multipliers $\boldsymbol{\lambda}$ represent the traction on the boundary, it is therefore denoted explicitly by $\tilde{\mathbf{t}}$. Hence, equation (18) can be re-written as

$$\Pi = \int_\Omega \frac{1}{2}\boldsymbol{\varepsilon}^{\mathrm{T}} \cdot \boldsymbol{\sigma}\, \mathrm{d}\Omega - \int_\Omega \mathbf{u}^{\mathrm{T}} \cdot \mathbf{b}\mathrm{d}\Omega - \int_{\Gamma_t} \tilde{\mathbf{u}}^{\mathrm{T}} \cdot \bar{\mathbf{t}}\mathrm{d}\Gamma + \int_\Gamma \tilde{\mathbf{t}}^{\mathrm{T}} \cdot (\tilde{\mathbf{u}} - \mathbf{u})\mathrm{d}\Gamma \tag{19}$$

The first term on the right hand side can be integrated by parts to become

$$\Pi = \int_\Gamma \frac{1}{2}\mathbf{t}^{\mathrm{T}} \cdot \mathbf{u}\mathrm{d}\Gamma - \int_\Omega \mathbf{u}^{\mathrm{T}} \cdot \mathbf{b}\mathrm{d}\Omega - \int_{\Gamma_t} \tilde{\mathbf{u}}^{\mathrm{T}} \cdot \bar{\mathbf{t}}\mathrm{d}\Gamma + \int_\Gamma \tilde{\mathbf{t}}^{\mathrm{T}} \cdot (\tilde{\mathbf{u}} - \mathbf{u})\mathrm{d}\Gamma - \int_\Omega \frac{1}{2}\nabla\boldsymbol{\sigma} \cdot \mathbf{u}\mathrm{d}\Omega \tag{20}$$

The starting integral relationship (19) which is an integral in the domain can now be reduced to an integral on the boundary by finding an analytical solution which makes the last integral in equation (20) equal to zero. The most desirable analytical solution is the fundamental solution, which satisfies the following equation:

$$\nabla\boldsymbol{\sigma}^{*}+\Delta^{i}=0 \tag{21}$$

where Δ^{i} is the Dirac delta function.

The displacement and traction vectors are approximated as a series of products of fundamental solutions (DeFigueiredo, 1991) $\mathbf{U}^{*}$, $\mathbf{T}^{*}$ and unknown parameters **s**. The boundary displacement and traction vectors are written as the product of known interpolation functions by unknown parameters (displacement and traction of boundary nodes), i.e.,

$$\mathbf{u}=\mathbf{U}^{*}\mathbf{s} \tag{22a}$$

$$\mathbf{t}=\mathbf{T}^{*}\mathbf{s} \tag{22b}$$

$$\tilde{\mathbf{u}}=\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{u}^{e} \tag{22c}$$

$$\tilde{\mathbf{t}}=\boldsymbol{\Psi}^{\mathrm{T}}\mathbf{t}^{e} \tag{22d}$$

Substituting equations (21) and (22) into equation (20), we can obtain

$$\Pi=-1/2\mathbf{s}^{\mathrm{T}}\mathbf{A}\mathbf{s}-\mathbf{t}^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{s}+\mathbf{t}^{\mathrm{T}}\mathbf{L}\mathbf{u}-\mathbf{u}^{\mathrm{T}}\mathbf{f}-\mathbf{s}^{\mathrm{T}}\mathbf{b} \tag{23}$$

where

$$\mathbf{A}=\int_{\Gamma}\mathbf{U}^{*}\mathbf{T}^{*}\mathrm{d}\Gamma \tag{24a}$$

$$\mathbf{G}=\int_{\Gamma}\boldsymbol{\Psi}\mathbf{U}^{*}\mathrm{d}\Gamma \tag{24b}$$

$$\mathbf{L}=\int_{\Gamma}\boldsymbol{\Psi}\boldsymbol{\Phi}^{\mathrm{T}}\mathrm{d}\Gamma \tag{24c}$$

$$\mathbf{f}=\int_{\Gamma}\boldsymbol{\Phi}\bar{\mathbf{t}}\mathrm{d}\Gamma \tag{24d}$$

$$\mathbf{b}=\int_{\Omega}\mathbf{U}^{*}\mathbf{b}\mathrm{d}\Omega \tag{24e}$$

The stationary conditions for Π can now be found by setting its first variation to zero. As this must be true for any arbitrary values of $\delta\mathbf{s}$, $\delta\mathbf{u}$ and $\delta\mathbf{t}$, one obtains:

$$\mathbf{Ku}=\mathbf{f}+\mathbf{d} \tag{25}$$

where

$$\mathbf{K}=\mathbf{R}^{\mathbf{T}}\mathbf{AR} \tag{26a}$$

$$\mathbf{R}=(\mathbf{G}^{\mathbf{T}})^{-\mathbf{1}}\mathbf{L} \tag{26b}$$

$$\mathbf{d}=\mathbf{R}^{\mathbf{T}}\mathbf{b} \tag{26c}$$

It can be proved that matrix **A** is symmetric, and hence the matrix **K**. It is possible to conclude from equation (25) that this hybrid displacement boundary formulation leads to an equivalent stiffness approach. The matrix **K** may be viewed as a symmetric stiffness matrix, but the above integrals are only needed to perform on boundaries, and the domain needs not to be discretized.

4 Coupling of EFG and HBE

4.1 Continuity conditions at coupled interfaces

Consider a problem consisting of two domains Ω^1 and Ω^2, shown in Figure 1, joined by an interface Γ_I. The EFG formulation is used in Ω^1 and the HBE formulation is used in Ω^2 . Continuity conditions on Γ_I must be satisfied, i. e.

$$\tilde{\mathbf{u}}_I^{\ 1}=\tilde{\mathbf{u}}_I^{\ 2} \tag{27}$$

$$\mathbf{F}_I^1+\mathbf{F}_I^2=0 \tag{28}$$

where $\tilde{\mathbf{u}}_I^{\ 1}$ and $\tilde{\mathbf{u}}_I^{\ 2}$ are the displacements on Γ_I for Ω^1 and Ω^2, $\mathbf{F}_I^1$ and $\mathbf{F}_I^2$ are the forces on Γ_I for Ω^1 and Ω^2, respectively.

Because the shape functions of EFG are derived using MLS, $\mathbf{u}^h$ in equation (5) differs with the displacement $\mathbf{u}$ at point $\mathbf{x}$. It is not possible to couple EFG and HBE directly along Γ_I.

4.2 Coupling EFG with HBE via modified variational form

A sub-functional is introduced to enforce the compatibility condition (27) by means of Lagrange multiplier λ on the interface boundary

$$\Pi_I = \int_{\Gamma_I} \boldsymbol{\gamma} \cdot (\tilde{\mathbf{u}}_I^{\,1} - \tilde{\mathbf{u}}_I^{\,2}) \mathrm{d}\Gamma = \int_{\Gamma_I} \boldsymbol{\gamma}\, \tilde{\mathbf{u}}_I^{\,1} \mathrm{d}\Gamma - \int_{\Gamma_I} \boldsymbol{\gamma}\, \tilde{\mathbf{u}}_I^{\,2} \mathrm{d}\Gamma = \Pi_I^1 - \Pi_I^2 \tag{29}$$

In equation (29), Π_I^1 and Π_I^2 are the boundary integration along the EFG side and the HBE side. Introducing Π_I^1 and Π_I^2 separately into functions (13) and (18), generalized functional forms can be written as

$$\Pi_{EFG} = \int_{\Omega} \frac{1}{2} \boldsymbol{\varepsilon}^{\mathrm{T}} \cdot \boldsymbol{\sigma}\, \mathrm{d}\Omega - \int_{\Omega} \mathbf{u}^{\mathrm{T}} \cdot \mathbf{b} \mathrm{d}\Omega - \int_{\Gamma_t} \mathbf{u}^{\mathrm{T}} \cdot \bar{\mathbf{t}} \mathrm{d}\Gamma - \int_{\Gamma_u} \lambda^{\mathrm{T}}{}_{EFG} \cdot (\mathbf{u} - \bar{\mathbf{u}}) \mathrm{d}\Gamma + \int_{\Gamma_I} \gamma^{\mathrm{T}} \cdot \tilde{\mathbf{u}}_I^{\,1} \mathrm{d}\Gamma \tag{30}$$

$$\Pi_{HBE} = \int_{\Omega} \frac{1}{2} \boldsymbol{\varepsilon}^{\mathrm{T}} \cdot \boldsymbol{\sigma}\, \mathrm{d}\Omega - \int_{\Omega} \mathbf{u}^{\mathrm{T}} \cdot \mathbf{b} \mathrm{d}\Omega - \int_{\Gamma_t} \tilde{\mathbf{u}}^{\mathrm{T}} \cdot \bar{\mathbf{t}} \mathrm{d}\Gamma + \int_{\Gamma} \lambda^{\mathrm{T}}{}_{HBE} \cdot (\tilde{\mathbf{u}} - \mathbf{u}) \mathrm{d}\Gamma - \int_{\Gamma_I} \gamma^{\mathrm{T}} \cdot \tilde{\mathbf{u}}_I^{\,2} \mathrm{d}\Gamma \tag{31}$$

In these variational formulations the domains of EFG and HBE are connected via Lagrange multiplier $\boldsymbol{\gamma}$.

In EFG domain, $\mathbf{u}$ is given by equation (5). $\boldsymbol{\gamma}$ is given by interpolation functions $\boldsymbol{\Lambda}$ and value of $\boldsymbol{\gamma}^I$

$$\boldsymbol{\gamma} = \boldsymbol{\Lambda}^{\mathrm{T}} \boldsymbol{\gamma}^I \tag{32}$$

$\boldsymbol{\Lambda}$ can be the interpolation of HBE. Substituting equations (5), (14) and (32) into equation (30), and using the stationary condition, the following EFG equations can be obtained

$$\begin{bmatrix} \mathbf{K}_{EFG} & \mathbf{G} & \mathbf{B} \\ \mathbf{G}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \lambda \\ \gamma \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_{EFG} + \mathbf{d}_{EFG} \\ \mathbf{q} \\ \mathbf{0} \end{Bmatrix} \tag{33}$$

where $\mathbf{K}_{EFG}$, $\mathbf{G}$, $\mathbf{f}_{EFG}$ and $\mathbf{b}_{EFG}$ are defined by equation (16), $\mathbf{B}$ is defined as

$$\mathbf{B} = \int_{\Gamma_I} \boldsymbol{\Lambda} \boldsymbol{\Phi}_{EFG}^{\mathrm{T}} \mathrm{d}\Gamma \tag{34}$$

Integrating first term on right hand side of equation (31) by parts, substituting equations (21),(22) and (32) into equation (31), and using the stationary condition, lead to the following HBE equations

$$\begin{bmatrix} \mathbf{K}_{HBE} & -\mathbf{H} \\ -\mathbf{H}^{\mathrm{T}} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\gamma} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_{HBE} + \mathbf{d}_{HBE} \\ \mathbf{0} \end{Bmatrix} \tag{35}$$

where $\mathbf{K}_{HBE}$, $\mathbf{f}_{HBE}$ and $\mathbf{d}_{HBE}$ are defined by equations (24) and (26). $\mathbf{H}$ is defined as

$$\mathbf{H} = \int_{\Gamma_I} \boldsymbol{\Lambda}\boldsymbol{\Phi}_{HBE}{}^{\mathrm{T}} \mathrm{d}\Gamma \tag{36}$$

Because two domains are connected along the interface boundary Γ_I , assembling of equations (33) and (35) yields a linear system of the following form

$$\begin{bmatrix} \mathbf{K}_{EFG} & \mathbf{0} & \mathbf{G} & \mathbf{B} \\ \mathbf{0} & \mathbf{K}_{HBE} & \mathbf{0} & -\mathbf{H} \\ \mathbf{G}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{B}^{\mathrm{T}} & -\mathbf{H}^{\mathrm{T}} & \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{EFG} \\ \mathbf{u}_{HBE} \\ \boldsymbol{\lambda} \\ \boldsymbol{\gamma} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_{EFG} + \mathbf{d}_{EFG} \\ \mathbf{f}_{HBE} + \mathbf{d}_{HBE} \\ \mathbf{q} \\ \mathbf{0} \end{Bmatrix} \tag{37}$$

The coupling conditions (27) and (28) are satisfied via the above technique.

5. Numerical results

Three cases have been studied in order to examine the coupled EFG/HBE method in two-dimensional elastostatics.

5.1 Cantilever beam

The coupled method is first applied to study the cantilever beam problem. Consider a beam of length L and height D subjected to a parabolic traction at the free end as shown in Figure 2. The beam has a unit thickness and a plane stress problem is considered. The analytical solution is available and can be found in a textbook by Timoshenko and Goodier (1970).

The parameters of the beam are taken as E=3.0×10^7, ν=0.3, D=12, L=48, and P=1000. The beam is separated into two parts. HBE is used in the part on the left where the essential boundary is included. EFG is used in the part on the right. The nodal arrangement is shown in Figure 3, and background mesh of 6×8 is used in EFG domain. In each integration cell, 4×4 Gauss quadrature is used to evaluate the stiffness matrix of the EFG. Only 100 nodes in total are used in the coupled method.

Figure 4 illustrates the comparison between the shear stress calculated analytically and by the coupled method at the section of x=L/2. The plot shows an excellent agreement between the analytical and numerical results. The computational result by coupled method using interface elements (I.E.) is also shown in the same figure. There is clear evidence that the accuracy of the coupled method using the modified variational formulation (M.V.F.) is higher than that of using the interface element method.

The displacement along the interface boundary is shown in the Table 1. It is shown that the compatibility is satisfied very well using the present modified variational formulation method.

5.2 Hole in an infinite plate

A plate with a circular hole subjected to a unidirectional tensile load of 1.0 in the x direction is considered. Due to symmetry, only the upper right quadrant (size 10×10) of the plate is modeled as shown in Figure 5. When the condition b/a>5 is satisfied, the solution of finite plate is very closed to that of the infinite plate (Roark and Young, 1975). Plane strain condition is assumed, and E=1.0×10^3, ν=0.3. Symmetry conditions are imposed on the left and bottom edges, and the inner boundary of the hole is traction free. The tensile load in the x direction is imposed on the right edge. The exact solution for the stresses of infinite plate is

$$\sigma_x(x,y) = 1 - \frac{a^2}{r^2}\{\frac{3}{2}\cos 2\theta + \cos 4\theta\} + \frac{3a^4}{2r^4}\cos 4\theta \quad (38)$$

$$\sigma_y(x,y) = -\frac{a^2}{r^2}\{\frac{1}{2}\cos 2\theta - \cos 4\theta\} - \frac{3a^4}{2r^4}\cos 4\theta \quad (39)$$

$$\sigma_{xy}(x,y) = -\frac{a^2}{r^2}\{\frac{1}{2}\sin 2\theta + \sin 4\theta\} + \frac{3a^4}{2r^4}\sin 4\theta \quad (40)$$

where (r, θ) are the polar coordinates and θ is measured counter-clockwise from the positive x axis. The plate is divided into two domains, where EFG and HBE are applied, respectively.

As the stress is most critical, detailed results on stress are presented here. The stress σ_x at x=0 obtained by the coupled method are plotted in Figure 6. The result are obtained using two kinds of nodal arrangement. The nodal arrangement of 65 nodes is shown in Figure 5. It can be observed from Figure 6 that the coupled method yields satisfactory results for the problem considered. The convergence of the present method is also demonstrated in this figure. As the number of nodes increases, the results obtained approaches to the analytical solution. Compared to the EFG method, fewer nodes are needed in the present coupled method. A previous research indicates that 231 nodes are needed in EFG method to obtained the results of same accuracy as those obtained by the present method where only 144 nodes are required.

5.3 Semi-infinite foundation

In this example the coupled method is used in soil-structure interaction problem. A structure stands on a semi-infinite soil foundation is shown in Figure 7. The infinite soil foundation can be treated in practice in either of the following three ways: by truncating the semi-infinite plane at a finite distance (approximate method), using a fundamental solution appropriate to the semi-space problem rather than a free-space Green's function in BEM, and using infinite element in FEM. The first approximate method is used herein. The present EFG/HBE method, the EFG/BE method (using interface elements), EFG and FE methods are used for the calculation, and the results are compared and investigated in details.

As shown in Figure 7, Region 2 represents the semi-infinite foundation and is given a semi-circular shape of a very large diameter in relation to Region 1 that represents the structure. Boundary conditions to restrain rigid body movements are applied. Region 1 is the EFG domain and Region 2 is the HBE domain. The nodal arrangement of the coupled EFG/HBE and EFG/BE methods is shown in Figure 8. The nodal arrangement of EFG for the entire domain is shown in Figures 9 and 10. Two loading cases shown in Figure 10 are computed: Case 1 considers five concentrated vertical loads on the top of the structure and case 2 considers an additional horizontal load acting at the right corner.

The displacement results on top of the structure are given in Table 2. The FEM result obtained by Brebbia and Georgiou (1979) is also included in the same table. The results obtained by the present method are in very good agreement with those obtained using other methods including the FE and EFG methods for the entire domain. The present method uses much fewer nodes to model foundation. Only 30 nodes are used in the HBE method compared to 120 nodes used in the EFG for the foundation.

6. Conclusions

A coupled EFG/HBE method has been presented in this paper. The Lagrange multiplier is used in a modified functional for EFG and HBE to enforce the compatibility condition along the interface. The discrete system equations of the coupled method are derived. Numerical examples have demonstrated effectiveness of the present coupled EFG/HBE method for elastostatics.

The method allows the advantages of both EFG and HBE methods to be used. The merits using the EFG/HBE are as follows:

(a) The computation cost is much lower because of the significant reduction on the node numbers, as well as the reduction of area integration in constructing system matrices.

(b) Imposition of essential boundary condition becomes easy.

(c) The method provides a potential effective numerical tool in many practical problems, such as fluid-structure interaction problems, infinite or semi-infinite problems, cracks propagation problems in a relatively big body, and so on.

Compared with the EFG/BE developed earlier by the authors, the present method has advanced in the following counts:

(a) The coupled system equations have been formulated in a more general manner. System matrices obtained are symmetric in EFG/HBE, and there is no need for an operation on matrix symmetrization. This improves both the accuracy of the results and the efficiency of computation.

(b) The shape functions obtained have higher order of continuity. This translates to higher accuracy in result obtained.

(c) There is no need for interface elements. This can further simplify the mesh generation. No special treatment is required for the interface between the EFG and HBE domain.

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Figure Captions:

Figure 1 Domain division into EFG and HBE regions

Figure 2 Cantilever beam

Figure 3 Nodal arrangement

Figure 4 Shear stress τ_{xy} at the section x=L/2 of the beam

Figure 5 Nodes in a plate with a central hole subjected to unidirectional tensile load in the x direction

Figure 6 Comparison of stress σ_x at x=0 for problem shown in Figure 5

Figure 7 A structure standing on a semi-infinite soil foundation

Figure 8 Nodal arrangement for the coupled EFG/HBE method

Figure 9 Nodal arrangement for the EFG method

Figure 10 Detailed nodal arrangement for the EFG method and load cases

Table 1 Vertical displacement along the interface boundary (cantilever beam)

Node (y)	EFG/HBE (I. E.)*	EFG/HBE(M.V.F.)* EFG side	EFG/HBE(M.V.F.)* HBE side	Exact
5.75	-4.73203E-03	-4.73090E-03	-4.73093E-03	-4.68750E-03
5.00	-4.72797E-03	-4.72617E-03	-4.72619E-03	-4.68302E-03
4.00	-4.72344E-03	-4.72050E-03	-4.72059E-03	-4.67802E-03
3.00	-4.71970E-03	-4.71664E-03	-4.71670E-03	-4.67414E-03
2.00	-4.71704E-03	-4.71419E-03	-4.71422E-03	-4.67136E-03
1.00	-4.71542E-03	-4.71257E-03	-4.71261E-03	-4.66969E-03
0.00	-4.71488E-03	-4.71199E-03	-4.71203E-03	-4.66914E-03

* EFG/HBE (I. E.): coupled EFG/HBE method using interface element

EFG/HBE (M. V. F.): coupled EFG/HBE method using Modified Variational Formulation

Table 2 Vertical displacements along top of the structure on the semi-infinite foundation

	Displacements ($\times 10^{-4}$)			
Load case 1				
Node No.	FE	EFG	EFG/BE (I. E.)*	EFG/HBE (M.V.F)*
1	1.41	1.42	1.42	1.41
2	1.34	1.34	1.33	1.33
3	1.32	1.32	1.32	1.32
4	1.34	1.34	1.33	1.33
5	1.41	1.42	1.42	1.41
Load case 2				
1	-3.39	-3.43	-3.58	-3.41
2	-0.97	-1.01	-1.04	-1.03
3	1.35	1.35	1.34	1.35
4	3.61	3.67	3.68	3.69
5	6.00	6.04	6.13	6.11

* EFG/BE (I. E.): coupled EFG/BE method using interface element

EFG/HBE (M. V. F.): coupled EFG/HBE method using Modified Variational Formulation