World Applied Sciences Journal 16 (3): 320-328, 2012
ISSN 1818-4952

Coupling of Homotopy Perturbation, Laplace Transform and Padé Approximants for Nonlinear Oscillatory Systems

[1]*Mehmet Merdan,* [2]*Ahmet Yildirim,*
[1]*Ahmet Gökdogan and* [3]*Syed Tauseef Mohyud-din*

[1]Department of Mathematic Engineering, Gümüshane University, 29100-Gümüshane, Turkey
[2]Department of Mathematics, Ege University, 35100, İzmir, Turkey
[3]Department of Mathematics, HITEC University Taxila Cantt Pakistan

Abstract: In this article, homotopy perturbation method coupled with Laplace transform and Padé 0approximants is applied on the re-formulated nonlinear oscillatory systems. Numerical results and graphical represenations explicitly reveal the complete reliability and effeicincy of the suggested algorithm.

Key words: Padé approximants · Homotopy perturbation method · Non-linear oscillatory systems

INTRODUCTION

This paper is devoted to the study of reliable and efficient applications of three very powerful tools namely, homotopy perturbation method, Laplace transform and Padé approximants [1-27] for re-formulated nonlinear oscillatory systems. It is observed that proposed algorithm is highly efficient and accurate. Moreover, suggested coupling is easier to implement and is free from number of inbuilt deficiencies in comparison with the existing techniques. The scheme has been successfully tested on Rayleigh, Van der Pol and Duffing equations. Numerical results and graphical represenations explicitly reveal the complete reliability and effeicincy of the suggested algorithm.

Padé Approximaton: A rational approximation to $f(x)$ on $[a,b]$ is the quotient of two polynomials $P_N(x)$ and $Q_N(x)$ of degrees N and M, respectively. We use the notation $R_{N,M}(x)$ to denote this quotient. The $R_{N,M}(x)$ Padé approximations to a function $f(x)$ are given by [1]

$$R_{N,M}(x) = \frac{P_N(x)}{Q_M(x)} \quad \text{for } a \le x \le b. \tag{2.1}$$

The method of Padé requires [22-27] that $f(x)$ and its derivative be continuous at $x = 0$. The polynomials used in (2.1) are

$$P_N(x) = p_0 + p_1 x + p_2 x^2 + ... + p_N x^N \tag{2.2}$$

$$Q_M(x) = 1 + q_1 x + q_2 x^2 + ... + q_M x^M \tag{2.3}$$

The polynomials in (2.2) and (2.3) are constructed so that $f(x)$ and $R_{N,M}(x)$ agree at $x = 0$ and their derivatives up to $N+M$ agree at $x = 0$. In the case $f(x)$, the approximation is just the Maclaurin expansion for $f(x)$. For a fixed value of $N+M$ the error is smallest when $P_N(x)$ and $Q_M(x)$ have the same degree or when $Q_N(x)$ has degree one higher then $Q_N(x)$.

Notice that the constant coefficient of Q_M is $q_0 = 1$. This is permissible, because it notice be 0 and $R_{N,M}(x)$ is not changed when both $P_N(x)$ and $Q_M(x)$ are divided by the same constant. Hence the rational function $R_{N,M}(x)$ has $N+M+1$ unknown coefficients. Assume that $f(x)$ is analytic and has the Maclaurin's expansion

$$f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_k x^k + ..., \tag{2.4}$$

And from the difference $f(x)Q_M(x) - P_N(x) = Z(x)$:

$$\left[\sum_{i=0}^{\infty} a_i x^i\right]\left[\sum_{i=0}^{M} q_i x^i\right] - \left[\sum_{i=0}^{N} p_i x^i\right] = \left[\sum_{i=N+M+1}^{\infty} c_i x^i\right], \tag{2.5}$$

The lower index $j = N+M+1$ in the summation on the right side of (2.5) is chosen because the first $N+M$ derivatives of $f(x)$ and $R_{N,M}(x)$ are to agree at $x = 0$.

When the left side of (10) is multiplied out and the coefficients of the powers of x' are set equal to zero for $k = 0,1,2,....,N+M$, the result is a system of $N+M+1$ linear equations:

Corresponding Author: Mehmet Merdan, Department of Geomatics Engineering, Gümüshane University, 29100-Gümüshane, Turkey.

$$\begin{aligned}
&a_0 - p_0 = 0 \\
&q_1 a_0 + a_1 - p_1 = 0 \\
&q_2 a_0 + q_1 a_1 + a_2 - p_2 = 0 \\
&q_3 a_0 + q_2 a_1 + q_1 a_2 + a_3 - p_3 = 0 \\
&q_M a_{N-M} + q_{M-1} a_{N-M+1} + a_N - p_N = 0
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
&q_M a_{N-M+1} + q_{M-1} a_{N-M+2} + \ldots + q_1 a_N \quad + a_{N+2} = 0 \\
&q_M a_{N-M+2} + q_{M-1} a_{N-M+3} + \ldots + q_1 a_{N+1} \quad + a_{N+2} = 0 \\
&. \\
&. \\
&. \\
&q_M a_N + q_{M-1} a_{N+1} + \ldots + q_1 a_{N+M+1} \quad + a_{N+M} = 0
\end{aligned} \tag{2.7}$$

Notice that in each equation the sum of the subscripts on the factors of each product is the same and this sum increases consecutively from 0 to $N+M$. The M equations in (2.7) involve only the unknowns q_1,q_2 q_3....., q_M and must be solved first. Then the equations in (2.6) are used successively to find p_1,p_2 p_3....., p_N [1].

Homotopy Perturbation Method: To illustrate the homotopy perturbation method (HPM) for solving non-linear differential equations, He [7, 8, 20-27] considered the following non-linear differential equation:

$$A(u) = f(r), \quad r \in \Omega \tag{3.1}$$

subject to the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \tag{3.2}$$

where A is a general differential operator, B is a boundary operator, f(r) is a known analytic function, Γ is the boundary of the domain Ω and $\frac{\partial}{\partial n}$ denotes differentiation along the normal vector drawn outwards from Ω. The operator A can generally be divided into two parts M and N. Therefore, (3.1) can be rewritten as follows:

$$M(u) + N(u) = f(r), \quad r \in \Omega \tag{3.3}$$

He [9,10] constructed a homotopy $v(r,p): \Omega x[0,1] \to \Re$ which satisfies

$$H(v,p) = (1-p)\left[M(v) - M(u_0)\right] + p\left[A(v) - f(r)\right] = 0, \tag{3.4}$$

which is equivalent to

$$H(v,p) = M(v) - M(u_0) + pM(v_0) + p\left[N(v) - f(r)\right] = 0, \tag{3.5}$$

where $p \in [0,1]$ is an embedding parameter and u_0 is an initial approximation of (3.1). Obviously, we have

$$\begin{aligned}
&H(v,0) = M(v) - M(u_0) = 0, \\
&H(v,1) = A(v) - f(r) = 0.
\end{aligned} \tag{3.6}$$

The changing process of p from zero to unity is just that of H(v,p) from $M(v) - M(v_0)$ to $A(v) - f(r)$. In topology, this is called deformation and $M(v) - M(v_0)$ and $A(v) - f(r)$ are called homotopic. According to the homotopy perturbation method, the parameter p is used as a small parameter and the solution of Eq. (3.4) can be expressed as a series in p in the form

$$v = v_0 + pv_1 + p^2 v_2 + p^3 v_3 + \ldots \tag{3.7}$$

When $p \to 1$, Eq. (3.4) corresponds to the original one, Eqs. (3.3) and (3.7) become the approximate solution of Eq. (3.3), i.e.

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + v_3 + \ldots \tag{3.8}$$

The convergence of the series in Eq. (3.8) is discussed by He in [7, 8].

Applications

Example 1 (Duffing Equation): The Duffing equation is a nonlinear second-order differential equation. Consider the equation

$$\frac{d^2 y}{dt^2} + y + \varepsilon y^3 = 0, \tag{4.1}$$

the initial conditions are chosen to be $y(0) = a$ and $y'(0) = 0$. By setting $y_1 = y$ and introducing the new variable $y_2 = y'$ the second-order equation is converted to a first-order system

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 \\ \dfrac{dy_2}{dt} = -y_1 - \varepsilon y_1^3 \end{cases} \tag{4.2}$$

with the initial conditions:

$y_1(0) = m_1$, $y_2(0) = m_2$, Throughout this paper, we set $a = 1$ and $\varepsilon = 0.1$.

In this section, we will apply the homotopy perturbation method to nonlinear ordinary differential systems (4.2).

Homotopy Perturbation Method to Duffing Equation: According to homotopy perturbation method, we derive a correct functional as follows:

$$(1-p)(\dot{v}_1-\dot{x}_0)+p(\dot{v}_1-v_2)=0,$$
$$(1-p)(\dot{v}_2-\dot{y}_0)+p\left(\dot{v}_2+v_1+\varepsilon v_1^3\right)=0, \quad (4.3)$$

where "dot" denotes differentiation with respect to t and the initial approximations are as follows:

$$v_{1,0}(t)=x_0(t)=y_1(0)=m_1,$$
$$v_{2,0}(t)=y_0(t)=y_2(0)=m_2. \quad (4.4)$$

and

$$v_1=v_{1,0}+pv_{1,1}+p^2v_{1,2}+p^3v_{1,3}+...,$$
$$v_2=v_{2,0}+pv_{2,1}+p^2v_{2,2}+p^3v_{2,3}+..., \quad (4.5)$$

where $v_{i,j}, i,j$ = 1,2,3,.... are functions yet to be determined. Substituting Eqs.(4.4) and (4.5) into Eq. (4.3) and arranging the coefficients of "p" powers, we have

$$\left(\dot{v}_{1,1}-m_2\right)p+\left(\dot{v}_{1,2}-v_{2,1}\right)p^2+\left(\dot{v}_{1,3}-v_{2,2}\right)p^3+...=0,$$
$$\left(\dot{v}_{2,1}+m_1+\varepsilon m_1^3\right)p+\left(\dot{v}_{2,2}+v_{1,1}+3\varepsilon m_1^2v_{1,1}\right)p^2 \quad (4.6)$$
$$+\left(\dot{v}_{2,3}+v_{1,2}+3\varepsilon\left(m_1^2v_{1,2}+m_1v_{1,1}^2\right)\right)p^3+...=0,$$

In order to obtain the unknowns $v_{i,j}(t), i,j$ = 1,2,3,.... we must construct and solve the following system which includes nine equations with nine unknowns, considering the initial conditions

$$v_{i,j}(0)=0, i,j=1,2,3,$$
$$\dot{v}_{1,1}-m_2=0,\ \dot{v}_{1,2}-v_{2,1}=0,\ \dot{v}_{1,3}-v_{2,2}=0,$$
$$\dot{v}_{2,1}+m_1+\varepsilon m_1^3=0,\ \dot{v}_{2,2}+v_{1,1}+3\varepsilon m_1^2v_{1,1}=0, \quad (4.7)$$
$$\dot{v}_{2,3}+v_{1,2}+3\varepsilon\left(m_1^2v_{1,2}+m_1v_{1,1}^2\right)=0,$$

From Eq. (3.8), if the three terms approximations are sufficient, we will obtain:

$$y_1(t)=\lim_{p\to 1}v_1(t)=\sum_{k=0}^{3}v_{1,k}(t),$$
$$y_2(t)=\lim_{p\to 1}v_2(t)=\sum_{k=0}^{3}v_{2,k}(t), \quad (4.8)$$

therefore

$$y_1(t)=m_1+m_2t+\frac{1}{2}\left[-m_1-\varepsilon m_1^3\right]t^2+\frac{1}{6}\left[-m_2-3\varepsilon m_1^2m_2\right]t^3$$
$$y_2(t)=m_2+\left(-m_1-\varepsilon m_1^3\right)t+\frac{1}{2}\left[-m_2-3\varepsilon m_1^2m_2\right]t^2 \quad (4.9)$$
$$+\frac{1}{6}\left[-3\varepsilon m_1^2\left(-m_1-\varepsilon m_1^3\right)-6\varepsilon m_1m_2^2\right]t^3$$

Here
$y_1(0)=1$, $y_2(0)=0$, for the four-component model.

A few first approximations for $y_1(t)$ are calculated and presented below:
Six terms approximations:

$$y_1(t)=1\text{-}.55t^2+.05933333332t^4\text{-}.005606944443t^6. \quad (4.10)$$

In this section, we apply Laplace transformation to (4.10), which yields

$$L\left(y_1(s)\right)=\frac{1}{s}-\frac{1.1}{s^3}+\frac{1.43}{s^5}-\frac{4.036999999}{s^7} \quad (4.11)$$

For simplicity, let $s=\frac{1}{t}$; then

$$L(y_1(t))=t\text{-}1.1t^3+1.43t^5\text{-}4.036999999t^7 \quad (4.12)$$

Padé approximant [4/4]of (4.12) and substituting $t=\frac{1}{s}$, we obtain [4/4] in terms of s. By using the inverse Laplace transformation, we obtain

$$y(t)=.002694117079\cos(3.181882839t)$$
$$+.9973058828\cos(1.037121782t) \quad (4.13)$$

In Table 1 we show the differences between the 6-term HPM and the the Padé approximations solutions

Example 2 (The Vander Pol Equation): The Vander Pol equation is a non-linear second-order differential equation. Consider the equation

Table 1: Differences between the 6-term HPM and the the Padé approximations solutions for the The Duffing equation when $\varepsilon = 0.1$.

t	Diff
0	1.2100e-010
0.1	1.1783e-010
0.2	1.7363e-009
0.3	4.7610e-008
0.4	4.7339e-007
0.5	2.7954e-006
0.6	1.1881e-005
0.7	4.0220e-005
0.8	1.1522e-004
0.9	2.9042e-004
1	6.6153e-004

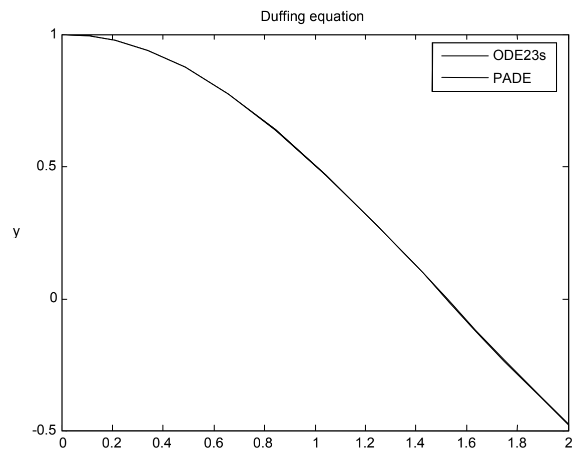

Fig. 1: Local changes of y for $\alpha = 1$ and $\varepsilon = 0.1$.

$$\frac{d^2y}{dt^2} + y - \varepsilon\left(1 - y^2\right)\frac{dy}{dt} = 0, \tag{4.14}$$

the initial conditions are chosen to be $y(0) = a$ and $y'(0) = 0$. By setting $y_1 = y$ and introducing the new variable $y_2 = y'$, the second-order equation is converted to a first-order system

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 \\ \dfrac{dy_2}{dt} = -y_1 + \varepsilon\left(1 - y_1^2\right)y_2 \end{cases} \tag{4.15}$$

With the initial conditions:
$y_1(0) = n_1$, $y_2(0) = n_2$, Throughout this paper, we set $a = 1$ and $\varepsilon = 0.1$.

In this section, we will apply the homotopy perturbation method to nonlinear ordinary differential systems (4.15).

Homotopy Perturbation Method to Vander Pol Equation: According to homotopy perturbation method, we derive a correct functional as follows:

$$\begin{aligned} &(1-p)\left(\dot{v}_1 - \dot{x}_0\right) + p\left(\dot{v}_1 - v_2\right) = 0, \\ &(1-p)\left(\dot{v}_2 - \dot{y}_0\right) + p\left(\dot{v}_2 + v_1 - \varepsilon\left(1 - v_1^2\right)v_2\right) = 0, \end{aligned} \tag{4.16}$$

where "dot" denotes differentiation with respect to t and the initial approximations are as follows:

$$\begin{aligned} v_{1,0}(t) &= x_0(t) = y_1(0) = n_1, \\ v_{2,0}(t) &= y_0(t) = y_2(0) = n_2. \end{aligned} \tag{4.17}$$

and

$$\begin{aligned} v_1 &= v_{1,0} + pv_{1,1} + p^2v_{1,2} + p^3v_{1,3} + ..., \\ v_2 &= v_{2,0} + pv_{2,1} + p^2v_{2,2} + p^3v_{2,3} + ..., \end{aligned} \tag{4.18}$$

Where $v_{i,j}, i,j = 1,2,3,....$ are functions yet to be determined. Substituting Eqs. (4.17) and (4.18) into Eq. (4.16) and arranging the coefficients of "p" powers, we have

$$\begin{aligned} &\left(\dot{v}_{1,1} - n_2\right)p + \left(\dot{v}_{1,2} - v_{2,1}\right)p^2 + \left(\dot{v}_{1,3} - v_{2,2}\right)p^3 + ... = 0, \\ &\left(\dot{v}_{2,1} + \left(n_1 - \varepsilon n_2\right) + \varepsilon n_1^2 n_2\right)p + \left(\dot{v}_{2,2} + \left(v_{1,1} - \varepsilon v_{2,1}\right) + \varepsilon\left(n_1^2 v_{2,1} + 2n_1 n_2 v_{1,1}\right)\right)p^2 \\ &+ \left(\dot{v}_{2,3} + \left(v_{1,2} - \varepsilon v_{2,2}\right) + \varepsilon\left(n_1^2 v_{2,2} + 2n_1 n_2 v_{1,2} + 2n_1 v_{1,1} v_{2,1} + v_{1,1}^2 n_2\right)\right)p^3 + ... = 0, \end{aligned} \tag{4.19}$$

In order to obtain the unknowns $v_{i,j}(t), i,j = 1,2,3,....$ we must construct and solve the following system which includes nine equations with nine unknowns, considering the initial conditions

$$\begin{aligned} &v_{i,j}(0) = 0, i, j = 1,2,3, \\ &\dot{v}_{1,1} - m_2 = 0, \dot{v}_{1,2} - v_{2,1} = 0, \dot{v}_{1,3} - v_{2,2} = 0, \\ &\dot{v}_{2,1} + \left(n_1 - \varepsilon n_2\right) + \varepsilon n_1^2 n_2 = 0, \dot{v}_{2,2} + \left(v_{1,1} - \varepsilon v_{2,1}\right) + \varepsilon\left(n_1^2 v_{2,1} + 2n_1 n_2 v_{1,1}\right) = 0, \\ &\dot{v}_{2,3} + \left(v_{1,2} - \varepsilon v_{2,2}\right) + \varepsilon\left(n_1^2 v_{2,2} + 2n_1 n_2 v_{1,2} + 2n_1 v_{1,1} v_{2,1} + v_{1,1}^2 n_2\right) = 0, \end{aligned} \tag{4.20}$$

From Eq. (3.8), if the three terms approximations are sufficient, we will obtain:

$$\begin{aligned} y_1(t) &= \lim_{p \to 1} v_1(t) = \sum_{k=0}^{3} v_{1,k}(t), \\ y_2(t) &= \lim_{p \to 1} v_2(t) = \sum_{k=0}^{3} v_{2,k}(t), \end{aligned} \tag{4.21}$$

$$y_1(t) = n_1 + n_2 t + \frac{1}{2}\left[-n_1 + \varepsilon n_2 - \varepsilon n_1^2 n_2\right]t^2 + \frac{1}{6}\left[-n_2 + \left(\varepsilon - \varepsilon n_1^2\right)\left(-n_1 + \varepsilon n_2 - \varepsilon n_1^2 n_2\right) - 2\varepsilon n_2^2 n_1\right]t^3$$

$$\begin{aligned} y_2(t) &= n_2 + \left(-n_1 + \varepsilon n_2 - \varepsilon n_1^2 n_2\right)t \\ &+ \frac{1}{2}\left[-n_2 + \left(\varepsilon - \varepsilon n_1^2\right)\left(-n_1 + \varepsilon n_2 - \varepsilon n_1^2 n_2\right) - 2\varepsilon n_2^2 n_1\right]t^2 \\ &+ \frac{1}{6}\left[\begin{matrix} -\varepsilon n_2 - 2\varepsilon^2 n_2^2 n_1 + \varepsilon n_1^2 n_2 - 2\varepsilon^2 n_2^2 n_1^3 - 2\varepsilon n_2^2 \\ +\left(-1 + \varepsilon^2 - 2\varepsilon^2 n_1^2 + \varepsilon^2 n_1^4 - 6\varepsilon n_1 n_2\right)\left(-n_1 + \varepsilon n_2 - \varepsilon n_1^2 n_2\right) \end{matrix}\right]t^3 \end{aligned} \tag{4.22}$$

.
.
.

Here
$y_1(0) = 1$ and $y_2(0) = 1$ for the four-component model.

A few first approximations for $y_1(t)$ are calculated and presented below:
Six terms approximations:

$$y_1(t) = 1.-.5t^2 + .04166666668t^4 - .005t^5 - .001388888889t^6 \tag{4.23}$$

In this section, we apply Laplace transformation to (4.23), which yields

$$L\left(y_1(s)\right) = \frac{1}{s} - \frac{1}{s^3} + \frac{1}{s^5} - \frac{0.6}{s^6} - \frac{1}{s^7} \tag{4.24}$$

For simplicity, let $s = \frac{1}{t}$; then

$$L(y_1(t)) = t - t^3 + t^5 - 0.6t^6 - t^7 \tag{4.25}$$

Padé approximant [4/4] of (4.25) and substituting $t = \frac{1}{s}$, we obtain [4/4] in terms of s. By using the inverse Laplace transformation, we obtain

$$\begin{aligned} y(t) &= e^{-.1004422537t}\left[.1678240288\sin(1.00174051t) + .9861656731\cos(1.00174051t)\right] \\ &+ .3193065996e^{-1.307856989t} - .01809633309e^{1.508741496t} \end{aligned} \tag{4.26}$$

In Table 2 we show the differences between the 6-term HPM and the the Padé approximations solutions

Example 3 (Rayleigh Differential Equation): The Rayleigh equation is a non-linear second-order differential equation. Consider the equation

$$\frac{d^2y}{dt^2} - \mu\left(1 - \frac{1}{3}\left(\frac{dy}{dt}\right)^2\right)\frac{dy}{dt} + y = 0, \tag{4.27}$$

the initial conditions are chosen to be $y(0) = \alpha$ and $y'(0) = 0$. By setting $y_1 = y$ and introducing the new variable $y_2 = y'$, the second-order equation is converted to a first-order system

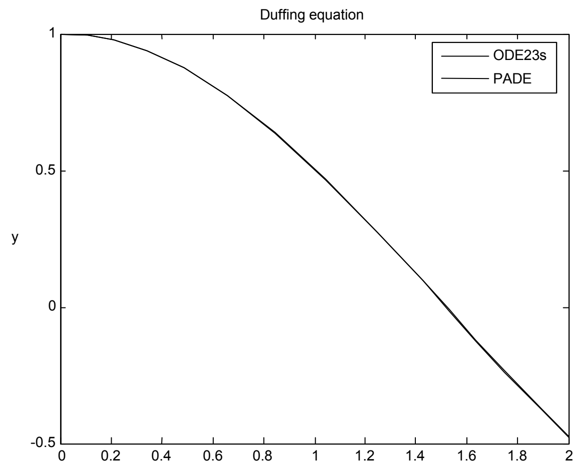

Fig. 2: Local changes of y for $\alpha = 1$ and $\varepsilon = 0.1$.

Table 2: Differences between the 6-term HPM and the the Padé approximations solutions for the The Vander Pol equation when $\varepsilon = 0.1$.

t	Diff
0	3.0000e-011
0.1	2.6393e-011
0.2	1.7350e-011
0.3	9.2888e-010
0.4	9.0588e-009
0.5	5.1941e-008
0.6	2.1424e-007
0.7	7.0359e-007
0.8	1.9538e-006
0.9	4.7684e-006
1	1.0499e-005

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 \\ \dfrac{dy_2}{dt} = -y_1 + \mu\left(1 - \dfrac{1}{3}y_2^2\right)y_2 \end{cases} \tag{4.28}$$

with the initial conditions:
$y_2(0) = r_1$, $y_2(0) = r_2$. Throughout this paper, we set $\alpha = 1$ and $\mu = 0.1$.

In this section, we will apply the homotopy perturbation method to nonlinear ordinary differential systems (4.27).

Homotopy Perturbation Method to Rayleigh Differential Equation: According to homotopy perturbation method, we derive a correct functional as follows:

$$\begin{aligned} &(1-p)(\dot{v}_1 - \dot{x}_0) + p(\dot{v}_1 - v_2) = 0, \\ &(1-p)(\dot{v}_2 - \dot{y}_0) + p\left(\dot{v}_2 + v_1 - \mu\left(1 - \frac{1}{3}v_2^2\right)v_2\right) = 0, \end{aligned} \tag{4.29}$$

Where "dot" denotes differentiation with respect to t and the initial approximations are as follows:

$$\begin{aligned} &v_{1,0}(t) = x_0(t) = y_1(0) = r_1, \\ &v_{2,0}(t) = y_0(t) = y_2(0) = r_2. \end{aligned} \tag{4.30}$$

and

$$\begin{aligned} v_1 &= v_{1,0} + pv_{1,1} + p^2 v_{1,2} + p^3 v_{1,3} + ..., \\ v_2 &= v_{2,0} + pv_{2,1} + p^2 v_{2,2} + p^3 v_{2,3} + ..., \end{aligned} \tag{4.31}$$

Where $v_{i,j}$, i,j = 1,2,3,...are functions yet to be determined. Substituting Eqs.(4.30) and (4.31) into Eq. (4.29) and arranging the coefficients of "p" powers, we have

$$\begin{aligned} &\left(\dot{v}_{1,1} - r_2\right)p + \left(\dot{v}_{1,2} - v_{2,1}\right)p^2 + \left(\dot{v}_{1,3} - v_{2,2}\right)p^3 + ... = 0, \\ &\left(\dot{v}_{2,1} + \left(r_1 - \mu r_2\right) + \frac{\mu r_2^3}{3}\right)p + \left(\dot{v}_{2,2} + \left(v_{1,1} - \mu v_{2,1}\right) + \frac{\mu}{3}\left(3r_2^2 v_{2,1}\right)\right)p^2 \\ &+\left(\dot{v}_{2,3} + \left(v_{1,2} - \mu v_{2,2}\right) + \frac{\mu}{3}\left(3r_2^2 v_{2,2} + 3v_{2,1}^2 r_2\right)\right)p^3 + ... = 0, \end{aligned} \tag{4.32}$$

In order to obtain the unknowns $v_{i,j}(t)$, i,j = 1,2,3, we must construct and solve the following system which includes nine equations with nine unknowns, considering the initial conditions

$$\begin{aligned} &\dot{v}_{1,1} - r_2 = 0, \dot{v}_{1,2} - v_{2,1} = 0, \dot{v}_{1,3} - v_{2,2} = 0, \\ &\dot{v}_{2,1} + \left(r_1 - \mu r_2\right) + \frac{\mu r_2^3}{3} = 0, \dot{v}_{2,2} + \left(v_{1,1} - \mu v_{2,1}\right) + \frac{\mu}{3}\left(3r_2^2 v_{2,1}\right) = 0, \\ &\dot{v}_{2,3} + \left(v_{1,2} - \mu v_{2,2}\right) + \frac{\mu}{3}\left(3r_2^2 v_{2,2} + 3v_{2,1}^2 r_2\right) = 0, \end{aligned} \tag{4.33}$$

From Eq. (3.8), if the three terms approximations are sufficient, we will obtain:

$$\begin{aligned} y_1(t) &= \lim_{p \to 1} v_1(t) = \sum_{k=0}^{3} v_{1,k}(t), \\ y_2(t) &= \lim_{p \to 1} v_2(t) = \sum_{k=0}^{3} v_{2,k}(t), \end{aligned} \tag{4.34}$$

therefore

$$\begin{aligned} y_1(t) &= r_1 + r_2 t + \frac{1}{2}\left[-r_1 + \mu r_2 - \frac{1}{3}\mu r_2^3\right]t^2 \\ &+\frac{1}{6}\left[-r_2 + \left(\mu - \mu r_2^2\right)\left(-r_1 + \mu r_2 - \frac{1}{3}\mu r_2^3\right)\right]t^3 \end{aligned}$$

$$\begin{aligned} y_2(t) &= r_2 + \left(-r_1 + \mu r_2 - \frac{1}{3}\mu r_2^3\right)t \\ &+\frac{1}{2}\left[-r_2 + \left(\mu - \mu r_2^2\right)\left(-r_1 + \mu r_2 - \frac{1}{3}\mu r_2^3\right)\right]t^2 \\ &+\frac{1}{6}\left[\begin{array}{l} -\mu r_2 + \mu r_2^3 - 2\mu r_2\left(-r_1 + \mu r_2 - \frac{1}{3}\mu r_2^3\right)^2 \\ +\left(-1 + \mu^2 - 2\mu^2 r_2^2 + \mu^2 r_2^3\right)\left(-r_1 + \mu r_2 - \frac{1}{3}\mu r_2^3\right) \end{array}\right]t^3 \end{aligned} \tag{4.35}$$

.
.
.

Here
$y_1(0) = 1$ and $y_2(0) = 0$ for the four-component model.
A few first approximations for $y_1(t)$ are calculated and presented below:
Six terms approximations:

$$y_1(t) = 1.-.5t^2 - .01666666667t^3 + .04125t^4 + .003325t^5 - .001152916667t^6 \quad (4.36)$$

In this section, we apply Laplace transformation to (4.36), which yields

$$L(y_1(s)) = \frac{1}{s} - \frac{1}{s^3} - \frac{.1}{s^4} + \frac{.99}{s^5} + \frac{.399}{s^6} - \frac{.8301}{s^7} \quad (4.37)$$

For simplicity, let $s = \frac{1}{t}$; then

$$L(y_1(t)) = t - t^3 - 0.1t^4 + .99t^5 + .399t^6 - .8301000002t^7 \quad (4.38)$$

Padé approximant [4/4] of (4.38) and substituting $t = \frac{1}{s}$, we obtain [4/4] in terms of s.

By using the inverse Laplace transformation, we obtain

$$\begin{aligned} y(t) = e^{.07393392516t}\left[-.0861732109\sin(.9952062118t) + 1.001863054\cos(.9952062118t)\right] \\ -.004335516653e^{-1.552673173t} + .002472462663e^{2.004805321t} \end{aligned} \quad (4.39)$$

In Table 3 we show the differences between the 6-term HPM and the the Padé approximations solutions

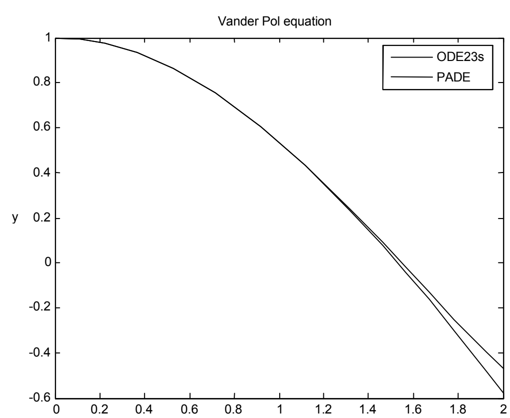

Fig. 3: Local changes of y for $\alpha = 1$ and $\mu = 0.1$.

Table 3: Differences between the 6-term HPM and the the Padé approximationssolutions for the the Rayleigh differential equation when $\mu = 0.1$.

t	Diff
0	1.0000e-011
0.1	1.2685e-011
0.2	1.1388e-010
0.3	2.3799e-009
0.4	2.3779e-008
0.5	1.4377e-007
0.6	6.2835e-007
0.7	2.1931e-006
0.8	6.4921e-006
0.9	1.6946e-005
1	4.0054e-005

CONCLUSIONS

In this paper, we apply homotopy perturbation method coupled with Laplace transform and Padé approximants on the re-formulated nonlinear oscillatory systems. Numerical results and graphical represenations explicitly reveal the complete reliability and efficiency of the suggested algorithm.

Note: The computations associated with the examples in this paper were performed using Maple 7 and Matlab 7.

REFERENCES

1. Baker, G.A., 1975. Essentials of Pad´e Approximants, Academic Press, London.
2. Baker, G.A. and P. Graves-Morris, 1996. Pade´ approdmants, Cambridge: Cambridge University Press.
3. Robertson, H.H., 1996. The solution of a set of reaction rate equations, in: J. Wals (Ed.), Numerical Analysis: An Introduction, Academic Press, London.
4. Jordan, D.W. and P. Smith, 1999. Nonlinear Ordinary Differential Equations, third ed. Oxford University Press.
5. Simmons, G.F., 1972. *Differential Equations with Applications and Historical Notes*, McGraw-Hill.

6. He, J.H., 1999. Homotopy perturbation technique, Comput Methods Appl Mech Engrg., 178: 257-62.
7. He, J.H., 2000. A coupling method of a homotopy technique and a perturbation technique for non-linear problems, Int. J. Non-linear Mech., 35(1): 37-43.
8. He, J.H., 1998. Approximate analytical solution for seepage flow with fractional derivatives in porous media, Computer Methods in Applied Mechanics and Engineering, 167(1-2): 57-68.
9. He, J.H., 1998. Approximate solution of nonlinear differential equations with convolution product nonlinearities, Computer Methods in Applied Mechanics and Engineering, 167(1-2): 69-73.
10. He, J.H., 2006. Some asymptotic methods for strongly nonlinear equations, International J. Modern Physics B, 20(10): 1141-1199.
11. Mickens RE., 1981. A uniformly valid asymptotic solution for $\frac{d^2y}{dt^2}+y=a+\varepsilon y^2$. J Sound Vibrat 76: 150-2.
12. Van Woerkom, P.T., 1982. Comments on 'A uniformly valid asymptotic solution for $\frac{d^2y}{dt^2}+y=a+\varepsilon y^2$. J. Sound Vibrat, 80: 157-8.
13. Kelly, S.G., 1982. Comments on A uniformly valid asymptotic solution for. J. Sound Vibrat 80: 155-6.
14. Atadan, A.S. and K. Huseyin, 1982. A note on A uniformly valid asymptotic solution for $\frac{d^2y}{dt^2}+y=a+\varepsilon y^2$. J. Sound Vibrat, 85: 129-31.
15. [Lord Rayleigh] J.W., 1945. Strutt, Theory of sound, 1, Dover, reprint.
16. Moustafa El-Shahed, 2007. Application of differential transform method to non-linear oscillatory systems, Communications in Nonlinear Science and Numerical Simulation.
17. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Some relatively new techniques for nonlinear problems, Mathematical Problems in Engineering, Hindawi, 2009 (2009); Article ID 234849, 25: doi:10.1155/2009/234849.
18. Mohyud-Din, S.T., M.A. Noor, K.I. Noor and M.M. Hosseini, 2010. Solution of singular equations by He's variational iteration method, International J. Nonlinear Sciences and Numerical Simulation, 11(2): 81-86.
19. Mohyud-Din, S.T., M.A. Noor, K.I. Noor and M.M. Hosseini, 2010. Variational iteration method for re-formulated partial differential equations, International J. Nonlinear Sciences and Numerical Simulation, 11(2): 87-92.
20. Mohyud-Din, S.T., M.A. Noor, K.I. Noor and M.M. Hosseini, 2010. On the coupling of He's polynomials and Laplace transformation, International J. Nonlinear Sciences and Numerical Simulation, 11(2): 93-96.
21. Mohyud-Din, S.T., 2010. On the conversion of partial differential equations, Zeitschrift für Naturforschung A- A J. Physical Sci., 65a: 896-900.
22. Mohyud-Din, S.T., 2010. Variational Iteration Method for Hirota-Satsuma Model Using He's Polynomials, Zeitschrift für Naturforschung A, A J. Physical Sci., 65(6-7): 525-528.
23. Mohyud-Din, S.T., 2010. Modified variational iteration method for integro-differential equations and coupled systems, Zeitschrift für Naturforschung A, A J. Physical Sci., 65a: 277-284.
24. Mohyud-Din, S.T., A. Yildirim and G. Demirli, 2011. Analytical solution of wave system in R^n with coupling controllers, International J. Numerical Methods for Heat and Fluid Flow, Emerald, 21(2): 198-205.
25. Mohyud-Din, S.T., A. Yildirim and G. Demirili, 2010. Traveling wave solutions of Whitham-Broer-Kaup equations by homotopy perturbation method, J. King Saud University (Science) Elsevier, 22: 173-176.
26. Mohyud-Din, S.T., M.A. Noor and K.I. Noor, 2009. Solving second-order singular problems using He's polynomials, World Applied Sciences J., 6(6): 769-775.
27. Mohyud-Din, S.T., A. Yildirim, M.M. Hosseini and M. Usman, 2010. An iterative algorithm for higher-dimensional IBVPs with variable co-efficient, World Applied Sciences J., 11(2): 159-164.