Couplings in coupled channels versus wave functions: application to the $X(3872)$ resonance

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We perform an analytical study of the scattering matrix and bound states in problems with many physical coupled channels. We establish the relationship of the couplings of the states to the different channels, obtained from the residues of the scattering matrix at the poles, with the wave functions for the different channels. The couplings basically reflect the value of the wave functions around the origin in coordinate space. In the concrete case of the $X(3872)$ resonance, understood as a bound state of $D^0\bar{D}^{*0}$ and D^+D^{*-} (and c.c.[1]), with the $D^0\bar{D}^{*0}$ loosely bound, we find that the couplings to the two channels are essentially equal leading to a state of good isospin $I = 0$ character. This is in spite of having a probability for finding the $D^0\bar{D}^{*0}$ state much larger than for D^+D^{*-} since the loosely bound channel extends further in space. The analytical results, obtained with exact solutions of the Schrödinger equation for the wave functions, can be useful in general to interpret results found numerically in the study of problems with unitary coupled channels methods.

I. INTRODUCTION

The $X(3872)$ resonance, observed by Belle [1] and confirmed by CDFII, D0 and BaBar collaborations [2–4], has been the object of intense debate from the theoretical point of view (see recent workshop on charm exotics at Badhonef [5]). Although different tentative explanations to its nature have been provided [6–12] the idea most supported recently is that it corresponds to a loosely bound state of $D\bar{D}^*$ [12, 14–27] or slightly unbound, virtual $D\bar{D}^*$ state [13, 28]. However, the energy of the resonance is very close to the $D^0\bar{D}^{*0}$ threshold, with the eventual charged components D^+D^{*-} bound by about 8 MeV. The binding of the $D^0\bar{D}^{*0}$ could be so small as to render the relatively very bound charged components irrelevant, at least from the probability point of view, given the fact that the loosely bound component would extend much further in space than the charged components. This is the idea behind many works [17, 19, 21–23, 25]. However it was found in [13] that the couplings of the resonance to the charged and neutral components were practically identical, implying a near $I = 0$ nature of the resonance as experimentally established. A pure $D^0\bar{D}^{*0}$ component would have an equal admixture of $I = 0$ and $I = 1$ and, according to [13] would produce a ratio of

$$\frac{\mathcal{B}(X \to J/\psi\pi^+\pi^-)}{\mathcal{B}(X \to J/\psi\pi^+\pi^-\pi^0)}, \qquad (1)$$

much larger than experiment. Indeed, in [29] several works based on pion exchange as the source for the $D\bar{D}^*$

binding are analysed thoroughly, stressing the importance of taking into account the neutral and charged components to properly study isospin violation in the $X(3872)$. There seems to be a contradiction between the intuitive idea of a dominance of the loosely bound component and the fact that experiment demands clearly an important contribution from the charged components. The clarification of this puzzle and establishing the meaning of the couplings in terms of wave functions is the purpose of this present work. While in the theoretical calculations one normally uses field theoretical methods to evaluate observables [12], without resorting to wave functions, the clarification of the puzzle forces one to face this problem solving the Schrödinger equation for the wave functions in coupled channels.

For simplicity, we will assume that the $X(3872)$ mass is below both the D^+D^{*-} and $D^0\bar{D}^{*0}$ thresholds. The work proceeds as follows: in the next section we make a brief summary of [13] to expose the problem. In sections III and IV we solve the Schrödinger equation in the case of one and two channels. In section V we extend the findings to the case of many channels. In section VI we come back to the $X(3872)$ and comment on its decay to J/ψ plus two and three pions, in section VII we comment on the independence of the results with the choice of the potential and in section VIII we outline our conclusions.

II. THE $X(3872)$ WITHIN COUPLED CHANNELS $D^0\bar{D}^{*0}$ AND D^+D^{*-}

In [12] the $X(3872)$ was plausibly explained as an $I = 0$ dynamically generated state in coupled channels with positive C-parity. However, in that work the charged and neutral D mesons were put with the same mass, in which case one had a good isospin symmetry. In [13] the

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[1]From now on, when we refer to $D^0\bar{D}^{*0}$, D^+D^{*-} or $D\bar{D}^*$ we are actually referring to the combination of these states with their complex conjugate in order to form a state with positive C-parity.

masses were taken different and a small isospin breaking was produced. Summarizing the approach of [12] we call channels 1 and 2 the $D^0\bar{D}^{*0}$ and D^+D^{*-}. It was found in this work, using the hidden gauge Lagrangians adapted to the $SU(4)$ flavor symmetry, containing explicit breaking of the symmetry, that the potential in coupled channels, in s-wave, was very close to the type

$$V^{\text{FT}} = \begin{pmatrix} v^{\text{FT}} & v^{\text{FT}} \\ v^{\text{FT}} & v^{\text{FT}} \end{pmatrix}, \tag{2}$$

the label FT standing for field theoretical approach, and v^{FT} is, in principle, a function of the invariant mass s (see Eqs. (7) and (12) of Ref. [13]). To describe the dynamics of the $X(3872)$, which is placed quite close to the $D\bar{D}^*$ threshold, it is sufficient to take the potential given in Eq. (7) of Ref. [13]) at threshold, neglecting in the potential all isospin breaking corrections induced by the difference of masses between charged and neutral mesons[1]

$$v^{\text{FT}} = -\frac{m_D m_{D^*}}{f_D^2} \tag{3}$$

with $f_D \sim 165$ MeV, the $D-$meson decay constant, and m_D and m_{D^*} averages of the neutral and charged D and D^* meson masses, respectively. The above interaction, should be considered with an ultraviolet cutoff in momentum space of natural size for hadron interactions, $\Lambda < 1$ GeV. In numerical calculations we use $m_{D^0} = 1865$ MeV, $m_{\bar{D}^{*0}} = 2007$ MeV, $m_{D^+} = m_{D^-} = 1870$ MeV and $m_{D^{*+}} = m_{D^{*-}} = 2010$ MeV.

The Bethe-Salpeter equation in coupled channels in the on-shell factorization approach stemming from the use of the N/D method [30–32] is given by:

$$T^{\text{FT}} = (1 - V^{\text{FT}} G^{\text{FT}})^{-1} V^{\text{FT}} \tag{4}$$

where both V^{FT} and T^{FT} are on-shell[2], and G^{FT} is the diagonal loop function for the two intermediate D and $\bar{D}^*$ meson propagators. In the particular case of Eq. (2), Eq. (4) is trivially written as

$$T^{\text{FT}} = \frac{V^{\text{FT}}}{1 - v^{\text{FT}} G_{11}^{\text{FT}} - v^{\text{FT}} G_{22}^{\text{FT}}} \tag{6}$$

We will assume that the T^{FT}-matrix develops a pole, in the first Riemann sheet and below both thresholds, for the $X(3872)$ resonance and the couplings g_i^{FT} to the

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[1] We will keep those isospin breaking corrections in the loop function G that will be introduced below. As it was discussed at length in Ref. [13], they turn out to be quite relevant.

[2] The normalization is fixed thanks to the relation between the scattering matrix and the differential center of mass cross section,

$$\left.\frac{d\sigma}{d\Omega}\right|_{\text{CM}} = \frac{1}{64\pi^2 s} \left|T^{\text{FT}}\right|^2. \tag{5}$$
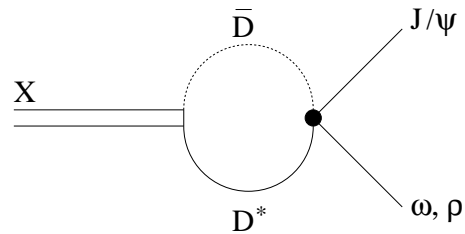

FIG. 1. X decay mechanism into $J/\psi\omega(\rho)$ assuming the X to be a $D\bar{D}^*$ molecule.

channels $D\bar{D}^*$ are defined such that in the vicinity of the pole

$$T_{ij}^{\text{FT}} = \frac{g_i^{\text{FT}} g_j^{\text{FT}}}{s - s_R} \tag{7}$$

with s_R the squared mass of the resonance, which allows one to obtain the couplings via

$$g_i^{\text{FT}} g_j^{\text{FT}} = \lim_{s \to s_R} (s - s_R) T_{ij}^{\text{FT}} = \left.\frac{-1}{\frac{d}{ds}\left(G_{11}^{\text{FT}} + G_{22}^{\text{FT}}\right)}\right|_{s=s_R} \tag{8}$$

$$= [g^{\text{FT}}]^2 \tag{9}$$

All the couplings are equal in this case and, as we see in Eq. (9), they are independent of v^{FT} and only depend on the derivative of the loop function G^{FT}. One is tempted to interpret the couplings as the components of the wave function, and since the couplings are equal we would have

$$|X(3872)\rangle \propto |D^0\bar{D}^{*0}\rangle + |D^+D^{*-}\rangle \tag{10}$$

which represents a pure isospin $I = 0$ state. Such an interpretation has some basis since the equality of the couplings is what makes the state behave as an $I = 0$ state in the field theoretical approach. Indeed, think of the ratio of Eq. (1). The two pion and the three pion states in the decay of the $X(3872)$ correspond to a ρ and an ω respectively, according to experiment [33, 34]. In a field theoretical approach the mechanism for this decay is depicted in Fig. 1, and the ratio for ρ and ω decays would be given by[3]

$$R_{\rho/\omega} = \left(\frac{g_1^{\text{FT}} G_{11}^{\text{FT}} - g_2^{\text{FT}} G_{22}^{\text{FT}}}{g_1^{\text{FT}} G_{11}^{\text{FT}} + g_2^{\text{FT}} G_{22}^{\text{FT}}}\right)^2. \tag{11}$$

The plus and minus signs in the numerator and denominator of Eq. (11) are simply a consequence of the fact that $J/\psi\rho$ has $I = 1$ while $J/\psi\omega$ has $I = 0$. If we had equal masses for the charged and neutral D mesons, the numerator of Eq. (11) vanishes and the decay $X \to J/\psi\rho$ is forbidden, since it violates isospin. If the masses of the two channels are different, even taking the two couplings

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[3] Technically, there can be a multiplicative coefficient in Eq. (11), but in the model used in [13] it is equal to 1.
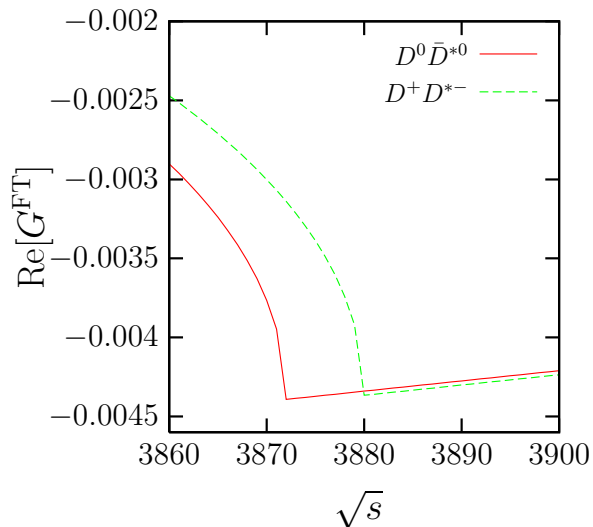
FIG. 2. Loop function G^{FT} for $D^0\bar{D}^{*0}$ and D^+D^{*-} from [13].

equal, as we found in Eq. (9), the numerator of Eq. (11) does not vanish due to the difference in the loop function for each channel, see Fig. 2, and the decay $X \to J/\psi\rho$ is allowed. One could interpret this by saying that the $X(3872)$ is an $I = 0$ state but the intermediate loops in the decay violate isospin. However, the free Hamiltonian (including masses) of the $D^0\bar{D}^{*0}$ and D^+D^{*-} system does not commute with isospin, which implies that the $X(3872)$ does not have a well defined isospin. The decays of the $X(3872)$ resonance can provide information on this isospin mixture. In this work we will look at its $J/\psi\rho$ and $J/\psi\omega$ decays, assuming transition operators of zero (short) range. Within this scheme, we will show that the isospin violation in these decays is linked to the different probability amplitudes of finding the $D^0\bar{D}^{*0}$ or D^+D^{*-} meson pairs, which form the X molecule, at short relative distances. This is the same as stating that the $D^0\bar{D}^{*0}$ and D^+D^{*-} wave functions around the origin will determine the isospin violations for these decays. Hence the couplings should not be understood as a measure of the wave function components. We shall come back to this issue in what follows.

III. COUPLING AND WAVE FUNCTION IN THE ONE CHANNEL CASE

We will first study the non-relativistic dynamics of a bound state generated by the interaction of two particles of masses m_1 and m_2, respectively.

A. The Lippmann Schwinger equation

We need a potential V and to illustrate our results, which are general, we can take some easy form for it (other forms will be analyzed in section VII). We choose a separable function in momentum space with the modulating factor being a simple step function, Θ. Thus our potential, already projected in $s-$wave, is assumed to be[4]

$$\langle \vec{p}\,'|V|\vec{p}\rangle = V(\vec{p}\,',\vec{p}) = v\,\Theta(\Lambda - p)\Theta(\Lambda - p') \quad (12)$$

where p and p' stand for $|\vec{p}|$ and $|\vec{p}\,'|$ and Λ is a cutoff in momentum space.

Let the Hamiltonian be $H = H_0 + V$ with H_0 the free Hamiltonian. In this case the non-relativistic Lippmann Schwinger equation can be written as

$$T = V + V\frac{1}{E - H_0}T \quad (13)$$

or also as

$$T = V + V\frac{1}{E - H}V \quad (14)$$

Taking Eq. (13) we can write:

$$\langle \vec{p}|T|\vec{p}\,'\rangle = \langle \vec{p}|V|\vec{p}\,'\rangle +$$
$$\int_{k<\Lambda} d^3k\, \frac{\langle \vec{p}|V|\vec{k}\rangle}{E - m_1 - m_2 - \frac{\vec{k}^2}{2\mu}}\langle \vec{k}|T|\vec{p}\,'\rangle \quad (15)$$

where μ is the reduced mass of the two particles that interact $[1/\mu = 1/m_1 + 1/m_2]$.

Eq. (15) has solution

$$\langle \vec{p}|T|\vec{p}\,'\rangle = \Theta(\Lambda - p)\Theta(\Lambda - p')\,t \quad (16)$$

which can also be seen from Eq. (14), with t given by

$$t = v + v\,G\,t, \qquad t = \frac{v}{1 - vG} \quad (17)$$

$$G = \int_{p<\Lambda} d^3p\, \frac{1}{E - m_1 - m_2 - \frac{\vec{p}^2}{2\mu}} \quad (18)$$

We can see that Eq. (17) is like the on-shell factorized equation (no integral left) of Eq. (4), and G is indeed, up to a factor[5], the non-relativistic reduction of the loop

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[4] We use normalization $\langle \vec{p}|\vec{x}\rangle = e^{-i\vec{p}\cdot\vec{x}}/(2\pi)^{\frac{3}{2}}$ which means $\int d^3x|\vec{x}\rangle\langle\vec{x}| = \int d^3p|\vec{p}\rangle\langle\vec{p}| = \mathbf{1}$ so that for a *local* potential we have $\langle \vec{p}\,'|V|\vec{p}\rangle = \int d^3x e^{-i(\vec{p}\,'-\vec{p})\cdot\vec{x}}V(\vec{x})/(2\pi)^3$.

[5] The non-relativistic reduction of the scattering matrix, potential and two particle propagator loop function introduced in previous Sect. II are related to those defined here by

$$v^{FT} = 32\pi^3\mu\sqrt{s}\,v, \qquad T^{FT} = 32\pi^3\mu\sqrt{s}\,t \quad (19)$$

function of two particle propagators regularized with a cutoff Λ, as is also usually done in the studies of hadron interactions with the on-shell Bethe Salpeter equation [35]. The name Bethe Salpeter equation was adopted in [35] because there, relativistic meson propagators are used. For very weak bindings, it is sufficient to use the Lippmann Schwinger equation, which is what we do here.

The poles of Eq. (17) occur for

$$1 - vG = 0 , \qquad (22)$$

which will occur for some value $E_\alpha < m_1 + m_2$ where we have a bound state. If the energy of the state is known, the above equation fixes the cut-off Λ, or conversely, if the cut-off is fixed, one can predict the energy of the state.

In the work of [12] the theoretical approach is based on the underlying hidden gauge formalism [36–38]. One assumes that it provides the potential V within a scale of momenta which is of the order of the cut off assumed here. Yet, one can assume certain uncertainties in V which can be compensated with changes in the range Λ in order to obtain the proper binding, as we show below. Indeed, using Eq. (22), invariance under renormalization leads to

$$\frac{d}{d\Lambda}(vG) = 0 \qquad (23)$$

$$\frac{1}{v}\frac{dv}{d\Lambda} = -\frac{1}{G}\frac{dG}{d\Lambda} \qquad (24)$$

and hence for a fixed E_α, the bare potential should have a dependence on the cutoff to compensate that of the loop function G.

B. The couplings of the state

The coupling in this case is defined as g such that in the vicinity of the pole the scattering matrix behaves as

$$t = \frac{g^2}{E - E_\alpha} \qquad (25)$$

and hence

$$g^2 = \lim_{E\to E_\alpha}(E - E_\alpha)t = -\left(\frac{dG}{dE}\right)^{-1}_{E=E_\alpha} \qquad (26)$$

$$G^{FT} = \frac{G}{32\pi^3\mu\sqrt{s}} \qquad (20)$$

with

$$G^{FT} = i\int\frac{d^4q}{(2\pi)^4}\frac{1}{q^2 - m_1^2 + i\epsilon}\frac{1}{(P-q)^2 - m_2^2 + i\epsilon} \qquad (21)$$

and $P_\mu P^\mu = s$.

where Eq. (17) has been used for t and the l'Hôpital's rule has been applied in the second equation.

The integral for the G function defined in Eq. (18) can be performed analytically and we obtain

$$G(E_\alpha) = -8\mu\pi\left(\Lambda - \gamma\arctan\left(\frac{\Lambda}{\gamma}\right)\right) \qquad (27)$$

$$\gamma = \sqrt{2\mu E_B^\alpha} \qquad (28)$$

where $0 < E_B^\alpha = m_1 + m_2 - E_\alpha$ is the binding energy of the state α. The above equation allows us, using Eq. (26), to write the coupling g as

$$g^2 = \frac{\gamma}{8\pi\mu^2\left(\arctan\left(\frac{\Lambda}{\gamma}\right) - \frac{\gamma\Lambda}{\gamma^2+\Lambda^2}\right)}. \qquad (29)$$

We note that g has a very smooth dependence on the cutoff Λ, which can even be removed ($\Lambda \to \infty$). The coupling is mostly determined by the binding energy (see Eq. (51) below), specially in the limit in which the latter one is much smaller than the cutoff, and hence g is, to great extent, renormalization scheme independent.

C. The wave function

The Schrödinger equation is given by:

$$H|\psi\rangle = E|\psi\rangle \qquad (30)$$

where ψ is an eigenfunction of H, the full Hamiltonian. We can write:

$$(H_0 + V)|\psi\rangle = E|\psi\rangle \qquad (31)$$

$$|\psi\rangle = \frac{1}{E - H_0}V|\psi\rangle \qquad (32)$$

which has the solution

$$\langle\vec{p}|\psi\rangle = \int d^3k\int d^3k'\langle\vec{p}|\frac{1}{E - H_0}|\vec{k}'\rangle$$
$$\times\langle\vec{k}'|V|\vec{k}\rangle\langle\vec{k}|\psi\rangle \qquad (33)$$
$$= v\frac{\Theta(\Lambda - p)}{E - m_1 - m_2 - \frac{\vec{p}^2}{2\mu}}\int_{k<\Lambda}d^3k\langle\vec{k}|\psi\rangle \qquad (34)$$

which gives us the wave function. Integrating Eq. (34) over d^3p, we obtain

$$1 - vG(E) = 0 \qquad (35)$$

which is the condition to find the pole given in Eq. (22).

In Eq. (34), we determined the state wave function up to a constant, $\int_{k<\Lambda}d^3k\langle\vec{k}|\psi\rangle$, which can be fixed from the normalization condition. Let $E_\alpha < m_1 + m_2$ be the solution of the above quantization equation, its wave function will satisfy

$$\int d^3p\,|\langle\vec{p}|\psi\rangle|^2 = 1 \qquad (36)$$

Note, that the wave function can be normalized because we are dealing with a bound state whose energy is below $m_1 + m_2$. From the above equation, one easily finds

$$1 = v^2 \int_{p<\Lambda} d^3p \left(\frac{1}{E_\alpha - m_1 - m_2 - \frac{\vec{p}^2}{2\mu}} \right)^2$$
$$\times \left| \int_{k<\Lambda} d^3k \langle \vec{k}|\psi \rangle \right|^2 \quad (37)$$

and hence, it follows

$$\left| v \int_{k<\Lambda} d^3k \langle \vec{k}|\psi \rangle \right|^2 = -\left(\frac{dG}{dE} \right)^{-1}_{E=E_\alpha}. \quad (38)$$

We can now use the form of Eq. (14) to solve the T matrix. We would have

$$T = V + \sum_{m,m'} V|m\rangle \langle m| \frac{1}{E-H} |m'\rangle \langle m'|V \quad (39)$$

where $|m\rangle$ and $|m'\rangle$ are complete sets of eigenstates of H. In the vicinity of the pole at $E = E_\alpha$ we care only for the contribution of channel α,

$$\langle \vec{p}|T|\vec{p}' \rangle \sim \langle \vec{p}|V|\alpha \rangle \frac{1}{E-E_\alpha} \langle \alpha|V|\vec{p}' \rangle$$
$$= \int d^3k \int d^3k' \langle \vec{p}|V|\vec{k} \rangle \langle \vec{k}|\alpha \rangle$$
$$\times \frac{1}{E-E_\alpha} \langle \alpha|\vec{k}' \rangle \langle \vec{k}'|V|\vec{p}' \rangle$$
$$= \frac{\left| v \int_{k<\Lambda} d^3k \langle \vec{k}|\alpha \rangle \right|^2}{E-E_\alpha} \Theta(\Lambda - p)\Theta(\Lambda - p') \quad (40)$$

exhibiting the form of Eq. (16) from where we defined t. We can obtain the residue of t as

$$g^2 = \lim_{E \to E_\alpha} (E - E_\alpha)T = \left| v \int_{k<\Lambda} d^3k \langle \vec{k}|\alpha \rangle \right|^2$$
$$= -\left(\frac{dG}{dE} \right)^{-1}_{E=E_\alpha} \quad (41)$$

where we have used Eq. (38) to get the last equality (α stands for ψ here) and we note that this is the same result as in Eq. (26).

The wave function in coordinate space can be equally evaluated:

$$\langle \vec{x}|\psi \rangle = \int d^3p \langle \vec{x}|\vec{p} \rangle \langle \vec{p}|\psi \rangle$$
$$= \int \frac{d^3p}{(2\pi)^{3/2}} e^{i\vec{p}.\vec{x}} \langle \vec{p}|\psi \rangle. \quad (42)$$

Using Eq. (34) we find[6]

$$\langle \vec{x}|\psi \rangle = g\sqrt{\frac{2}{\pi}} \frac{1}{r} \text{Im} \int_0^\Lambda dp\, p \frac{e^{ipr}}{E_\alpha - m_1 - m_2 - \frac{\vec{p}^2}{2\mu}} \quad (43)$$

For large values of r this function goes as

$$\langle \vec{x}|\psi \rangle_{r\to\infty} \sim \frac{A}{\sqrt{4\pi r}} e^{-\gamma r} \quad (44)$$

where

$$\gamma = \sqrt{2\mu E_B^\alpha}, \quad (45)$$
$$A = -2\mu g\sqrt{2\pi} \{1 + \mathcal{O}(1/\Lambda)\}. \quad (46)$$

The exponential fall-off at large distances is controlled by the binding energy of the state α and the coupling of the state. This behavior follows the general rule of bound states outside the interaction region and, as we see, is largely independent of the cut-off Λ. For the same reason, A carries information on the interaction region.

D. The meaning of the coupling in terms of wave functions

By means of Eq. (41) we know that (assuming a real wave-function for the bound state)

$$g = v \int_{k<\Lambda} d^3k \langle \vec{k}|\alpha \rangle, \quad (47)$$

and from Eq. (42) we can obtain the value of the wave function at the origin in coordinate space:

$$\langle \vec{x} = \vec{0}|\psi \rangle \equiv \psi(\vec{0}) = \int \frac{d^3p}{(2\pi)^{3/2}} \langle \vec{p}|\psi \rangle \quad (48)$$

and so

$$g = (2\pi)^{3/2} G^{-1}(E_\alpha)\psi(\vec{0}) \quad (49)$$

where we have also used the condition for the bound state $1 - vG = 0$. Since g hardly depends on the cutoff Λ, the wave function at the origin inherits the linear dependence on Λ exhibited by $G(E_\alpha)$ in Eq. (27). Yet, if v is known, the binding energy also fixes $G(E_\alpha)$ from the condition $vG(E_\alpha) = 1$.

Now we define

$$\hat{\psi} = gG(E_\alpha) = (2\pi)^{3/2}\psi(\vec{0}). \quad (50)$$

This constant will appear often in what follows. This is an important result concerning the problem at stake: the coupling, up to the factor $G(E_\alpha)$, is a measure of the wave function in coordinate space at the origin.

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[6] It is possible to write Eq. (43) in terms of the analytical functions sine integral and cosine integral.

$X(3872)$ is a bound state of the system, its mass ($E_\alpha \leq M_1 < M_2$) will be obtained by requiring that the denominator of Eq. (63) will vanish (pole of the $t-$matrix in the first Riemann sheet and below all thresholds). Clearly when $E_\alpha \to M_1$ the $D^0 \bar{D}^{*0}$ is loosely bound and the $D^+ D^{*-}$ is bound by about 8 MeV. The explicit expressions for $G_{ii}(E_\alpha)$ are given by Eq. (27) with

$$\gamma_i = \sqrt{2\mu_i E^\alpha_{Bi}} \tag{72}$$
$$E^\alpha_{Bi} = M_i - E_\alpha. \tag{73}$$

Let us now pay attention to the couplings of the bound state to each of the two channels. Since all elements of the matrix v are equal, both couplings g_1 and g_2 are the same:

$$g_1^2 = g_2^2 \equiv g^2 = \lim_{E \to E_\alpha} (E - E_\alpha) t_{ij}$$
$$= -\left(\frac{dG_{11}}{dE} + \frac{dG_{22}}{dE} \right)^{-1}\Bigg|_{E=E_\alpha} \tag{74}$$

On the other hand, as in Eq. (58), we obtain

$$g^{\mathrm{FT}} = \left(64\pi^3 \bar{\mu} E_\alpha^2\right)^{\frac{1}{2}} g \tag{75}$$

with $\bar{\mu}$, the average of the μ_1 and μ_2 reduced masses, as discussed in the footnote 7. By using Eqs. (74) and (75), we find not only a qualitative, but also a quantitative good agreement with the results for the coupling shown in Fig. 3 of Ref. [13]. For instance, if the neutral channel is bound by 1 (0.1) MeV, the denominator of Eq. (63) will vanish for a value of the cutoff Λ of around 680 (653) MeV, with $\hat{v}$ given by Eq. (71). This leads to a coupling g^{FT} of the order of 5400 (3200) MeV, in reasonable agreement with the result in Ref. [13].

In the limit when $E^\alpha_{B1} \to 0$, we have $\frac{dG_{11}}{dE}\big|_{E=E_\alpha} \to \infty$ and we find

$$g_1^2 = g_2^2 \sim \frac{\gamma_1}{4\pi^2 \mu_1^2}, \qquad E^\alpha_{B1} \to 0 \tag{76}$$

thus, both couplings go to zero as $\sqrt{\gamma_1}$.

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which guaranties that poles are placed in the same position in both approaches. Finally, note that a potential in the field theory approach of the form in Eq. (2) does not lead to a quantum mechanics potential of the form assumed in Eq. (61), unless that both reduced masses μ_1 and μ_2 are taken to be equal in Eq. (69). This is an excellent approximation for the case of the coupled channels $D^0 \bar{D}^{*0}$ and $D^+ D^{*-}$, and we have done so here when relating Quantum Field Theory and Quantum Mechanics quantities. Thus, we have taken $\mu_1 \sim \mu_2 \sim \bar{\mu}$, with $\bar{\mu}$ some average reduce mass. Hence, the Quantum Mechanics potential, $\hat{v}$, near threshold and for an ultraviolet cutoff of natural size for hadron interactions $\Lambda < 1$ GeV, can be now approximated by

$$\hat{v} = -\frac{1}{32\pi^3 f_D^2} \tag{71}$$

as deduced from Eq. (3), with the further approximation $\bar{\mu}\sqrt{s} \approx m_D m_{D^*}$.

B. The wave function

For the bound state we have now a two component wave function, representing each of the $D^0 \bar{D}^{*0}$ and $D^+ D^{*-}$ channels, and the Schrödinger equation reads

$$(H_0 + V)|\psi\rangle = E|\psi\rangle \tag{77}$$
$$|\psi\rangle = \begin{pmatrix} |\psi_1\rangle \\ |\psi_2\rangle \end{pmatrix} \tag{78}$$

The solution to this equation is given by

$$|\psi\rangle = \frac{1}{E - H_0} V|\psi\rangle \tag{79}$$

$$\langle \vec{p}|\psi\rangle = \begin{pmatrix} \frac{1}{E-M_1-\vec{p}^2/2\mu_1} & 0 \\ 0 & \frac{1}{E-M_2-\vec{p}^2/2\mu_2} \end{pmatrix}$$
$$\times \int d^3k \langle \vec{p}|V|\vec{k}\rangle \langle \vec{k}|\psi\rangle \tag{80}$$

which represents two coupled channels equations

$$\langle \vec{p}|\psi_1\rangle = \hat{v} \frac{\Theta(\Lambda - p)}{E - M_1 - \frac{\vec{p}^2}{2\mu_1}}$$
$$\times \int_{k<\Lambda} d^3k \left(\langle \vec{k}|\psi_1\rangle + \langle \vec{k}|\psi_2\rangle \right) \tag{81}$$
$$\langle \vec{p}|\psi_2\rangle = \hat{v} \frac{\Theta(\Lambda - p)}{E - M_2 - \frac{\vec{p}^2}{2\mu_2}}$$
$$\times \int_{k<\Lambda} d^3k \left(\langle \vec{k}|\psi_1\rangle + \langle \vec{k}|\psi_2\rangle \right) \tag{82}$$

which require to know $\int_{k<\Lambda} d^3k \langle \vec{k}|\psi_i\rangle$ for its solution. To evaluate this latter magnitude let us integrate over $\vec{p}$ in Eq. (80) and we get

$$\int_{p<\Lambda} d^3p \langle \vec{p}|\psi\rangle = G\,v \int_{p<\Lambda} d^3p \langle \vec{p}|\psi\rangle \tag{83}$$

an algebraic equation that requires for its solution

$$\det(1 - G\,v) = 1 - \hat{v} G_{11} - \hat{v} G_{22} = 0 \tag{84}$$

This equation is satisfied for the poles, $E = E_\alpha$, of the t matrix corresponding to bound states (see Eq. (63)). Eq. (83) can be now be solved and we obtain

$$\int_{p<\Lambda} d^3p \langle \vec{p}|\psi_2\rangle = \frac{\hat{v} G^\alpha_{22}}{1 - \hat{v} G^\alpha_{22}} \int_{p<\Lambda} d^3p \langle \vec{p}|\psi_1\rangle$$
$$= \frac{G^\alpha_{22}}{G^\alpha_{11}} \int_{p<\Lambda} d^3p \langle \vec{p}|\psi_1\rangle \tag{85}$$

where $G^\alpha_{ii} = G_{ii}(E = E_\alpha)$. Thus, Eqs. (81) and (82) can be written as

$$\langle \vec{p}|\psi_1\rangle = \frac{1}{G^\alpha_{11}} \frac{\Theta(\Lambda - p)}{E_\alpha - M_1 - \frac{\vec{p}^2}{2\mu_1}} \int_{k<\Lambda} d^3k \langle \vec{k}|\psi_1\rangle \tag{86}$$

$$\langle \vec{p}|\psi_2\rangle = \frac{1}{G^\alpha_{11}} \frac{\Theta(\Lambda - p)}{E_\alpha - M_2 - \frac{\vec{p}^2}{2\mu_2}} \int_{k<\Lambda} d^3k \langle \vec{k}|\psi_1\rangle \tag{87}$$

If we define the partial probability

$$P_i = \langle \psi_i | \psi_i \rangle = \int d^3 p |\langle \vec{p} | \psi_i \rangle|^2 \qquad (88)$$

we can further use the total normalization condition

$$1 = P_1 + P_2 \equiv \int_{p<\Lambda} d^3 p \left\{ |\langle \vec{p} | \psi_1 \rangle|^2 + |\langle \vec{p} | \psi_2 \rangle|^2 \right\}$$

$$= -\left(\frac{1}{[G_{11}^\alpha]^2} \frac{dG_{11}}{dE} \bigg|_{E=E_\alpha} + \frac{1}{[G_{11}^\alpha]^2} \frac{dG_{22}}{dE} \bigg|_{E=E_\alpha} \right)$$

$$\times \left| \int_{k<\Lambda} d^3 k \langle \vec{k} | \psi_1 \rangle \right|^2 \qquad (89)$$

from where, using Eq. (74)

$$\left| \int_{p<\Lambda} d^3 p \langle \vec{p} | \psi_1 \rangle \right|^2 = [G_{11}^\alpha]^2 \, g^2 \qquad (90)$$

and hence

$$P_1 = -g^2 \frac{dG_{11}}{dE} \bigg|_{E=E_\alpha}, \quad P_2 = -g^2 \frac{dG_{22}}{dE} \bigg|_{E=E_\alpha} \qquad (91)$$

$$\frac{P_1}{P_2} = \frac{\mu_1^2 \gamma_2}{\mu_2^2 \gamma_1} \frac{\arctan\left(\frac{\Lambda}{\gamma_1}\right) - \frac{\gamma_1 \Lambda}{\gamma_1^2 + \Lambda^2}}{\arctan\left(\frac{\Lambda}{\gamma_2}\right) - \frac{\gamma_2 \Lambda}{\gamma_2^2 + \Lambda^2}}$$

$$= \frac{\gamma_2}{\gamma_1} \left[1 + \mathcal{O}(\Lambda^{-1}) \right] = \sqrt{\frac{E_{B2}^\alpha}{E_{B1}^\alpha}} \left[1 + \mathcal{O}(\Lambda^{-1}) \right] \quad (92)$$

The neglected terms are finite range corrections, which in our case are represented by the finite cut-off. On the other hand, assuming real wave functions, Eq. (90) together with Eq. (85) lead to

$$g\, G_{11}^\alpha = \int_{p<\Lambda} d^3 p \langle \vec{p} | \psi_1 \rangle \qquad (93)$$

$$g\, G_{22}^\alpha = \int_{p<\Lambda} d^3 p \langle \vec{p} | \psi_2 \rangle \qquad (94)$$

The wave functions in coordinate space would be again given by means of Eq. (43) using μ_1 for ψ_1 and μ_2 for ψ_2 and substituting $m_1 + m_2$ by M_i for each component. At long distances this means, see Eq. (44),

$$\langle \vec{x} | \psi_1 \rangle_{r \to \infty} \sim \frac{A_1}{\sqrt{4\pi} r} e^{-\gamma_1 r} \qquad (95)$$

$$\langle \vec{x} | \psi_2 \rangle_{r \to \infty} \sim \frac{A_2}{\sqrt{4\pi} r} e^{-\gamma_2 r} \qquad (96)$$

where we have used Eq. (73). Once again $\int_{p<\Lambda} d^3 p \langle \vec{p} | \psi_i \rangle$ can be interpreted (up to a constant factor $(2\pi)^{\frac{3}{2}}$) as the wave function at the origin, as done in Eq. (48) and then Eq. (93) and Eq. (94) can be rewritten as

$$g G_{11}^\alpha = (2\pi)^{3/2} \psi_1(\vec{0}) = \hat{\psi}_1 \qquad (97)$$

$$g G_{22}^\alpha = (2\pi)^{3/2} \psi_2(\vec{0}) = \hat{\psi}_2 \qquad (98)$$
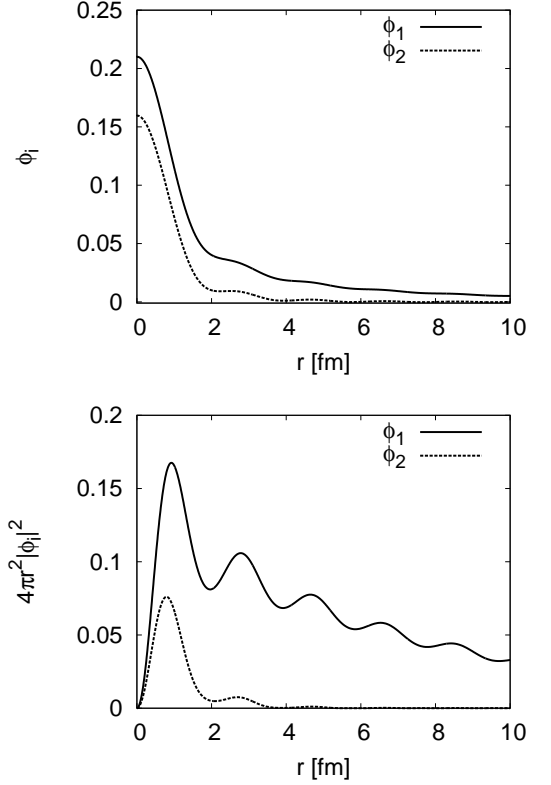
FIG. 3. Neutral and charged Wave function components for a $D^0 \bar{D}^{*0}$ binding energy of 0.1 MeV. In the upper panel one can see the value of the wave functions at the origin for both channels, in the lower panel we plot the probability density for each channel.

¿From the above expressions, we trivially find the ratio of wave functions at the origin

$$\frac{\hat{\psi}_2}{\hat{\psi}_1} = \frac{G_{22}^\alpha}{G_{11}^\alpha} = (\hat{v} G_{11}^\alpha)^{-1} - 1 \qquad (99)$$

and their departure from unity provides a tangible measure of the isospin breaking in the interaction region. On the other hand,

$$\frac{d}{d\Lambda} \left(\hat{\psi}_2 / \hat{\psi}_1 \right) = \frac{\pi}{2} \frac{\mu_2}{\mu_1} \frac{\gamma_2 - \gamma_1}{\Lambda^2} + \mathcal{O}(1/\Lambda^4) \qquad (100)$$

which shows that, though both $\hat{\psi}_1$ and $\hat{\psi}_2$ depend greatly on the cutoff, such a dependence is much reduced in their ratio.

In Fig. 3 we show the two wave function components for the problem solved in [13] with only two channels and the binding energy for $D^0 \bar{D}^{*0}$ of 0.1 MeV. The oscillations are caused by the sharp cut-off Λ introduced to regularize the Lippmann-Schwinger equation. Eqs. (97) and (98) and the asymptotic behavior of the wave functions given by Eqs. (95) and (96) clearly show that, in the limit of zero binding energy of the $D^0 \bar{D}^{*0}$ component, this wave function extends up to infinity while the

D^+D^{*-} component is restricted in space because of the 8 MeV binding. The probability to have the $D^0\bar{D}^{*0}$ component becomes much larger than that of the D^+D^{*-} component (see Eq. (92)) and we could think of the $X(3872)$ as a $D^0\bar{D}^{*0}$ molecule. While technically correct from the point of view of probabilities, this interpretation is misleading concerning physical processes, like decays, because these require Hamiltonians of short range, zero range ordinarily in effective field theory, such that what matters in these processes is the wave function at the origin. For instance, what will determine the $I = 0$ character of the wave function will be the $\psi_1(\vec{0})$ and $\psi_2(\vec{0})$ magnitudes, not the probability integrated in all coordinate space. Hence, Eqs. (97) and (98) indicate an isospin breaking, with respect to the $I = 0$ combination, given by the differences between G_{11}^α and G_{22}^α (see Fig. 2).

Eq. (11), related to the decay process depicted in Fig. 1, has now an intuitive representation to the light of Eqs. (97) and (98). The amplitude for

$$X(3872) \to J/\psi\rho(\omega) \qquad (101)$$

is given by

$$\begin{aligned}\mathcal{M} &= g_1 G_{11}^\alpha F_1 + g_2 G_{22}^\alpha F_2 \\ &= (2\pi)^{3/2}(\psi_1(\vec{0})F_1 + \psi_2(\vec{0})F_2) \qquad (102)\end{aligned}$$

where F_1 and F_2 are isospin factors for the vertices $D^0\bar{D}^{*0} \to J/\psi\rho(\omega)$ and $D^+D^{*-} \to J/\psi\rho(\omega)$, $F_2/F_1 = -1$ for the ρ and $F_2/F_1 = 1$ for the ω. Certainly this should be the case for many channels and we shall see the generalization in the next section.

Eq. (102) can alternatively be interpreted as

$$\begin{aligned}\mathcal{M} &= \int d^3p \, \langle\vec{p}|\psi_1\rangle t_{D^0\bar{D}^{*0}\to J/\psi\rho(\omega)} \\ &+ \int d^3p \, \langle\vec{p}|\psi_2\rangle t_{D^+D^{*-}\to J/\psi\rho(\omega)} \qquad (103)\end{aligned}$$

assuming that the range of the t amplitudes is very short compared to the extension of the wave functions, essentially that the t are constant functions in momentum space, as one has in field theory vertices stemming from a contact Lagrangian.

V.   GENERALIZATION TO MANY CHANNELS

We have now

$$\langle\vec{p}\,'|V|\vec{p}\rangle \equiv v\Theta(\Lambda - \vec{p})\Theta(\Lambda - \vec{p}\,') \qquad (104)$$

where v is a $N \times N$ matrix with N the number of channels.

The expressions that we have obtained can be generalized to many channels and we derive here some useful expressions.

A.   The couplings

We can write for the T matrix

$$T = \frac{\mathcal{A}v}{\det(1 - vG)} \qquad (105)$$

where $\mathcal{A}$ is defined as

$$\mathcal{A} = [\det(1 - vG)](1 - vG)^{-1} \qquad (106)$$

This matrix is introduced to single out the source of the pole in all channels which is given by the condition

$$\det(1 - vG) = 0 \qquad (107)$$

We have now

$$\begin{aligned}g_i g_j &= \lim_{E\to E_\alpha} (E - E_\alpha)T_{ij} \\ &= \left.\frac{(\mathcal{A}v)_{ij}}{\frac{d}{dE}\det(1 - vG)}\right|_{E=E_\alpha} \qquad (108) \\ \frac{g_j}{g_i} &= \left.\frac{(\mathcal{A}v)_{ij}}{(\mathcal{A}v)_{ii}}\right|_{E=E_\alpha} \qquad (109)\end{aligned}$$

We can see that g_j/g_i is a ratio of two matrix elements of matrices without singularities. This means that if $g_i \to 0$ as a consequence of having the binding in channel i going to zero, then all the couplings to the other channels coupled to channel i will also go to zero. This is also obvious from Eq. (108) since $g_j^2 \to 0$ because the denominator contains $\frac{dG_{ii}^\alpha}{dE}$ and, one has $\frac{dG_{ii}^\alpha}{dE} \to \infty$ for all cases.

B.   Wave functions

Eqs. (81) and (82) can be generalized as

$$\begin{aligned}\langle\vec{p}|\psi_i\rangle &= \Theta(\Lambda - p)\frac{1}{E - M_i - \frac{\vec{p}^2}{2\mu_i}} \\ &\times \sum_j v_{ij} \int_{k<\Lambda} d^3k\langle\vec{k}|\psi_j\rangle \qquad (110)\end{aligned}$$

which upon integration leads to

$$\begin{aligned}\int_{p<\Lambda} d^3p\langle\vec{p}|\psi_i\rangle &= G_{ii} \sum_j v_{ij} \\ &\times \int_{k<\Lambda} d^3k\langle\vec{k}|\psi_j\rangle \qquad (111)\end{aligned}$$

which in the language of Eq. (97) and Eq. (98) reads

$$\begin{aligned}\hat{\psi}_i &= G_{ii} \sum_j v_{ij}\hat{\psi}_j \\ \hat{\psi} &= Gv\hat{\psi} \qquad (112)\end{aligned}$$

where Eq. (112) is written in matrix form and it requires that $\det(1-vG) = 0$ for its solution, which is guaranteed

for the bound eigenstate. Eq. (112) can also be written as ($G_\alpha^{-1} = G^{-1}(E_\alpha)$)

$$G_\alpha^{-1}\hat\psi = v\hat\psi \tag{113}$$

which allows to rewrite the equation for the wave functions Eq. (110) as

$$\langle\vec{p}|\psi\rangle = \text{diag}\left(\frac{\Theta(\Lambda - p)}{E_\alpha - M_i - \frac{\vec{p}^2}{2\mu_i}}\right) G_\alpha^{-1}\hat\psi \tag{114}$$

which gives the wave function in momentum space in terms of the wave function in coordinate space at the origin. These equations are the generalization of Eq. (86) and Eq. (87) together with Eq. (85).

Let us now use the normalization condition

$$\sum_i \langle\psi_i|\psi_i\rangle = \int d^3p \sum_i |\langle\vec{p}|\psi_i\rangle|^2$$
$$= -\sum_i \frac{dG_{ii}}{dE}\frac{1}{G_{ii}^2}\bigg|_{E=E_\alpha}\hat\psi_i^{\;2} = 1 \tag{115}$$

We can now take advantage of Eq. (114) to define the couplings in terms of the $\hat\psi_i$. For this we use the version of Eq. (14) for the Lippmann Schwinger equation, recalling that close to the pole of the eigenfunction of the Hamiltonian, $|\psi\rangle$, associated to the energy E_α, only this state contributes in the sum over eigenstates of H, and we find

$$T_{ij} = v_{ij} + \sum_{mn} v_{im} \int_{k<\Lambda} d^3k \langle\vec{k}|\psi_m\rangle$$
$$\times \frac{1}{E - E_\alpha} \int_{k'<\Lambda} d^3k' \langle\vec{k}\,'|\psi_n\rangle v_{nj} \tag{116}$$

which means that

$$g_i g_j = \sum_{mn} v_{im}\hat\psi_m v_{nj}\hat\psi_n$$
$$= G_{ii}^{-1}\hat\psi_i G_{jj}^{-1}\hat\psi_j\bigg|_{E=E_\alpha} \tag{117}$$

from where we conclude that

$$g_i = \hat\psi_i/G_{ii}^\alpha$$
$$g_i G_{ii}^\alpha = \hat\psi_i \tag{118}$$

as we found in Eq. (97) and Eq. (98) in the two channel problem. This allows to reinterpret Eq. (115) in terms of the couplings and we find

$$\sum_i g_i^2 \frac{dG_{ii}}{dE}\bigg|_{E=E_\alpha} = -1 \tag{119}$$

which is the generalization of Eq. (74).

Eq. (119) is interesting because when one channel becomes loosely bound then the loop derivative for this channel goes to infinity while the other derivatives remain finite. In this limit we get, if channel 1 is loosely bound

$$\lim_{E_B\to 0} g_1^2 \frac{dG_{11}}{dE}\bigg|_{E=E_\alpha} = -1$$
$$g_1^2 = -\left(\frac{dG_{11}}{dE}\right)^{-1}_{E=E_\alpha} \tag{120}$$

which is the same result obtained for one channel in Eq. (41). Thus, in this limit the coupling of the loosely bound state goes to zero as the binding energy goes to zero. On the other hand, Eq. (109) guarantees that all the other couplings will also go to zero since the matrix $\mathcal{A}v$ is not singular. This result was also found in [42] although derived in a different way.

VI. DECAY WIDTH OF THE $X(3872)$

After these clarifications we would like to go back to the ratio of the decay width of the $X(3872)$ to $J/\psi\rho$ and $J/\psi\omega$ of Eq. (11). As discussed in [13], the ratio of widths was given by the square of Eq. (11) times the factor to correct for the phase space of ρ decaying to two pions and the ω decaying to three pions:

$$\frac{\mathcal{B}(X \to J/\psi\pi\pi)}{\mathcal{B}(X \to J/\psi\pi\pi\pi)} = \left(\frac{G_{11}^\alpha - G_{22}^\alpha}{G_{11}^\alpha + G_{22}^\alpha}\right)^2 \frac{\int_0^\infty q\mathcal{S}(s, m_\rho, \Gamma_\rho)\,\Theta(m_X - m_{J/\psi} - \sqrt{s})\,ds}{\int_0^\infty q\mathcal{S}(s, m_\omega, \Gamma_\omega)\,\Theta(m_X - m_{J/\psi} - \sqrt{s})\,ds} \frac{\mathcal{B}_\rho}{\mathcal{B}_\omega} \tag{121}$$

where $\mathcal{B}_\rho$ and $\mathcal{B}_\rho$ are the branching fractions of ρ decaying into two pions (~ 100 %) and ω decaying into three pions (~ 89 %), q is the center of mass momentum of the outgoing meson pair in each channel and value of s, and $\mathcal{S}(s, m, \Gamma)$ is the spectral function of the mesons given by:

$$\mathcal{S}(s, m, \Gamma) = -\frac{1}{\pi}\text{Im}\left(\frac{1}{s - m^2 + i\Gamma m}\right) \tag{122}$$

In [13] it was found, using dimensional regularization for the loops,

$$\frac{\mathcal{B}(X \to J/\psi\pi^+\pi^-\pi^0)}{\mathcal{B}(X \to J/\psi\pi^+\pi^-)} = 1.4 \tag{123}$$

which is compatible with the experimental value 1.0 ± 0.4 from [34].

Now let us assume that we take seriously that there is only one channel, the $D^0\bar{D}^{*0}$. Then the ratio of the

widths is

$$R_{\rho/\omega}^{(D^0\bar{D}^{*0})} = \left(\frac{\hat{\psi}_1 t_{D^0\bar{D}^{*0}\to J/\psi\rho}}{\hat{\psi}_1 t_{D^0\bar{D}^{*0}\to J/\psi\omega}}\right)^2 = 1 \qquad (124)$$

which is about 30 times bigger than the value obtained for this ratio (0.032) in [13][8] . When we take into account the phase space for the decay into ρ and ω and the ρ and ω branching ratios into two and three pions, with the ratio in Eq. (124) we find

$$\frac{\mathcal{B}(X \to J/\psi\pi^+\pi^-\pi^0)}{\mathcal{B}(X \to J/\psi\pi^+\pi^-)} = 0.05 \qquad (126)$$

which is about a factor 20 times smaller than experiment.

It is thus clear that the charged components of the wave function have played an essential role bringing this branching ratio close to experiment and this stresses that the wave functions at the origin for each channel, and not the probabilities of finding the state in a single channel alone, is what determines the isospin nature of the state in coupled channels. Indeed, the $X(3872)$ wave function would read

$$\langle\vec{r}|\psi\rangle = \frac{\psi_1(\vec{r}) + \psi_2(\vec{r})}{\sqrt{2}}\chi_{I=0}$$
$$+ \frac{\psi_1(\vec{r}) - \psi_2(\vec{r})}{\sqrt{2}}\chi_{I=1} \qquad (127)$$

with $\chi_{I=0,1}$ scalar and vector isospin wave function spinors. In the charge basis used in Section IV, the isospin wave functions are

$$\chi_{I=0} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}, \quad \chi_{I=1} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}. \quad (128)$$

As mentioned in the introduction, the $X(3872)$ does not have well defined isospin because the free Hamiltonian (including masses) of the $D^0\bar{D}^{*0}$ and D^+D^{*-} system does not commute with isospin. The mixing depends on the relative distance $\vec{r}$ between the pseudoscalar and vector mesons. Thus with transition operators of zero range, one easily understands that the ratio of branching fractions of Eq. (1) is determined by the ratio $\left[\left(1 - \hat{\psi}_2/\hat{\psi}_1\right)/\left(1 + \hat{\psi}_2/\hat{\psi}_1\right)\right]^2$, which in turn gives the ratio of isospin 1 to isospin 0 probabilities at $\vec{r} = 0$. As we shall see in the next section (see $R_{\rho/\omega}$ of Table I), this ratio is of the order of 2%.

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[8] Note that

$$\left(\frac{G_{11}^\alpha - G_{22}^\alpha}{G_{11}^\alpha + G_{22}^\alpha}\right)^2 = \left(\frac{1 - \hat{\psi}_2/\hat{\psi}_1}{1 + \hat{\psi}_2/\hat{\psi}_1}\right)^2 \qquad (125)$$

and thanks to Eq. (100) the ratio of wave functions at the origin depends little on the ultraviolet cutoff Λ.

VII. RESULTS FOR OTHER FORMS OF THE POTENTIAL

One might think that the results obtained are specific of the type of the potential chosen in Eq. (60), but the results are actually very general. To show that this is the case, we use other potentials. Let us consider a separable potential where we substitute the sharp cut off by a form factor

$$\langle\vec{p}'|V|\vec{p}\rangle \equiv v f(\vec{p}) f(\vec{p}') \qquad (129)$$

where v is a $N \times N$ matrix with N the number of channels. The results of Sect. V follow nearly identically substituting the $\Theta(\Lambda - p)$ by $f(\vec{p})$. Eq. (105)-Eq. (109) are the same, but G is now given by

$$G_{ii} = \int d^3p f^2(\vec{p})\frac{1}{E - M_i - \frac{\vec{p}^2}{2\mu_i}} \qquad (130)$$

and the wave functions are now given by

$$\langle\vec{p}|\psi_i\rangle = f(\vec{p})\frac{1}{E_\alpha - M_i - \frac{\vec{p}^2}{2\mu_i}}\sum_j v_{ij}$$
$$\times \int d^3k f(\vec{k})\langle\vec{k}|\psi_j\rangle \qquad (131)$$

Eq. (111)-Eq. (113) follow, but now

$$\hat{\psi}_i = \int d^3k f(\vec{k})\langle\vec{k}|\psi_i\rangle \qquad (132)$$

which allows to write Eq. (131) as

$$\langle\vec{p}|\psi\rangle = \text{diag}\left(\frac{f(\vec{p})}{E_\alpha - M_i - \frac{\vec{p}^2}{2\mu_i}}\right)G_\alpha^{-1}\hat{\psi} \qquad (133)$$

and again we find Eq. (118)

$$g_i = \hat{\psi}_i/G_{ii}^\alpha$$
$$g_i G_{ii}^\alpha = \hat{\psi}_i \qquad (134)$$

and Eq. (119)-Eq. (120) also follow.

Everything is identical as before, but now $\hat{\psi}$ is not, up to a factor $(2\pi)^{\frac{3}{2}}$, the wave function at the origin (it would be if we removed $f(\vec{p})$ from Eq. (132)). To see the meaning of $\hat{\psi}$ we write $f(\vec{p})$ in terms of its Fourier Transform

$$f(\vec{p}) = \frac{1}{(2\pi)^{3/2}}\int d^3x \hat{f}(\vec{x})e^{i\vec{p}\cdot\vec{x}} \qquad (135)$$

and the wave function of Eq. (133) also in terms of its Fourier Transform

$$\psi_i(\vec{p}) = \frac{1}{(2\pi)^{3/2}}\int d^3x e^{-i\vec{p}\cdot\vec{x}}\psi_i(\vec{x}) \qquad (136)$$

Then upon integrating explicitly over $\vec{k}$ in Eq. (132) we find

$$\hat{\psi}_i = \int d^3x\, \psi_i(\vec{x})\hat{f}(\vec{x}) \qquad (137)$$

We performed explicit calculations using a gaussian form for $f(\vec{p})$

$$f(\vec{p}) = e^{-\frac{1}{2}\vec{p}^2/\Lambda^2}$$
$$\hat{f}(\vec{x}) = \Lambda^3 e^{-\frac{1}{2}\vec{x}^2\Lambda^2} \quad (138)$$

and a Lorentz form

$$f(\vec{p}) = \frac{\Lambda^2}{\Lambda^2 + \vec{p}^2}$$
$$\hat{f}(\vec{x}) = \sqrt{\frac{\pi}{2}}\Lambda^2 \frac{e^{-|\vec{x}|\Lambda}}{|\vec{x}|} \quad (139)$$

We can see that $\hat{f}(\vec{x})$ has a range of $1/\Lambda \sim 0.2 - 0.3$ fm, a range much smaller than the extension of the wave function. Thus $\hat{\psi}_i$ gives the average of the wave function in the vicinity of the origin, while in the case of the sharp cut off one finds exactly the wave function at the origin.

One sees again that this average value of the wave function at the origin is what governs the decay process of the resonance. Indeed, if we go to Eq. (103) to get the amplitudes for the decay mechanism of Fig. 1 we would find:

$$\mathcal{M} = \int d^3p \, \langle\vec{p}|\psi_1\rangle f(\vec{p}) \, t_{D^0\bar{D}^{*0}\to J/\psi\rho(\omega)}$$
$$+ \int d^3p \, \langle\vec{p}|\psi_2\rangle f(\vec{p}) \, t_{D^+D^{*-}\to J/\psi\rho(\omega)} \quad (140)$$

where we have taken for the $D\bar{D}^* \to J/\psi\rho(\omega)$ transition amplitude, for consistency, $f(\vec{p})t_{D\bar{D}^*\to J/\psi\rho(\omega)}$ with t a constant. Eq. (140) reads

$$\mathcal{M} = g_1 G_{11}^\alpha t_{D^0\bar{D}^{*0}\to J/\psi\rho(\omega)}$$
$$+ g_2 G_{22}^\alpha t_{D^+D^{*-}\to J/\psi\rho(\omega)} \quad (141)$$

as we obtained before, and according to Eq. (134) now $g_i G_{ii} = \hat{\psi}_i$.

It is interesting to compare the results with the form factor with those we had before. We proceed as follows, we take the same strength for the potential as before and determine Λ to get the neutral channel bound $D^0\bar{D}^{*0}$ at the same energy. In Table I we show the values of g, $\hat{\psi}$ and Λ for the three approaches: the sharp cut off, a gaussian form factor and a lorentzian form factor. We see that the differences for g and $\hat{\psi}_i$, the relevant magnitudes in the $X(3872)$ decay, are small. Note that the value of the $R_{\rho/\omega}$ of [13] was obtained using dimensional regularization and zero binding energy for the $D^0\bar{D}^{*0}$ channel. If the calculation is made in the same scheme but with E_B=0.1 MeV one obtains $R_{\rho/\omega}$=0.025, which differs from the values on Table I. The bulk of this difference can be attributed to differences between using hard cut off in momentum space or dimensional regularization. Indeed, using the relativistic treatment of [13], but using a hard cut off, provides a value for $R_{\rho/\omega}$=0.020 at E_B =0.1 MeV.

We also plot in Fig. 4 the wave functions obtained with the three approaches and, as we see, they hardly differ

above $0.5 - 0.6$ fm. The differences at the origin cancel in $\hat{\psi}_i$, where a smearing around the origin is involved, and particularly in the ratios $\hat{\psi}_1/\hat{\psi}_2$.

The robustness of the results against changes in the form factor suggests that a large cut-off approximation should work fairly well. Indeed, in this limit all cut-off procedures merge into a single one, and taking into account that $G_{ii}(E_\alpha) - G_{ii}(M_i)$ is finite we get

$$G_{ii}(E_\alpha) = -8\pi\mu_i\bigg[\int_0^\infty dp[f(\vec{p})]^2 - \frac{\pi}{2}\gamma_i$$
$$+ \mathcal{O}(\Lambda^{-1})\bigg] \quad (142)$$

which shows that what matters is the integrated strength of $[f(\vec{p})]^2$ and corresponds to using a common subtraction constant with a different cut-off interpretation,

$$\int_0^\infty dp[f(\vec{p})]^2 = \Lambda_{\text{Sharp}} = \frac{\sqrt{\pi}}{2}\Lambda_{\text{Gauss}} = \frac{\pi}{4}\Lambda_{\text{Lorentz}} \quad (143)$$

This identification provides a simple rule relating the different cut-offs Λ_{Sharp}, Λ_{Gauss} and Λ_{Lorentz} which works very well as can be checked from Table I. Keeping the leading terms in Eq. (142) and taking $\mu_1 = \mu_2 = \bar{\mu}$ we obtain the following remarkably simple analytical results by using the bound state condition, Eq. (84),

$$\int_0^\infty dp[f(\vec{p})]^2 = \frac{\pi}{4}(\gamma_1 + \gamma_2) + \frac{2\pi^2 f_D^2}{\bar{\mu}} \quad (144)$$
$$R_{\rho/\omega} = \frac{(\gamma_1 - \gamma_2)^2\bar{\mu}^2}{64 f_D^4 \pi^2} \quad (145)$$

which yields

$$\int_0^\infty dp[f(\vec{p})]^2 = 665\text{MeV}, \quad R_{\rho/\omega} = 0.025 \quad (146)$$

Next we want to connect the present results with those obtained in [12, 13] in the relativistic approach. Actually, by matching the relativistic one-loop integral calculated within dimensional regularization with scale ν (called μ in [12]) with the non-relativistic propagator in the heavy meson limit $m_D, m_{D^*} \gg \gamma_1, \gamma_2$ (see Eqs. (20), (21) and (142)), we get[9]

$$\int_0^\infty dp[f(\vec{p})]^2 = -\frac{1}{4}\bigg[(m_D + m_{D^*})\alpha_H$$
$$+ m_D \log(\frac{m_D^2}{\nu^2}) + m_{D^*} \log(\frac{m_{D^*}^2}{\nu^2})\bigg] \quad (147)$$

where α_H is a dimensionless subtraction constant which depends on the scale ν. For $\nu = 1.5$ GeV the subtraction

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[9] We are using averaged values of masses for D and D^* in this formula. This is appropriate since no differences of masses appear in the formula. This allows the matching between relativistic and non-relativistic expressions.

TABLE I. Comparative results for different potentials for a $D^0\bar{D}^{*0}$ binding energy of 0.1 MeV.

Form Factor	Λ [MeV]	g^{FT} [MeV]	$\psi_1(\vec{0})/\psi_2(\vec{0})$	$\hat{\psi}_1/\hat{\psi}_2$	$\hat{\psi}_1$	$\hat{\psi}_2$	$R_{\rho/\omega}$
Sharp	653	3202	1.31	1.31	3.29	2.50	0.018
Gauss	731	3238	1.20	1.29	3.30	2.56	0.016
Lorentz	834	3254	1.17	1.28	3.31	2.58	0.015

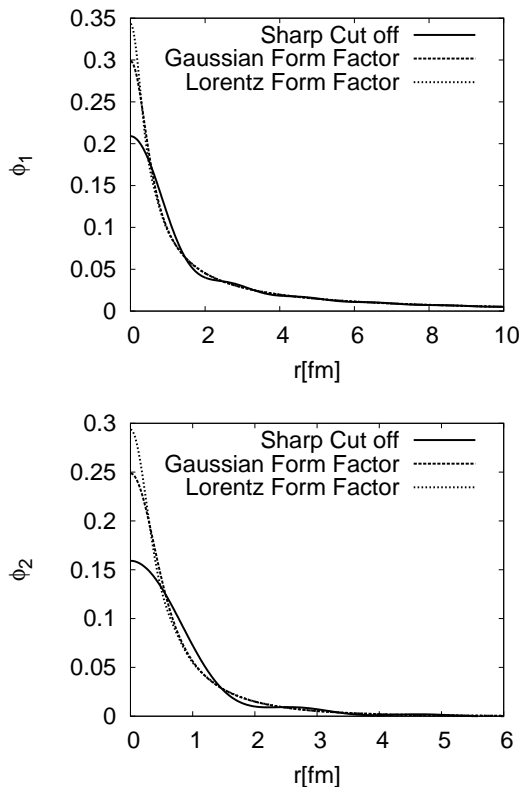
FIG. 4. Wave functions for different form factors in the potential.

constant used in Ref. [13] was $\alpha_H = -1.185$ yielding $\int_0^\infty dp[f(\vec{p})]^2 = 651\,\mathrm{MeV}$ in fairly good agreement with Table I (Λ_{Sharp}). Neglecting the finite cut-off corrections as in Eq. (142) one obtains now $R_{\rho/\omega} = 0.026$.

VIII. CONCLUSIONS

With a view to the structure of the $X(3872)$ as a possible coupled channel bound state of mostly the $D^0\bar{D}^{*0}$ and D^+D^{*-} and other minor channels, we have studied the meaning of the couplings, which one determines from the residues of the scattering matrix at the poles, in terms of wave functions for the different channels. We have done the study in one channel, then in two channels suited to the $X(3872)$ resonance and then we have generalized the results to many channels. Interesting relationships are obtained which shed light on the field theoretical approaches to reactions from the perspective of wave functions. Essentially we find that the couplings are proportional to the value of the wave function at the origin in coordinate space or the averaged value within the range of the interaction. They are not sensitive to the wave function at long distances which is governed by the binding energy, and we also find that this averaged value of the wave function at the origin, $\hat{\psi} = gG$, is the only information that is needed when dealing with short range processes, like those provided in terms of contact Lagrangians in field theory. We also found that the values $\hat{\psi}_i$ were very stable against assumed shapes of the potential once the binding energy is fixed fulfilling the quantization condition $det(1 - VG) = 0$.

We also find that, when one channel becomes loosely bound, the couplings to all coupled channels go to zero. Even if in terms of probabilities the loosely bound channel, whose wave function extends up to infinity, has the largest probability, what matters in the reactions is the averaged values of the wave functions at the origin that determine the dynamics of the processes and the underlying symmetries like isospin. The isospin violation in particular is tied to the ratio of wave functions around the origin $\hat{\psi}_1/\hat{\psi}_2$ (for short range processes), which goes to a finite limit when the binding of the ψ_1 component goes to zero.

When coming to the $X(3872)$ case, which can correspond to the $D^0\bar{D}^{*0}$ channel very loosely bound and the D^+D^{*-} bound by about 8 MeV, we find that the wave functions at the origin for the two channels are similar, suggesting that one has a state with $I = 0$, with small isospin breaking, even if the probability to find the $D^0\bar{D}^{*0}$ component in the full space is much larger than for the D^+D^{*-} component. A precise measure of the isospin admixture is given by the ratio $\hat{\psi}_1/\hat{\psi}_2$, which is very stable and has a value around 1.3, the value of 1 corresponding to a pure $I = 0$ state where the decay $X \to J/\psi\rho$ would be forbidden.

The consideration of the charged D^+D^{*-} component to describe physical processes is so important that if it is neglected one finds a ratio of $\frac{\mathcal{B}(X \to J/\psi\pi^+\pi^-)}{\mathcal{B}(X \to J/\psi\pi^+\pi^-\pi^0)}$ twenty times bigger than experiment.

The work done has also an academical component. Some useful expressions, as well as exact analytical solutions for the wave functions in coupled channels have been given. The work also shows a different perspective on the on-shell approach to the scattering matrix based on the N/D method used in all modern works of chiral dynamics in coupled channels, by means of which the coupled Bethe Salpeter integral equations become algebraic ones. The suitable choice of the potential in momentum space that we made gives rise to the same equations as in the field theoretical on-shell approach. The analytical expressions found can be very useful to give alternative

interpretations of results found in the chiral unitary approach, or in general in unitary coupled channels methods in many physical processes.

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