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Ref.TH.1598-CERN

COUPLINGS OF THREE EXCITED PARTICLES IN THE DUAL RESONANCE MODEL

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ABSTRACT

The vertex operators associated with the transverse states of DRM are explicitly written. As an application of this result the couplings of three excited "transverse" states are explicitly computed.

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Ref.TH.1598-CERN
15 December 1972

1. INTRODUCTION

In the framework of particle physics, the dual resonance model (DRM) appears as a highly unconventional theory. Actually it is a set of rules for constructing scattering amplitudes for any number of external particles, without reference to any Lagrangian or equation of motion. After some inquiry into the structure of these rules, a tremendous amount of symmetry and group theoretical structure shows up, as for instance, the infinite number of gauge identities which follow from the requirement of duality [1].

This gave the hope that a fully algebraic treatment of DRM may be possible, indeed. The program would be then that of getting the spectrum and the couplings, i.e., all the relevant things, in a completely group theoretical way.

Up to now, this program has been only partially successful. A complete algebraization of the spectrum problem was achieved at least for the critical value of the space-time dimension $D = 26$; its structure has been grasped and the presence of ghosts excluded [2]. A class of positive norm states ("transverse states") has been introduced [3] and they have been recognized to be a complete basis in the physical subspace for the critical dimension $D = 26$.

The problem of couplings remained still unsolved. One could dream that a generalized Wigner-Eckart theorem holds, where the couplings are the "Clebsch-Gordan" coefficients of some group. However, no explicit formula for such couplings was yet derived, so it was very difficult to carry out the task of finding the group one dreamed of!

Actually, the operator form derived by Sciuto [4] was available for the three excited particles vertex, but the actual coupling constants between on-mass-shell states were never computed.

In this paper we move a first step in the direction outlined above and we give the explicit couplings among any three transverse states. That solves completely the problem for the critical number of space-time dimensions.

The formula for the couplings between three transverse states characterized by the internal numbers $\left\{n_i^{(1)}, \varepsilon_i^{(1)}\right\}$, $\left\{n_i^{(2)}, \varepsilon_i^{(2)}\right\}$, $\left\{n_i^{(3)}, \varepsilon_i^{(3)}\right\}$ and belonging to the levels N_1, N_2, N_3 is given by Eq. (5.14) where x_i are parameters depending on the masses and M_i is the number of photons which are present in the i^{th} excited particle.

The coupling is the product of two factors: an invariant factor and a covariant factor. The first one is built up by a product of terms, one for each photon entering in the definition of the excited state. Each term is essentially a binomial coefficient which depends upon the "photon quantized momentum" $n_i^{(h)}$ and the tachyon momenta $p^{(h)}$. The covariant factor is a sum over all the combinations of photons; each term takes into account the two-photon interactions by a product of simple terms and the single-photon contributions by a product of other terms containing the transverse components of the momenta. A group theoretical structure can be suspected, but we are still unable to elaborate on that matter.

This paper is organized as follows. Section 2 contains a review of the properties of the spectrum of DRM. The main properties of the vertex operators associated with the transverse states are discussed in Section 3. The formula for the couplings is explicitly derived in Section 4 for the collinear case and extended in Section 5 to the general case. Section 6 is devoted to few examples. In the Appendix we derive some useful relations.

2. REVIEW OF THE PROPERTIES CONCERNING THE SPECTRUM OF DRM

In this Section, for the sake of completeness we will sketch some general properties of DRM in the most symmetric case with $\alpha(0) = 1$. Let us start with the scattering amplitude for n ground state particles:

$$A(p_1 \dots p_n) = \lim_{z_{n-1}\to 1} \int \prod_{i=2}^{n-2} dz_i\, \vartheta(z_{i+1}-z_i) \langle 0,-p_1| \prod_{i=2}^{n-1} \frac{V_0(p_i,z_i)}{z_i} |0,p_n\rangle \tag{2.1}$$

where

$$V_0(p,z) = \,:e^{i\sqrt{2}\,p\cdot Q(z)}: \tag{2.2}$$

is the vertex operator associated with the ground state particle and $Q_\mu(z)$ is given in terms of the harmonic oscillators:

$$Q_\mu(z) = q_{o\mu} + i p_{o\mu} \log z + \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \left[a_{n,\mu} z^{n} + a^{+}_{n,\mu} z^{-n} \right] \qquad (2.3)$$

The vertex operator $\mathcal{V}_0(p,z)$ associated with the ground particle is related to the tachyon state $|0,p\rangle$ through the "asymptotic" conditions

$$\lim_{z\to 0} \langle 0| \frac{\mathcal{V}_0(p,z)}{z} = \langle 0,-p| \quad ; \quad \lim_{z\to\infty} z\, \mathcal{V}_0(p,z)|0\rangle = |0,p\rangle \qquad (2.4)$$

In addition $\mathcal{V}_0(p,z)$ satisfies the following important properties

$$\mathcal{V}_0^{+}(p,z) = \mathcal{V}_0\left(-p, \frac{1}{z}\right) \qquad (2.5)$$

$$\left[L_n, \frac{\mathcal{V}_0(p,z)}{z} \right] = \frac{d}{dz}\left(z^{-n}\, \mathcal{V}_0(p,z) \right) \qquad (2.6)$$

$$\mathcal{V}_0(p_1,z_1)\mathcal{V}_0(p_2,z_2) = \mathcal{V}_0(p_2,z_2)\mathcal{V}_0(p_1,z_1)\, e^{-2i\pi p_1\cdot p_2\, \varepsilon\left(1-\frac{z_1}{z_2}\right)} \qquad (2.7)$$

where $\varepsilon(x) = x/|x|$ and L_n are the well-known gauge operators which satisfy the following infinite algebra:

$$\left[L_n, L_m \right] = (m-n) L_{n+m} + \frac{D}{12} n (n^2-1)\, \delta_{n+m;0} \qquad (2.8)$$

where D is the number of space-time dimensions.

The transformation properties of $\mathcal{V}_0(p,z)$ under the subgroup $O(2,1)$ formed by the operators L_1, L_0, L_{-1} give rise to the invariance of the integrand function in (2.1) under simultaneous projective transformations on the variables z_i:

$$z_i \longrightarrow z_i' = \frac{\alpha z_i + \beta}{\gamma z_i + \delta} \quad , \; i = 1, \dots n \tag{2.9}$$

As a consequence of this projective invariance and of the relation (2.7) it is easy to prove the invariance of the scattering amplitude (2.1) under cyclic and anticyclic permutations of the external particles. The covariance of $\mathcal{V}_0$ under the more general conformal transformations allows one to connect amplitudes related by non-cyclic permutations of the external particles.

Starting from the amplitude (2.1) the factorization properties of DRM can be easily studied. The spectrum of the intermediate states which factorize the n point amplitude at a certain level $\alpha(M^2) = 1+M^2 = N$, can be explicitly given in the space of the infinite harmonic oscillators. The manifest Lorentz covariance of DRM forces the presence in the spectrum of negative norm states which may give rise to negative cross-sections violating basic principles. However, the gauge group of L_n provides a mechanism for the elimination of those negative norm states by requiring that the intermediate on-mass shell states which are effectively coupled, must satisfy the following conditions [5]

$$\begin{aligned} (L_0 + 1)\,|\psi\rangle &= 0 \\ L_n\,|\psi\rangle &= 0 \quad , \; n = 1, 2, \dots \end{aligned} \tag{2.10}$$

Using the vertex operator associated with the "photon" state appearing at the level $N = 1$ it has been possible to construct a new set of harmonic oscillators $A_{i,n}$ which satisfy the conditions (2.10) once they are applied to a physical state [3]:

$$A_{i,n} = \frac{1}{2\pi i} \oint \frac{dy}{y}\, P_i(y)\, e^{i\sqrt{2} K Q(y)} \tag{2.11}$$

with the restriction

$$\sqrt{2}\, p_0 \cdot K = n \tag{2.12}$$

where n is an integer number and

$$P_\mu(z) = z \frac{d}{dz} Q_\mu(z) = i p_{o\mu} + \sum_{n=1}^{\infty} \sqrt{n} \left[a_{n,\mu} z^n - a^+_{n,\mu} z^{-n} \right] \tag{2.13}$$

The integral is evaluated along a small circle centered in the origin. The four-vector k is lightlike and it has been chosen to be moving along the axis $(D-1)$. The index i is then running from 1 to $(D-2)$ along the subspace orthogonal to the vector $\vec{k}$.

The transverse operators $A_{i,n}$ satisfy the algebra of the non-relativistic harmonic oscillators [6]

$$\left[A_{n,i} , A_{m,j} \right] = n \, \delta_{i,j} \, \delta_{n+m;0} \tag{2.14}$$

and they commute with the gauge operators L_m

$$\left[L_m , A_{n,i} \right] = 0 \tag{2.15}$$

From the transverse harmonic oscillators it is possible to construct an infinite set of states

$$\langle 0,-p| \prod_i A_{n_i,j_i} \, Z_{\{n_i\}} = \langle \{ n_i, \varepsilon_i \} | \tag{2.16}$$

where n_i can assume all the positive integer values and $|0,p\rangle$ is the ground particle state whose momentum is constrained to satisfy the condition (2.12). The normalization factor $Z_{\{n_i\}}$ is given by

$$Z_{\{n_i\}} = \prod_k \frac{1}{\sqrt{\lambda_k!}} \prod_i \frac{1}{\sqrt{n_i}} \tag{2.17}$$

where λ_h is the multiplicity of the operator $A_{i,h}$ in the product (2.16).

The states (2.15) are orthogonal to each other:

$$< \{n_i, \varepsilon_i\} \mid \{m_i, \varepsilon_i'\} > = \delta_{\{n_i, \varepsilon_i\};\{n_i, \varepsilon_i'\}} \tag{2.18}$$

and satisfy the gauge conditions

$$L_m \mid \{n_i, \varepsilon_i\} > = 0 \qquad m = 1, 2, 3 \ldots \tag{2.19}$$

The transverse states (2.16) form a complete and orthogonal basis in the subspace of the physical states in the case $D = 26$; hence, for $D = 26$, all the negative norm states required by the manifest Lorentz covariance are decoupled from the physical subspace [2)]. When $D < 26$ the transverse states span a subset of the physical subspace and additional states must be included to recover the entire physical subspace. However, if $D < 26$, the physical subspace is also ghost free.

We conclude this section by recalling that the conditions (2.10) are satisfied also by zero norm states which are at the same time also decoupled from the physical subspace. As in the case of quantum electrodynamics they are important to ensure the required properties of the spectrum under Lorentz transformations.

3. VERTEX OPERATORS RELATED TO THE TRANSVERSE STATES

As we have seen in the previous section, the properties of the spectrum of DRM are quite well known. However, to recover the full content of DRM and have an understanding of its possible deeper implications, one must also study those features of DRM connected with the couplings among excited states or more generally of the n point amplitudes where the external particles are any physical state of the spectrum.

Following the general approach of Ref. 7) we will construct in this section the vertex operators associated with the transverse states. Using these operators which have the same properties as $\mathcal{V}_0(z,p)$, the n point amplitude for excited particles is constructed in exactly the same way as in the case of ground particles. This way of proceeding is very interesting because it shows explicitly how the DRM satisfies the bootstrap requirement of a complete "democracy" within the physical particles.

Starting from $\mathcal{V}_0(p,z)$ and the "photon" vertex operators we can construct the following expression

$$\mathcal{V}_{\{n_i,\varepsilon_i\}}(\Pi,z) = Z_{\{n_i,\varepsilon_i\}} \mathcal{V}_0(p,z) \prod_n \oint_z \frac{dz_n}{z_n} P_{in}(z_n) e^{i\sqrt{2} k_n Q(z_n)} \tag{3.1}$$

The integrals over z_n are evaluated along a curve of the complex plane z_n containing the point z. The singularity of the integrand function for $z_n = z$ is a pole of order n which arises from the procedure of bringing all the operators with positive frequency on the right-hand side and all the operators with negative frequency on the left-hand side, provided that the following relations are satisfied:

$$2p \cdot k_i = -n_i \tag{3.2}$$

The previous conditions make the integrals in (3.1) unambiguously defined. The light-like vectors k_i are taken to be aligned along a common direction which, for example, can be chosen to be along the axis $(D-1)$. The momentum Π of the operators (3.1) is related to p and k_i by the conservation equation

$$\Pi = p + \sum_i k_i \tag{3.3}$$

We discuss now some properties of the expression (3.1). It is easy to check the validity of the following limits:

$$\lim_{z \to 0} \langle 0 | \frac{\mathcal{V}_{\{n_i,\varepsilon_i\}}(\Pi,z)}{z} = \langle \{n_i,\varepsilon_i\}; -\Pi | \tag{3.4}$$

$$\lim_{z\to\infty} z\, \mathcal{V}_{\{n_i,\varepsilon_i\}}(\pi,z)|0\rangle = |\{n_i,\varepsilon_i\},\pi\rangle \tag{3.5}$$

They follow immediately from the definition of the transverse states.

The covariance under the group of L_m's

$$\left[L_m, \frac{\mathcal{V}_{\{n_i,\varepsilon_i\}}(\pi,z)}{z}\right] = \frac{d}{dz}\left(z^{-m}\, \mathcal{V}_{\{n_i,\varepsilon_i\}}(\pi,z)\right) \tag{3.6}$$

follows immediately from the covariance of $\mathcal{V}_0(p,z)$ under the gauge operators and from the fact that the L_m's commute with the integrals in (3.1) because of the commutation relations

$$\left[L_m, \frac{P_i(z)}{z} e^{i\sqrt{2}kQ(z)}\right] = \frac{d}{dz}\left(z^{-m} P_i(z) e^{i\sqrt{2}kQ(z)}\right) \tag{3.7}$$

Finally, the following two other properties can be immediately checked

$$\mathcal{V}^{+}_{\{n_i,\varepsilon_i\}}(\pi,z) = (-1)^N\, \mathcal{V}_{\{n_i,\varepsilon_i\}}\left(-\pi, \frac{1}{z}\right) \tag{3.8}$$

$$\begin{aligned}\mathcal{V}_{\{n_i,\varepsilon_i\}}(\pi_1,z_1)\, \mathcal{V}_{\{n'_i,\varepsilon'_i\}}(\pi_2,z_2) &= \mathcal{V}_{\{n'_i,\varepsilon'_i\}}(\pi_2,z_2)\, \mathcal{V}_{\{n_i,\varepsilon_i\}}(\pi_1,z_1)\,\cdot \\ &\cdot e^{-2\pi i\, \pi_1\cdot\pi_2\, \varepsilon\left(1-\frac{z_1}{z_2}\right)} + \text{other terms}\end{aligned} \tag{3.9}$$

where N is the level of the state $\{n_i, \varepsilon_i\}$. The presence of the factor $(-1)^N$ in the case of the excited particles is a consequence of the fact that the physical states are eigenstates of the twisting operator with eigenvalue equal to $(-1)^N$.

The first term in (3.9) comes out from the commutation between only the exponential factors which appear in the expression (3.1). The other terms involving δ functions or derivatives of δ functions, do not need to be evaluated explicitly because the expression (3.9) is used only in expressions of the type (2.1) where these additional terms do not give any contribution. Following the analogy with the case of the tachyon vertex it is natural now to identify $\mathcal{V}_{\{n_i, \varepsilon_i\}}(\pi, z)$ with the vertex operator associated with the transverse state $|\{n_i, \varepsilon_i\}\rangle$.

In fact, on the one hand, these vertex operators give back the transverse states in the "asymptotic" limits $z \to 0$ or $z \to \infty$ and all the properties of the spectrum can be easily recovered. On the other hand using those operators $\mathcal{V}_{\{n_i, \varepsilon_i\}}(\pi, z)$ it is possible to evaluate the scattering amplitude with any transverse state as an external particle.

In fact, using the general properties (3.5), (3.6), (3.7) and (3.9) of the vertex operators it is straightforward to show that the scattering amplitude among transverse states is given by

$$\lim_{z_{n-1}\to 1} \int \prod_{i=2}^{n-2} dz_i\, \vartheta(z_{i+1}-z_i) \langle -\pi_1, \{n_i^{(1)}, \varepsilon_i^{(1)}\}| \prod_{j=2}^{n-1} \frac{\mathcal{V}_{\{n_i^{(j)}, \varepsilon_i^{(j)}\}}(\pi_j, z_j)}{z_j} |\pi_n, \{n_i^{(n)}, \varepsilon_i^{(n)}\}\rangle \tag{3.10}$$

which is the obvious generalization of the n point amplitude for ground particles.

As a consequence of the above general properties of the vertex operators it is easy to show the invariance of (3.11) under cyclic and anticyclic permutations of the external particles.

To conclude this section let us evaluate the two and three-point functions between excited particles. Using the invariance under translations and dilatations the two-point function can be written

$$\langle 0| \frac{\mathcal{V}_{\alpha_1}(z_1)}{z_1} \frac{\mathcal{V}_{\alpha_2}(z_2)}{z_2} |0\rangle = \frac{\delta_{\alpha_1;\alpha_2}}{(z_1 - z_2)^2} \tag{3.11}$$

The dependence on the variables z_i of the three-point function is also determined by the projective invariance. In fact, using the invariance under dilatations, translations and inversions it is easy to evaluate

$$\langle 0| \frac{V_{\alpha_1}(z_1)}{z_1} \frac{V_{\alpha_2}(z_2)}{z_2} \frac{V_{\alpha_3}(z_3)}{z_3} |0\rangle = \frac{W_{\alpha_1\alpha_2\alpha_3}}{(z_3-z_1)(z_3-z_2)(z_2-z_1)} \qquad (3.12)$$

The numbers $W_{\alpha_1\alpha_2\alpha_3}$ which are the couplings among the three transverse states, will be explicitly evaluated in the next sections.

4. THE THREE-POINT FUNCTION IN THE COLLINEAR CASE

In this Section we want to find an explicit expression for the vertex of three physical excited states. The physical states we shall consider are the transverse states discussed in the preceding Section, which, for critical space time dimensions $D=26$ span the whole physical space. For $D<26$ we shall only consider a subset of all the physical vertex functions. For simplicity we shall also restrict ourselves to the collinear case, in which the momenta of the three excited states and also the momenta of all their constituent tachyon and photon operators are aligned along the same space direction. In this configuration, in fact, the vertex takes a very simple form. The restriction of parallel momenta will be released in the next Section, where we shall find the couplings in the general case. Thus, we consider the three-point function for the excited states $|\alpha_1, \Pi_1\rangle$, $|\alpha_2, \Pi_2\rangle$ and $|\alpha_3, \Pi_3\rangle$ where Π_1, Π_2 and Π_3 are the incoming momenta, satisfying $\Pi_1+\Pi_2+\Pi_3=0$, and α_1, α_2 and α_3 label the internal quantum numbers and the polarization states in the following way

$$\alpha_i \equiv \{ n_1^{(i)}, n_2^{(i)}, \dots n_{M_i}^{(i)} ; \varepsilon_1^{(i)}, \varepsilon_2^{(i)} \dots \varepsilon_{M_i}^{(i)} \} \; , \; i=1,2,3 \qquad (4.1)$$

as discussed in the preceding Section.

The occupation numbers $n_r^{(i)}$ are such that

$$\sum_{r=1}^{M_i} n_r^{(i)} = N_i = 1 - \Pi_i^2 \qquad (i=1,2,3) \tag{4.2}$$

We also need, as auxiliary variables, the momenta and the Koba-Nielsen variables associated with the tachyon and the photon operators, constituents of each excited state and we call $p^{(i)}$ and a_i the tachyon variables and $k_r^{(i)}$ and $z_r^{(i)}$ the photon variables of the i^{th} excited particle as shown in Fig. 1. They satisfy

$$\begin{aligned} p^{(i)2} &= 1 \\ k_r^{(i)2} &= 0 \\ 2p^{(i)} \cdot k_r^{(i)} &= -n_r^{(i)} \end{aligned} \tag{4.3}$$

In the collinear configuration we take all the photon momenta proportional to the vector e, such that $e^2 = 0$, $e_+ = 1$, $e_- = \underline{e} = 0$ so that we have

$$\begin{aligned} k_r^{(i)} &= -n_r^{(i)} p_+^{(i)} e \\ k_r^{(i)} \cdot k_s^{(j)} &= 0 \\ 2p_+^{(i)} p_-^{(i)} &= 1 \end{aligned} \tag{4.4}$$

where we have used the light cone variables $p_\pm = 1/\sqrt{2}(p_{D-1} \pm p_0)$, with the photon momenta along the $(D-1)$ axis.

The symmetric three-point function is given by

$$\begin{aligned} W_{\alpha_1\alpha_2\alpha_3} &= (a_2 - a_1)(a_3 - a_1)(a_3 - a_2) \cdot \\ &\cdot \langle 0| \frac{V_{\alpha_1}(\Pi_1, a_1)}{a_1} \frac{V_{\alpha_2}(\Pi_2, a_2)}{a_2} \frac{V_{\alpha_3}(\Pi_3, a_3)}{a_3} |0\rangle = \\ &= \langle -\Pi_1, \alpha_1 | V_{\alpha_2}(\Pi_2, 1) | \Pi_3, \alpha_3 \rangle \end{aligned} \tag{4.5}$$

where the vertex operators for the excited particles have the expression

$$\mathcal{V}_{\alpha_i}(\Pi_i, a_i) = Z_i\, \mathcal{V}_0(p^{(i)}, a_i) \prod_{r=1}^{M_i} \frac{1}{2\pi i} \oint_{a_i} \frac{dz_r^{(i)}}{z_r^{(i)}} P_{j_r}(z_r^{(i)}) e^{i\sqrt{2}k_r^{(i)} Q(z_r^{(i)})} \tag{4.6}$$

Putting this into (4.5) we observe that the transverse components of the operators $P_j(z_r)$ commute with all the tachyon vertices and with the exponentials of the photon vertices so that we can factorize the vacuum expectation value of the product of the $P_j(z_r)$ operators. What is left is essentially the integrand of the n point function for ground states and we obtain

$$W_{\alpha_1\alpha_2\alpha_3} = Z_1 Z_2 Z_3 \prod_{\substack{i,j=1\\ i<j}}^{3} \left(a_j - a_i\right)^{2p^{(i)}\cdot p^{(j)}+1}$$

$$\prod_{r=1}^{M} \left(\frac{1}{2\pi i} \oint_{a^{(r)}} \frac{dz_r}{z_r}\right) \langle 0| \prod_{r=1}^{M} P_{j_r}(z_r) |0\rangle \prod_{i=1}^{3} \prod_{r=1}^{M} \left|z_r - a_i\right|^{2p^{(i)}\cdot k_r} \tag{4.7}$$

where $M = M_1+M_2+M_3$ and we have used the compact notation

$$\begin{aligned}
&k_r = k_r^{(1)}, \quad z_r = z_r^{(1)}, \quad a^{(r)} = a_1, \quad r = 1, \ldots M_1 \\
&k_{M_1+r} = k_r^{(2)}, \quad z_{M_1+r} = z_r^{(2)}, \quad a^{(M_1+r)} = a_2, \quad r = 1, \ldots M_2 \\
&k_{M_1+M_2+r} = k_r^{(3)}, \quad z_{M_1+M_2+r} = z_r^{(3)}, \quad a^{(M_1+M_2+r)} = a_3, \quad r = 1, \ldots M_3
\end{aligned} \tag{4.8}$$

The vacuum expectation value in (4.7) is obviously zero for odd M. Since each transverse photon carries a helicity ± 1, the vanishing of the collinear vertex for an odd number of photons is a consequence of helicity conservation. For even M that factor can be expressed by means of the Wick theorem as a sum of products of factors like $\langle 0|P_i(z_r)P_j(z_s)|0\rangle$ for each couple of photons, summed over the independent photon permutations. Each of these factors is

$$\langle 0| P_i(z_r) P_j(z_s) |0\rangle = \langle 0| \left[P_i^{(+)}(z_r), P_j^{(-)}(z_s)\right] |0\rangle = -\delta_{ij} \frac{z_r z_s}{(z_r - z_s)^2} \tag{4.9}$$

and therefore we have

$$<0|\prod_{r=1}^{M} P_{j_r}(z_r)|0> = (-1)^{\frac{M}{2}} \prod_{r=1}^{M} z_r \cdot$$

$$\cdot \sum_{\{r_1,\dots r_M\}} \prod_{k=1}^{M/2} \frac{\delta_{j_{r_{2k-1}};\, j_{r_{2k}}}}{\left(z_{r_{2k-1}} - z_{r_{2k}}\right)^2} \tag{4.10}$$

where the sum is extended over the (M-1)!! permutations of the indices $\{1,2,\dots,M\}$ corresponding to different photon pairings. We see that the couplings between photon variables only arise from the above factors, so that for each term of the sum in (4.10) the multiple integral in (4.7) factorizes into M/2 double integrals, one for each couple of photons. Furthermore, since the expression (4.7) is projective invariant, we may arrange things in such a way that each factor is individually projective invariant. In conclusion, we can rewrite (4.7) in the following form

$$W_{\alpha_1\alpha_2\alpha_3} = Z_1 Z_2 Z_3 (-1)^{M/2} \sum_{\{r_1,\dots r_M\}} \prod_{k=1}^{M/2} \delta_{j_{r_{2k-1}} j_{r_{2k}}} \mathcal{D}(p^{(1)},p^{(2)},p^{(3)},k_{r_{2k-1}},k_{r_{2k}}) \tag{4.11}$$

where

$$\mathcal{D}(p^{(1)},p^{(2)},p^{(3)},k_r,k_s) = (a_2-a_1)^{2p^{(3)}(k_r+k_s)} (a_3-a_2)^{2p^{(1)}(k_r+k_s)} \frac{1}{(2\pi i)^2}$$

$$\cdot (a_3-a_1)^{2p^{(2)}(k_r+k_s)} \oint_{a^{(r)}} dz_r \oint_{a^{(s)}} dz_s (z_r-z_s)^{-2} \prod_{i=1}^{3} |z_r-a_i|^{2p^{(i)}\cdot k_r} |z_s-a_i|^{2p^{(i)}\cdot k_s} \tag{4.12}$$

The D functions defined above are the basic ingredients for the vertex and are essentially the discontinuities of the five-point functions corresponding to the three tachyons and two photons. Using momentum conservation and the property $(p^{(1)}+p^{(2)}+p^{(3)})\cdot k_{r,s} = 0$ it is easily verified that (4.12) is invariant under simultaneous projective transformations of the variables a_i, z_r and z_s. As a consequence the D functions are individually independent of the values of a_1, a_2 and a_3 which can be chosen in the most convenient way.

There are two different kinds of D functions: the first one, that we call D_1, when the integrals are around the same point and the second one, that we call D_2, when the integrals are around two different points. In the first case let $a^{(r)} = a^{(s)} = a_{i_1}$ and we shall calculate (4.12) in the limit $a_{i_1} \to 0$, $a_{i_2} \to 1$, $a_{i_3} \to \infty$, where (i_1, i_2, i_3) is a cyclic permutation of $(1,2,3)$. We then have

$$\mathcal{D}_1(n, n', p{\cdot}k, pk') = \frac{1}{(2\pi i)^2} \oint_0 dz \oint_0 dz' \, (z-z')^{-2} z^{-n} z'^{-n'} \cdot (1-z)^{2p'k} (1-z')^{2p'k'} \tag{4.13}$$

where $p = p^{(i_1)}$, $p' = p^{(i_2)}$, $k = k_r^{(i_1)}$, $k' = k_s^{(i_1)}$ and we used the conditions (4.3) with $n = -2p{\cdot}k = n_r^{(i_1)}$ $n' = -2p{\cdot}k' = n_s^{(i_1)}$. If we suppose the integral over z to be carried first, we may take $|z| < |z'|$ and use the expansion

$$(z-z')^{-2} = \frac{1}{zz'} \sum_{\ell=1}^{\infty} \ell \left(\frac{z}{z'}\right)^{\ell} \tag{4.14}$$

to obtain

$$\mathcal{D}_1 = (-1)^{n+n'} \sum_{\ell=1}^{n} \ell \binom{2p'{\cdot}k}{n-\ell} \binom{2p'{\cdot}k'}{n'+\ell} \tag{4.15}$$

We now observe that in virtue of (4.4) we have

$$2p'{\cdot}k = -2n\, p'_-\, p_+ = -n \frac{p_+}{p'_+} = -n x \tag{4.16}$$

and similarly $2p'.k' = -n'x$. In this case the sum in (4.15) can be explicitly performed as shown in the Appendix and we get

$$\mathcal{D}_1 = (-1)^{n+n'} \binom{-nx}{n} \binom{-n'x}{n'} \frac{nn'}{n+n'} \; \frac{1+x}{x} \tag{4.17}$$

Since D_1 is clearly symmetric in (n,n'), it is also independent of the order of the integrations in (4.13), as it was to be expected. In the previous general notations the above result reads

$$\mathcal{D}(p^{(1)}, p^{(2)}, p^{(3)}, k_r^{(i)}, k_s^{(i)}) =$$
$$= (-1)^{n_2^{(i)}+n_s^{(i)}} \binom{-n_2^{(i)} \frac{p_+^{(i)}}{p_+^{(i+1)}}}{n_2^{(i)}} \binom{-n_s^{(i)} \frac{p_+^{(i)}}{p_+^{(i+1)}}}{n_s^{(i)}} \frac{n_2^{(i)} n_s^{(i)}}{n_r^{(i)} + n_s^{(i)}} \frac{p_+^{(i)} + p_+^{(i+1)}}{p_+^{(i)}} \tag{4.18}$$
$$(i = 1, 2, 3)$$

We now consider the case of D_2, where the two photons belong to different excited states. With reference to (4.12) let $a^{(r)} = a_{i_1}$, $a^{(s)} = a_{i_2}$ and we consider as before the limit $a_{i_1} \to 0$, $a_{i_2} \to 1$, $a_{i_3} \to \infty$. We then obtain

$$\mathcal{D}_2(n, n', p'k, pk') = \frac{1}{(2\pi i)^2} \oint_0 dz \oint_1 dz' (z-z')^{-2} z^{-n} \cdot$$
$$\cdot (1-z)^{2p'k} z'^{\,2pk'} (z'-1)^{-n'} = \tag{4.19}$$
$$= \sum_{\ell=1}^{n} (-1)^{n-\ell} \ell \binom{2p'k}{n-\ell} \binom{2pk'-\ell-1}{n'-1}$$

where we set $p = p^{(i_1)}$, $p' = p^{(i_2)}$, $k = k_r^{(i_1)}$, $k' = k_s^{(i_2)}$, $n = -2p\cdot k = n_r^{(i_1)}$, $n' = -2p'\cdot k' = n_s^{(i_2)}$. If we now consider that $2p'\cdot k = -np_+/p'_+ = -nx$ and similarly $2pk' = -n'/x$, the sum in (4.19) can again be explicitly performed (see the Appendix) and we obtain

$$\mathcal{D}_2 = (-1)^{n+1} \binom{-nx}{n} \binom{-\frac{n'}{x}}{n'} \frac{nn'(1+x)}{nx+n'} \tag{4.20}$$

which in the general notation for the D function reads

$$\mathcal{D}(p^{(1)}, p^{(2)}, p^{(3)}, k_r^{(i)}, k_s^{(i+1)}) =$$
$$= (-1)^{n_2^{(i)}+1} \binom{-n_r^{(i)} \frac{p_+^{(i)}}{p_+^{(i+1)}}}{n_r^{(i)}} \binom{-n_s^{(i+1)} \frac{p_+^{(i+1)}}{p_+^{(i)}}}{n_s^{(i+1)}} \frac{n_2^{(i)} n_s^{(i+1)} (p_+^{(i)} + p_+^{(i+1)})}{n_2^{(i)} p_+^{(i)} + n_s^{(i+1)} p_+^{(i+1)}} \tag{4.21}$$

We may put (4.18) and (4.21) in a more symmetric form. We first observe that momentum conservation along the $-$ and the $+$ directions gives, by (4.2) and (4.4)

$$\frac{1}{p_+^{(1)}} + \frac{1}{p_+^{(2)}} + \frac{1}{p_+^{(3)}} = 0 \tag{4.22}$$

$$m_1^2\, p_+^{(1)} + m_2^2\, p_+^{(2)} + m_3^2\, p_+^{(3)} = 0 \tag{4.23}$$

where $m_i^2 = -\pi_i^2$ are the squared masses of the excited particles (integer numbers). We next introduce the parameters

$$x_i = \frac{p_+^{(i)}}{\left(p_+^{(1)}\, p_+^{(2)}\, p_+^{(3)}\right)^{1/3}} \tag{4.24}$$

These satisfy the relations

$$\begin{aligned} & x_1\, x_2\, x_3 = 1 \\ & x_i + x_j = -\left(x_i x_j\right)^2 \quad , \quad i \neq j \\ & \sum_{i=1}^{3} m_i^2\, x_i = 0 \end{aligned} \tag{4.25}$$

and they can therefore be expressed as functions of the external masses.

After some algebraic manipulations and using the relation

$$\begin{pmatrix} -n \dfrac{x_i}{x_{i-1}} \\ n \end{pmatrix} = (-1)^{n+1}\, \frac{x_{i+1}}{x_{i-1}} \begin{pmatrix} -n \dfrac{x_i}{x_{i+1}} \\ n \end{pmatrix} \tag{4.26}$$

we can write the D functions in the form

$$\begin{aligned} D\left(p^{(1)}, p^{(2)}, p^{(3)}, k_2^{(i)}, k_s^{(j)}\right) = (-1)^{n_2^{(i)} + n_s^{(j)} + 1} \, \cdot \\ \cdot \begin{pmatrix} -n_r^{(i)} \dfrac{x_i}{x_{i+1}} \\ n_2^{(i)} \end{pmatrix} \begin{pmatrix} -n_s^{(j)} \dfrac{x_j}{x_{j+1}} \\ n_s^{(j)} \end{pmatrix} \frac{n_2^{(i)} x_i\, x_{i+1}\, n_s^{(j)} x_j\, x_{j+1}}{n_2^{(i)} x_i + n_s^{(j)} x_j} \end{aligned} \tag{4.27}$$

which is valid both for $i = j$ and for $i \neq j$. As a result, the D functions factorize very nicely in the following way: to each photon with momentum $k_r^{(i)}$ is associated a factor

$$(-1)^{n_r^{(i)}} n_r^{(i)} x_i x_{i+1} \binom{-n_r^{(i)} \frac{x_i}{x_{i+1}}}{n_r^{(i)}} \tag{4.28}$$

and to each couple of photons with momenta $k_r^{(i)}$ and $k_s^{(j)}$ is associated a factor

$$-\left(n_r^{(i)} x_i + n_s^{(j)} x_j \right)^{-1} \tag{4.29}$$

Following these rules we can finally write the vertex (4.11) in the very compact form

$$W_{\alpha_1 \alpha_2 \alpha_3} = Z_1 Z_2 Z_3 \, (-1)^{N_1+N_2+N_3} \prod_{i=1}^{3} x_i^{-M_{i+1}} \cdot$$

$$\cdot \prod_{r=1}^{M_i} n_r^{(i)} \binom{-n_r^{(i)} \frac{x_i}{x_{i+1}}}{n_r^{(i)}} \sum_{\{r_1 \dots r_M\}} \prod_{k=1}^{M/2} \delta_{j_{r_{2k-1}};\, j_{r_{2k}}} \left(n_{r_{2k-1}} x^{(r_{2k-1})} + n_{r_{2k}} x^{(r_{2k})} \right)^{-1} \tag{4.30}$$

where under the sum we used for the indices of n_r and $x^{(r)}$ a notation analogous to (4.8) for k_r and $a^{(r)}$, respectively.

The formula (4.30) is manifestly symmetric under cyclic permutations of the three excited particles. Under anticyclic permutations the three-point function changes into the three-point function for the twisted states, so it must take a factor $(-1)^{N_1+N_2+N_3}$. This property is satisfied by (4.30), as it can be easily seen by changing in each photon factor (4.28) the variables (x_i, x_{i+1}) into the variables (x_i, x_{i-1}) and using the relation (4.26).

5. THE THREE-POINT FUNCTION IN THE GENERAL CASE

We want now to extend the result of the preceding section to the general case of non-collinear momenta. We can devise two different ways to represent the physical states of the excited particles. The first way consists in taking the tachyon momentum and the photon momenta all parallel to the momentum $\vec{\pi}$ of the particle and the polarization vectors of the photons orthogonal to this direction. This gives an intrinsic representation for the transverse states of the particle of momentum $\vec{\pi}$. The second way makes reference, for any excited particle, to a fixed reference frame. In this frame the photon momenta are all taken in the same direction, say the $(D-1)$ axis, and the polarization vectors span the $D-2$ dimensional space orthogonal to this direction, independent of the direction of $\vec{\pi}$. For $D=26$ these two descriptions are complete and equivalent, the states of one type being related to the states of the other type by a suitable rotation operator [8], apart from the contribution of zero norm spurious states, which, however, do not contribute to the amplitude.

In our case, and more generally for the n point function, the second type of description is much more convenient since in this case the scalar product of any two-photon momenta is zero and many factors in the Koba-Nielsen integrand are absent. With the same notations as in Section 4, the kinematics is now the following. Equations (4.4) are replaced by

$$K_z^{(i)} = - \frac{n_r^{(i)}}{2p_-^{(i)}} \, e$$

$$K_r^{(i)} \cdot K_s^{(j)} = 0 \qquad (5.1)$$

$$2 p_+^{(i)} p_-^{(i)} + p_\perp^{(i)2} = 1$$

while Eqs. (4.22) and (4.23) become

$$\sum_{i=1}^{3} p_-^{(i)} = 0$$

$$\sum_{i=1}^{3} \frac{m_i^2 + p_\perp^{(i)2}}{p_-^{(i)}} = 0 \qquad (5.2)$$

We shall also use the parameters

$$x_i = \frac{\left(p_-^{(1)}\, p_-^{(2)}\, p_-^{(3)}\right)^{1/3}}{p_-^{(i)}} \tag{5.3}$$

which reduce to those of (4.24) in the collinear case $(p_\perp^{(i)} = 0)$. The new parameters satisfy the relations

$$\begin{aligned} & x_1 x_2 x_3 = 1 \\ & x_i + x_j = -(x_i x_j)^2 , \qquad (i \neq j) \\ & \sum_{i=1}^{3} \left(m_i^2 + \pi_{i\perp}^2\right) x_i = 0 \end{aligned} \tag{5.4}$$

by which they result in being functions of the longitudinal masses $m_i^2 + \pi_i^2$ only. By means of these parameters the scalar products of the tachyon and photon momenta can be expressed as follows:

$$2p^{(i)} \cdot k_z^{(j)} = -n_z^{(j)} \frac{x_j}{x_i} \tag{5.5}$$

which is formally identical to the collinear case.

We are now ready to calculate the three-point function (4.5) corresponding to Fig. 1. In the present case the operators $P_{j_r}(z_r)$ do not commute with the tachyon vertex operators. However, by carrying the destruction operator parts of the tachyon vertices to the right and the creation operator parts to the left we can still put the amplitude $W \alpha_1 \alpha_2 \alpha_3$ in the form (4.7) where the vacuum expectation value is replaced by

$$\langle 0 | \prod_{z=1}^{M} \left[P_{j_z}(z_z) + i\sqrt{2}\, z_z \sum_{i=1}^{3} \frac{p_{j_z}^{(i)}}{a_i - z_z} \right] | 0 \rangle \tag{5.6}$$

A general term in the expansion of (5.6) will contain the vacuum expectation value of 2K P_j operators times the product of M-2K terms containing the transverse momenta. Accordingly, the multiple integral (4.7) factorizes into the product of K double integrals, leading to the D functions of the collinear case, and M-2K single integrals. We now consider these last factors.

To take into account the momentum conservation of the excited particles in a symmetric way, we introduce the "quark" momenta $q^{(1)}$, $q^{(2)}$ and $q^{(3)}$ such that

$$\Pi_i = q^{(i-1)} - q^{(i+1)} \quad , \quad (i = 1,2,3) \tag{5.7}$$

with the cyclic notation for the indices, as shown in Fig. 2.

The contribution to $W_{\alpha_1 \alpha_2 \alpha_3}$ in (4.7) coming from a single photon with momentum $k_r^{(i)}$ and polarization along j and from the second term in (5.6) is then the following

$$\begin{aligned} F_j\left(p^{(1)}, p^{(2)}, p^{(3)}, k_r^{(i)}\right) &= i\sqrt{2}\,(a_2-a_1)^{2p^{(3)}\cdot k_r^{(i)}}(a_3-a_2)^{2p^{(1)}\cdot k_r^{(i)}}\cdot \\ &\cdot (a_3-a_1)^{2p^{(2)}\cdot k_r^{(i)}} \frac{1}{2\pi i}\oint_{a_i} dz \prod_{k=1}^{3} |z-a_k|^{2p^{(k)}\cdot k_r^{(i)}} \cdot \\ &\cdot \left[q_j^{(1)} \frac{a_3-a_2}{(a_3-z)(a_2-z)} + q_j^{(2)} \frac{a_1-a_3}{(a_1-z)(a_3-z)} + q_j^{(3)} \frac{a_2-a_1}{(a_2-z)(a_1-z)} \right] \end{aligned} \tag{5.8}$$

Since this expression is projective invariant, it can be evaluated in the limit $a_i \to 0$, $a_{i+1} \to 1$, $a_{i+2} \to \infty$. To be more definite we work out explicitly the case $i = 1$. In this case (5.8) becomes

$$\begin{aligned} F_j\left(p^{(k)}, k_z^{(1)}\right) &= i\sqrt{2}\,\frac{1}{2\pi i}\oint_0 dz\, z^{-n_z^{(1)}} (1-z)^{-n_r^{(1)}\frac{x_1}{x_2}} \cdot \\ &\cdot \left(\frac{q_j^{(1)}}{1-z} + \frac{q_j^{(2)}}{z} - \frac{q_j^{(3)}}{z(1-z)} \right) = i\sqrt{2}\, q_j^{(1)} (-1)^{n_z^{(1)}-1} \cdot \\ &\cdot \binom{-n_r^{(1)}\frac{x_1}{x_2}-1}{n_z^{(1)}-1} + i\sqrt{2}\, q_j^{(2)} (-1)^{n_z^{(1)}} \binom{-n_r^{(1)}\frac{x_1}{x_2}}{n_r^{(1)}} - i\sqrt{2}\, q_j^{(3)} (-1)^{n_r^{(1)}} \binom{-n_r^{(1)}\frac{x_1}{x_2}-1}{n_r^{(1)}} = \\ &= i\sqrt{2}\,(-1)^{n_z^{(1)}} \binom{-n_z^{(1)}\frac{x_1}{x_2}}{n_z^{(1)}} \left(q_j^{(1)}\frac{x_2}{x_1} + q_j^{(2)} + q_j^{(3)}\frac{x_2}{x_3} \right) \end{aligned} \tag{5.9}$$

where in the last step we used the properties of the binomial coefficients and the first two relations (5.4), for which $1+x_2/x_1 = -x_2/x_3$.

Using the cyclic symmetry of (5.8) the above result is immediately generalized and we have

$$F_j\left(p^{(1)}, p^{(2)}, p^{(3)}, k_r^{(i)}\right) = i\sqrt{2}\,(-1)^{n_2^{(i)}} x_{i+1} \binom{-n_2^{(i)}\,\frac{x_i}{x_{i+1}}}{n_2^{(i)}} Q_j \tag{5.10}$$

where we set

$$Q_j = \sum_{k=1}^{3} \frac{q_j^{(k)}}{x_k} \tag{5.11}$$

By (5.4) and (5.7) the vector Q can also be reexpressed in terms of any two momenta of the excited particles:

$$Q = \frac{\Pi_{i+1}}{x_i} - \frac{\Pi_i}{x_{i+1}} \quad , \quad (i = 1, 2, 3) \tag{5.12}$$

We observe that the binomial coefficients associated with the single photons in (5.10) are the same as those associated with the coupled photons [see Eq. (4.28)].

To conclude our analysis of the non-collinear vertex, we remember from the preceding section that each couple of photons with momenta $k_r^{(i)}$ and $k_s^{(j)}$ and polarizations along j_r and j_s, coupled through the factor $\langle 0|P_{j_r}(z_r)P_{j_s}(z_s)|0\rangle$ in the expansion of (5.6), contributes to $W_{\alpha_1\alpha_2\alpha_3}$ by a factor [see (4.9), (4.11) and (4.12)]

$$-\,\delta_{j_r j_s}\, D\left(p^{(1)}, p^{(2)}, p^{(3)}, k_r^{(i)}, k_s^{(j)}\right) \tag{5.13}$$

where D is given by (4.27).

Finally, we have for the vertex function of three excited particles the general expression

$$W_{\alpha_1\alpha_2\alpha_3} = Z_1 Z_2 Z_3 (-1)^{N_1+N_2+N_3} \prod_{i=1}^{3} x_{i+1}^{M_i} \prod_{r=1}^{M_i} \binom{-n_r^{(i)} \frac{x_i}{x_{i+1}}}{n_r^{(i)}} \cdot$$

$$\cdot \sum_{K=1}^{[\frac{M}{2}]} \sum_{\{r_1 .. r_M\}} \prod_{h=1}^{K} \delta_{j_{r_{2h-1}}, j_{r_{2h}}} \frac{n_{r_{2h-1}} x^{(r_{2h-1})} n_{r_{2h}} x^{(r_{2h})}}{n_{r_{2h-1}} x^{(r_{2h-1})} + n_{r_{2h}} x^{(r_{2h})}} \cdot \tag{5.14}$$

$$\cdot (i\sqrt{2})^{M-2K} \prod_{\ell=2K+1}^{M} Q_{j_{r_\ell}}$$

where $[M/2]$ is the largest integer $\leq M/2$; the second summation is extended over the $\binom{M}{2K}(2K-1)!!$ independent permutations of the photon indices $\{1,2,\ldots,M\}$ corresponding to all possible ways of forming K couples, and the other notations are the same as in Eq. (4.30).

Eq. (5.14) summarizes the following rules: i) the vertex amplitude is the product of three relevant factors: the proper normalization constants for the three excited states, an invariant factor and a covariant factor; ii) the invariant factor is in turn the product of M factors, one for each internal quantum number $n_r^{(i)}$ in (4.1), independent of the polarization states. The factor associated with $n_r^{(i)}$ is

$$(-1)^{n_r^{(i)}} x_{i+1} \binom{-n_r^{(i)} \frac{x_i}{x_{i+1}}}{n_r^{(i)}} \tag{5.15}$$

iii) the covariant factor is a tensor of rank M, each index corresponding to the transverse component of one polarization vector, and is the sum of all possible covariants which can be made in terms of Kronecker deltas and Q components. In each term of the sum each Kronecker delta $\delta_{j_r j_s} = (\varepsilon_r^{(i)} \cdot \varepsilon_s^{(j)})$ brings a factor

$$\frac{n_r^{(i)} x_i \, n_s^{(j)} x_j}{n_r^{(i)} x_i + n_s^{(j)} x_j} \delta_{j_r j_s} \tag{5.16}$$

and each Q component $Q_{j_t} = (\epsilon_t^{(h)} \cdot Q)$ brings a factor

$$i\sqrt{2}\, Q_{j_t} \tag{5.17}$$

It is immediately seen that (5.14) is symmetric under cyclic permutations of the excited states. Under anticyclic permutations it takes the right factor $(-1)^{N_1+N_2+N_3}$ as it follows from (4.26) and from the fact that Q is odd.

The expression for the three-point function we have derived above is remarkably simple and is suggestive of some underlying algebraic structure, from which it could perhaps be derived in a more direct way. In any case, our vertex can be a useful starting point in the investigation of the presence of new algebraic structures, like the existence of conserved quantum numbers, which would be very useful for a new classification of the physical states.

Furthermore, the technique of the transverse states can be applied to other physical reactions, to find explicit expressions for the four and n point functions for excited particles.

6. EXAMPLES

In this last section we evaluate explicitly some couplings in the collinear case..

i) Particles $i = 1,3$ are tachyons $[n_j^{(i)} = 0,\ i = 1,3]$ and $i = 2$ is an excited state with $n_i^{(n)} = 1$ for $i = 1,\dots,N$ (N is even). The three-point function for these states is given by

$$\frac{(-1)^N}{\sqrt{N!}} \left(-\frac{p_+^{(2)}}{p_+^{(3)}}\right)^N \left(\frac{p_+^{(3)}}{2p_+^{(1)}}\right)^{\frac{N}{2}} \sum_{\{i_1,\dots i_N\}} \prod_{k=1}^{N/2} \delta_{\alpha_{i_{2k-1}};\alpha_{i_{2k}}} = \tag{6.1}$$

$$= \frac{1}{\sqrt{N!}} \left(\frac{p_+^{(2)}\, p_+^{(2)}}{2p_+^{(3)}\, p_+^{(1)}}\right)^{\frac{N}{2}} \sum_{\{i_1\dots i_N\}} \prod_{k=1}^{N/2} \delta_{\alpha_{i_{2k-1}};\alpha_{i_{2k}}}$$

The momentum conservation reads:

$$\frac{1}{p_+^{(1)}} + \frac{1}{p_+^{(2)}} + \frac{1}{p_+^{(3)}} = 0 \tag{6.2}$$

$$(N-1)\, p_+^{(2)} - p_+^{(1)} - p_+^{(3)} = 0 \tag{6.3}$$

Using these two equations it is easy to show that

$$\frac{p_+^{(2)}\, p_+^{(2)}}{2\, p_+^{(3)}\, p_+^{(1)}} = -\frac{1}{2\,(N-1)} \tag{6.4}$$

Therefore, the three-point function reduces to

$$\frac{1}{\sqrt{N!}} \left[-\frac{1}{2(N-1)} \right]^{\frac{N}{2}} \sum_{\{i_1 .. i_N\}} \prod_{k=1}^{N/2} \delta_{\alpha_{i_{2k-1}}\, ;\, \alpha_{i_{2k}}} \tag{6.5}$$

The number of terms present in the sum is $(N-1)!!$. In the case where the Lorentz indices are all equal to each other one gets

$$\frac{1}{\sqrt{N!}} \left[-\frac{1}{2(N-1)} \right]^{N/2} (N-1)!! \tag{6.6}$$

which, for $N \to \infty$ is decreasing more than exponentially.

ii) Let us consider the case where particle 1 is a tachyon and particles 2 and 3 are excited states with only one photon with Lorentz indices i_2 and i_3 respectively.

The three-point function is given in this case by

$$(-1)^{N_2+N_3} \sqrt{N_2}\, \sqrt{N_3} \begin{pmatrix} -N_2 \frac{p_+^{(2)}}{p_+^{(3)}} \\ N_2 \end{pmatrix} \begin{pmatrix} -N_3 \frac{p_+^{(3)}}{p_+^{(1)}} \\ N_3 \end{pmatrix} \frac{\delta_{i_2;\, i_3}\, p_+^{(3)}}{N_2\, p_+^{(2)} + N_3\, p_+^{(3)}} \tag{6.7}$$

with the following conservation equations:

$$\frac{1}{p_+^{(1)}} + \frac{1}{p_+^{(2)}} + \frac{1}{p_+^{(3)}} = 0 \tag{6.8}$$

$$-p_+^{(1)} + (N_2-1)\,p_+^{(2)} + (N_3-1)\,p_+^{(3)} = 0 \tag{6.9}$$

These equations give a simple solution in the case $N_3 = N_2 - 1 = N - 1$

$$\frac{p_+^{(2)}}{p_+^{(1)}} = x = \frac{a}{\sqrt{N-1}} \quad ; \quad \frac{p_+^{(3)}}{p_+^{(1)}} = y = -\frac{1}{1 + a\sqrt{N-1}} \tag{6.10}$$

$a = \pm 1$ is a consequence of the second degree equation which has been solved.

In this case the three-point function becomes

$$\sqrt{N(N-1)}\,(-1)^{N-1}\,\delta_{i_2;i_3}\,\frac{\Gamma(1+N+Nx)}{N!\,\Gamma(Nx+2)}\;\frac{\Gamma((N-1)(1+y))}{(N-1)!\,\Gamma((N-1)y)} \tag{6.11}$$

ACKNOWLEDGEMENTS

Two of us (M.A. and S.F.) wish to thank for the hospitality extended to them at the Theoretical Study Division of CERN, where part of this work has been done. We thank D. Olive for reading the manuscript.

APPENDIX

Here we want to calculate explicitly the summations which appear in (4.15) and (4.19) and give the D_1 and D_2 functions respectively. In the first case we have to calculate

$$S = \sum_{\ell=1}^{n} \ell \binom{-nx}{n-\ell}\binom{-n'x}{n'+\ell} \tag{A.1}$$

Let us consider the sum S_r of the last r terms

$$\begin{aligned} S_r &= \sum_{\ell=n-r+1}^{n} \ell \binom{-nx}{n-\ell}\binom{-n'x}{n'+\ell} = \\ &= \sum_{\ell=0}^{r-1} (n-\ell) \binom{-nx}{\ell}\binom{-n'x}{n+n'-\ell} \end{aligned} \tag{A.2}$$

We have, in particular

$$S_1 = n\binom{-n'x}{n+n'} = \binom{-nx}{1}\binom{-n'x}{n+n'-1}\frac{n'(x+1)+n-1}{(n+n')x} \tag{A.3}$$

$$\begin{aligned} S_2 &= \binom{-nx}{1}\binom{-n'x}{n+n'-1}\left[\frac{n'(1+x)+n-1}{(n+n')x} + n-1\right] = \\ &= \binom{-nx}{2}\binom{-n'x}{n+n'-2}\frac{2\left[n'(1+x)+n-2\right]}{(n+n')x} \end{aligned} \tag{A.4}$$

Then we can prove by induction that

$$S_r = \binom{-nx}{r}\binom{-n'x}{n+n'-r}\frac{r\left[n'(x+1)+n-r\right]}{(n+n')x} \tag{A.5}$$

In fact, we have

$$S_{r+1} = S_r + (n-r)\binom{-nx}{r}\binom{-n'x}{n+n'-r} = \tag{A.6}$$

$$= \binom{-nx}{r+1}\binom{-n'x}{n+n'-r-1}\frac{r+1}{-nx-r}\;\frac{-n'x-n-n'+r+1}{n+n'-r}\;\cdot$$

$$\cdot\left[\frac{r\left[n'(x+1)+n-r\right]}{(n+n')x}+n-r\right] =$$

$$= \binom{-nx}{r+1}\binom{-n'x}{n+n'-r-1}\frac{(r+1)\left[n'(1+x)+n-r-1\right]}{(n+n')x}$$

Then, from (A.5) for $r=n$ we have

$$S = S_n = \binom{-nx}{n}\binom{-n'x}{n'}\frac{nn'(1+x)}{x(n+n')} \tag{A.7}$$

In a similar way we proceed for the second summation which is

$$T = \sum_{\ell=1}^{n}(-1)^{n-\ell}\,\ell\binom{-nx}{n-\ell}\binom{-\frac{n'}{x}-\ell-1}{n'-1} \tag{A.8}$$

We consider the sum T_r of the last r terms

$$T_r = \sum_{\ell=0}^{r-1}(-1)^{\ell}(n-\ell)\binom{-nx}{\ell}\binom{-\frac{n'}{x}+\ell-n-1}{n'-1} \tag{A.9}$$

In particular for $r=1,2$ we have:

$$T_1 = -\binom{-nx}{1}\binom{-\frac{n'}{x}-n}{n'-1}\frac{n'\left(\frac{1}{x}+1\right)+n-1}{nx+n'} \tag{A.10}$$

$$T_2 = \binom{-nx}{2}\binom{-\frac{n'}{x}-n+1}{n'-1}\frac{2\left[n'\left(\frac{1}{x}+1\right)+n-2\right]}{nx+n'} \tag{A.11}$$

In general we can prove by induction that we have

$$T_r = (-1)^r \binom{-nx}{r} \binom{-\frac{n'}{x} - n + r - 1}{n' - 1} \frac{r\left[n'\left(\frac{1}{x}+1\right)+n-r\right]}{nx+n'} \tag{A.12}$$

Then, for $r = n$ we get

$$T = T_n = (-1)^n \binom{-nx}{n} \binom{-\frac{n'}{x} - 1}{n' - 1} \frac{nn'\left(1+\frac{1}{x}\right)}{nx+n'} = \tag{A.13}$$

$$= (-1)^{n-1} \binom{-nx}{n} \binom{-\frac{n'}{x}}{n'} \frac{nn'(1+x)}{nx+n'}$$

R E F E R E N C E S

1) For a general review of these symmetry properties see e.g.,
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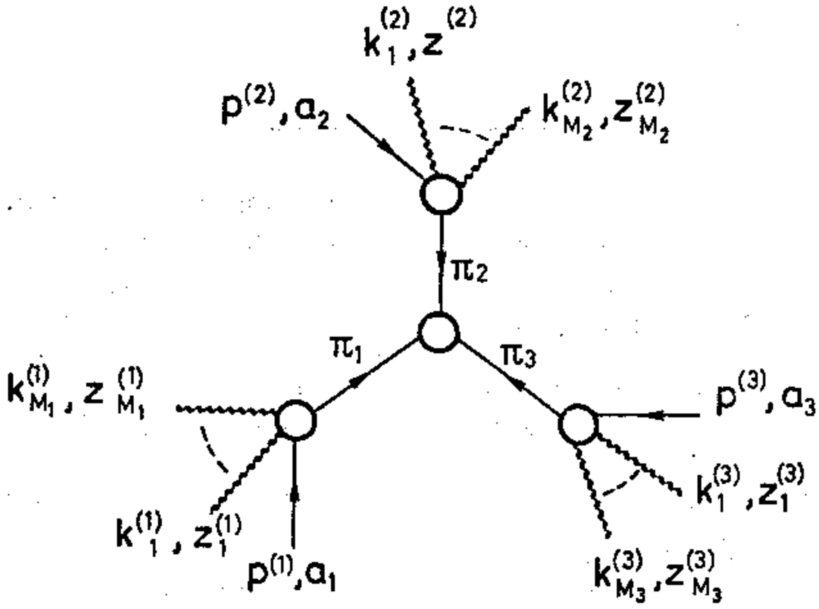

FIG.1

Coupling between these transverse states

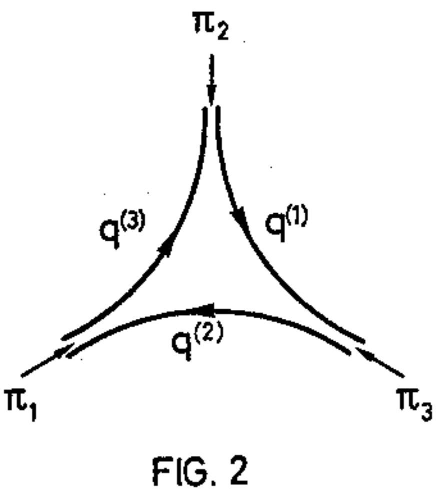

FIG. 2

Quark picture of the three-point function