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Covariance of Euclidean Fermi Fields

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Two sorts of free Euclidean Fermi fields, one represented in a definite metric Fock space and the other in an indefinite metric Fock space, are constructed as collections of "Euclidean" fields of Fermi oscillators. The Euclidean covariance of these fields are proved and it is shown that they are unitarily equivalent to Osterwalder and Schrader's and Ek's, respectively.

§ 1. Introduction

It is widely recognized that the quantum field theory is the quantum mechanics of the system with an infinitely many degrees of freedom, and conversely the quantum mechanics of an oscillator is the quantum field theory without space dimensions. This point of view is achieved by the observation that fields in the canonical quantization are decomposed into oscillators, while the latter are quantized in accordance with Bose or Fermi statistics.

Presumably we may extend the same viewpoint also to Euclidean fields, in particular, to Euclidean Fermi fields with which we will be concerned here. The construction of Euclidean Fermi fields by Osterwalder and Schrader[1)] was made on the basis of the Euclidean covariance, not stimulated by the resemblance of Euclidean Fermi fields and "Euclidean" Fermi oscillators. In fact, following their theory, we feel some discord with Fermi oscillators in reducing the theory of quantized fields to the quantum theory by eliminating spatial degrees of freedom.

In § 2 we summarize briefly the quantum theory of Fermi oscillators and, after the Wick rotation, calculate the Schwinger function. In this miniature theory for Fermi fields we set up a rule for assigning "Euclidean" fields to Fermi oscillators with an imaginary time such that the Euclidean fields are anticommuting and provide the correct Schwinger function.

In § 3, after decomposing the free Dirac field into a collection of Fermi oscillators, we outline the well-known results of the Schwinger function and problems related to the Euclidean covariance. The present authors never claim that that section is new, but only wish to devote it to the explanation of notations used in the present paper. In the succeeding section the free Euclidean Fermi fields are constructed according to the rule established in § 3. It turns out that these fields are different from those constructed by Osterwalder and Schrader, but they have the same two-point functions including the Schwinger function as Osterwalder and Schrader's.

Suggested by the result of the preceding section, we define in § 5 a class of general Euclidean Fermi fields which contains ours and Osterwalder and Schrader's as special cases. We prove the covariance of Euclidean Fermi fields in this general form. Moreover, we show the unitary equivalence of any two sets of Euclidean Fermi fields in the class, which is carried out in § 6.

We can also construct Euclidean Fermi fields by the so-called reconstruction theory starting with the set of two-point functions given by Osterwalder and Schrader. In this process, which is described in § 7, the class of general Euclidean Fermi fields defined in § 5 is reconstructed.

If the Schwinger function is modified so that it becomes hermitian, we obtain Ek's formulation of Euclidean Fermi fields[2] represented in an indefinite Fock space. In the final section we give the corresponding rule in this case to associate "Euclidean" fields with Fermi oscillators and prove the covariance of the Euclidean Fermi fields.

§ 2. Fermi oscillator

A linearized form of the equation for classical harmonic oscillator is given by

$$\left(-i\sigma_3\frac{d}{dt}+\omega\sigma_0\right)\psi(t)=0\,, \tag{2·1}$$

where the σ_i's ($i=1, 2, 3$) are, as usual, Pauli's spin matrices and σ_0 the 2×2 unit matrix. The name of the *Fermi oscillator* will be given to a system described by this equation when it is quantized according to the Fermi statistics. In harmony with the hole theory we write a solution of (2·1) in the form

$$\psi(t)=u_1a_1(t)+u_2a_2^*(t) \tag{2·2}$$

and put*)

$$\psi^\dagger(t)\equiv\psi^*(t)\sigma_3=u_1^\dagger a_1^*(t)+u_2^\dagger a_2(t). \tag{2·3}$$

Here $u_1=(1\ 0)^T$ and $u_2=(0\ 1)^T$ are column vectors and $u_r^\dagger=u_r^*\sigma_3$, $r=1, 2$, row vectors. $a_r^\#(t)$ satisfies the equation

$$i\frac{d}{dt}a_r^\#(t)=\pm\omega a_r^\#(t). \tag{2·4}$$

The upper sign on the right-hand side, RHS for short, of (2·4) is attached to $a_r^\#(t)=a_r(t)$ and the lower sign to $a_r^\#(t)=a_r^*(t)$. The solution of this equation

*) It is customary to write $\bar{\psi}=\psi^*\sigma_3$, but in the present paper the bar will be reserved to denote the complex conjugate.

is $a_r^{\#}(t)=\exp(\mp i\omega t)a_r^{\#}$ with $a_r^{\#}=a_r^{\#}(0)$ being the initial data. In order to quantize the system to obtain Fermi oscillators we only have to replace $a_r^{\#}$ by the operators, denoted by the same letters as c-number entities, satisfying the canonical anticommutation relations (referred to as CAR hereafter):

$$\{a_r, a_s^*\}=\delta_{rs}, \qquad \{a_r, a_s\}=\{a_r^*, a_s^*\}=0.$$

The Hamiltonian of the system is given by $H=\omega(a_1^*a_1+a_2^*a_2)$.

We move to the "Euclidean" theory by means of the *Wick rotation* $t\to -it$. Then all the operators wear a cap, so that for instance $a_r(t)$ in (2·2) and (2·3) becomes $\hat{a}_r(t)=\exp(-tH)a_r\exp(tH)=\exp(-\omega t)a_r$. Such operators are sometimes referred to as *imaginary time operators*. It should be remarked that $\hat{a}_r^*(t)=\exp(-tH)a_r^*\exp(tH)$ is not the adjoint to $\hat{a}_r(t)$. The vacuum expectation of the time-ordered product of $\hat{\psi}(t)$ and $\hat{\psi}^\dagger(s)$ is calculated as

$$(\Omega, T(\hat{\psi}(t)\hat{\psi}^\dagger(s))\Omega)=\frac{1}{2\pi}\int\frac{ip\sigma_3+\omega\sigma_0}{p^2+\omega^2}e^{-ip(t-s)}dp\equiv\chi(t, s). \tag{2·5}$$

This is the Schwinger function for the imaginary time fields of Fermi oscillators. LHS of (2·5) is to be understood as a matrix M whose element is given by $M_{\alpha\beta}=(\Omega, T(\hat{\psi}_\alpha(t)\hat{\psi}_\beta^\dagger(s))\Omega)$. Similar conventions will frequently be used in the following.

The "Euclidean" fields of Fermi oscillators are obtained from the imaginary time fields by the rule that the imaginary time operators $\hat{a}_r(t)$ and $\hat{a}_r^*(t)$, $r=1, 2$, are replaced by

$$\begin{aligned} A_r^1(t)&=(2\pi)^{-1/2}\int\nu(p)e^{-ipt}[a_r(-p)+ib_r^*(p)]dp, \\ A_r^2(t)&=(2\pi)^{-1/2}\int\bar{\nu}(p)e^{-ipt}[a_r^*(p)+ib_r(-p)]dp, \end{aligned} \tag{2·6}$$

respectively, where $\nu(p)=(-ip+\omega)^{-1/2}$. $a_r^{\#}(p)$ and $b_r^{\#}(p)$ are annihilation-creation operators, or more precisely annihilation-creation operator-valued distributions, satisfying CAR. This rule has been inspired by the *Grassmann-algebraic* formulation of the path integral for Fermi fields, which is given in a separate paper of the present authors.[3] Then the Euclidean fields are represented by

$$\Psi^\rho(t)=(2\pi)^{-1/2}\int e^{-ipt}\,\tilde{\Psi}^\rho(p)dp, \qquad \rho=1, 2, \tag{2·7}$$

$\Psi^1(t)$ coming from $\hat{\psi}(t)$ and $\Psi^2(t)$ from $\hat{\psi}^\dagger(t)$. Explicitly we have

$$\begin{aligned} \tilde{\Psi}^1(p)&=u_1\nu(p)[a_1(-p)+ib_1^*(p)]+u_2\bar{\nu}(p)[a_2^*(p)+ib_2(-p)], \\ \tilde{\Psi}^2(p)&=u_1^\dagger\bar{\nu}(p)[a_1^*(p)+ib_1(-p)]+u_2^\dagger\nu(p)[a_2(-p)+ib_2^*(p)]. \end{aligned} \tag{2·8}$$

The vacuum expectation value of Euclidean fields will be denoted by $\langle\cdots\rangle_0$.

It is a simple exercise to ascertain that the Euclidean fields (2·7) provide the following relations:

$$\{\Psi^1(t),\ \Psi^2(s)\}=0 \qquad \text{for all } t,\ s\,, \tag{2·9}$$

$$\langle \Psi^1(t)\,\Psi^2(s)\rangle_0=\chi(t,\ s), \tag{2·10}$$

$$\{\Psi^\rho(t),\ \Psi^{\sigma *}(s)\}=2\delta_{\rho\sigma}\sigma_0\frac{1}{2\pi}\int\frac{e^{-ip(t-s)}}{(p^2+\omega^2)^{1/2}}dp\,, \tag{2·11}$$

$$\langle \Psi^\rho(t)\,\Psi^{\sigma *}(s)\rangle_0=\delta_{\rho\sigma}\sigma_0\frac{1}{2\pi}\int\frac{e^{-ip(t-s)}}{(p^2+\omega^2)^{1/2}}dp\,. \tag{2·12}$$

The zero on RHS of (2·9) means the 2×2 null matrix. We easily see that these relations are miniatures of Osterwalder and Schrader's.[1]

§ 3. Decomposition of Dirac fields into Fermi oscillators

We start with the Dirac equation of the form

$$i\frac{\partial}{\partial t}\tilde{\psi}(t,\boldsymbol{p})=H(\boldsymbol{p})\tilde{\psi}(t,\boldsymbol{p}), \tag{3·1}$$

where $\tilde{\psi}(t,\boldsymbol{p})$ is the Fourier transform of the Dirac field $\psi(t,\boldsymbol{x})$ in the space variables and $H(\boldsymbol{p})$ is an hermitian matrix given by

$$H(\boldsymbol{p})=\gamma^0\Big(\sum_{j=1}^{3}\gamma^j p^j+m\Big)=\begin{pmatrix} m & \boldsymbol{p\sigma} \\ \boldsymbol{p\sigma} & -m\end{pmatrix}; \tag{3·2}$$

we have used the convention $\boldsymbol{p\sigma}=\sum_{j=1}^{3}p^j\sigma^j$ and the Pauli representation for the γ-matrices $\gamma^0=\sigma_3\otimes\sigma_0$ and $\gamma^j=i\sigma_2\otimes\sigma_j$, where $\otimes$ means the Kronecker product of matrices. The matrix $H(\boldsymbol{p})$ can be diagonalized by the unitary matrix

$$\Sigma(\boldsymbol{p})=[2\omega(\boldsymbol{p})(\omega(\boldsymbol{p})+m)]^{-1/2}\begin{pmatrix} \omega(\boldsymbol{p})+m & \boldsymbol{p\sigma} \\ -\boldsymbol{p\sigma} & \omega(\boldsymbol{p})+m\end{pmatrix}, \tag{3·3}$$

where $\omega(\boldsymbol{p})=(\boldsymbol{p}^2+m^2)^{1/2}$. In fact we have $\Sigma(\boldsymbol{p})H(\boldsymbol{p})\Sigma^*(\boldsymbol{p})=\omega(\boldsymbol{p})\gamma^0$.

If we define

$$\begin{aligned} a_r(t,\boldsymbol{p})&=(\Sigma(\boldsymbol{p})\tilde{\psi}(t,\boldsymbol{p}))_r &&\text{for } r=1,2\,,\\ a_r(t,\boldsymbol{p})&=(\Sigma(-\boldsymbol{p})\tilde{\psi}(t,-\boldsymbol{p}))_r &&\text{for } r=3,4 \end{aligned} \tag{3·4}$$

and their adjoints, and write $a(t,\boldsymbol{p})=(a_1(t,\boldsymbol{p}),\ a_2(t,\boldsymbol{p}),\ a_3(t,\boldsymbol{p}),\ a_4(t,\boldsymbol{p}))^T$, then, by (3·1), $a^{\#}(t,\boldsymbol{p})$ satisfies the equation

$$i\frac{\partial}{\partial t}a^{\#}(t,\boldsymbol{p})=\pm\,\omega(\boldsymbol{p})a^{\#}(t,\boldsymbol{p}), \tag{3·5}$$

which is completely in correspondence with (2·4). Therefore the definition (3·4) provides the decomposition of the Dirac field into Fermi oscillators, each having the angular frequency $\omega(\boldsymbol{p})$.

The imaginary time fields can be written in the form

$$\hat{\psi}(t, \boldsymbol{x})=(2\pi)^{-3/2}\int e^{-ipx}\{\sum_{r=1,2}u_r(-\boldsymbol{p})\hat{a}_r(t, -\boldsymbol{p})+\sum_{r=3,4}u_r(-\boldsymbol{p})\hat{a}_r^*(t, \boldsymbol{p})\}d\boldsymbol{p} \tag{3·6}$$

and

$$\hat{\psi}^\dagger(t, \boldsymbol{x})=(2\pi)^{-3/2}\int e^{-ipx}\{\sum_{r=1,2}u_r^\dagger(\boldsymbol{p})\hat{a}_r^*(t, \boldsymbol{p})+\sum_{r=3,4}u_r^\dagger(\boldsymbol{p})\hat{a}_r(t, -\boldsymbol{p})\}d\boldsymbol{p}\ , \tag{3·7}$$

where $u_r(\boldsymbol{p})$ is the r-th column of $\Sigma^*(\boldsymbol{p})$, $u_r^\dagger(\boldsymbol{p})$ the r-th row of $\Sigma(\boldsymbol{p})\gamma^0$, and

$$a_r^\#(t, \boldsymbol{p})=e^{-tH}a_r^\#(\boldsymbol{p})e^{tH}=e^{\mp\omega(p)t}a_r^\#(\boldsymbol{p}).$$

The $a_r^\#(\boldsymbol{p})$'s are CAR operators and the Hamiltonian H is given by

$$H=\int\omega(\boldsymbol{p})\sum_{r=1}^{4}a_r^*(\boldsymbol{p})a_r(\boldsymbol{p})d\boldsymbol{p}\ .$$

Before proceeding to the construction of the Euclidean Fermi fields, we want to make a digression on the homogeneous Euclidean transformation $SO(4)$. Notations and conventions are similar to those of Osterwalder and Schrader,[1)] but there is one important difference due to the difference of the Wick rotation; they made use of $t\to it$ instead of $t\to -it$. Let $x=(x^0, \boldsymbol{x})\in \boldsymbol{R}^4$ be a Euclidean four-vector, and denote the inner product of $x, y\in\boldsymbol{R}^4$ by xy simply, otherwise it may lead to a misunderstanding. We define an hermitian matrix $\hat{p}^E$ by the inner product of a vector $p\in\boldsymbol{R}^4$ and the Euclidean γ-matrices, γ^E, given by $(\gamma^0, -i\boldsymbol{\gamma})$:

$$\hat{p}^E=p^0\gamma^0-i\sum_{j=1}^{3}p^j\gamma^j=\begin{pmatrix}p^0\sigma_0 & -i\boldsymbol{p\sigma}\\ i\boldsymbol{p\sigma} & -p^0\sigma_0\end{pmatrix}. \tag{3·8}$$

It is not difficult to see that for the Euclidean rotation $\hat{p}^E$ undergoes the transformation

$$S(A, B)\hat{p}^E S^*(A, B)=(R(A, B)p\,\widehat{)}^E\ , \tag{3·9}$$

where $R(A, B)$ is an element of $SO(4)$ written in terms of $(A, B)\in SU(2)\times SU(2)$, the universal covering group of $SO(4)$, and $S(A, B)$ is a unitary matrix given by

$$S(A, B)=\frac{1}{2}\begin{pmatrix}A+\bar{B} & A-\bar{B}\\ A-\bar{B} & A+\bar{B}\end{pmatrix}. \tag{3·10}$$

With the notation introduced above, the Schwinger function calculated for the imaginary time fields (3·6) and (3·7) reads

$$(\Omega,\ T(\hat{\psi}(x)\hat{\psi}^{\dagger}(y))\Omega)=\frac{1}{(2\pi)^4}\int\frac{ip^E+m}{p^2+m^2}e^{-ip(x-y)}dp\ , \qquad (3\cdot11)$$

where we have suppressed the 4×4 unit matrix multiplying on the mass term m.

§4. Euclidean Fermi fields

In constructing the Euclidean Fermi fields we only have to follow the recipe in §2 of associating Fermi oscillators with their Euclidean fields. Namely we replace $\hat{a}_r(t,\boldsymbol{p})$ and $\hat{a}_r{}^*(t,\boldsymbol{p})$ in (3·6) and (3·7) by

$$A_r{}^1(x^0,\boldsymbol{p})=(2\pi)^{-1/2}\int\nu(p)e^{-ip^0x^0}[a_r(-p^0,\boldsymbol{p})+ib_r{}^*(p^0,\boldsymbol{p})]dp^0 \qquad (4\cdot1)$$

and

$$A_r{}^2(x^0,\boldsymbol{p})=(2\pi)^{-1/2}\int\bar{\nu}(p)e^{-ip^0x^0}[a_r{}^*(p^0,\boldsymbol{p})+ib_r(-p^0,\boldsymbol{p})]dp^0\ , \qquad (4\cdot2)$$

respectively. Here $\nu(p)=(-ip^0+\omega(\boldsymbol{p}))^{-1/2}$, and $a_r{}^\#(p)=a_r{}^\#(p^0,\boldsymbol{p})$ and $b_r{}^\#(p)=b_r{}^\#(p^0,\boldsymbol{p})$ are two anticommuting sets of operators, each of which obeys CAR:

$$\{a_r(p),\ a_s{}^*(q)\}=\delta_{rs}\delta(p-q),\quad \{a_r(p),\ a_s(q)\}=\{a_r{}^*(p),\ a_s{}^*(q)\}=0 \qquad (4\cdot3)$$

and similar relations for $b_r{}^\#(p)$. In this way we get the Euclidean fields

$$\Psi^\rho(x)=(2\pi)^{-2}\int e^{-ipx}\tilde{\Psi}^\rho(p)dp\ ,\qquad \rho=1,2\ , \qquad (4\cdot4)$$

where

$$\begin{aligned}\tilde{\Psi}^1(p)&=\Sigma(\boldsymbol{p})W(p)[c(-p)-\gamma^0d^*(p)],\\ \tilde{\Psi}^2(p)&=\bar{\Sigma}(\boldsymbol{p})\bar{W}(p)[\gamma^0d(-p)+c^*(p)].\end{aligned} \qquad (4\cdot5)$$

We have written both $\Psi^1(x)$ and $\Psi^2(x)$ as column vectors by putting

$$\begin{aligned}c(p)&=(c_1(p),\ c_2(p),\ c_3(p),\ c_4(p))^T\\ &=(a_1(p^0,\boldsymbol{p}),\ a_2(p^0,\boldsymbol{p}),\ ib_3(p^0,-\boldsymbol{p}),\ ib_4(p^0,-\boldsymbol{p}))^T\end{aligned}$$

and

$$\begin{aligned}d(p)&=(d_1(p),\ d_2(p),\ d_3(p),\ d_4(p))^T\\ &=(ib_1(p^0,-\boldsymbol{p}),\ ib_2(p^0,-\boldsymbol{p}),\ a_3(p^0,\boldsymbol{p}),\ a_4(p^0,\boldsymbol{p}))^T\end{aligned}$$

so that CAR remains unchanged,

$$\{c_r(p),\ c_s{}^*(q)\}=\{d_r(p),\ d_s{}^*(q)\}=\delta_{rs}\delta(p-q). \qquad (4\cdot6)$$

Further

$$W(p)=\begin{pmatrix} \bar{\nu}(p)\sigma_0 & 0 \\ 0 & \nu(p)\sigma_0 \end{pmatrix} \tag{4·7}$$

and the following relations for $\Sigma(\boldsymbol{p})$ have been used

$$\Sigma^*(-\boldsymbol{p})=\Sigma(\boldsymbol{p}) \qquad \text{and} \qquad \gamma^0\Sigma^T(\boldsymbol{p})\gamma^0=\bar{\Sigma}(\boldsymbol{p}). \tag{4·8}$$

The unitarity of $\Sigma(\boldsymbol{p})$ and the relations pertaining to $\Sigma(\boldsymbol{p})$ and $W(p)$,

$$\Sigma(\boldsymbol{p})W(p)W^*(-p)\Sigma^*(-\boldsymbol{p})=\frac{i\not{p}^E+m}{p^2+m^2}, \tag{4·9}$$

$$W(p)W^*(p)=(p^2+m^2)^{-1/2}, \tag{4·10}$$

are sufficient to see that the fields constructed above are really Euclidean Fermi fields. Indeed we are led to

$$\{\Psi^1(x),\ \Psi^2(y)\}=0 \qquad \text{for all } x, y, \tag{4·11}$$

$$\langle \Psi^1(x)\, \Psi^2(y)\rangle_0=(\Omega,\ T(\hat{\psi}(x)\hat{\psi}^\dagger(y))\Omega)\equiv\chi_{12}(x, y). \tag{4·12}$$

$\langle\cdots\rangle_0$ in (4·12) signifies the expectation value with respect to the Euclidean vacuum and RHS is given by (3·11). In addition to these relations we obtain

$$\langle \Psi^\rho(x)\, \Psi^\rho(y)\rangle_0=0, \tag{4·13}$$

$$\{\Psi^\rho(x),\ \Psi^{\sigma *}(y)\}/2=\langle \Psi^\rho(x)\, \Psi^{\sigma *}(y)\rangle_0=\chi_{\rho\bar{\sigma}}(x, y), \tag{4·14}$$

where

$$\chi_{\rho\bar{\sigma}}(x, y)=\delta_{\rho\sigma}\frac{1}{(2\pi)^4}\int\frac{e^{-ip(x-y)}}{(p^2+m^2)^{1/2}}dp \tag{4·15}$$

is a multiple of the unit matrix.

Though the Euclidean Fermi fields $\Psi^\rho(x)$, $\rho=1, 2$, whose Fourier transforms are given by (4·5), are different from Osterwalder and Schrader's, the relations (4·11)~(4·15) are identical with theirs. This fact allows us to convince ourselves that the two sets of Euclidean Fermi fields must be unitarily equivalent, and that this is indeed the case will be proved in § 6.

§ 5. Euclidean covariance

In this section we first generalize the arguments in the preceding section to find a wide class of Euclidean Fermi fields having the same two-point functions as Osterwalder and Schrader's. The Euclidean covariance will be proved as a general statement to this class of Fermi fields.

Let $X(p)$ and $U(p)$ be two arbitrary 4×4 *unitary* matrices. We determine other two matrices $Y(p)$ and $V(p)$ by the conditions

$$\begin{aligned} i\not{p}^E+m&=(p^2+m^2)^{1/2}X(p)\,Y^*(-p),\\ i\not{p}^E+m&=-(p^2+m^2)^{1/2}\,U(p)\,V^*(-p). \end{aligned} \tag{5·1}$$

Since $(i\not{p}^E+m)/(p^2+m^2)^{1/2}$ is unitary, we see that $Y(p)$ and $V(p)$, too, are *unitary*. Our choice of these matrices was

$$\begin{aligned} X(p)&=Y(p)=(p^2+m^2)^{1/4}\Sigma(\boldsymbol{p})W(p),\\ -U(p)&=V(p)=(p^2+m^2)^{1/4}\Sigma(\boldsymbol{p})W(p)\gamma^0\,. \end{aligned}$$

Let us define the fields $\Phi^\rho(x)$, $\rho=1,\ 2$, whose Fourier transformation like (4·4) is given by

$$\begin{aligned} \tilde{\Phi}^1(p)&=(p^2+m^2)^{-1/4}[X(p)c(-p)+U(p)d^*(p)],\\ \tilde{\Phi}^2(p)&=(p^2+m^2)^{-1/4}[\bar{V}(p)d(-p)+\bar{Y}(p)c^*(p)]. \end{aligned} \tag{5·2}$$

Then it is almost immediate to see that the two-point functions and the anticommutators are the same as Osterwalder and Schrader's.

Now we turn to the covariance of the Euclidean Fermi fields. We are going to prove the following:

THEOREM *The Fermi fields $\Phi^\rho(x)$ defined by (5·2) show the Euclidean covariance.*

More precisely, for a Euclidean transformation $(A, B;\, a)$, where $(A, B)\in SU(2)\times SU(2)$ and $a\in \boldsymbol{R}^4$, there exists a unitary operator $U(A, B;\, a)$ such that $\Phi^\rho(x)$, $\rho=1,\ 2$, undergo the transformations

$$\begin{aligned} U(A,B;\,a)\Phi^1(x)U(A,B;\,a)^{-1}&=S^*(A,B)\Phi^1(R(A,B)x+a),\\ U(A,B;\,a)\Phi^2(x)U(A,B;\,a)^{-1}&=S^T(A,B)\Phi^2(R(A,B)x+a). \end{aligned} \tag{5·3}$$

Here $S(A, B)$ and $R(A, B)$ have already been defined in § 3.

Proof Consider unitary matrices

$$\begin{aligned} W(A,B|p)&=Y^T(p)S^T(A,B)\,\bar{Y}(R(A,B)p),\\ Z(A,B|p)&=U^*(p)S^*(A,B)\,U(R(A,B)p) \end{aligned} \tag{5·4}$$

and define a canonical transformation

$$\begin{aligned} c(p)\to c'(p)&=\bar{W}(A,B|p)c(R(A,B)p),\\ d(p)\to d'(p)&=\bar{Z}(A,B|p)d(R(A,B)p) \end{aligned} \tag{5·5}$$

and their adjoints. Since it does not mix annihilation and creation operators, this canonical transformation is unitarily implementable,[4] that is, there exists a unitary operator $U(A, B)$ such that the relations

$$c'^{\#}(p) = U(A, B)c^{\#}(p)U(A, B)^{-1}, \quad d'^{\#}(p) = U(A, B)d^{\#}(p)U(A, B)^{-1} \tag{5·6}$$

hold. A canonical transformation

$$c(p) \to e^{ipa}c(p), \qquad d(p) \to e^{ipa}d(p) \tag{5·7}$$

for $a \in \boldsymbol{R}^4$ is also unitarily implementable. Let $U(a)$ be a unitary operator which implements this transformation and put

$$U(A, B; a) = U(a)U(A, B). \tag{5·8}$$

It follows from (5·4) that

$$\bar{W}(A, B|-p) = X^*(p)S^*(A, B)X(R(A, B)p), \quad \bar{Z}(A, B|-p) = V^T(p)S^T(A, B)\bar{V}(R(A, B)p), \tag{5·9}$$

since the consistency with (5·4) can be verified by (3·9).

Now we are ready to derive the relations

$$U(A, B; a)\tilde{\Phi}^1(p)U(A, B; a)^{-1} = e^{-i(R(A,B)p)a}S^*(A, B)\tilde{\Phi}^1(R(A, B)p), \quad U(A, B; a)\tilde{\Phi}^2(p)U(A, B; a)^{-1} = e^{-i(R(A,B)p)a}S^T(A, B)\tilde{\Phi}^2(R(A, B)p), \tag{5·10}$$

which are equivalent to (5·3). This completes the proof of the theorem.

§ 6. Unitarily equivalent fields

We have shown in the preceding section that any set of unitary matrices $X(p)$, $Y(p)$, $U(p)$, $V(p)$ satisfying the condition (5·1) provide Fermi fields $\Phi^\rho(x)$, $\rho=1, 2$, which are Euclidean covariant and have the same two-point functions and anticommutators as Osterwalder and Schrader's. Here we will prove the unitary equivalence of these sets of fields.

Proposition *Let* $(X(p), Y(p), U(p), V(p))$ *and* $(X'(p), Y'(p), U'(p), V'(p))$ *be two sets of unitary matrices for which the condition (5·1) is satisfied. Then the corresponding fields* $(\Phi^1(x), \Phi^2(x))$ *and* $(\Phi'^1(x), \Phi'^2(x))$ *are unitarily equivalent.*

Proof Define

$$\Xi(p) = Y^T(p)\bar{Y}'(p) \quad \text{and} \quad Z(p) = U^*(p)U'(p), \tag{6·1}$$

then these matrices are unitary and by (5·1) we have

$$\bar{\Xi}(-p)=X^*(p)X'(p) \quad \text{and} \quad \bar{Z}(-p)=V^*(p)\bar{V}'(p). \tag{6·2}$$

The canonical transformation given by

$$c(p)\to\bar{\Xi}(p)c(p), \qquad d(p)\to\bar{Z}(p)d(p) \tag{6·3}$$

and their adjoints is evidently unitarily implementable, and hence there exists a unitary operator U such that

$$Uc(p)U^{-1}=\bar{\Xi}(p)c(p) \quad \text{and} \quad Ud(p)U^{-1}=\bar{Z}(p)d(p) \tag{6·4}$$

together with their adjoints hold.

The unitarity of the matrices concerned and the relations (6·1) and (6·2) are sufficient to assure the validity of the relations

$$U\tilde{\Phi}^{\rho}(p)U^{-1}=\tilde{\Phi}'^{\rho}(p), \qquad \rho=1,2, \tag{6·5}$$

where $\tilde{\Phi}^{\rho}(p)$ and $\tilde{\Phi}'^{\rho}(p)$ are the Fourier transforms of $\Phi^{\rho}(x)$ and $\Phi'^{\rho}(x)$, respectively (see (5·2)).

Osterwalder and Schrader's construction of the Euclidean Fermi fields corresponds to

$$\begin{aligned} X(p)&=iV(p)=(p^2+m^2)^{-1/4}S^E(-p)W(|p|),\\ Y(p)&=-iU(p)=(p^2+m^2)^{-1/4}S^E(p)\bar{W}(|p|). \end{aligned} \tag{6·6}$$

Here $|p|=((p^0)^2+\boldsymbol{p}^2)^{1/2}$ and

$$S^E(p)=\left(\frac{p^0+|p|}{2|p|}\right)^{1/2}\begin{pmatrix} 1 & \dfrac{i\boldsymbol{p\sigma}}{p^0+|p|} \\ \dfrac{i\boldsymbol{p\sigma}}{p^0+|p|} & 1 \end{pmatrix} \tag{6·7}$$

is a unitary matrix which diagonalizes $\not{p}^E$:

$$S^{E*}(p)\not{p}^E S^E(p)=|p|\gamma^0\,. \tag{6·8}$$

The matrix $W(|p|)$ is defined by

$$W(|p|)=\begin{pmatrix} (-i|p|+m)^{1/2}\sigma_0 & 0 \\ 0 & (i|p|+m)^{1/2}\sigma_0 \end{pmatrix}. \tag{6·9}$$

By (6·8) we easily see that the matrices (6·1) satisfy the condition (5·1). This shows that the Euclidean Fermi fields constructed in § 3 are unitarily equivalent to Osterwalder and Schrader's.

Roughly speaking, the unitary matrix $S^E(p)$ corresponds to the matrix $\Sigma(\boldsymbol{p})$ in § 3. In the limit $\boldsymbol{p}\to 0$ the former tends to the unit matrix multiplied by $\theta(p^0)$, while the latter tends to the unit matrix. From this we may say that Osterwalder and Schrader's formulation of the Euclidean Fermi fields has a correspondence to the Fermi oscillators rather different from ours.

§ 7. Two-point functions and the reconstruction of the free fields

We have shown in the preceding two sections that any Euclidean Fermi fields unitarily equivalent to Osterwalder and Schrader's have the same two-point functions $(4\cdot 11)\sim(4\cdot 15)$. In this section we will prove the converse statement.[5]

Reconstruction Theorem *Any Euclidean free Fermi fields* $\Psi^{\rho}(x)$, $\rho=1,2$, *which have the same two-point functions as* $(4\cdot 11)\sim(4\cdot 15)$ *are unitarily equivalent to Osterwalder and Schrader's.*

Let $X_l(x,y)$, $l=1,2$, be defined by

$$X_1(x,y)=\begin{pmatrix}\chi_{1\bar{1}}(x,y) & \chi_{12}(x,y)\\ \chi_{\bar{2}\bar{1}}(x,y) & \chi_{\bar{2}2}(x,y)\end{pmatrix} \tag{7·1}$$

and

$$X_2(x,y)=\begin{pmatrix}\chi_{\bar{1}1}(x,y) & \chi_{\bar{1}\bar{2}}(x,y)\\ \chi_{21}(x,y) & \chi_{2\bar{2}}(x,y)\end{pmatrix}, \tag{7·2}$$

both of which are 8×8 hermitian matrices, $X_l^*(x,y)=X_l(y,x)$. The Fourier transform of $X_l(x,y)$ is given by

$$X_l(x,y)=(2\pi)^{-4}\int e^{-ip(x-y)}\tilde{X}_l(p)\,dp$$

with

$$\tilde{X}_1(p)=(p^2+m^2)^{-1/2}\begin{pmatrix}1 & \tau(p)\\ \tau^*(p) & 1\end{pmatrix} \tag{7·3}$$

and

$$\tilde{X}_2(p)=(p^2+m^2)^{-1/2}\begin{pmatrix}1 & -\tau^T(p)\\ -\bar{\tau}(p) & 1\end{pmatrix}, \tag{7·4}$$

where $\tau(p)=(i\not{p}^E+m)/(p^2+m^2)^{1/2}$ is a unitary matrix and $\tau^*(p)=\tau(-p)$. Eventually we set

$$X(x,y)=\begin{pmatrix}0 & X_2(x,y)\\ X_1(x,y) & 0\end{pmatrix}. \tag{7·5}$$

Let $\underline{f}$ be an element of $\boldsymbol{C}^{16}\otimes\mathcal{S}(\boldsymbol{R}^4)$ of the form

$$\underline{f}(x)=[f_1(x),f_2(x),f_3(x),f_4(x)];\qquad f_j(x)\in\boldsymbol{C}^4\otimes\mathcal{S}(\boldsymbol{R}^4)$$

and for the fields $\Psi^{\rho\#}$ under consideration we write

$$\Psi(\underline{f}) = \Psi^{1*}(f_1) + \Psi^2(f_2) + \Psi^1(f_3) + \Psi^{2*}(f_4), \tag{7·6}$$

which is linear in $\underline{f}$, then we have

$$\langle \Psi(\underline{f})\, \Psi(\underline{g})\rangle_0 = \int \underline{f}^T(x) X(x, y) \underline{g}(y) dx dy$$

$$= \int (p^2+m^2)^{-1/2} (\tilde{f}_1(-p) - \tau^*(p) \tilde{f}_2(-p))^T (\tilde{g}_3(p) - \tau^T(p) \tilde{g}_4(p)) dp$$

$$+ \int (p^2+m^2)^{-1/2} (\tilde{f}_3(-p) + \bar{\tau}(p) \tilde{f}_4(-p))^T (\tilde{g}_1(p) + \tau(p) \tilde{g}_2(p)) dp\,. \tag{7·7}$$

We agree to say that the fields $\Psi^\rho(x)$ are *free* if the many-point functions for the fields are given as follows:

$$X(x_k, \cdots, x_1, y_1, \cdots, y_k) = \sum_{\pi \in P_k} (\mathrm{sgn}\, \pi) X(x_1, y_{\pi(1)}) \cdots X(x_k, y_{\pi(k)}),$$

$$X(x_k, \cdots, x_1, y_1, \cdots, y_l) = 0 \qquad \text{if } k \neq l\,. \tag{7·8}$$

Denote $\boldsymbol{C}^8 \otimes \mathcal{S}(\boldsymbol{R}^4)$ by K and let $\xi = [f_1, f_2] \in K$ with $f_j \in \boldsymbol{C}^4 \otimes \mathcal{S}(\boldsymbol{R}^4)$. K may be considered a Hilbert space if we introduce into it an inner product given by

$$(\xi, \eta) = ([f_1, f_2], [g_1, g_2]) = (f_1, g_1) + (f_2, g_2), \tag{7·9}$$

where

$$(f_j, g_j) = \int (p^2+m^2)^{-1/2} \tilde{f}_j^*(p) \tilde{g}_j(p) dp\,. \tag{7·10}$$

The representation $u(A, B; a)$ of the covering group of the inhomogeneous $SO(4)$ defined by

$$(u(A, B; a)[f, g])(x)$$

$$= [S^T(A^{-1}, B^{-1}) f(R(A, B)^{-1}(x-a)),\ S^*(A^{-1}, B^{-1}) g(R(A, B)^{-1}(x-a))] \tag{7·11}$$

is unitary.

Let $\mathcal{F}$ be a Fermi Fock space over K. It is the completion of the sum of exterior products of K,

$$\Lambda(K) = \bigoplus_m \Lambda^m(K), \qquad \Lambda^0(K) = \boldsymbol{C} \tag{7·12}$$

by the inner product defined by

$$(\xi_1\wedge\cdots\wedge\xi_m,\ \eta_1\wedge\cdots\wedge\eta_n)=\delta_{mn}\det((\xi_i,\ \eta_j)), \tag{7·13}$$

where $\xi_1\wedge\cdots\wedge\xi_m\in\Lambda^m(K)$. The unitary representation $U(A, B; a)$ in $\mathcal{F}$ can be realized as a linear extension of

$$U(A, B; a)(\xi_1\wedge\cdots\wedge\xi_m)=(u(A, B; a)\xi_1)\wedge\cdots\wedge(u(A, B; a)\xi_m). \tag{7·14}$$

The creation operator $B^*(\xi)$, $\xi\in K$, is defined by

$$B^*(\xi)(\xi_1\wedge\cdots\wedge\xi_m)=\xi\wedge\xi_1\wedge\cdots\wedge\xi_m\ . \tag{7·15}$$

Then its adjoint, the annihilation operator $B(\xi)$, is given by

$$B(\xi)(\xi_1\wedge\cdots\wedge\xi_m)=\sum_i(-1)^{i+1}(\xi,\ \xi_i)\xi_1\wedge\cdots\wedge\xi_{i-1}\wedge\xi_{i+1}\wedge\cdots\wedge\xi_m\ , \tag{7·16}$$

and as usual the anticommutation relations

$$\{B(\xi), B^*(\eta)\}=(\xi, \eta), \qquad \{B(\xi), B(\eta)\}=\{B^*(\xi), B^*(\eta)\}=0 \tag{7·17}$$

are satisfied. Obviously we have

$$U(A, B; a)B^\#(\xi)U(A, B; a)^{-1}=B^\#(u(A, B; a)\xi) \tag{7·18}$$

for $\xi\in K$, $(A, B)\in SU(2)\times SU(2)$ and $a\in \boldsymbol{R}^4$.

Let T be an operator on $\boldsymbol{C}^4\otimes\mathcal{S}(\boldsymbol{R}^4)$ defined by

$$T:\ f(x)\to(2\pi)^{-2}\int e^{-ipx}T(p)\tilde{f}(p)dp \tag{7·19}$$

with $T(p)$ being a 4×4 matrix, then there follows

$$\bar{T}:\ \ f(x)\to(2\pi)^{-2}\int e^{-ipx}\bar{T}(-p)\tilde{f}(p)dp\ . \tag{7·20}$$

Let X and U be any such operators with *unitary* matrices $X(p)$ and $U(p)$, respectively, and define an operator $\Phi(\underline{f})$, $\underline{f}\in\boldsymbol{C}^{16}\otimes\mathcal{S}(\boldsymbol{R}^4)$, by

$$\Phi(\underline{f})=B([U^T(\bar{f}_1-\bar{\tau}\bar{f}_2),\ X^*(\bar{f}_3+\tau\bar{f}_4)])+B^*([U^T(f_3-\bar{\tau}f_4),\ X^*(f_1+\tau f_2)]). \tag{7·21}$$

Then we have

$$\langle\Phi(\underline{f})\Phi(\underline{g})\rangle_0=\langle\ \Psi(\underline{f})\ \Psi(\underline{g})\rangle_0\ .$$

Since the two-point functions for free fields coincide with (7·7), so do the many-point functions correspondingly. Therefore the mapping

$$V:\quad \Psi(\underline{f}_1)\cdots\ \Psi(\underline{f}_n)\Omega_\Psi\to\Phi(\underline{f}_1)\cdots\Phi(\underline{f}_n)\Omega_\Phi$$

is unitary, and $\Psi(x)$ and $\Phi(x)$ are unitarily related by this operator.

Let us define

$$\Phi^1(f)=\Phi([0,0,f,0])=B([0,X^*\bar{f}])+B^*([U^Tf,0]),$$
$$\Phi^2(f)=\Phi([0,f,0,0])=B([-U^T\bar{\tau}\bar{f},0])+B^*([0,X^*\tau f]). \tag{7·22}$$

Evidently $\Phi^1(f)$ and $\Phi^2(g)$ are anticommuting with each other. $\Phi^1(x)$ and $\Phi^2(x)$ will be shown to have the form (5·2). To this end we first notice that $B^\#([f,g])$ can be represented by means of the annihilation-creation operators $c_r{}^\#(p)$ and $d_r{}^\#(p)$ satisfying the CAR (4·6) as follows:

$$B([f,g])=\int(p^2+m^2)^{-1/4}\{\tilde{f}^*(-p)d(p)+\tilde{g}^*(-p)c(p)\}dp\,,$$
$$B^*([f,g])=\int(p^2+m^2)^{-1/4}\{\tilde{f}^T(-p)d^*(p)+\tilde{g}^T(-p)c^*(p)\}dp\,. \tag{7·23}$$

It is easy to see that

$$\{B([f_1,g_1]),B^*([f_2,g_2])\}=([f_1,g_1],[f_2,g_2]),$$

which is nothing else but (7·17). From (7·11), (7·18), (7·19) and (7·20) there follow the transformation rules

$$U(A,B;a)c^*(p)U(A,B;a)^{-1}=e^{-ipa}S^T(A,B)c^*(R(A,B)p),$$
$$U(A,B;a)d^*(p)U(A,B;a)^{-1}=e^{-ipa}S^T(A,B)d^*(R(A,B)p). \tag{7·24}$$

These rules are identical with those given by (5·5) and (5·7).

In this way we have

$$\Phi^1(f)=\int(p^2+m^2)^{-1/4}\{(X^T(-p)\tilde{f}(p))^Tc(p)+(U^T(p)\tilde{f}(-p))^Td^*(p)\}dp\,,$$
$$\Phi^2(f)=\int(p^2+m^2)^{-1/4}\{-(U^*(p)\tau(p)\tilde{f}(p))^Td(p)$$
$$+(X^*(-p)\tau(-p)\tilde{f}(-p))^Tc^*(p)\}dp \tag{7·25}$$

or equivalently

$$\Phi^1(x)=\int e^{-ipx}(p^2+m^2)^{-1/4}[X(p)c(-p)+U(p)d^*(p)]dp\,,$$
$$\Phi^2(x)=\int e^{-ipx}(p^2+m^2)^{-1/4}[-\bar{\tau}(p)\bar{U}(-p)d(-p)+\bar{\tau}(p)\bar{X}(-p)c^*(p)]dp\,. \tag{7·26}$$

On considering the condition (5·1) we find that the Euclidean Fermi fields of general class (5·2) have been reconstructed.

§ 8. Euclidean Fermi fields in an indefinite metric state space

As is well known, the doubling of the Fermi fields in the Euclidean formulation is due to the fact that the Schwinger function (2·5) or (3·11) is *not hermitian* in contrast to the case of scalar fields where it is hermitian. Another way to get rid of such a trouble is the use of an *indefinite* inner product state space, as was made by Ek.[2)]

Our approach in this direction is as follows. In the quantum mechanical level, (2·6) should be replaced by the rule

$$\hat{a}_r(t)\to A_r^1(t)=(2\pi)^{-1/2}\int\nu(p)e^{-ipt}[a_r(-p)+b_r^*(p)]dp\,,$$
$$\hat{a}_r^*(t)\to A_r^2(t)=(2\pi)^{-1/2}\int\bar{\nu}(p)e^{-ipt}[a_r^*(p)+b_r(-p)]dp\,. \tag{8·1}$$

This time, though the operators $a_r^{\#}(p)$ are usual, $b_r^{\#}(p)$ are *not* ordinary CAR operators but satisfy

$$\{b_r(p),\, b_s^*(q)\}=-\delta_{rs}\delta(p-q),\quad \{b_r(p),\, b_s(q)\}=\{b_r^*(p),\, b_s^*(q)\}=0\,. \tag{8·2}$$

The appearance of the minus sign on the right-hand side enforces the introduction of an indefinite inner product state space. However, it was shown in Ref. 3) that from the viewpoint of the Grassmann-algebraic formulation of the path integral *the rule* (8·1) *is more natural than* (2·6). One more change is necessary to get an hermitian Schwinger function; *we modify the definition of the adjoint field to* $\psi^{\dagger\prime}=\psi^*\sigma_3\sigma_1$, instead of $\psi^\dagger=\psi^*\sigma_3$. Then, along the same line as in § 2, we reach "Euclidean" fields $\Psi^1(t)$ and $\Psi^{2\prime}(t)$ (the substitute of $\Psi^2(t)$), and make their linear combination to obtain

$$\Psi(t)=(\Psi^1(t)+\Psi^{2\prime *}(t))/\sqrt{2} \tag{8·3}$$

and its hermitian adjoint. It is not difficult to see that

$$\{\Psi(t),\, \Psi^*(s)\}=0 \qquad \text{for all } t,\, s \tag{8·4}$$

and

$$\langle\Psi(t)\,\Psi^*(s)\rangle_0=\chi(t,\, s)\sigma_1\,, \tag{8·5}$$

where $\chi(t, s)$ is defined in (2·6). The RHS of (8·5) is clearly hermitian. Since we do not need any linear combination of $\Psi^1(t)$ and $\Psi^{2\prime}(t)$ other than (8·3), the "number of degrees of freedom" is not increased.

An analogous construction in the field theory defines Euclidean Fermi fields in an indefinite metric state space, yielding[3)]

$$\Psi(x)=(2\pi)^{-2}\int e^{-ipx}\,\tilde{\Psi}(p)dp \tag{8·6}$$

with

$$\tilde{\Psi}(p)=\Sigma(\boldsymbol{p})W(p)((\gamma^5+\gamma^0)/\sqrt{2})[c(-p)-d^*(p)]. \tag{8·7}$$

Here $\gamma^5=\sigma_1\otimes\sigma_0$ and it plays the role of σ_1 in the quantum mechanics to define the new adjoint field. $\Sigma(\boldsymbol{p})$ and $W(p)$ are the same as above (see (3·3) and (4·7)). $c^\#(p)$ and $d^\#(p)$ are "anomalous" CAR operators such that

$$\begin{aligned}\{c_r(p),\ c_s{}^*(q)\}&=(\gamma^0)_{rs}\delta(p-q),\\ \{d_r(p),\ d_s{}^*(q)\}&=-(\gamma^0)_{rs}\delta(p-q)\end{aligned} \tag{8·8}$$

and the remaining anticommutators are all vanishing.

$\Psi(x)$ and $\Psi^*(y)$ anticommute for all x, y, and the Schwinger function in this case is given by

$$\langle\ \Psi(x)\ \Psi^*(y)\rangle_0=\frac{1}{(2\pi)^4}\int\frac{(i\hat{p}^E+m)\gamma^5}{p^2+m^2}e^{-ip(x-y)}dp\ , \tag{8·9}$$

which is hermitian. This Schwinger function is essentially the same as Ek's.[2)]

As in § 5 we can extend (8·6) to $\Phi(x)$ by putting its Fourier transform in the form with *unitary* $X(p)$ and $Y(p)$

$$\tilde{\Phi}(p)=(p^2+m^2)^{-1/4}[X(p)c(-p)+Y(p)d^*(p)]. \tag{8·10}$$

The only additional condition on $X(p)$ and $Y(p)$ required by (8·9) and the anticommutativity is that they should satisfy

$$X(p)\gamma^0X^*(p)=Y(p)\gamma^0Y^*(p)=\tau(p)\gamma^5\ , \tag{8·11}$$

which corresponds to (5·1).

Let us examine the Euclidean covariance of the field given by (8·10). Since $\gamma^5S(A,B)\gamma^5=S(A,B)$ we have

$$S(A,B)\tau(p)\gamma^5S^*(A,B)=\tau(R(A,B)p)\gamma^5\ . \tag{8·12}$$

Consider the transformation

$$\begin{aligned}c(p)&\to e^{i(R(A,B)p)a}\bar{W}(A,B|p)c(R(A,B)p),\\ d(p)&\to e^{i(R(A,B)p)a}\bar{Z}(A,B|p)d(R(A,B)p),\end{aligned} \tag{8·13}$$

where

$$\begin{aligned}W(A,B|p)&=X^T(-p)S^T(A,B)\bar{X}(-R(A,B)p),\\ Z(A,B|p)&=Y^*(p)S^*(A,B)Y(R(A,B)p).\end{aligned} \tag{8·14}$$

The condition for the transformation (8·13) to be canonical is

$$W(A,B|p)\gamma^0W^*(A,B|p)=\gamma^0\ ,$$

$$Z(A, B|p)\gamma^0 Z^*(A, B|p) = \gamma^0 . \tag{8·15}$$

This condition is assured by (8·11) and (8·12). Furthermore, since the transformation (8·13) is not of mixing type, it is implemented by a unitary operator $U(A, B; a)$.[6] Combining these results we obtain

$$U(A, B; a)\tilde{\Phi}(p)U(A, B; a)^{-1} = e^{-i(R(A,B)p)a} S^*(A, B)\tilde{\Phi}(R(A, B)p), \tag{8·16}$$

the transformation law for the Euclidean Fermi field under the inhomogeneous $SO(4)$. Finally we add that the unitary equivalence of different Euclidean fields of the same class subject to the condition (8·11) can also be proved similarly.

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