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Covariant differential calculus on the quantum hyperplane#

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Abstract

We develop a differential calculus on the quantum hyperplane covariant with respect to the action of the quantum group $GL_q(n)$.

This is a concrete example of noncommutative differential geometry. We describe the general constraints for a noncommutative differential calculus and verify that the example given here satisfies all these constraints. We also discuss briefly the integration over the quantum plane.

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1. INTRODUCTION.

It is known, through the work of Woronowicz[1], that one can define a consistent differential calculus on the noncommutative space of a quantum group. Thus quantum groups provide a concrete example of noncommutative differential geometry[2]. In this paper we give a simpler example of noncommutative differential geometry by establishing a differential calculus on the quantum (hyper-) plane. This example is simpler because the operators O_i^j defined in Section 2 map the variables linearly, while for a quantum group the map is non linear. As a consequence all formulas are very simple and can be given quite explicitly in terms of the R matrix of $GL_q(n)$.

The quantum plane is defined, according to Kobyzev and Manin[3], in terms of n variables (coordinates) x^i, $i = 1, 2, \ldots n$, which satisfy the commutation relations

$$x^i x^j = q\, x^j x^i, \qquad i<j, \qquad (1.1)$$

where q is a complex number* . One also introduces variables ξ^i which satisfy

$$\xi^i \xi^j = -\frac{1}{q}\, \xi^j \xi^i, \qquad i<j. \qquad (1.2)$$

As explained by those authors, the quantum matrices of $GL_q(n)$ can be defined as linear transformations of the variables x^i and ξ^i which preserve the commutation relations (1.1) and (1.2) (the elements of the matrices must be taken as commuting with x^i and with ξ^i).

For q=1 the x's commute and the ξ's anticommute. It is therefore natural to ask whether one can interpret the ξ's as the differentials of the x's

$$dx^i = \xi^i. \qquad (1.3)$$

To define the differential calculus it is necessary then to give the commutation relations between x's and ξ's. In Section 3 we give these commutation relations and also those of the now noncommuting derivatives with the x's, with the ξ's and with themselves. All these relations are expressed in a simple form in terms of the R matrix of $GL_q(n)$ and the entire calculus is covariant under the action of this quantum group, in the same sense as (1.1) and (1.2) are.

In Section 4 we write explicitly all relations for the two dimensional case (n=2) and explain that the calculus can be considered as a deformation of the phase space for a quantum mechanical system.

Before considering the quantum plane we describe, in Section 2, the general constraints for a noncommutative differential calculus.It is remarkable that, when the operators O_i^j map linearly the variables, the Yang-Baxter braid equation emerges naturally from the requirement of consistency.

It is possible to define pseudodifferential operators, which are functions of the variables and of the derivatives of the quantum plane and which act on the plane like the infinitesimal transformations of $GL_q(n)$. These operators satisfy the commutation relations of the deformed Lie algebra. We shall describe these results in a forthcoming publication.

In this paper we deal mainly with the real quantum plane, but the analysis can be generalized to the complex quantum plane, in which case one must give also the commutation relations between the variables and their conjugates. The matrices of the complex quantum group are required to preserve the quantum structure of the complex plane, in the sense explained above for the real

* Manin uses $\frac{1}{q}$ where we use q, following the usage of the Leningrad school [4].

plane. This gives, for general n, the commutation relations between the elements of the matrix and their complex conjugates, a generalization of a result given by Woronowicz and Podleś[5] for $SL_q(2, C)$. These results will also be described in a forthcoming publication.

It is a pleasure to dedicate this paper to Raymond Stora on the occasion of his 60th birthday. This is especially appropriate because of his help during the development of our work through discussions, advice and even computations.

2. NONCOMMUTATIVE DIFFERENTIAL CALCULUS.

Consider variables x^i, for $i = 1, 2, ...n$, belonging to an associative algebra, which satisfy the commutation relations

$$r^{ij}(x) = x^i x^j - B^{ij}{}_{kl}\, x^k x^l = 0. \tag{2.1}$$

We assume that these commutation relations are sufficient to order in some standard way an arbitrary monomial (a product of the basic variables elevated to arbitrary powers). Functions of the variables x^i are defined as formal power series. We wish to define an exterior differential d satisfying the usual properties such as

$$d^2 = 0 \tag{2.2}$$

and the Leibniz rule

$$d(fg) = (df)g + f\, dg. \tag{2.3}$$

In general the differentials of the basic variables

$$\xi^i = dx^i \tag{2.4}$$

will not commute with the variables or with functions of them. We introduce linear operators $O_k{}^j$ which satisfy

$$f(x)\, \xi^j = \xi^k\, O_k{}^j\, f(x) \tag{2.5}$$

(sum over repeated indices). These operators define the differential calculus and must satisfy certain consistency conditions. Applying (2.5) to the product of two functions f(x) and g(x)

$$\begin{aligned} f\, g\, \xi^j &= f\, \xi^k\, O_k{}^j\; g = \xi^i\, (O_i{}^k f)\, (O_k{}^j g) \\ &= \xi^i\, O_i{}^j\, (fg), \end{aligned} \tag{2.6}$$

we see that it must be

$$O_i{}^j\, (fg) = (O_i{}^k f)\, (O_k{}^j g). \tag{2.7}$$

This formula, plus the linearity, shows that it is sufficient to know the action of the operators $O_i{}^j$ on the basic variables. The action on any function, defined as a formal power series, is then determined. Applied to the basic variables (2.5) gives their commutation relations with the differentials

$$x^i\, \xi^j = \xi^k\, O_k{}^j\, x^i. \tag{2.8}$$

One can introduce derivatives

$$\partial_i \equiv \frac{\partial}{\partial x^i}\ , \quad \partial_i\, x^j = \delta^j_i \tag{2.9}$$

in the standard way through

$$d = \xi^i\, \partial_i\,. \tag{2.10}$$

In general the derivatives do not satisfy the simple Leibniz rule of commutative algebra. Indeed

$$\begin{aligned} d(fg) &= (\xi^i \partial_i\; f)\, g + f\, \xi^i \partial_i\; g \\ &= (\xi^i \partial_i\; f)\, g + \xi^k\, (O_k{}^i f)\, \partial_i\; g = \xi^k\, \partial_k\, (fg). \end{aligned} \tag{2.11}$$

Therefore

$$\partial_k (fg) = (\partial_k f)\, g + (O_k{}^i f)\, \partial_i g. \qquad (2.12)$$

From the associativity of the algebra of functions

$$\partial_i (f(gh)) = \partial_i ((fg)h) \qquad (2.13)$$

we obtain again (2.7).

As already mentioned, the operators $O_i{}^j$ must satisfy certain consistency conditions. If one differentiates the left hand side of the basic relations (2.1), one must require that the result vanish, at least as a consequence of the basic relations themselves. We shall call an equality, which is valid in virtue of the basic relations, a weak equality and employ the symbol $\cong$. Using

$$\partial_m (x^k x^l) = \delta^k{}_m\, x^l + O_m{}^l\, x^k \qquad (2.14)$$

we find

$$\partial_m r^{ij} = (\delta^i{}_k\, \delta^j{}_l - B^{ij}{}_{kl})\, (\delta^k{}_m\, x^l + O_m{}^l\, x^k) \cong 0. \qquad (2.15)$$

This is a consistency condition for the operators $O_i{}^j$, which we call the "linear condition".

The basic relations are valid also when multiplied by a function

$$r^{ij}(x)\, f(x) = 0 \qquad (2.16)$$

Differentiating this we see that it must be

$$(\partial_m r^{ij})\, f + (O_m{}^n\, r^{ij})\, \partial_n f = 0 \qquad (2.17)$$

In virtue of the linear relation the first term vanishes. Since the function f is arbitrary, it must be

$$O_m{}^n\, r^{ij} = (\delta^i{}_k\, \delta^j{}_l - B^{ij}{}_{kl})\, (O_m{}^p\, x^k)\, (O_p{}^n\, x^l) \cong 0. \qquad (2.18)$$

We shall call this the "quadratic consistency condition" for the operators $O_i{}^j$. In terms of differentials the linear condition follows simply from

$$dr^{ij}(x) \cong 0, \qquad (2.19)$$

while the quadratic consistency condition comes from

$$r^{ij}\, \xi^n = (\xi^m\, O_m{}^n\, r^{ij}) \cong 0. \qquad (2.20)$$

Quantum groups provide an example of noncommutative differential calculus. The variables x^i are the quantized group parameters. In this case the expressions $O_i{}^j\, x^k$ are non linear functions of the variables x and it is convenient, following Woronowicz[1], to work with the right (or left) invariant differential forms which satisfy simpler commutation relations with the group parameters than the differentials of the parameters.

Another example is the quantum (hyper-) plane, which we describe in the next section, where the expressions $O_i{}^j\, x^k$ are linear in the variables. With that example in mind we shall assume for the rest of this section that

$$O_i{}^j\, x^k = C^{kj}{}_{il}\, x^l \qquad (2.21)$$

are linear functions of the variables, the C's being suitable numerical coefficients.

Using the standard tensor product notation, one can then write the linear condition (2.15) as

$$(E_{12} - B_{12})\, (E_{12} + C_{12}) = 0, \qquad (2.22)$$

where E is the unit matrix. This is now a strong relation. The quadratic condition becomes

$$(E_{12} - B_{12})\, C_{23}\, C_{12}\, x_2 x_3 \equiv 0. \qquad (2.23)$$

Using (2.1), in the form

$$x_2 x_3 - B_{23}\, x_2 x_3 \equiv 0, \qquad (2.24)$$

this can be written as

$$(B_{12}\, C_{23}\, C_{12} - C_{23}\, C_{12}\, B_{23})\, x_2 x_3 \equiv 0. \qquad (2.25)$$

This equation is satisfied if the Yang-Baxter equation

$$B_{12}\, C_{23}\, C_{12} = C_{23}\, C_{12}\, B_{23} \qquad (2.26)$$

holds.

We have the commutation relations (2.1) among the variables and those between variables and differentials, given by (2.8) or

$$x^i \xi^j = C^{ij}{}_{kl}\, \xi^k x^l. \qquad (2.27)$$

The Leibniz rule (2.3) gives the commutation relations between the variables and the derivatives, considered as operators,

$$\partial_j x^i = \delta^i{}_j + C^{ik}{}_{jl}\, x^l \partial_k. \qquad (2.28)$$

Can one complete the algebra by giving commutation relations between derivatives and differentials ? We try a relation of the form

$$\partial_j \xi^i - D^{ik}{}_{jl}\, \xi^l \partial_k = 0, \qquad (2.29)$$

where the tensor D is to be determined.

If we multiply the left hand side by x^r from the right, we can commute x^r through to the left, using (2.28) and the inverse of (2.27), i.e.

$$\xi^k x^l = (C^{-1})^{kl}{}_{ij}\, x^i \xi^j. \qquad (2.30)$$

One finds terms linear in ξ which must cancel separately. This requires $D = C^{-1}$, i.e.

$$D^{ij}{}_{kl}\, C^{kl}{}_{rs} = C^{ij}{}_{kl}\, D^{kl}{}_{rs} = \delta^i{}_r\, \delta^j{}_s. \qquad (2.31)$$

With this choice for D one finds, after some simplifications

$$(\partial_j \xi^i - D^{ik}{}_{jl}\, \xi^l \partial_k)\, x^r =$$
$$D^{ir}{}_{st}\, C^{su}{}_{jv}\, x^v (\partial_u \xi^t - D^{tm}{}_{un}\, \xi^n \partial_m). \qquad (2.32)$$

So far our new relation (2.29) appears to be consistent with our previous relations.

One must still perform various consistency checks. For instance, multiply (2.27) from the left with ∂_l and commute this derivative through to the right using (2.28) and (2.29). With a little algebra one can see that this requires the Yang-Baxter equation

$$C_{12}\, C_{23}\, C_{12} = C_{23}\, C_{12}\, C_{23}. \qquad (2.33)$$

Differentiating (2.27) we find the commutation relation among the differentials

$$\xi^i \xi^j = - C^{ij}{}_{kl}\, \xi^k \xi^l \qquad (2.34)$$

i.e.

$$(E_{12} + C_{12})\, \xi_1 \xi_2 = 0. \qquad (2.35)$$

Finally we may ask for the commutation relations among the derivatives. We try

$$\partial_i \partial_j - F^{lk}{}_{ji}\, \partial_k \partial_l = 0 \qquad (2.36)$$

i.e.

$$\partial_2 \partial_1 - \partial_2 \partial_1\, F_{12} = 0. \qquad (2.37)$$

Multiplying this equation from the right by x^r and commuting x^r through to the left we find terms linear in the derivatives which must cancel separately. This requires

$$(E_{12} + C_{12})(E_{12} - F_{12}) = 0. \qquad (2.38)$$

It is not hard to see that the remaining terms will also cancel if

$$C_{12}\, C_{23}\, F_{12} = F_{23}\, C_{12}\, C_{23}. \qquad (2.39)$$

3. CALCULUS ON THE QUANTUM PLANE.

The quantum (hyper-)plane provides a simple example in which all consistency conditions necessary for a differential calculus are satisfied. The variables satisfy the commutation relations[3]

$$x^i x^j = q\, x^j x^i, \qquad i < j. \qquad (3.1)$$

These relations can be cast in the form (2.1) by means of the R-matrix for the quantum group $GL_q(n)$, which is[4]

$$R^{ij}{}_{kl} = \delta^i{}_k\, \delta^j{}_l\, (1 + (q-1)\, \delta^{ij}) + (q - \frac{1}{q})\, \delta^j{}_k\, \delta^i{}_l\, \theta(i-j), \qquad (3.2)$$

where

$$\theta(i-j) = \begin{cases} 1 & i>j \\ 0 & i\leq j \end{cases} \qquad (3.3)$$

It is more convenient to work with

$$\hat{R} = PR \qquad (3.4)$$

where P is the permutation matrix. Explicitly

$$\hat{R}^{ij}{}_{kl} = R^{ji}{}_{kl} \qquad (3.5)$$

It is easy to check that (3.1) is equivalent to

$$(E_{12} - \frac{1}{q}\hat{R}_{12})\, x_1\, x_2 = 0. \qquad (3.6)$$

The matrix $\hat{R}$ is symmetric

$$\hat{R}^{ij}{}_{kl} = \hat{R}^{kl}{}_{ij} \qquad (3.7)$$

and satisfies the equation (E is again the unit matrix)

$$(E - \frac{1}{q}\hat{R})(E + q\hat{R}) = 0, \qquad (3.8)$$

which shows that its eigenvalues are q and $-\frac{1}{q}$. Equivalently, one can write

$$\hat{R}^2 = E + (q - \frac{1}{q})\hat{R} \qquad (3.9)$$

or

$$\hat{R}^{-1} = \hat{R} + (\frac{1}{q} - q)\, E. \qquad (3.10).$$

The matrices

$$\mathcal{A} = E - \frac{1}{q}\hat{R} \qquad (3.11)$$

and

$$\mathcal{S} = E + q\hat{R} \qquad (3.12)$$

are orthogonal projections (relative to the eigenvalues $-\frac{1}{q}$ and q respectively)

$$\mathcal{A}\mathcal{S} = \mathcal{S}\mathcal{A} = 0 \qquad (3.13)$$

$$\mathcal{A}^2 = (1 + \frac{1}{q^2})\, \mathcal{A}, \qquad (3.14)$$

$$\mathcal{S}^2 = (1 + q^2)\, \mathcal{S}, \qquad (3.15)$$

which could be easily normalized† . Finally we recall that, in terms of $\hat{R}$, the Yang-Baxter equation takes the form

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}. \qquad (3.16)$$

It is now obvious that, if we take

$$B = F = \frac{1}{q}\hat{R}, \quad C = q\hat{R}, \qquad (3.17)$$

all consistency conditions described in the previous section are satisfied. In particular, the Yang-Baxter equations (2.26), (2.33) and (2.39) reduce to the single equation (3.16). If one computes the commutation relations for the operation d one finds, as expected,

$$dx - xd = \xi \qquad (3.18)$$

and

$$d\xi + \xi d = 0, \qquad (3.19)$$

but

$$d\,\partial_i = q^2\,\partial_i\,d. \qquad (3.20)$$

This perhaps unexpected result is, however, perfectly consistent. For instance

$$\begin{aligned} d^2 &= d\,\xi^i\,\partial_i = -\xi^i\,d\,\partial_i \\ &= -q^2\,\xi^i\,\partial_i\,d = -q^2 d^2 \end{aligned} \qquad (3.21)$$

which is consistent with (2.2).

† The projections $\mathcal{A}$ and $\mathcal{S}$ are the quantum analogues of the classical antisymmetrizer and symmetrizer for tensors with two indices. For the two-dimensional case, n=2, Michael Schlieker and Markus Scholl have developed a complete quantum tensor calculus in which these projections play a crucial role[6].

There is a different choice which also satisfies all consistency conditions. Just as above, we take

$$B = F = \frac{1}{q}\hat{R}, \qquad (3.22)$$

but now

$$C = \frac{1}{q}\hat{R}^{-1}. \qquad (3.23)$$

The existence of this second solution is not surprising. The equations

$$(E_{12} - \frac{1}{q}\hat{R}_{12}(q))\,x_1\,x_2 = 0 \qquad (3.24)$$

and

$$(E_{12} + q\,\hat{R}_{12}(q))\,\xi_1\,\xi_2 = 0 \qquad (3.25)$$

go into themselves by the exchange of x_1 with x_2 and q with $\frac{1}{q}$, in virtue of the identity

$$\hat{R}_{12}(\tfrac{1}{q}) = \hat{R}_{21}{}^{-1}(q). \qquad (3.26)$$

Here we have introduced explicitly the dependence of the matrix $\hat{R}(q)$ upon the quantum parameter. The "intermediary" equation

$$x_1\,\xi_2 - q\,\hat{R}_{12}(q)\,\xi_1\,x_2 = 0 \qquad (3.27)$$

goes into

$$x_2\,\xi_1 - \frac{1}{q}\hat{R}_{21}{}^{-1}(q)\,\xi_2\,x_1 = 0 \qquad (3.28)$$

i.e.

$$\xi_2\,x_1 - q\,\hat{R}_{21}(q)\,x_2\,\xi_1 = 0. \qquad (3.29)$$

In the following we shall work with the first solution (3.17).

One can see immediately that all relations of the differential calculus on the quantum plane described above are covariant under the action of a linear transformation

$$x \to A\,x, \quad \xi \to A\xi, \tag{3.30}$$

or in components

$$x^i = a^i{}_j\, x^j, \quad \xi^i \to a^i{}_j\, \xi^j, \tag{3.31}$$

where the matrix elements $a^i{}_j$ of the matrix A are assumed to commute with the variables x^i and with ξ^i, provided they satisfy

$$A_1 A_2 \hat{R}_{12} = \hat{R}_{12} A_1 A_2. \tag{3.32}$$

This is just the condition for the matrices to belong to the quantum group $GL_q(n)$. The calculus is invariant under the action of the quantum group. The Yang-Baxter equation for $\hat{R}$, necessary for the quantum group property, emerges also as a consistency condition for the calculus.

Let us now consider the real quantum plane

$$\overline{x^i} = x^i. \tag{3.33}$$

Since

$$\overline{x^i x^j} = \overline{x^j}\ \overline{x^i} = x^j\, x^i, \tag{3.34}$$

it must be

$$\bar{q} = \frac{1}{q}. \tag{3.35}$$

In this case (3.26) can be written as

$$\overline{\hat{R}_{12}} = \hat{R}_{21}{}^{-1} \tag{3.36}$$

We may choose the differentials ξ^i to be real

$$\overline{\xi^i} = \xi^i, \tag{3.37}$$

in which case d is imaginary. One can check that the entire scheme goes into itself under complex conjugation provided one also takes

$$\bar{\partial}_i = -\,q^{2(n-i+1)}\,\partial_i, \quad i=1, 2, \ldots n. \tag{3.38}$$

For this computation, which we shall not reproduce here, one must invert some of the commutation relations. This is easily done if one observes that the matrix X which satisfies

$$X^{ri}{}_{sj}\,(\hat{R}^{-1})^{jk}{}_{il} = (\hat{R}^{-1})^{ri}{}_{sj}\,X^{jk}{}_{il} = \delta^r{}_l\,\delta^k{}_s \tag{3.39}$$

is given by

$$X^{ri}{}_{sj} = (\hat{R})^{ir}{}_{js}\,q^{2(r-j)} = (\hat{R})^{ir}{}_{js}\,q^{2(s-i)}. \tag{3.40}$$

Furthermore (sum over repeated indices)

$$X^{ik}{}_{jk} = \delta^i{}_j\, q\, q^{2(i-1)} \tag{3.41}$$

and

$$X^{ki}{}_{kj} = \delta^i{}_j\, q\, q^{2(n-i)}. \tag{3.42}$$

The complex conjugation defined above is an involution, its square is the identity. For the real quantum plane the matrix elements $a^i{}_j$ in (3.31) must, of course be real. Finally, we observe that

$$\overline{q^{n-i+1}\,\partial_i} = -\,q^{n-i+1}\,\partial_i \tag{3.43}$$

are pure imaginary.

4. THE TWO-DIMENSIONAL CASE

In this section, as an illustration, we give explicitly the commutation relations of the differential calculus on the two-dimensional quantum plane. They are obtained immediately from the formulas of the previous section. The R matrix is

$$R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{4.1}$$

and the matrix $\hat{R} = PR$ is

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q-q^{-1} & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}. \tag{4.2}$$

We denote the variables x and y, the differentials

$$dx = \xi\,, \quad dy = \eta \tag{4.3}$$

and the derivatives

$$\frac{\partial}{\partial x} = \partial_x\,, \quad \frac{\partial}{\partial y} = \partial_y. \tag{4.4}$$

In addition to the well known relations

$$xy = qyx \tag{4.5}$$

and

$$\xi\eta = -\frac{1}{q}\eta\xi \tag{4.6}$$

we have the commutation relation between the derivatives

$$\partial_x \partial_y = \frac{1}{q}\partial_y \partial_x, \tag{4.7}$$

the relations between variables and differentials *

$$\begin{aligned} x\,\xi &= q^2\,\xi\,x \\ x\,\eta &= q\,\eta\,x + (q^2-1)\,\xi\,y \\ y\,\xi &= q\,\xi\,y \\ y\,\eta &= q^2\,\eta\,y \end{aligned} \tag{4.8}$$

and those between variables and derivatives.

$$\begin{aligned} \partial_x\,x &= 1 + q^2\,x\,\partial_x + (q^2-1)\,y\,\partial_y \\ \partial_x\,y &= q\,y\,\partial_x \\ \partial_y\,x &= q\,x\,\partial_y \\ \partial_y\,y &= 1 + q^2\,y\,\partial_y. \end{aligned} \tag{4.9}$$

To complete the scheme we give the relations between derivatives and differentials.

$$\begin{aligned} \partial_x\,\xi &= \frac{1}{q^2}\,\xi\,\partial_x \\ \partial_x\,\eta &= \frac{1}{q}\eta\,\partial_x \\ \partial_y\,\xi &= \frac{1}{q}\xi\,\partial_y \\ \partial_y\,\eta &= \frac{1}{q^2}\eta\,\partial_y + (\frac{1}{q^2}-1)\,\xi\partial_x. \end{aligned} \tag{4.10}$$

The exterior differential

* These explicit equations (4.8) for the two-dimensional plane were derived first, from the requirements of consistency and covariance under the action of the quantum group $GL_q(2)$. The observation that they can be written in the form $x_2\,\xi_1 = q\,R_{12}\,\xi_1\,x_2$ is due to Arne Schirrmacher.

$$d = \xi \partial_x + \eta \partial_y \tag{4.11}$$

satisfies all usual relations, such as (2.2) and (2.3). However, as in the general case

$$d\partial_x = q^2 \partial_x d \quad , \quad d\partial_y = q^2 \partial_y d. \tag{4.12}$$

All above relations correspond to the choice (3.17). Had we taken (3.23) some of the relations would change. For instance, instead of (4.8) one would have

$$\begin{aligned} \xi x &= q^2 x \xi \\ \xi y &= qy\xi + (q^2-1) x\eta \\ \eta x &= qx\eta \\ \eta y &= q^2 y\eta. \end{aligned} \tag{4.13}$$

The relation between these two solutions has been explained in the previous section. For the rest of this section we shall work with the first solution (4.10).

Using the commutation relation (4.5) one can order monomials so that the variable y is before the variable x, for instance. One can then compute the derivative of an ordered monomial, using (4.9), and reorder the result. One finds

$$\begin{aligned} \partial_y(y^n x^m) &= y^{n-1} x^m \frac{1-q^{2n}}{1-q^2} \\ \partial_x(y^n x^m) &= y^n x^{m-1} q^n \frac{1-q^{2m}}{1-q^2} . \end{aligned} \tag{4.14}$$

By linearity, these formulas give the derivatives of ordered formal power series. Viceversa, one could take this as the definition of the derivatives. It is easy to see that the relations (4.9), (4.10) and then (4.8) follow. In this way one sees very clearly that the calculus developed above cannot lead to inconsistencies. A similar argument can be given for the higher dimensional case of the previous section.

It is known that the relations (4.5) and (4.6) are preserved under the action of a $GL_q(2)$ quantum matrix whose elements commute with x, y, ξ and η and, viceversa, that this requirement defines a quantum matrix. The other commutation relations given above are also preserved, the calculus is "invariant" under the action of the quantum group.

For the real quantum plane x, y, ξ and η are real, $|q| = 1$ and

$$\overline{\partial_x} = -q^4 \partial_x \quad , \quad \overline{\partial_y} = -q^2 \partial_y. \tag{4.15}$$

This conjugation operation changes the entire scheme into itself and is an involution. We notice that $q^2\partial_x$ and $q\partial_y$ are pure imaginary. If we define

$$p_x = -i\hbar q^2 \partial_x \quad , \quad p_y = -i\hbar q \partial_y, \tag{4.16}$$

the quantities x, y, p_x and p_y are real and provide a one parameter deformation of the quantum mechanical phase space for a two-dimensional system (a two-parameter deformation of the classical phase space).

It is possible to define an integral over the real quantum plane which satisfies the quantum Stokes' theorem

$$\int d\,\omega_1 = 0, \tag{4.17}$$

where ω_1 is a one-form which satisfies certain regularity conditions. Since a two-form can always be written as

$$\omega_2 = \xi\eta\, f(x,y), \tag{4.18}$$

one can therefore define an integral for functions of x and y. We use for the integral of functions the notation $<f>$. Stokes' theorem (4.17) implies that

$$<\partial_x f> = <\partial_y f> = 0, \qquad (4.19)$$

where ∂_x and ∂_y are the quantum derivatives.

The integral is essentially defined by (4.19), plus standard properties such as linearity etc.. Up to a q-dependent factor, it turns out to be equal to the integral of the classical function associated with the quantum function by ordering it, for instance by moving the variable y to the left of the variable x. Therefore the conditions for the existence of the quantum integral are the same as those for the existence of the associated classical integral.

The q-dependent factor is the same for all integrands and can be chosen so that

$$\overline{<f>} = <\bar{f}>. \qquad (4.20)$$

One can then verify that

$$<\bar{f} f> \geq 0 \qquad (4.21)$$

and vanishes if and only if

$$f \equiv 0. \qquad (4.22)$$

The Hermitean inner product

$$<f \mid g> = <\bar{f} g> \qquad (4.23)$$

makes the space of quantum functions into a quantum Hilbert space. Formal Hermitean conjugation is defined as usual

$$<f \mid Tg> = <T^{\dagger} f \mid g>. \qquad (4.24)$$

One can verify that, for the operators x, y, ∂_x and ∂_y (or p_x and p_y), and for functions of them, Hermitean conjugation coincides with the involution defined above for the algebra in (4.15) and (4.16). Questions of convergence and domains for the quantum Hilbert space can be reduced to analogous questions for the classical Hilbert space of square integrable functions.

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