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Covariant Quantization of the Electromagnetic Field in the Landau Gauge*)

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A new covariant quantization of the free electromagnetic field is proposed corresponding to the fact that a massless vector bound state in the Bethe-Salpeter formalism is accompanied by a dipole ghost. The Lorentz condition is formulated as an operator identity.

§1. Introduction

As is well known, a covariant quantization of the electromagnetic field was formulated by Gupta and Bleuler.[1] Since the Lorentz condition is not consistent with the canonical commutation relation, the former is regarded as a supplementary condition which holds only for certain "physical states". Correspondingly, the photon propagator does not satisfy the Lorentz condition.

It is often convenient theoretically to use the photon propagator in the Landau gauge, which satisfies the Lorentz condition. It has been customary, however, that the Landau gauge is introduced *ad hoc* after the quantization. The difficulty of quantizing the electromagnetic field in the Landau gauge consists in the fact that if the Lorentz condition is regarded as an operator identity, then the corresponding commutation relation is necessarily inconsistent with d'Alembert equation of the free electromagnetic field. In the quantization of Rohrlich and Strocchi[2] in the Landau gauge, this difficulty is camouflaged by using a product of distributions which are not associative. On the other hand, Just[3] has recently proposed a theory in which only the interacting field can be covariantly quantized by means of Lehmann's spectral representation. He has forbidden one to use the interaction representation in which the quantum electrodynamics has been most successful.

The purpose of the present paper is to propose a consistent, covariant quantization of the free electromagnetic field in the Landau gauge. In our previous work[4] on the Bethe-Salpeter equation for equal-mass particles, we have found that a massless vector bound state is accompanied by a dipole ghost[5] instead of

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a ghost, namely the corresponding Green's function then has a double pole at zero energy. Since we have not introduced any artificial assumption in the Bethe-Salpeter formalism, the above result suggests that it will be natural to use a dipole ghost in the quantization of a massless vector field.

As we remarked above, when the Lorentz condition is regarded as an operator identity, $\Box A_\mu$ cannot vanish identically, where A_μ is the free electromagnetic field. But we can easily see that the commutation relation can be consistent with an ansatz

$$\Box A_\mu = \eta \partial_\mu B, \quad \Box B = 0, \tag{1·1}$$

where B is an auxiliary scalar field, η being a constant.

As will be seen later, B is related to the residue of a double pole of the Green's function. The longitudinal photon becomes a dipole ghost accompanied by B, and the scalar photon is reduced to the longitudinal one because of the Lorentz condition. The d'Alembert equation holds only in the sense of the expectation value for "physical states":

$$\langle \Box A_\mu \rangle_{\text{phys}} = 0, \tag{1·2}$$

where physical states are the states which contain no dipole ghosts.

Our method of quantization may also be applied to the weak gravitational field by using a "tripole" ghost.

§2. Quantization

We use the following metric tensor:

$$g_{00} = -g_{jj} = 1, \quad (j=1, 2, 3),$$
$$g_{\mu\nu} = 0 \quad \text{for } \mu \neq \nu. \tag{2·1}$$

We start from the following Lagrangian density:

$$L_0 = \frac{1}{4} (\partial^\mu A^\nu - \partial^\nu A^\mu)(\partial_\mu A_\nu - \partial_\nu A_\mu)$$
$$- \eta B \partial^\mu A_\mu + \frac{1}{2} (\partial^\mu B)(\partial_\mu B), \tag{2·2}$$

where $\eta \neq 0$ is an arbitrary real constant having the dimension of mass. It is evident that L_0 is invariant under the gauge transformation

$$A_\mu \to A_\mu + \partial_\mu \Lambda,$$
$$B \to B$$
$$\Box \Lambda = 0, \quad (\Box \equiv \partial^\mu \partial_\mu), \tag{2·3}$$

where Λ may be an operator.

The equations of motion follow from (2·2):

$$-\partial^\mu(\partial_\mu A_\nu-\partial_\nu A_\mu-\eta g_{\mu\nu}B)=0, \tag{2·4}$$

$$-\eta\partial^\mu A_\mu-\partial^\mu\partial_\mu B=0. \tag{2·5}$$

Operating ∂^ν on (2·4), we have

$$\Box B=0. \tag{2·6}$$

Then (2·5) reduces to

$$\partial^\mu A_\mu=0. \tag{2·7}$$

Substituting (2·7) in (2·4), we find

$$\Box A_\mu=\eta\partial_\mu B. \tag{2·8}$$

Our fundamental equations are (2·6), (2·7) and (2·8). From (2·6) and (2·8), we see

$$\Box^2 A_\mu=0. \tag{2·9}$$

The commutation relation of A_μ must be consistent with (2·7) and (2·9). It is uniquely (up to a coefficient) determined as follows:

$$[A_\mu(x), A_\nu(x')]=i\widetilde{D}_{\mu\nu}(x-x'), \tag{2·10}$$

where

$$\widetilde{D}_{\mu\nu}(x)\equiv i(2\pi)^{-3}\int d^4k\varepsilon(k_0)\,[g_{\mu\nu}\delta(k^2)+k_\mu k_\nu\delta'(k^2)]\,e^{-ikx} \tag{2·11}$$

with $k^2\equiv k^\mu k_\mu$. The integral of the first term is well known. The integration in the second term can also be easily carried out. We then obtain

$$\widetilde{D}_{\mu\nu}(x)=(1/2\pi)\varepsilon(x_0)\left[g_{\mu\nu}\delta(x^2)-\frac{1}{4}\partial_\mu\partial_\nu\theta(x^2)\right]. \tag{2·12}$$

The second term of (2·12) may be determined also by the requirement

$$\partial^\mu\widetilde{D}_{\mu\nu}(x)=0. \tag{2·13}$$

From (2·10) with (2·11) we have

$$\begin{aligned}[\Box A_\mu(x), A_\nu(x')]&=-(2\pi)^{-3}\int d^4k\,\varepsilon(k_0)k_\mu k_\nu\delta(k^2)e^{-ik(x-x')}\\&=i\partial_\mu\partial_\nu D(x-x'),\end{aligned} \tag{2·14}$$

where

$$D(x)\equiv-(1/2\pi)\varepsilon(x_0)\delta(x^2). \tag{2·15}$$

Therefore, (2·8) leads us to

$$[\partial_\mu B(x), A_\nu(x')]=i\eta^{-1}\partial_\mu\partial_\nu D(x-x'), \tag{2·16}$$

$$[\partial_\mu B(x), \partial_\nu' B(x')] = 0. \tag{2·17}$$

Integrating (2·16) and (2·17), we obtain

$$[A_\mu(x), B(x')] = -i\eta^{-1}\partial_\mu D(x-x'), \tag{2·18}$$

$$[B(x), B(x')] = 0. \tag{2·19}$$

We can translate the commutation relations (2·10), (2·18) and (2·19) into those in the momentum space. Let*)

$$A_\mu(x) = (2\pi)^{-3/2}\int d^4k\, \theta(k_0)\, [a_\mu(k)e^{-ikx} + a_\mu^*(k)e^{ikx}],$$

$$B(x) = (2\pi)^{-3/2}\int d^4k\, \theta(k_0)\, [b(k)e^{-ikx} + b^*(k)e^{ikx}]. \tag{2·20}$$

Then our equations of motion are rewritten as

$$k^2 b(k) = k^2 b^*(k) = 0, \tag{2·21}$$

$$k^\mu a_\mu(k) = k^\mu a_\mu^*(k) = 0, \tag{2·22}$$

$$k^2 a_\mu(k) = i\eta k_\mu b(k),$$

$$k^2 a_\mu^*(k) = -i\eta k_\mu b^*(k). \tag{2·23}$$

Our commutation relations become

$$[a_\mu(k), a_\nu(k')] = 0, \tag{2·24}$$

$$[a_\mu(k), a_\nu^*(k')] = -\delta^4(k-k')\,[g_{\mu\nu}\delta(k^2) + k_\mu k_\nu \delta'(k^2)], \tag{2·25}$$

$$[a_\mu(k), b(k')] = 0, \tag{2·26}$$

$$[a_\mu(k), b^*(k')] = i\eta^{-1}\delta^4(k-k')k_\mu\delta(k^2), \tag{2·27}$$

$$[b(k), b(k')] = 0, \tag{2·28}$$

$$[b(k), b^*(k')] = 0. \tag{2·29}$$

As is seen from (2·18) or (2·27), two free fields A_μ and B do not commute with each other. This fact is closely related to the existence of a dipole ghost as was shown in Froissart's model.[6] We note that a similar formulation was used also by Fujii and Kamefuchi[7] for a different purpose.

§3. States

As usual, we define the vacuum Ω by

$$a_\mu(k)\Omega = 0, \quad b(k)\Omega = 0 \tag{3·1}$$

*) We understand that $a_\mu(k) = 0$ and $b(k) = 0$ if k does not lie on (or infinitesimally near) the light cone. Hence $\theta(k_0)$ is Lorentz-invariant.

for all k_μ with $k_0>0$. The norm of Ω is normalized to unity, namely

$$\Omega^*\Omega=1. \tag{3·2}$$

Next, we consider one-particle states. For a given k_μ, there always exist four vectors $e^{(\alpha)}{}_\mu$, $(\alpha=0,1,2,3)$, such that

$$e^{(\alpha)\mu}e_\mu^{(\beta)}=g_{\alpha\beta},\ \sum_\alpha e_\mu^{(\alpha)}e_\nu^{(\alpha)}=g_{\mu\nu}, \tag{3·3}$$

$$e^{(1)\mu}k_\mu=e_\mu^{(2)}k_\mu=0. \tag{3·4}$$

We write $k_\alpha\equiv e^{(\alpha)\mu}k_\mu$ and $a_\alpha(k)\equiv e^{(\alpha)\mu}a_\mu(k)$ for simplicity of notation. Then (3·4) is rewritten as

$$k_1=k_2=0. \tag{3·5}$$

Let

$$\Psi_\alpha(k)\equiv a_\alpha{}^*(k)\,\Omega,$$

$$\Phi(k)\equiv b^*(k)\,\Omega. \tag{3·6}$$

The Lorentz condition $k^\mu a_\mu{}^*(k)=0$ reduces to

$$k_3 a_3{}^*(k)=k_0 a_0{}^*(k). \tag{3·7}$$

Since $k_0>0$, $a_0{}^*(k)$ is not independent of $a_3{}^*(k)$. Hence we have only to consider four one-particle states $\Psi_j(k)$, $(j=1,2,3)$, and $\Phi(k)$. Their norms are

$$\Psi_1{}^*(k')\Psi_1(k)=\Psi_2{}^*(k')\Psi_2(k)=\delta^4(k-k')\delta(k^2),$$

$$\Psi_3{}^*(k')\Psi_3(k)=\delta^4(k-k')\,[\delta(k^2)-(k_3)^2\delta'(k^2)],$$

$$\Phi^*(k)\Phi(k)=0, \tag{3·8}$$

on account of (2·25), (2·29) and (3·1). Of course, (3·8) should be understood in the following way. Let a wave-packet state be

$$\Psi_1(\varphi)\equiv\int d^4k\varphi(k)\Psi_1(k). \tag{3·9}$$

Then

$$\Psi_1{}^*(\varphi)\Psi_1(\varphi)=\int d^4k\delta(k^2)\,|\varphi(k)|^2. \tag{3·10}$$

Thus $\Psi_1(k)$ and $\Psi_2(k)$ are positive-norm states corresponding to transverse photons, while $\Phi(k)$ is a zero-norm state. The sign of the norm of $\Psi_3(k)$ is not definite. Furthermore, (2·23) and (2·21) yield

$$k^2\Psi_1(k)=k^2\Psi_2(k)=k^2\Phi(k)=0,$$

$$k^2\Psi_3(k)=-i\eta k_3\Phi(k). \tag{3·11}$$

We can, therefore, regard $\Psi_3(k)$ as a dipole ghost and $\Phi(k)$ as the state which is related to the residue of a double pole of the Green's function. In other words, $\Phi(k)$ and $\Psi_3(k)$ correspond to Φ_0 and $\Phi_{\rm Dip}$, respectively, in Heisenberg's notation.[5] The result obtained here is quite analogous to that in the Bethe-Salpeter formalism for a massless vector bound state.[4]

The free electromagnetic field has to satisfy the d'Alembert equation at least in the sense of the expectation value. For this purpose, we should exclude the dipole ghost $\Psi_3(k)$ as an unphysical state. Let $M_n{}^*$ be an arbitrary monomial of creation operators which contains no dipole-ghost creation operator $a_3{}^*(k)$ [or $a_0{}^*(k)$]. Then we call a state of the form

$$\Psi \equiv \alpha\Omega + \sum_n \alpha_n M_n{}^* \Omega \tag{3·12}$$

a "physical state", where α and α_n are arbitrary complex numbers, and the summation $\sum_n$ generally contain integrations over four-momenta. In other words, a physical state is a state which can be constructed from Ω without using $a_3{}^*(k)$ [or $a_0{}^*(k)$]. We call the totality of the physical states (after completion) "Hilbert Space I". The norm in it is positive semi-definite. For a physical state Ψ, it is not difficult to show

$$\Psi^* \Box A_\mu(x) \Psi = 0. \tag{3·13}$$

We postulate that when the interaction is switched off the state must belong to Hilbert Space I. We have thus constructed a covariant quantum theory of the free electromagnetic field in the Landau gauge.

Finally, the interaction Lagrangian can be introduced as usual:

$$L_I = -j^\mu A_\mu, \tag{3·14}$$

where j_μ denotes an electric current. Then (2·6) and (2·7) remain unchanged if

$$\partial^\mu j_\mu = 0, \tag{3·15}$$

while (2·8) is replaced by

$$\Box A_\mu = \eta \partial_\mu B - j_\mu . \tag{3·16}$$

It is subject to future investigation to check the consistency of our theory when an interaction is present.

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