Covering Arrays on Graphs

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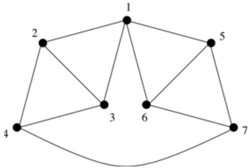

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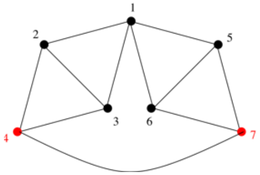

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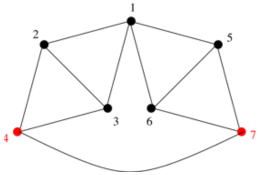

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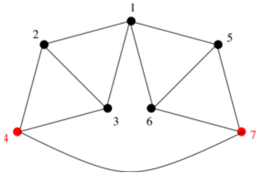

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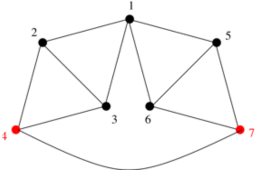

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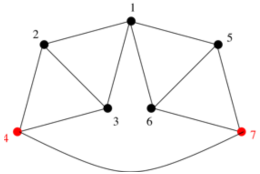

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What are all rows that can go into a covering array?

When are the rows adjacent?

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- Vertices are all partitions of $\{1, 2, ..., n\}$ into k parts.
- Two partitions $P = \{P_1, \ldots, P_k\}$ and $Q = \{Q_1, \ldots, Q_k\}$ are adjacent if
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(Called this *qualitatively independent* .)

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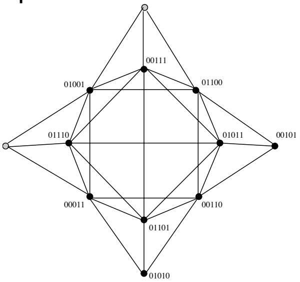

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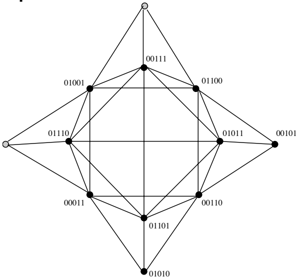

By construction, it is possible to build a covering array on $QI(n, k)$ with n columns and a k alphabet.

Why is $QI(n, k)$ Interesting?

Theorem (Meagher and Stevens - 2002)

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Theorem (Meagher and Stevens - 2002)

A covering array on a graph G with n columns and alphabet k exists if and only if there is a graph homomorphism

$$G \to QI(n, k).$$

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Is this set the largest independent set in $QI(k^2, k)$?

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- If k is a prime power then this is the largest independent set, because we have cliques of size $k+1$.

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Table of eigenvalues:

$$\left(\begin{array}{ccccc|c} 1 & 27 & 162 & 54 & 36 & 1 \\ 1 & 11 & -6 & 6 & -12 & 27 \\ 1 & 6 & -6 & -9 & 8 & 48 \\ 1 & -3 & 12 & -6 & -4 & 84 \\ 1 & -3 & -6 & 6 & 2 & 120 \end{array}\right)$$

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What is this association scheme and does it work for general k?

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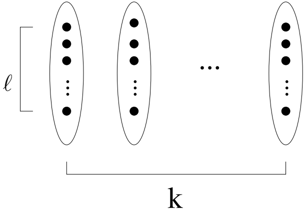

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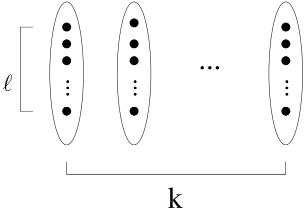

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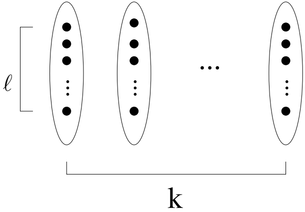

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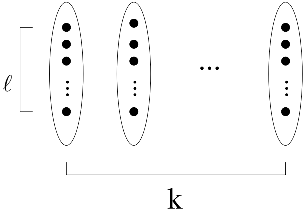

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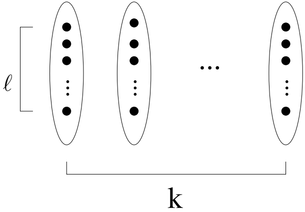

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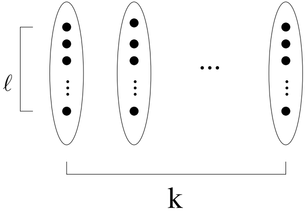

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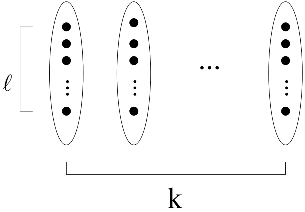

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$QI(k^2, k)$ is in an association scheme only if $k = 3$.

We actually found all subgroups G of $\mathrm{Sym}(n)$ such that $\operatorname{ind}_{\mathrm{Sym}(n)}(1_G)$ is multiplicity free.

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3. What are the interesting features of the association schemes from the subgroups of $\mathrm{Sym}(n)$?