Covering Graphs with Few Complete Bipartite Subgraphs *

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Abstract. We consider computational problems on covering graphs with bicliques (complete bipartite subgraphs). Given a graph and an integer k, the *biclique cover problem* asks whether the edge-set of the graph can be covered with at most k bicliques; the *biclique partition problem* is defined similarly with the additional condition that the bicliques are required to be mutually edge-disjoint. The *biclique vertex-cover problem* asks whether the vertex-set of the given graph can be covered with at most k bicliques, the *biclique vertex-partition problem* is defined similarly with the additional condition that the bicliques are required to be mutually vertex-disjoint. All these four problems are known to be NP-complete even if the given graph is bipartite. In this paper we investigate them in the framework of parameterized complexity: do the problems become easier if k is assumed to be small? We show that, considering k as the parameter, the first two problems are fixed-parameter tractable, while the latter two problems are not fixed-parameter tractable unless P = NP.

Keywords. bicliques, parameterized complexity, covering and partitioning problems.

1 Introduction

Let G be a simple undirected graph and let $\mathcal{S}$ be a set of (not necessarily vertex-induced) subgraphs of G. The set $\mathcal{S}$ is a *cover* of G of *size* $|\mathcal{S}|$ if every edge of

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G is contained in at least one of the subgraphs in $\mathcal{S}$. The set $\mathcal{S}$ is a *vertex-cover* of G if every vertex of G is contained in at least one of the subgraphs in $\mathcal{S}$. If all subgraphs in $\mathcal{S}$ are *bicliques*, that is, complete connected bipartite graphs, then we speak of a *biclique cover* or a *biclique vertex-cover*, respectively.

We consider the following four problems.

BICLIQUE COVER
Instance: A graph G and a positive integer k.
Question: Does G have a biclique cover of size at most k?

BICLIQUE PARTITION
Instance: A graph G and a positive integer k.
Question: Does G have a biclique cover of size at most k consisting of mutually *edge-disjoint* bicliques?

BICLIQUE VERTEX-COVER
Instance: A graph G and positive integer k.
Question: Does G have a biclique vertex-cover of size at most k?

BICLIQUE VERTEX-PARTITION
Instance: A graph G and positive integer k.
Question: Does G have a biclique vertex-cover of size at most k consisting of mutually *vertex-disjoint* bicliques?

We observe that the BICLIQUE VERTEX-COVER and BICLIQUE VERTEX-PARTITION problem are equivalent, since we can always make the bicliques of a biclique vertex-cover disjoint without increasing the size of the cover (and we can do so without introducing *trivial* bicliques, that is, bicliques having one vertex only). However, BICLIQUE COVER and BICLIQUE PARTITION are not equivalent. Take for example the bipartite graph with vertex set $U_1 \cup U_2$, $U_1 = \{x_1, x_2, x_3\}$, $U_2 = \{y_1, y_2, y_3\}$, and all possible edges between vertices in U_1 and U_2 except for the edges x_1y_3 and x_3y_1. This graph has a biclique cover of size 2, namely the biclique cover with bicliques $(\{x_1, y_1, x_2, y_2\}, \{x_1y_1, x_1y_2, x_2y_1, x_2y_2\})$ and $(\{x_2, y_2, x_3, y_3\}, \{x_2y_2, x_2y_3, x_3y_2, x_3y_3\})$. However, any biclique cover that consists of mutually edge-disjoint bicliques has size at least 3.

One can consider variants of the above problems where solutions must consist of nontrivial bicliques only. However, minimal solutions for BICLIQUE COVER and BICLIQUE PARTITION clearly do not contain trivial bicliques, and it is easy to see that for BICLIQUE VERTEX-COVER and BICLIQUE VERTEX-PARTITION there is always a minimal solution where only isolated vertices are contained in trivial bicliques. Hence, the computational complexities of the four problems do not change if solutions must avoid trivial bicliques.

The BICLIQUE COVER problem arises in both theoretical and practical areas for more than thirty years. From a theoretical point of view, the BICLIQUE COVER problem is equivalent to the set basis problem [22] and related to boolean algebraic forms associated with graphs and combinatorial optimization problems. There, the minimum number of bicliques necessary to cover all the edges of a graph G is also called the *bipartite dimension* of G, which is considered to be an interesting graph property on its own. For more details we refer to [1, 3, 7]. From a more practical perspective, bicliques are used to model the rectangle cover problem that asks if a rectilinear polygon can be expressed as the union of a minimum number of rectangles [16]. Both the BICLIQUE COVER and the BICLIQUE PARTITION problem play a significant role in the analysis of so-called HLA reaction matrices used in biology [19]. Other practical applications lie in artificial intelligence and data mining. In Formal Concept Analysis, context is structured into a set of concepts with binary relations. It turns out that each concept corresponds to a so-called closed item set in data mining and, by representing the binary relations as bipartite graphs, to a maximal biclique. See [24] for more details. Applications of BICLIQUE VERTEX-COVER include data mining, e-commerce, information retrieval and network management. In all these applications, large bipartite graphs are analyzed in order to discover so-called cross associations corresponding to bicliques [13].

All four problems are computationally hard problems: BICLIQUE COVER is NP-complete and remains NP-hard for chordal bipartite graphs [18, 21]. The BICLIQUE PARTITION problem is also already NP-complete for bipartite graphs [15]. Very recently, Heydari et al. [13] showed that BICLIQUE VERTEX-PARTITION, and consequently, BICLIQUE VERTEX-COVER are NP-complete for bipartite graphs.

In this paper we investigate the questions of whether these problems become easier if the given upper bound k on the number of bicliques in the cover is assumed to be small. We undertake this investigation in the framework of parameterized complexity as developed by Downey and Fellows [6], considering the upper bound k on the number of bicliques in the cover as the parameter. We give some basic background of parameterized complexity in Section 2.2. In principle, the problems under consideration can fall into any of the following three categories.

1. For every fixed k the problem can be solved in polynomial time where the order of the polynomial is independent of k; in this case we say that the problem is *fixed-parameter tractable*.
2. For every fixed k the problem can be solved in polynomial time but the order of the polynomial grows with k.
3. For some fixed k the problem is NP-hard.

Problems that fall into the second category can be further categorized by means of the complexity classes W[1], W[2], $\ldots$, XP (see Section 2.2). In the literature, a similar study has been performed for the problems CLIQUE COVER and CLIQUE PARTITION. These NP-complete problems ask if a given graph has a cover consisting of at most k cliques or k mutually edge-disjoint cliques, respectively. Both CLIQUE COVER [10, 12] and CLIQUE PARTITION [17] are fixed-parameter tractable. The problem CLIQUE VERTEX-COVER (or PARTITION INTO CLIQUES) asks if the vertices of a given graph can be covered by at most k cliques. This problem is NP-complete for each fixed $k \geq 3$ and polynomial-time solvable for $k \leq 2$ [9, GT15].

New Results

Our results show that the problems under consideration fall into all three of the above categories, spanning a wide range of parameterized complexities.

1. *Problems* Biclique Cover *and* Biclique Partition *are fixed-parameter tractable.*

We show these results in Section 3. We make use of *kernelization*, that is, we give an algorithm that reduces an instance of Biclique Cover or Biclique Partition in polynomial time into an equivalent instance where the number of vertices is bounded in terms of the parameter k.

2. *For* $k \leq 2$ *the problem* Biclique Vertex-Cover *can be solved in polynomial time for bipartite graphs. For every fixed* $k \geq 3$ *the problem* Biclique Vertex-Cover *is* NP*-complete and remains* NP*-hard for bipartite graphs.*

As the problem Biclique Vertex-Cover is equivalent to the problem Biclique Vertex-Partition, the above result is also valid for the latter problem. In Section 4.1 we establish the NP-completeness result by a reduction from an NP-complete variant of the List-Coloring problem. Note that our NP-completeness result is stronger than the one in [13] as we assume that $k \geq 3$ is a constant and not part of the input. In Section 4.2 we show the polynomial case $k = 2$. The result for this case follows directly from a stronger result on a graph homomorphism problem defined on the complement graph (we explain this in detail in Section 4.2).

In view of the NP-hardness it makes sense to study the more restricted problem b-Biclique Vertex-Cover where the bicliques in the cover are bicliques where at least one of the bipartite sets contains at most b vertices. Indeed, in Section 4.3, we show that this restriction moves the problem from the third to the second of the above categories:

3. *For every fixed* $b \geq 1$ *the problem* b-Biclique Vertex-Cover *is* W[2]*-complete and remains* W[2]*-hard for bipartite graphs.*

2 Preliminaries

2.1 Graph Theoretic Terminology

For graph theoretic terminology not defined in this paper, we refer the reader to standard text books [2, 5]. In this paper we consider connected simple graphs $G = (V, E)$. The set of neighbors of a vertex v in a graph G is denoted by $N_G(v)$, and we set $N_G(T) = \bigcup_{v \in T} N_G(v)$ for $T \subset V$ (we often omit the subscript G if it is clear from the context which graph G is considered). A set $D \subseteq V$ is a *dominating set* of G if every vertex of G is either in D or has a neighbor in D. The *distance* $d_G(u, v)$ between two vertices u and v in G is the number of edges of a shortest path from u to v. The *diameter* $\text{diam}(G)$ of G is the maximum distance over all pairs of vertices of G; $\text{diam}(G) = \infty$ if G is disconnected.

If $V' \subseteq V$, we denote by $G[V']$ the subgraph of G induced by V'. We write $G = ((V_1, V_2), E)$ for a bipartite graph $G = (V, E)$ having the vertex bipartition $V = V_1 \cup V_2$. We say that $G = ((V_1, V_2), E)$ is a *biclique* if G is connected and E contains all possible edges between vertices in V_1 and vertices in V_2. A biclique $((U_1, U_2), E)$ is a *star centered at a vertex u* if $U_1 = \{u\}$ or $U_2 = \{u\}$.

2.2 Parameterized Complexity

We give some basic background on parameterized complexity; for a detailed discussion we refer the reader to other sources [6, 20]. In parameterized complexity theory, we consider the problem input as consisting of two parts; that is, a pair (I, k), where I is the main part and k (usually an integer given in unary) is the parameter. We say a problem is *fixed parameter tractable* if an instance (I, k) can be solved in time $O(f(k) + n^c)$ or $O(f(k)n^c)$, where f denotes a computable function and c denotes a constant that is independent of the parameter k. Therefore, such an algorithm may provide an efficient solution to the problem if the parameter is reasonably small. We denote by FPT the class of all fixed-parameter tractable decision problems.

A well known technique to show that a parameterized problem Π is fixed-parameter tractable is to find a *reduction to a problem kernel* (this is also called *kernelization*). It replaces an instance (I, k) of Π with a reduced instance (I', k') of Π (called *problem kernel*) such that

(i) $k' \leq k$ and $|I'| \leq g(k)$ for some computable function g;
(ii) the reduction from (I, k) to (I', k') is computable in polynomial time;
(iii) (I, k) is a yes-instance of Π if and only if (I', k') is a yes-instance of Π.

It is well known that a parameterized problem is fixed-parameter tractable if and only if it is kernelizable [12, 14, 20].

Parameterized complexity offers a completeness theory, similar to the theory of NP-completeness, that allows the accumulation of strong theoretical evidence that some parameterized problems are not fixed-parameter tractable. This completeness theory is based on a hierarchy of complexity classes W[1], W[2], ..., XP. Each class contains all parameterized decision problems that can be reduced to a certain fixed parameterized decision problem under *fpt-reductions*. An fpt-reduction from a parameterized problem Π to a parameterized problem Π' is an algorithm that computes for every instance (I, k) of Π an instance (I', k') of Π' in at most $f(k)|I|^c$ time for some computable function f and constant c such that

(i) $k' \leq h(k)$ for some computable function h, and
(ii) (I, k) is a yes-instance of Π if and only if (I', k') is a yes-instance of Π'.

This means that if Π' belongs to some parameterized complexity class W then Π also belongs to W. For instance, the class W[1] contains all parameterized problems that can be reduced to WEIGHTED 3-CNF-SATISFIABILITY by an fpt-reduction. The latter problem asks for a given instance $\mathcal{F}$ of 3CNF and a positive integer k , whether $\mathcal{F}$ can be satisfied by setting exactly k variables to true. The class XP consists of parameterized decision problems Π such that for each instance (I, k), it can be decided in $O(f(k)|I|^{g(k)})$ time whether $(I, k) \in \Pi$, where f, g are computable functions depending only k. That is, XP consists of parameterized decision problems which can be solved in polynomial time if the parameter is considered as a constant. The above classes form the chain FPT $\subseteq$ W[1] $\subseteq$ W[2] $\subseteq \ldots \subseteq$ XP where all inclusions are conjectured to be proper; FPT $\neq$ XP is known [6, 8].

3 Covering the Edges

As mentioned in the introduction, the decision problem corresponding to Biclique Cover is NP-complete even for bipartite graphs [21]. In this section we establish fixed-parameter tractability.

We start with two simple reduction rules that can be easily applied to simplify an instance of the Biclique Cover or Biclique Partition problem.

Rule 1. Given an instance (G, k) and a vertex $v \in V(G)$ of degree 0, then (G, k) is a yes-instance if and only if $(G - v, k)$ is a yes-instance.

Rule 2. Given an instance (G, k) and a pair of (non-adjacent) vertices u, v such that $N(u) = N(v)$, then (G, k) is a yes-instance if and only if $(G - \{v\}, k)$ is a yes-instance.

Clearly, the following is true.

Lemma 1 *Rules 1 and 2 are correct for both problems* Biclique Cover *and* Biclique Partition*, and can be applied in polynomial time.*

We say that an instance (G, k) is *reduced* (with respect to Rules 1 and 2) if these rules cannot be applied.

Theorem 2 (Kernelization) *If (G, k) is a reduced yes-instance of* Biclique Cover *or* Biclique Partition *then G has at most 3^k vertices. Furthermore, if G is bipartite, then it has at most 2^{k+1} vertices.*

Proof. Let (G, k) be a reduced instance of the Biclique Cover or Biclique Partition problem with biclique cover $\mathcal{S} = \{C_1, \ldots, C_l\}$ of size $l \leq k$, where $C_i = ((X_i, Y_i), E_i)$. We will argue similarly as Gramm et al. [10]. We assign to each vertex $v \in V(G)$ a vector $b_v \in \{0, 1, 2\}^l$ where the i-th component $b_{v,i} = 1$ if v is contained in X_i, $b_{v,i} = 2$ if v is contained in Y_i, and $b_{v,i} = 0$ otherwise. Since (G, k) is reduced, each vertex belongs to at least one biclique of $\mathcal{S}$. Consider an arbitrary but fixed vector $b \in \{0, 1, 2\}^l$. Let V_b be the set of vertices of G such that $b_u = b$ for all $u \in V_b$. Suppose V_b contains two distinct vertices x, y. Since $b_x = b_y$, it follows that x and y belong to the same partition classes of the same bicliques. Then $N(x) = N(y)$ and we obtain a contradiction. Hence $|V_b| = 1$. Therefore we conclude that G has at most $|\{0, 1, 2\}^l| \leq 3^k$ vertices.

If G is bipartite, we can define $b_{v,i} = 1$ if v is contained in C_i and $b_{v,i} = 0$ otherwise. Then we find that each set V_b must be complete (as otherwise we could apply rule 2 for two vertices x, y with $b_x = b_y$), and thus contains at most two vertices. This means that G has at most $2 \times |\{0,1\}^l| \leq 2^{k+1}$ vertices if it is bipartite. □

A direct consequence of Theorem 2 is that Biclique Cover and Biclique Partition are fixed-parameter tractable.

Corollary 3 *Both the* Biclique Cover *and the* Biclique Partition *problem can be solved in $O(f(k)+n^3)$ time where $f(k) = 3^{2k^2+3k}$ for non-bipartite graphs and $f(k) = 2^{2k^2+3k}$ for bipartite graphs.*

Proof. We represent a graph $G = (V, E)$ on $|V| = n$ vertices by its *adjacency matrix*, i.e., the $n \times n$ matrix $A = (a_{ij})$ with rows and columns indexed by the vertices of V such that $a_{uv} = 1$ if $uv \in E$ and $a_{uv} = 0$ otherwise. Then it takes $O(n^2)$ time to detect and remove all isolated vertices in G (Rule 1) and $O(n^3)$ time to verify if $N(u) = N(v)$ for any two vertices u and v (Rule 2). Then, by Theorem 2, we find a reduced graph G' with 3^k vertices and consequently $O(9^k)$ edges if it is non-bipartite and 2^{k+1} vertices and consequently $O(4^k)$ edges if it is bipartite in $O(n^3)$ time.

A brute force algorithm that solves the Biclique Partition problem with input (H, k), where H is a graph with m edges, guesses for each edge of H to which biclique it belongs and verifies if the resulting partition of $E(H)$ yields a biclique cover of size at most k. This takes $O(m^k)$ time. As any partition of $E(H)$ fixes the vertices of both bipartition classes of each biclique, we only have to verify if all the fixed sets of vertices indeed induce mutually edge-disjoint bicliques. This verification process takes $O(|V(H)|^3)$ time. We can do exactly the same for the Biclique Cover problem except that here we do not care if the bicliques are mutually edge-disjoint. Hence, for both problems, we find $f(k) = 3^{2k^2+3k}$ if G' is non-bipartite and $f(k) = 2^{2k^2+3k}$ otherwise. This finishes the proof of Corollary 3. □

4 Covering the Vertices

As we observed in Section 1 that Biclique Vertex-Cover and Biclique Vertex-Partition are equivalent, we will only consider the former problem in this section.

4.1 NP-Hardness

We now proceed to show that Biclique Vertex-Cover is NP-complete for fixed $k \geq 3$, even if the given graph is bipartite. We present a polynomial-time reduction from the following problem.

List-Coloring
Instance: A graph $G = (V, E)$ and a mapping L that assigns to every $v \in V$ a list $L(v)$ of colors allowed for v.
Question: Is there a coloring c of $V(G)$ such that $c(v) \in L(v)$ for each $v \in V$ and $c(u) \neq c(v)$ for each $uv \in E$?

If such a coloring c exists, then we call c an *L-coloring of G*, and we say that G is *L-colorable.* If the number of available colors $k = |\bigcup_{v \in V} L(v)|$ is fixed, then the problem is called k-List-Coloring. This problem is known to be NP-complete for bipartite graphs and $k \geq 3$ [11].

Our reduction proceeds as follows. Let (G, L) be an instance of k-List-Coloring where $G = ((U, V), E)$ is a bipartite graph. We assume that $\bigcup_{v \in V} L(v) = \{1, 2, \ldots, k\}$. We construct a graph H as follows (see Figure 1 for an example):

1. Let G^* be the bipartite complement of G; i.e., $V(G^*) = V(G) = U \cup V$ and $E(G^*) = \{\, uv : u \in U, v \in V, uv \notin E(G) \,\}$.
2. For $i = 1, \ldots, k$, introduce k new edges $u_i v_i$ for $i = 1, \ldots, k$ using $2k$ new vertices $u_i, v_i \notin V(G^*)$.
3. Now take G^* and the k edges $u_i v_i$. For every $x \in U$ and $i \in \{1, \ldots, k\}$, if $i \in L(x)$ add an edge xv_i. For every $y \in V$ and $i \in \{1, \ldots, k\}$, if $i \in L(y)$ add an edge yu_i. Call the resulting graph H. Thus, H is a bipartite graph containing G^* as a proper subgraph (note that $V(H) = (U \cup \{\, u_i : 1 \leq i \leq k \,\}) \cup (V \cup \{\, v_i : 1 \leq i \leq k \,\})$).

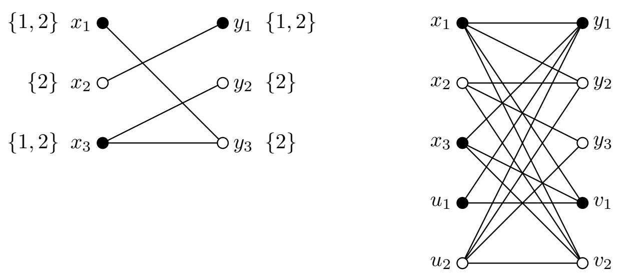

Fig. 1. A graph G with list assignment L and the graph H obtained from G. The L-coloring c of G with $c(x_1) = c(x_3) = c(y_1) = 1$ and $c(x_2) = c(y_2) = c(y_3) = 2$ and the corresponding biclique vertex-partition of H are indicated with black and white vertices.

In general, it is clear that H can be constructed in polynomial time and $|V(H)| = |V(G)| + 2k$. Furthermore, the following can be established (as illustrated in Figure 1).

Lemma 4 *G is L-colorable if and only if $V(H)$ can be covered by k bicliques.*

Proof. Suppose that G has an L-coloring c. Define a partition of $V(H)$ as follows. For $i = 1, \dots, k$, define

$$C_i := \{ v \in V(G) : c(v) = i \} \cup \{u_i, v_i\}.$$

Let $\mathcal{S} := \{H[C_1], \dots, H[C_k]\}$. Note that each vertex of H belongs to precisely one element of $\mathcal{S}$. We claim that $\mathcal{S}$ is a biclique vertex-cover of H. Choose an arbitrary element $H[C_i] \in \mathcal{S}$. Let $x, y \in V(G)$ be two vertices in $H[C_i]$ belonging to different classes in the vertex bipartition of H induced by the vertex bipartition of G. Clearly $xy \notin E(G)$ because $c(x) = c(y)$. Thus $xy \in E(G^*)$ which in turn implies that $xy \in E(H[C_i])$. Moreover, by definition of H, $u_i y \in E(H[C_i])$ for every $y \in C_i \cap V$, and $v_i x \in E(H[C_i])$ for every $x \in C_i \cap U$. Thus, $H[C_i]$ is a biclique of H. Thus, we conclude the set $\mathcal{S}$ is a biclique vertex-cover of G.

Now suppose that H has a biclique vertex-cover $\mathcal{S} = \{G_1, \dots, G_k\}$. The edges $u_i v_i$, $i = 1, \dots, k$ belong to distinct bicliques since $u_i v_j \notin E(H)$, $i, j \in \{1, \dots, k\}$, $i \neq j$. Hence we may assume, w.l.o.g., that $u_i v_i \in E(G_i)$, $i = 1, \dots, k$. Let $\overline{C}_i := V(G_i) - \{u_i, v_i\}$. The set $\{\overline{C}_1, \dots, \overline{C}_k\}$ defines k disjoint independent sets in G since $H[\overline{C}_i]$ is a biclique or a subset of U or a subset of V. Now define a function $\gamma : V(G) \to \bigcup_{v \in V(G)} L(v)$ as follows:

$$\gamma(v) := i \quad \text{if and only if} \quad v \in \overline{C}_i.$$

For every $v \in \overline{C}_i$ we have, by definition of H, $i \in L(v)$, and as we deduced above $\gamma(x) \neq \gamma(y)$ for $xy \in E(G)$. Thus γ defines an L-coloring of G. ☐

For every fixed k the problem Biclique Vertex-Cover belongs to NP. Since, as mentioned above, k-List-Coloring is NP-complete for bipartite graphs for $k \geq 3$, the reduction in Lemma 4 yields following result.

Theorem 5 Biclique Vertex-Cover *is* NP*-complete for every fixed* $k \geq 3$. *This also holds if only bipartite graphs are considered.*

Corollary 6 Biclique Vertex-Cover *is not fixed-parameter tractable unless* P = NP.

4.2 Polynomial Cases

Next we study the question of whether $k \geq 3$ is an optimal bound for the NP-hardness of Biclique Vertex-Cover. If $k = 1$ the problem is trivially solvable in polynomial time: G has a biclique vertex-cover of size one if and only if the complement graph $\overline{G}$ (which has vertex set $V(\overline{G}) = V(G)$ and edges uv whenever $uv \notin E(G)$) is disconnected or $|V| = 1$. The case $k = 2$ is still open. However, we can establish polynomial-time results for a special graph class that includes all bipartite graphs.

For this purpose we transform Biclique Vertex-Cover for $k = 2$ into an equivalent problem involving graph homomorphisms. We need the following definitions. Let G, H be two simple graphs. A mapping $h : V(G) \to V(H)$ is a *homomorphism from G to the reflexive closure of H* if for every edge $uv \in E(G)$ we have either $h(u) = h(v)$ or $h(u)h(v) \in E(H)$. The homomorphism h is *vertex-surjective* if for each $c \in V(H)$ there is some $v \in V(G)$ with $h(v) = c$. Let C_k denote the cycle on k vertices $c_1, \ldots, c_k$ where c_i and c_j are adjacent if and only if $|i - j| \equiv 1 (\text{mod } k)$. We make the following observation, which is easy to see.

Observation 7 *A graph G has a biclique vertex-cover consisting of two non-trivial vertex-disjoint bicliques if and only if there is a vertex-surjective homomorphism from the complement graph $\overline{G}$ to the reflexive closure of C_4.*

A *dominating edge* of a graph G is an edge xy with $N(x) \cup N(y) = V(G)$.

Theorem 8 *We can check in polynomial time whether a graph G allows a vertex-surjective homomorphism to the reflexive closure of C_4 if*

(i) G has a dominating edge, or
(ii) G has diameter not equal to two, or
(iii) G has bounded maximum degree, or
(iv) G is triangle-free.

Proof. Let $G = (V, E)$ be a graph. The following terminology is useful. Let h be a vertex-surjective homomorphism from G to the reflexive closure of C_4. If h maps a vertex $v \in V$ to c_i, we say that v has *color i*. This way h induces a coloring with exactly four different colors 1,2,3,4 such that neither color pair $(1, 3)$ nor $(2, 4)$ is used on the end vertices of an edge. We call h a *diagonal coloring.* Since any diagonal coloring corresponds to a vertex-surjective homomorphism from G to the reflexive closure of C_4 as well, we are done if we can decide in polynomial time if G has a diagonal coloring for cases (i)-(iv). We prove each case separately.

(i) Suppose xy is a dominating edge of G. Clearly, $\{x, y\}$ will be assigned two different colors by any diagonal coloring h of G.

Suppose such a coloring h exists. Then we may, w.l.o.g., assume that x has got color 1 and y has got color 2. We will show how we can check in polynomial time whether this precoloring can be extended to a full diagonal coloring of G. We call a set $U \subseteq V$ *colored* if every vertex in U has received a color. In a precoloring, we denote the set of all colored neighbors of a vertex u by $N^c(u)$, and we call a colored set U *j-chromatic* if the number of different colors in U equals j.

We proceed as follows. First we guess an uncolored vertex s not adjacent to x that we assign color 3 and an uncolored vertex t not adjacent to y that we assign color 4. Note that the number of guesses is bounded by $O(|V|^2)$. We apply the following rule as long as possible: if there exists an uncolored vertex u with 3-chromatic $N^c(u)$ then u can only get one possible color, which we then assign to u. Afterwards we check if there exists a vertex w with a 4-chromatic colored neighbor set. If so, then pair (s, t) was a wrong guess, because we cannot assign an appropriate color to w. We then guess another pair (s', t') that we assign color $3, 4$ respectively, and so on.

Suppose that for a particular pair (s, t) we have applied the above rule as long as possible and such a vertex w (with 4-chromatic $N^c(w)$) does not exist. Since xy is a dominating edge, we can partition the uncolored vertices of G into the following sets: sets $U_{i,j}$ consisting of vertices adjacent to vertices with color i and j for $(i, j) \in \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4)\}$ and sets U_i consisting of vertices *only* adjacent to color i for $i = 1, 2$. Then we extend the precoloring of F by assigning color 1 to the vertices in $U_{1,2} \cup U_{1,4} \cup U_{2,4} \cup U_1 \cup U_2$ and color 2 to the vertices in $U_{1,3} \cup U_{2,3}$. This proves case (i).

(ii) Suppose G does not have diameter 2. If $\text{diam}(G) = 1$, then G is a complete graph and does not have a diagonal coloring. Let $\text{diam}(G) \geq 3$. Then there exist vertices u, v in G with $d_G(u, v) = \text{diam}(G) \geq 3$. We can find such a pair u, v in polynomial time. We assign color 1 to u, color 2 to all neighbors of u, color 3 to all vertices of distance 2 from u, and color 4 to all remaining vertices in G. As this coloring is diagonal, we have shown case (ii).

(iii) Suppose G has maximum degree d for some fixed integer d. By (ii) we may assume that G has diameter 2. This means that V has at most $d^2 + 1$ vertices, which proves case (iii).

(iv) Suppose G is triangle-free. By (ii) we may assume that $\text{diam}(G) = 2$. If G has a *dominating vertex* u, i.e., $N(u) = V \backslash \{u\}$ then G does not have a diagonal coloring (since u would become adjacent to a forbidden color). Suppose G does not have a dominating vertex. We claim that G has a diagonal coloring if and only if $|V| \geq 4$.

Suppose G has a diagonal coloring c. As $|c(V)| = 4$, we obtain $|V| \geq 4$. To prove the reverse implication, suppose $|V| \geq 4$. Let $u \in V$ be a vertex with degree at least two. We color u by 1, one of its neighbors by 2, its remaining neighbors by 4 and all the other vertices by 3 (as u is not dominating, G has at least one vertex not adjacent to u). Since G is triangle-free, $N(u)$ is an independent set, and we have obtained a diagonal coloring of G. This proves case (iv) and completes the proof of Theorem 8. $\square$

Corollary 9 Biclique Vertex-Cover *for fixed $k = 2$ can be solved in polynomial time for the class of graphs that contain a pair of nonadjacent vertices with no common neighbor. In particular,* Biclique Vertex-Cover *for fixed $k = 2$ can be solved in polynomial time for bipartite graphs.*

Proof. The first statement immediately follows from Observation 7 and Theorem 8. So, let G be a bipartite graph with bipartition classes A, B. If G has two nonadjacent vertices $x \in A$ and $y \in B$, then we are done by the first statement. In the other case G is a biclique. This proves Corollary 9. $\square$

Remark 10. A homomorphism f from a graph G to a graph H is called *edge-surjective* or a *compaction* if for each $xy \in E(H)$ with $x \neq y$ there is some $uv \in E(G)$ with $f(u)f(v) = xy$. The problem that asks whether there exists a compaction from a given graph to the reflexive closure of C_4 is known to be NP-complete [23]. For a graph G with diameter 2 it is equivalent to asking if G allows a vertex-surjective homomorphism to the reflexive closure of C_4. This can be seen as follows.

We first note that any compaction, which is edge-surjective, is also vertex-surjective. To show the remaining implication, suppose f is a vertex-surjective homomorphism from G to the reflexive closure of C_4. Suppose $f(u) = c_1$ and $f(v) = c_2$. If $uv \in E$, then c_1 and c_2 are images of the end vertices of an edge. Otherwise, since G has diameter two, there exists a vertex s adjacent to u and v. Then $f(s) = c_1$ or $f(s) = c_2$. In the first case sv and in the second case su is the desired edge. We use the same arguments for edges c_2c_3, c_3c_4, and c_4c_1. Hence the reverse implication is valid too.

So far, we could only show that the problem that asks if G allows a compaction to the reflexive closure of C_4 stays NP-complete if we restrict the input graphs to graphs with diameter 3. We do this by slightly modifying the NP-completeness reduction given in [23]. As this is beyond the scope of this paper, we leave out the proof details.

Remark 11. Of related interest is the concept of *H-partitions* as studied by Dantas et al. [4]. Let H be a fixed graph with four vertices $h_1, \ldots, h_4$. An H-partition of a graph $G = (V, E)$ is a partition of V into four *nonempty* sets $X_1, \ldots, X_4$ such that whenever h_ih_j is an edge of H, then G contains the biclique $K = ((X_i, X_j), E_k)$. H-PARTITION denotes the problem of deciding whether a given graph admits an H-partition. Evidently, BICLIQUE VERTEX-COVER for $k = 2$ is equivalent to the problem $2K_2$-PARTITION where $2K_2$ denotes the graph on four vertices with two independent edges. $H = 2K_2$ is the only case for which the complexity of H-PARTITION is not known. All other cases are known to be solvable in polynomial time.

Remark 12. The BICLIQUE VERTEX-COVER problem for $k = 2$ is also equivalent to asking if $\overline{G}$ has a *disconnected cut*, i.e., a set $U \subseteq V$ such that both $\overline{G}[U]$ and $\overline{G}[V \backslash U]$ are disconnected. We are not aware of any previous work on the problem expressed this way.

4.3 Bounding One Side of the Bicliques

In the following we study the question of whether BICLIQUE VERTEX-COVER becomes easier when the number of vertices in one of the two classes of the vertex bipartition of bicliques is bounded. For a complete bipartite graph $K = ((U_1, U_2), E)$ we define $\beta(K) = \min\{|U_1|, |U_2|\}$. Clearly, $\beta(K) = 1$ if and only if K is a star. A *b-bounded biclique* is a biclique K such that $\beta(K) \leq b$. A *b-biclique vertex-cover* of a graph G is a set of b-bounded bicliques of G such that each vertex of G is contained in one of these bicliques.

Let b be a fixed positive integer. We consider the following parameterized problem.

b-Biclique Vertex-Cover
Instance: A graph G and a positive integer k.
Parameter: The integer k.
Question: Does there exist a b-biclique vertex-cover $\mathcal{S}$ of G such that $|\mathcal{S}| \leq k$?

It is not difficult to see that b-Biclique Vertex-Cover is in XP as follows. Let $G = (V, E)$ be a graph with n vertices and $k > 0$. We assume w.l.o.g. that G does not contain isolated vertices. We choose independently subsets $X_1, \dots, X_k \subseteq V$ of size at most b, there are $O(n^{kb})$ possibilities. For each choice $X_1, \dots, X_k$ we define $Y_1, \dots, Y_k$ where $Y_i = \bigcap_{x \in X_i} N(x)$. Then we check in polynomial time if every vertex of G is in at least one set X_i or Y_i, and if all Y_i are non-empty (note that $Y_i = \emptyset$ implies $|X_i| \geq 2$ because we assume G does not have isolated vertices). If both conditions are satisfied, then we have found a b-biclique vertex-cover of size at most k. Furthermore, if there exists a b-biclique cover of size at most k then one of the guesses will succeed.

Next we will identify the exact parameterized complexity of b-Biclique Vertex-Cover. In Lemma 14 we show that the b-Biclique Vertex-Cover problem is in W[2]. In Lemma 15 we show that the b-Biclique Vertex-Cover problem is W[2]-hard. These two lemmas together imply the following result.

Theorem 13 *The b-Biclique Vertex-Cover problem is W[2]-complete for every $b \geq 1$. This also holds if only bipartite graphs are considered.*

To show W[2]-membership we use the following parameterized problem known to be W[2]-complete [6].

Dominating Set
Instance: A graph G and a positive integer k.
Parameter: The integer k.
Question: Does there exist a dominating set of G of size at most k?

Lemma 14 *There is an fpt-reduction from b-Biclique Vertex-Cover to Dominating Set.*

Proof. Consider an instance (G, k) of b-Biclique Vertex-Cover. For a set $S \subseteq V(G)$ let $S' \subseteq V(G)$ denote the set of common neighbors of vertices in S, i.e., $S' = \bigcap_{v \in S} N(v)$. Furthermore, let $\mathcal{T}$ denote the set of subsets $S \subseteq V(G)$ with $1 \leq |S| \leq b$ and $S' \neq \emptyset$.

We construct a graph $H = (V', E')$ as follows. We let V' consist of $V(G)$, two new vertices z, z' and a new vertex v_S for every $S \in \mathcal{T}$. We let E' consist of $E(G)$ together with the edge zz' and all edges $v_S w$ for $w \in S \cup S' \cup \{z\}$, $S \in \mathcal{T}$. Note that H can be constructed in polynomial time as $|\mathcal{T}| = O(bn^b)$ where $n = |V(G)|$. We show that G has a b-biclique vertex-cover of size at most k if and only if H has a dominating set of size at most $k + 1$.

Let $\mathcal{S}$ be a b-biclique vertex-cover of G and $|\mathcal{S}| \leq k$. Note that $\mathcal{S} \subseteq \mathcal{T}$. For every $K \in \mathcal{S}$ we choose a vertex $x_K \in V(H)$ as follows. If K is trivial, i.e.,

$V(K) = \{v\}$, then we put $x_K = v$. Otherwise we put $x_K = v_S$. Evidently $D = \{ x_K : K \in S \} \cup \{z\}$ is a dominating set of H, and $|D| \leq k+1$.

Conversely, let D be a dominating set of H with $|D| \leq k+1$. We may assume, w.l.o.g., that $z \in D$ (otherwise $z' \in D$ and we can replace z' by z). For every $x \in D\backslash\{z\}$ we identify a biclique K_x of G as follows. If $x = v_S$ for some set $S \subseteq V(G)$ then we let $K_x = ((S, S'), E_{K_x})$. Otherwise, if $x \in V(G)$, then we define $K_x = ((\{x\}, N(x)), E_{K_x})$. We let $\mathcal{S} = \{ K_x : x \in D\backslash\{z\} \}$. Again it is easy to verify that $\mathcal{S}$ is a b-biclique vertex cover of G, and clearly $|\mathcal{S}| \leq |D| - 1 = k$. □

To show W[2]-hardness we use the following parameterized problem known to be W[2]-complete [6].

Hitting Set
Instance: A pair $(Q, \mathcal{C})$, where $Q = \{q_1, \ldots, q_m\}$ and $\mathcal{C} = \{C_1, \ldots, C_n\}$ with $C_i \subseteq S$ for $i = 1, \ldots, n$, and a positive integer k.
Parameter: The integer k.
Question: Does there exist a subset $H \subseteq Q$ with $|H| \leq k$, such that $H \cap C_i \neq \emptyset$ for $i = 1, \ldots, n$?

Lemma 15 *There is an fpt-reduction from* Hitting Set *to* b-Biclique Vertex-Cover *for bipartite graphs.*

Proof. Let $I = ((Q, \mathcal{C}), k)$ be an instance of Hitting Set, where $Q = \{q_1, \ldots, q_m\}$ and $\mathcal{C} = \{C_1, \ldots, C_n\}$. We transform I into an instance of b-Biclique Vertex-Cover as follows. First construct a bipartite graph $G = ((Q, \mathcal{C}), E)$ by letting $q_iC_j \in E(G)$ if and only if $q_i \in C_j$. Now add two new vertices z and z' to G, such that z is adjacent to every q_i and z' is adjacent to z only. Finally, for each vertex C_j add bk new vertices $v_{j_1}, \ldots, v_{j_{bk}}$ and add edges such that $N(v_{j_d}) := N(C_j)$, $d = 1, \ldots, bk$. Call the resulting graph G'. An example of a graph G' is given in Figure 2. Clearly, G' is bipartite. Let

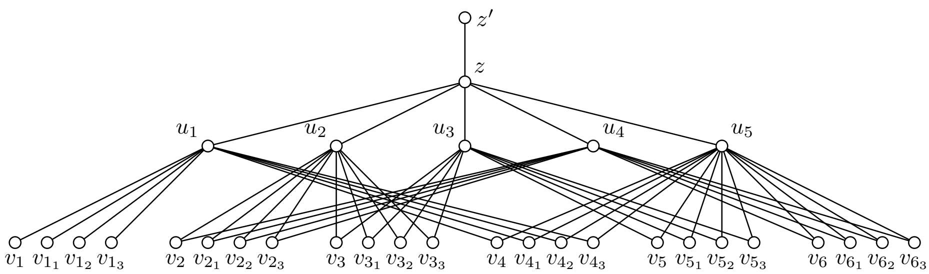

Fig. 2. The graph G' for the instance $((Q, \mathcal{C}), 3)$ and $b = 1$, where $Q = \{q_1, \ldots, q_5\}$ and $\mathcal{C} = \{C_1, \ldots, C_6\}$ with $C_1 = \{q_1\}$, $C_2 = \{q_2, q_4\}$, $C_3 = \{q_2, q_3\}$, $C_4 = \{q_1, q_5\}$, $C_5 = \{q_3, q_5\}$ and $C_6 = \{q_4, q_5\}$.

U', V' be the bipartition classes of $V(G')$, such that $z \in V'$, and consequently, $U' = Q \cup \{z'\}$. We show that $(Q, \mathcal{C})$ has a hitting set of size at most k if and only if G' has a b-biclique vertex-cover of size at most $k+1$.

Let H be a hitting set of $(Q, \mathcal{C})$ with $|H| \leq k$. Define $K_q := N_{G'}(q) \cup \{q\}$ for all $q \in H$. These $|H|$ stars together with the star that consists of z, z' and the elements of $Q \backslash H$ form a b-biclique vertex-cover of G' with size $|H| + 1 \leq k + 1$.

Conversely, suppose that G' has a b-biclique vertex-cover $\mathcal{S}$ of size at most $k+1$. We may assume, w.l.o.g., that $\mathcal{S}$ contains a star K_0 centered at the vertex z. Let $\mathcal{S}' := \mathcal{S} \backslash \{K_0\}$. For a biclique $K = ((X, Y), E_K) \in \mathcal{S}'$ we may assume, w.l.o.g., that $X \subseteq Q$ and $Y \subseteq V'$. Then $|X| \leq b$ or $|Y| \leq b$. Let Q' be the union of all vertices that are in a set X of at least one biclique $K = ((X, Y), E_K) \in \mathcal{S}'$ with $|X| \leq b$. We claim that $\mathcal{C} = N_G(Q')$. Suppose to the contrary that there is a vertex $C_j \in \mathcal{C} \backslash N_G(Q')$. Consider the set $V_j = \{C_j, v_{j_1}, \ldots, v_{j_{bk}}\}$. Since $C_j \notin N_G(Q')$, we have $V_j \cap N_{G'}(Q') = \emptyset$ by construction of G'. Thus, for each biclique $K = ((X', Y'), E_K) \in \mathcal{S}'$ containing an element $v \in V_j$, it follows that $|X'| > b$ and $|Y'| \leq b$. However, then $|\mathcal{S}'| \geq k + 1$, since $|V_j| > bk$. This means that $|\mathcal{S}| = |\mathcal{S}'| + 1 \geq k + 2$. This is a contradiction. Therefore, we obtain a set $H \subseteq Q$ that is a hitting set of $(Q, \mathcal{C})$ of size at most k by including in H precisely one vertex in $Q' \cap X$ for each $K = ((X, Y), E_K) \in \mathcal{S}'$. □

Remark 16. Since the non-parameterized Hitting Set problem, where k is just part of the input and not a parameter, is well known to be NP-hard, and since the reduction in the proof of Lemma 15 is in fact a polynomial-time reduction, it follows that the non-parameterized b-Biclique Vertex-Cover problem is NP-hard.

5 Final Remarks

We have classified the parameterized complexity of the problems Biclique Cover, Biclique Partition, Biclique Vertex-Cover, and Biclique Vertex-Partition: the first two are fixed-parameter tractable, the latter two are equivalent and not fixed-parameter tractable unless P = NP. It would be interesting to improve our algorithm for Biclique Cover and Biclique Partition. In particular, it would be interesting to improve on the 3^k kernel or to show that under plausible complexity theoretic assumptions a kernelization to a kernel of size polynomial in k is not possible. Our results for the Biclique Vertex-Cover problem are negative. It would be interesting to identify special graph classes for which the problem becomes fixed-parameter tractable, and to determine the complexity of Biclique Vertex-Cover for fixed $k = 2$.

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