Covering Points by Unit Disks of Fixed Location

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Abstract. Given a set $\mathcal{P}$ of points in the plane, and a set $\mathcal{D}$ of unit disks of fixed location, the *discrete unit disk cover* problem is to find a minimum-cardinality subset $\mathcal{D}' \subseteq \mathcal{D}$ that covers all points of $\mathcal{P}$. This problem is a geometric version of the general set cover problem, where the sets are defined by a collection of unit disks. It is still NP-hard, but while the general set cover problem is not approximable within $c \log |\mathcal{P}|$, for some constant c, the discrete unit disk cover problem was shown to admit a constant-factor approximation. Due to its many important applications, e.g., in wireless network design, much effort has been invested in trying to reduce the constant of approximation of the discrete unit disk cover problem. In this paper we significantly improve the best known constant from 72 to 38, using a novel approach. Our solution is based on a 4-approximation that we devise for the subproblem where the points of $\mathcal{P}$ are located below a line l and contained in the subset of disks of $\mathcal{D}$ centered above l. This problem is of independent interest.

1 Introduction

We consider the problem of covering a given set of points in the plane by a given set of unit disks. Formally, we are given a set of points $\mathcal{P}$ in the plane, and a set of disks $\mathcal{D} = \{D_1, D_2, ..., D_n\}$ of radius 1 and centers $\mathcal{O} = \{o_1, o_2, ..., o_n\}$. We would like to find a minimum-cardinality subset $\mathcal{D}' \subseteq \mathcal{D}$, such that for each point $p \in \mathcal{P}$ there exists a disk $D \in \mathcal{D}'$ that contains p. We call this problem *discrete unit disk cover*.

The discrete unit disk cover problem (DUDC) has numerous applications, in particular in wireless network design. We are given a set of potential planar locations for placing base stations, and a set of points in the plane representing static clients. All wireless base stations have the same transmission range, where a client can "hear" the signals of a base station if and only if he/she is located within a disk of radius 1 around the base station. We are required to choose a minimum set of base stations such that each client is served by one or more base stations of the chosen set.

The problem of covering a given set of points by unit disks where the disk center locations are not restricted to a given set of points but rather may be chosen at any point in the plane, is studied in [4,5]. A polynomial-time approximation scheme (PTAS) is given for this problem using a grid-shifting strategy.

T. Tokuyama (Ed.): ISAAC 2007, LNCS 4835, pp. 644–655, 2007.

The discrete case, studied in this paper, where the center locations are restricted to a given set, is harder to approach since in order to cover the points within a constant size square one might need more than a constant number of the given disks.

This problem is a geometric set cover problem, where the given sets are defined by unit disks. It is still NP-hard [6]. However, this geometric restriction on the sets allows us to achieve a constant factor approximation, while the general set cover problem is not approximable within $c \log |\mathcal{P}|$, for some constant c, [9]. Due to the importance of the discrete unit disk cover problem, a continuous attempt has been made to achieve a constant approximation algorithm with a good constant factor. Brönnimann and Goodrich [1] gave an ϵ-net based algorithm where the constant factor is not specified. A 108-approximation for the discrete unit disk cover problem was presented in [2]. Narayanappa and Vojtechovsky [8] later improved this constant to 72 and stated that this is the best constant that can be achieved using their technique. In this paper we show that this constant can be reduced to 38 using a new approach.

Our algorithm is based on the single line problem, in which there exists a separating line such that the points to be covered are all located on one side of the line and contained in unit disks centered on the other side of the line. The covering disks may be chosen from both sides of the line. We present a 4-approximation algorithm for this special case and use this solution for approximating the general case. We partition the plane by a grid of width 3/2 and apply the 4-approximation twice for each grid line (once for each direction). We then consider each grid cell separately in order to take care of the uncovered points.

2 A 4-Approximation for the Single Line Problem

2.1 Setting

Let l be a horizontal line. Let $\mathcal{U}$ denote the disks of $\mathcal{D}$ centered above l and let $\mathcal{L} = \mathcal{D} \setminus \mathcal{U}$. We first provide some notation for the arrangement formed by the disks of $\mathcal{U}$ below l. Let B denote the region below l covered by the disks of $\mathcal{U}$. A disk $D \in \mathcal{U}$ is called a *lower boundary disk* if it contributes an arc to the boundary of B, or equivalently, if there exists a point $p \in D \cap B$ that does not belong to any other disk. (Otherwise it is called a *non-boundary disk*.) We then call the region $D \cap B$ a *lower boundary segment* and the arc $circ(D) \cap B$ a *lower boundary arc* (see Figure 1).

Let $\mathcal{S}$ be the set of all lower boundary segments of $\mathcal{U}$. Consider the arrangement $\texttt{Cells}(\mathcal{S})$ formed by the segments in $\mathcal{S}$. Assume the boundary disks are indexed according to their left intersection point with l, and associate with each cell of $\texttt{Cells}(\mathcal{S})$ the set of the indices of the segments that contain it. The next lemma (whose proof is omitted for lack of space) states that for each cell in $\texttt{Cells}(\mathcal{S})$, the set of indices associated with it forms a consecutive set of indices $i, i+1, \ldots, j$ for some $i \leq j$. We call such a cell an *interval cell* and denote it by $\texttt{icell}(i, j)$. We then say that $\mathcal{S}$ forms a *semi-chain*.

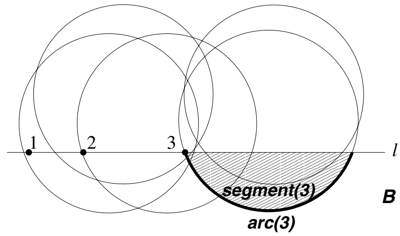

Fig. 1. Segments and arcs

Lemma 1. *Each cell of* $\texttt{Cells}(\mathcal{S})$ *is an interval cell.*

Consider the basic problem of covering the points of $\mathcal{P}$ that belong to B using only lower boundary disks. The following observation implies that the restriction to the set of lower boundary disks (rather than to the set $\mathcal{U}$) increases the size of the solution by a factor of at most 2.

Observation 1. *For any non-boundary segment s, there exist two consecutive boundary disks D_i, D_{i+1} that completely cover s. I.e.,* $\mathcal{P} \cap s \subseteq D_i \cup D_{i+1}$.

Proof. Take the disk D_i to be the boundary disk that appears immediately before s (according to the left intersection point with the line l). □

An interval cell is said to be *occupied* if it contains a point of $\mathcal{P}$. For an interval $[i, j] \subseteq [1, k]$ let $\texttt{Occ}(i, j)$ denote the set of points of $\mathcal{P}$ contained in the occupied cells that correspond to subintervals of $[i, j]$. The following greedy algorithm finds a cover $\mathcal{C}$ of the points of $\mathcal{P}$ that belong to B, using a minimum subset of boundary disks. Initially set $\mathcal{C} = \emptyset$. At each step of the algorithm let i be the largest index such that all the points of $\texttt{Occ}(1, i)$ are covered by the disks in $\mathcal{C} \cup D_i$, and add D_i to $\mathcal{C}$. It is straightforward to see that the cover $\mathcal{C}$ is indeed of minimum cardinality if the covering set must consist of boundary disks.

2.2 Assisted Covers

Let $\mathcal{S}$ be the semi-chain formed by the lower boundary segments. Consider a disk $\tilde{D}$ centered below l, that intersects B. An interval $[i, j] \subseteq [1, k]$ with $i < j$ is said to be *assisted* by $\tilde{D}$ if the set $\{D_i, D_j, \tilde{D}\}$ covers all the points in $\texttt{Occ}(i, j)$. We then say that $\{D_i, D_j, \tilde{D}\}$ is an *assisting set* for $[i, j]$. (For $j = i + 1$, we take the assisting set of $[i, j]$ to be $\{D_i, D_j\}$). A *left assisting pair* of an interval $[i, j]$ with $i < j$ is a pair $\{D_i, \tilde{D}\}$ where $\{D_i, D_j, \tilde{D}\}$ forms an assisting set for $[i, j]$. (For $j = i + 1$ or $i = k$ (the last disk), we take the left assisting pair of $[i, j]$ to be D_i, that is, each chain disk itself is considered to be a left assisting pair.) Given the semi-chain $\mathcal{S}$, an *assisted cover* for $\mathcal{S}$ is a family $\mathcal{F}$ of left assisting pairs that covers all the points of $\mathcal{P}$ contained in $\texttt{Cells}(\mathcal{S})$.

In the example shown in Figure 2, the intervals $[1, 3]$ and $[2, 4]$ are assisted by $\tilde{D}$. The assisting sets are $\{D_1, D_3, \tilde{D}\}$, $\{D_1, D_2\}$, $\{D_2, D_3\}$, $\{D_2, D_4, \tilde{D}\}$ and

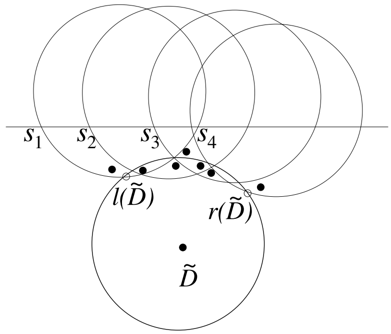

Fig. 2. $\tilde{D}$ assists the intervals $[1, 3]$ and $[2, 4]$

$\{D_3, D_4\}$. The left assisting pairs are $\{D_1, \tilde{D}\}$, $\{D_1\}$, $\{D_2\}$, $\{D_2, \tilde{D}\}$, $\{D_3\}$ and $\{D_4\}$. The family $\mathcal{F} = \{\{D_1\}, \{D_2, \tilde{D}\}, \{D_4\}\}$ forms an assisted cover.

Let $\mathcal{D}^*$ denote a minimum-cardinality (regular) cover of the points of $\mathcal{P}$ contained in B. Note that $\mathcal{D}^*$ can make use of all the disks in $\mathcal{D}$. Put $d^* = |\mathcal{D}^*|$. We now define the *minimum assisted cover* problem for $\mathcal{S}$, and show that a solution to this problem approximates d^*.

Minimum Assisted Cover Problem: Given a semi-chain $\mathcal{S}$, find a minimum-cardinality assisted cover $\mathcal{F}$ for $\mathcal{S}$.

For this problem we have the following lemma.

Lemma 2. *The minimum assisted cover problem has a polynomial-time solution.*

Proof. Given the semi-chain $\mathcal{S}$, the solution is constructed via an immediate reduction to the *minimum half-open interval cover* problem, defined as follows. Given a set $\mathcal{I}$ of points on the real line, and a family $\mathcal{J}$ of half-open intervals of the form $[a, b)$ (where a and b are both integers), find a minimum-cardinality family $\mathcal{J}' \subseteq \mathcal{J}$ that covers the points of $\mathcal{I}$ (assuming such a cover exists).

This one-dimensional problem can be solved easily using a greedy algorithm. The reduction is constructed as follows. For each point $p \in \mathcal{P} \cap B$ consider the largest index i such that s_i contains p. Let $\mathcal{I}$ be the set of indices corresponding to the points of $\mathcal{P} \cap B$. The family $\mathcal{J}$ consists of all half-open intervals $[i, j)$ such that for some $\tilde{D}$, $\{D_i, \tilde{D}\}$ is a left assisting pair for the interval $[i, j]$, including the half open intervals of the form $[i, i + 1)$ (and the interval $[k]$). $\square$

Let Single Line denote the procedure that solves the minimum assisted cover problem for the semi-chain $\mathcal{S}$ using the reduction defined above, obtaining a family $\mathcal{F}$ of left assisting pairs. Partition the disks participating in $\mathcal{F}$ into two sets: the set $\mathcal{U}'$ of disks that belong to $\mathcal{S}$ (centered above l) and the set $\mathcal{L}'$ of assisting disks. Clearly $\mathcal{U}' \cup \mathcal{L}'$ is a cover of the points of $\mathcal{P}$ contained in B. Our goal is to show that $|\mathcal{U}' \cup \mathcal{L}'| \leq 4d^*$.

Let us now analyze the sizes of the sets $\mathcal{U}'$ and $\mathcal{L}'$. We have the following lemmas.

Lemma 3. *The family* $\mathcal{F}$ *and the set* $\mathcal{U}'$ *obtained by invoking Procedure* Single Line *satisfy* $|\mathcal{F}| = |\mathcal{U}'|$.

Proof. This holds since the chain disks of the left assisting pairs in $\mathcal{F}$ are distinct. □

Lemma 4. *The sets* $\mathcal{U}'$ *and* $\mathcal{L}'$ *obtained by invoking Procedure* Single Line*, satisfy* $|\mathcal{L}'| \leq |\mathcal{U}'|$.

Proof. Follows from the definition of left assisting pairs; for each assisting disk taken into $\mathcal{L}'$, at least one additional disk is taken into $\mathcal{U}'$. □

Consider an assisting disk $\tilde{D}$. Let $l(\tilde{D})$ (respectively, $r(\tilde{D})$), denote the leftmost (respectively, rightmost) point at which $\tilde{D}$ intersects the boundary of $\mathcal{S}$. Let $left(\tilde{D})$ denote the index i such that D_i contains $l(\tilde{D})$. Let $right(\tilde{D})$ denote the index j such that D_j contains $r(\tilde{D})$. We refer to the disk $D_{left(\tilde{D})}$ (respectively, $D_{right(\tilde{D})}$) as the *left bounding disk* (respectively, *right bounding disk*) of $\tilde{D}$. In Figure 2, $left(\tilde{D}) = 1$ and $right(\tilde{D}) = 4$.

For two assisting disks $\tilde{D}$ and $\tilde{D}'$, we say that $\tilde{D}$ is *dominated* by $\tilde{D}'$ with respect to the interval $[i, j]$ if all points in $\texttt{Occ}(i, j)$ that are covered by $\tilde{D}$ are also covered by $\tilde{D}'$. An assisting disk $\tilde{D}$ is called *strong assisting* if its center point $o(\tilde{D})$ lies above at least one of the points $l(\tilde{D})$ or $r(\tilde{D})$ (defined above). Otherwise $\tilde{D}$ is called *weak assisting*. For an assisting disk $\tilde{D}$, let the *left-right arc* of $\tilde{D}$ denote the upper part of $circ(\tilde{D})$ enclosed between $l(\tilde{D})$ and $r(\tilde{D})$.

We now have the following observations.

Observation 2. *Consider an interval* $[i, j]$ *and two weak assisting disks* $\tilde{D}$ *and* $\tilde{D}'$ *such that* $left(\tilde{D}) \leq i$ *and* $left(\tilde{D}') \leq i$. *Then either* $circ(\tilde{D})$ *and* $circ(\tilde{D}')$ *intersect each other exactly once in* $\texttt{Cells}(i + 1, j - 1)$*, or there is a dominance relationship between* $\tilde{D}$ *and* $\tilde{D}'$ *with respect to* $[i + 1, j - 1]$.

Proof. By the definition of weak assisting disks, the left-right arcs of $\tilde{D}$ and $\tilde{D}'$ belong to the upper half-circles of $circ(\tilde{D})$ and $circ(\tilde{D}')$, respectively. Therefore these arcs intersect each other at most once within $\texttt{Cells}(\mathcal{S})$. If they do not intersect each other within $\texttt{Cells}(i+1, j-1)$ (see Figure 3(b)), then there must be a dominance relationship between them with respect to $[i+1, j-1]$. Moreover, as shown in Figure 3(a), if they do intersect each other within $\texttt{Cells}(i+1, j-1)$, then the subarc of the left-right arc of $\tilde{D}$ to the right of the intersection point is contained in $\tilde{D}'$ (or vice versa). □

Observation 3. *Let* $\tilde{D}$ *be a strong assisting disk, and suppose w.l.o.g. that* $o(\tilde{D})$ *is above* $r(\tilde{D})$. *Then* $\tilde{D}$ *intersects its right bounding disk above the line* l.

Proof. Let D be the right bounding disk of $\tilde{D}$, and let a and b denote the two intersection points of D and $\tilde{D}$, where $a = r(\tilde{D})$. By symmetry considerations, segments $\overline{o(\tilde{D}), a}$ and $\overline{o(D), b}$ are parallel. Therefore b must be above $o(D)$ which is above l. □

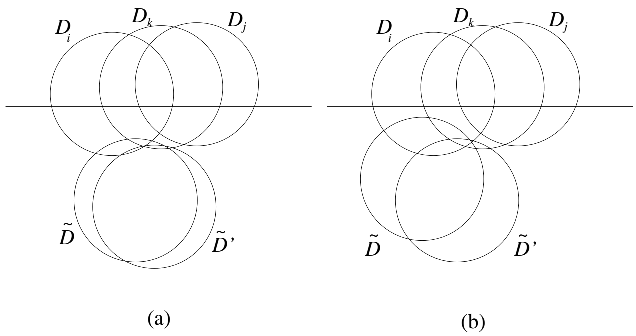

Fig. 3. Proof of Observation 2

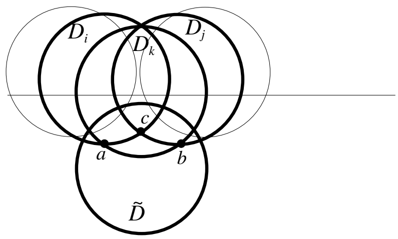

Fig. 4. $\{D_i, D_j, \tilde{D}\}$ is an assisting set for the interval $[i, j]$

Observation 4. *Consider an assisting disk $\tilde{D}$ that covers the intersection point of $arc(i)$ and $arc(j)$ of the chain $\mathcal{S}$, such that $left(\tilde{D}) \leq i$ and $right(\tilde{D}) \geq j$. Then $\{D_i, D_j, \tilde{D}\}$ is an assisting set for the interval $[i, j]$.*

Proof. Consider the disks D_i, D_j and D_k, where $i < k < j$. Consider an assisting disk $\tilde{D}$ that satisfies the conditions of Observation 4, i.e., $\tilde{D}$ contains the intersection point c of $arc(i)$ and $arc(j)$ and $left(\tilde{D}) \leq i$ and $right(\tilde{D}) \geq j$. Let a be the intersection point of D_i and D_k below l, and let b be the intersection point of D_k and D_j below l. As shown is Figure 4, $\tilde{D}$ intersects D_i in two points, with one point to the left of a and one point to the right of c. This implies that the arc between a and c is contained in $\tilde{D}$. Similarly, the arc between c and b is contained in $\tilde{D}$. Also $\tilde{D}$ intersects D_k in two points, with one point to the left of a and one point to the right of b. Therefore, all the area of $segment(k)$ outside $D_i \cup D_j$ is contained in $\tilde{D}$. □

We now show that the number of disks participating in the minimum assisted cover $\mathcal{F}$ is bounded by four times the size d^* of D^*, the minimum-cardinality cover of the points of $\mathcal{P}$ contained in B.

Lemma 5. *The set $\mathcal{U}'$ obtained by invoking Procedure* Single Line *satisfies $|\mathcal{U}'| \leq 2d^*$.*

Proof. Let $\mathcal{U}^*$ be the subset of $\mathcal{U}$ used in the optimal solution $\mathcal{D}^*$, and let $\mathcal{L}^*$ be the assisting disks used in $\mathcal{D}^*$. We now transform $\mathcal{U}^*$ into a set of boundary disks that, together with $\mathcal{L}^*$, forms an assisted cover.

If $\mathcal{U}^*$ contains non-boundary segments, then we replace each non-boundary segment s in $\mathcal{U}^*$ by the two boundary segments that contain it (see Observation 1). This manipulation results in a set $\mathcal{U}^{**}$ of boundary disks with $|\mathcal{U}^{**}| \leq 2|\mathcal{U}^*|$.

We now differentiate between the strong and weak assisting disks in $\mathcal{L}^*$. Set $\mathcal{U}^w = \emptyset$. Note that by Observation 2, the weak assisting disks $\tilde{D}$ in $\mathcal{L}^*$ can be ordered from left to right according to their leftmost intersection $l(\tilde{D})$ with the boundary of B. For each disk $\tilde{D}_i$ in this ordered set, if the left-right arcs of $\tilde{D}_i$ and $\tilde{D}_{i+1}$ intersect each other within B, then add a disk of $\mathcal{S}$ that contains this intersection point to the set $\mathcal{U}^w$. Otherwise, add $D_{right(\tilde{D}_i)}$ to $\mathcal{U}^w$.

For the strong assisting disks, let $\mathcal{U}^s$ be the set of their left and right bounding disks, i.e.,

$$\mathcal{U}^s = \{D_i \mid i = left(\tilde{D}) \ \text{ or } \ i = right(\tilde{D}) \ \text{ for some strong } \ \tilde{D} \in \mathcal{L}^*\} \ .$$

Consider the combined set of upper disks $\mathcal{U}^{**} \cup \mathcal{U}^w \cup \mathcal{U}^s$. We now show that this combined set forms together with $\mathcal{L}^*$ an assisted cover.

Consider two consecutive disks D_i and D_j in $\mathcal{U}^{**} \cup \mathcal{U}^w \cup \mathcal{U}^s$, under the ordering of the semi-chain $\mathcal{S}$. If there are occupied inner cells, i.e., if $\texttt{Occ}(i+1, j-1) \neq \emptyset$, then the points in these cells are covered in the optimal solution D^* by the disks of $\mathcal{L}^*$. We will show that a single disk of $\mathcal{L}^*$ is enough to cover the points in $\texttt{Occ}(i+1, j-1)$. Let $\mathcal{L}^*_{(i,j)}$ denote the set of disks in $\mathcal{L}^*$ that actually participate in covering the points in $\texttt{Occ}(i+1, j-1)$ in the optimal solution. (I.e., disks that are dominated with respect to $[i+1, j-1]$ are not taken into $\mathcal{L}^*_{(i,j)}$). If $\mathcal{L}^*_{(i,j)}$ includes a strong assisting disk $\tilde{D}$, we know that $left(\tilde{D})$ is outside the interval $[i+1, j-1]$ (because $D_{left(\tilde{D})} \in \mathcal{U}^s$ and $D_k \notin \mathcal{U}^s$ for $i < k < j$) and similarly $right(\tilde{D})$ is outside the interval $[i+1, j-1]$. Consider the following two cases. If both $left(\tilde{D})$ and $right(\tilde{D})$ are on the same side of $[i+1, j-1]$, then $\tilde{D}$ does not cover any internal points of $\texttt{Cells}(i+1, j-1)$ and thus cannot assist this interval. If $left(\tilde{D}) \leq i$ and $right(\tilde{D}) \geq j$, then by the definition of strong assisting disk and by Observation 4, we have that $\{D_i, D_j, \tilde{D}\}$ is an assisting set for this interval. Therefore, if such a strong assisting disks exists then no more disks are needed.

Otherwise, we know that $\mathcal{L}^*_{(i,j)}$ consists only of weak assisting disks and let $\tilde{D}$ be the leftmost disk in $\mathcal{L}^*_{(i,j)}$. But if more than one weak assisting disk is needed to cover the points in $\texttt{Occ}(i+1, j-1)$, then by Observation 2, the successor of $\tilde{D}$ in the weak assisting ordering, intersects $\tilde{D}$ within $\texttt{Cells}(i+1, j-1)$. This cannot happen since D_i and D_j are consecutive. (Note that the left-right arcs of $\tilde{D}$ and its successor are not disjoint, otherwise we would have added the right bounding disk of $\tilde{D}$).

We have shown that for each pair of consecutive disks $D_i, D_j \in \mathcal{U}^{**} \cup \mathcal{U}^w \cup \mathcal{U}^s$ there exists at most one assisting disk $\tilde{D} \in \mathcal{L}^*$ such that $\{D_i, D_j, \tilde{D}\}$ is an assisting set for the interval $[i, j]$. We can now claim that the set $\mathcal{L}^* \cup \mathcal{U}^{**} \cup \mathcal{U}^w \cup \mathcal{U}^s$ forms an assisted cover $\mathcal{F}'$ and the number of left assisting pairs in $\mathcal{F}'$ is at most $|\mathcal{U}^{**} \cup \mathcal{U}^w \cup \mathcal{U}^s| \leq 2(|\mathcal{U}^*| + |\mathcal{L}^*|)$.

As Procedure Single Line finds a minimum-cardinality assisted cover $\mathcal{F}$ for $\mathcal{S}$, recalling Lemma 3 we have that $|\mathcal{U}'| = |\mathcal{F}| \leq |\mathcal{F}'| \leq 2(|\mathcal{U}^*| + |\mathcal{L}^*|) = 2d^*$. □

Corollary 1. *The number of disks of $\mathcal{D}$ participating in $\mathcal{F}$ is at most $4d^*$.*

Proof. The number of disks participating in $\mathcal{F}$ is $|\mathcal{U}'| + |\mathcal{L}'| \leq 2|\mathcal{U}'| \leq 4d^*$. □

The following theorem summarizes the result of this section.

Theorem 1. *One can compute a 4-approximation for the Single Line Problem by invoking Procedure* Single Line.

3 A 38-Approximation Algorithm for DUDC

In this section we present an approximation algorithm for our main problem: Given a set $\mathcal{P}$ of points in the plane and a set $\mathcal{D}$ of unit disks, find a subset $\mathcal{D}' \subseteq \mathcal{D}$ of minimum cardinality, such that $\mathcal{P} \subseteq \cup_{D \in \mathcal{D}'} D$. We show that this algorithm computes a 38-approximation.

The algorithm first lays a regular grid over the input scene, such that the distance between two consecutive vertical lines (alternatively, horizontal lines) is 3/2. Let $\mathcal{V}$ (resp., $\mathcal{H}$) be the set of vertical lines (resp., horizontal lines) of the grid.

The algorithm consists of two stages. In the first stage, for each line $l \in \mathcal{V} \cup \mathcal{H}$ such that there exists a disk in $\mathcal{D}$ that is intersected by l, we apply the 4-approximation algorithm for the single line problem (presented in Section 2) twice; once for each side of l. For a more detailed description, assume w.l.o.g. that l is vertical and let $\mathcal{D}_l^l \subseteq \mathcal{D}$ (resp., $\mathcal{D}_l^r \subseteq \mathcal{D}$) be the subset of disks that are intersected by l and whose centers lie to the left (resp., to the right) of l. We apply the 4-approximation algorithm twice. Once to the set $\mathcal{D}_l^l$ (using the disks in $\mathcal{D} \setminus \mathcal{D}_l^l$ as assisting disks), in order to cover the points in $\mathcal{P}$ that lie in the union of the disks in $\mathcal{D}_l^l$ and to the right of l, and once to the set $\mathcal{D}_l^r$ in order to cover the points in $\mathcal{P}$ that lie in the union of the disks in $\mathcal{D}_l^r$ and to the left of l.

Consider now an arbitrary point $p \in \mathcal{P}$. If there exists a disk in $\mathcal{D}$ that contains p and whose center does not lie in the same grid-square as p, then p is already covered in the first stage of the algorithm. Let $\mathcal{Q} \subseteq \mathcal{P}$ be the subset of points that are not yet covered. Then, for each $p \in \mathcal{Q}$, p can only be contained in disks whose centers lie in the grid-square of p. Thus, in the second stage, we consider each (non-empty) grid-square separately. For each such square $\mathcal{S}$ (of side length 3/2), we would like to cover the points in $\mathcal{Q} \cap S$ by disks whose centers lie in $\mathcal{S}$. This can be done by applying the 6-approximation algorithm described in Section 4.

3.1 Analysis

It is clear that at the end of the second stage each point in $\mathcal{P}$ is covered. We now prove that the size of the subset (i.e., cover) computed by our algorithm is

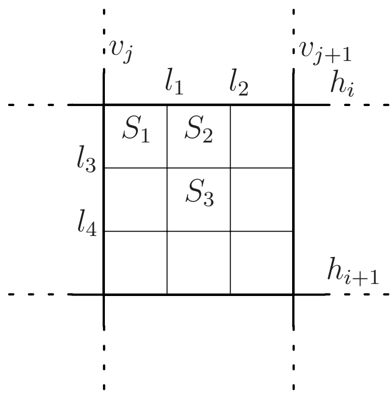

Fig. 5. The cell S

at most 38 times the size of an optimal cover. We first claim that a disk in $\mathcal{D}$ can participate in at most 8 applications of the algorithm of Section 2.

Claim. Let D be a disk in $\mathcal{D}$. Then, in the first stage of the above algorithm, D can participate in at most 8 applications of the algorithm of Section 2.

Proof. Let o be the center of D, and let S be the grid-square in which o lies. Divide S into 9 equal squares by lines l_1, l_2, l_3, and l_4, as depicted in Figure 5. We distinguish between three cases, depending on the location of o within S.

Case 1: $o \in S_1$ (or in any other corner sub-square of S). In this case D can participate as a non-assisting disk in at most 2 applications of the algorithm of Section 2 (namely, v_j (left) and h_i (up)). It can also participate as an assisting disk in at most 6 applications (namely, v_{j-1} (right), v_j (right), v_{j+1} (left), and h_{i-1} (down), h_i (down), h_{i+1} (up)). Thus in total D can participate in at most 8 applications of the algorithm of Section 2.
Case 2: $o \in S_3$ (i.e., in the middle sub-square of S). In this case D can participate as a non-assisting disk in 4 applications of the algorithm of Section 2 (namely, v_j (left), v_{j+1} (right), and h_i (up), h_{i+1} (down)). It can also participate as an assistant disk in 4 applications (namely, v_j (right), v_{j+1} (left), h_i (down), and h_{i+1} (up)). Thus in total D can participate in at most 8 applications.
Case 3: $o \in S_2$ (or in any other of the remaining sub-squares of S). In this case D can participate as a non-assisting disk in 3 applications of the algorithm of Section 2 (namely, v_j (left), v_{j+1} (right), and h_i (up)). It can also participate as an assisting disk in 5 applications of the algorithm of Section 2 (namely, v_j (right), v_{j+1} (left), and h_{i-1} (down), h_i (down), h_{i+1} (up)). Thus in total D can participate in at most 8 applications. □

Theorem 2. *The algorithm above computes a 38-approximation for* DUDC*.*

Proof. Consider a disk D in an optimal solution. By Claim 3.1 we know that D can contribute to the solution of at most eight single line problems and one single square problem. Since each of these problems is solved separately, and since the approximation ratio for the single line problem is 4 and for the single square problem is 6, we obtain that the approximation ratio of the algorithm above is $8 \times 4 + 1 \times 6 = 38$. □

Remark. The choice of grid-square size $3/2 \times 3/2$ seems to be optimal (in our approach). By increasing the grid-square size one can reduce the number of applications of the single line algorithm a disk participates in. For example, for a square size 2×2 this number is 6, and for a square of size 3 this number is only 4. However, the approximation ratio for the single square problem increases, and the final approximation ratio that is obtained is greater than 38. Trying to decrease the squares to, e.g., $\sqrt{2} \times \sqrt{2}$ increases the number of applications of the single line algorithm a disk participates in to 10, which already gives a final approximation ratio that is greater than 38.

4 A 6-Approximation for the Single $(\frac{3}{2} \times \frac{3}{2})$-Square Problem

Let S be a grid square of side length 3/2. In this section we devise a constant-factor approximation algorithm for covering the points $\mathcal{P}'$ of $\mathcal{P}$ that lie in S using the centers $\mathcal{O}'$ of $\mathcal{O}$ that lie in S; moreover we show that this constant is 6.

We divide the square S into nine equal squares by lines l_1, l_2, h_1 and h_2 as shown in Figure 6, and distinguish between the different cases according to the location of the center points of $\mathcal{O}'$ with respect to the nine subsquares. For each such case, we first check if there exists an optimal solution consisting of at most two centers, and if yes we return this solution. Otherwise, we apply the appropriate combination of claims from the following series of nine claims, and verify that by doing so we obtain an at most 6-approximation. For lack of space only the first two claims are included in this version.

Claim 1. *Any two centers o_i and o_j such that $o_i \in S_1$ and $o_j \in S_3$ satisfy $S_2 \subseteq D(o_i) \cup D(o_j)$, where $D(o)$ is a unit disk centered at o.*

Proof. Let p be a point in S_2, and assume w.l.o.g. that p lies in the right side of S_2. Then the disk $D(o_j)$ covers p. □

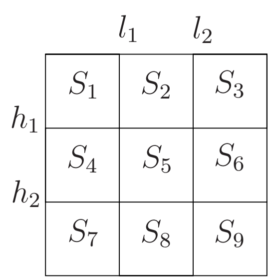

Fig. 6. Square S

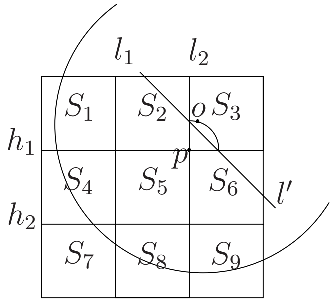

Fig. 7. Claim 2

Remark. Claim 1, as well as all subsequent claims, has several symmetric formulations.

Claim 2. *Consider the centers that lie in S_3 (alternatively S_1, S_7, or S_9). Then there exists a line l' such that all the centers in S_3 lie above l', and there exists a center o in S_3 such that all points in S_2 and S_6 above l' are covered by $D(o)$.*

Proof. Let o be the center in S_3 closest to the intersection point p of lines h_1 and l_2. Draw a disk centered at p of radius $d(p, o)$ (where $d(p, o)$ is the distance between p and o), and let l' be the line defined by the intersection points of this disk with the boundary of S_3. See Figure 7.

W.l.o.g. we show that $D(o)$ covers all points of S_6 above l'. Notice that the greatest distance between a point in the triangle formed in S_6 and a point on the arc is S_3 is determined by the intersection point of l' and the right side of S and the intersection point of l' and l_2; moreover this distance is actually the length of the diagonal of the squares S_i $(1 \leq i \leq 9)$ which is smaller than one. Therefore, regardless of the location of o on the arc, $D(o)$ covers all points of S_6 above l'. □

We now use the claims to obtain a 6-approximation for the singe $(3/2 \times 3/2)$-square problem. There are many cases to consider, depending on the location of the center points. We have generated all of the cases systematically, and have verified for each of them that an at most 6-approximation can be computed by applying the appropriate combination of claims from the series of claims. Due to the large number of cases and the resemblance between them, let us consider two cases for example.

Example 1. All the centers are in one square S_i $(1 \leq i \leq 9)$. In this case we optimally solve the problem of covering the points of $\mathcal{P}'$ using the centers in S_i, applying the algorithm of Lev-Tov [7] (that is not restricted to congruent disks).
Example 2. Assume all the centers are in squares S_1 and S_8. Apply Claim 2 to the square S_1 to obtain a line l' and a center $o \in S_1$, such that all the centers in S_1 are above l' and all points above l' in S_2 and in S_4 are covered by $D(o)$. We now apply the algorithm of Section 2 to the line l' using the centers in S_8 as

assisting centers. Finally, the remaining uncovered points are covered optimally using only the centers in S_8. It is easy to verify that the approximation factor in this case is 6 (actually, $5\frac{1}{3}$).

The following theorem summarizes the main result of this section.

Theorem 3. *One can compute a 6-approximation for the single* $(3/2 \times 3/2)$-*square problem.*

References

1. Brönnimann, H., Goodrich, M.T.: Almost optimal set covers in finite VC-dimension. Discrete Computational Geometry 14, 463–479 (1995)
2. Calinescu, G., Mandoiu, I.I., Wan, P.-J., Zelikovsky, A.: Selecting forwarding neighbors in wireless ad hoc networks. MONET 9(2), 101–111 (2004)
3. Clarkson, K.L., Varadarajan, K.: Improved approximation algorithms for geometric set cover. In: Proc. 21st ACM Sympos. Computational Geometry, pp. 135–141 (2005)
4. Gonzalez, T.: Covering a set of points in multidimensional space. Information Processing Letters 40, 181–188 (1991)
5. Hochbaum, D.S., Maas, W.: Approximation schemes for covering and packing problems in image processing and VLSI. J. ACM 32, 130–136 (1985)
6. Johnson, D.S.: The NP-completeness column: An ongoing guide. J. Algorithms 3, 182–195 (1982)
7. Lev-Tov, N.: Algorithms for Geometric Optimization Problems in Wireless Networks. Ph.D. Dissertation, Weizmann Institute of Science (2005)
8. Narayanappa, S., Vojtechovsky, P.: An improved approximation factor for the unit disk covering problem. In: Proc. 18th Canadian Conf. Computational Geometry, pp. 15–18 (2006)
9. Raz, R., Safra, S.: A sub-constant error-probability low-degree test, and a sub-constant error-probability PCP characterization of NP. In: Proc. 29th ACM Sympos. Theory of Computing, pp. 475–484 (1997)