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# Covering the integers by arithmetic sequences

by

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**1. Introduction.** Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}^+$ the set of positive reals. For $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$ we call

$$\alpha + \beta\mathbb{Z} = \{\ldots, \alpha - 2\beta, \alpha - \beta, \alpha, \alpha + \beta, \alpha + 2\beta, \ldots\}$$

an *arithmetic sequence with common difference* $\beta$. In the case $\alpha \in \mathbb{Z}$ and $\beta \in \mathbb{Z}^+$, $\alpha + \beta\mathbb{Z}$ is just the residue class $\alpha \bmod \beta$ with modulus $\beta$.

Let $m$ be a positive integer. A finite system

(1) $$\mathcal{A} = \{\alpha_s + \beta_s\mathbb{Z}\}_{s=1}^k \quad (\alpha_1, \ldots, \alpha_k \in \mathbb{R} \text{ and } \beta_1, \ldots, \beta_k \in \mathbb{R}^+)$$

of arithmetic sequences is said to be an (exact) *m-cover* of $\mathbb{Z}$ if it covers each integer at least (resp., exactly) $m$ times. Instead of "1-cover" and "exact 1-cover" we use the terms "cover" and "exact cover" respectively.

Since they were introduced by P. Erdős ([5]) in the early 1930's, covers of $\mathbb{Z}$ by (finitely many) residue classes have been studied seriously and many nice applications have been found. (Cf. sections A19, B21, E23, F13 and F14 of R. K. Guy [9].) For problems and results in this area we refer the reader to surveys of Erdős [7, 8], Š. Porubský [13] and Š. Znám [21]. Recently further progress was made by various authors.

If a finite system

(2) $$A = \{a_s + n_s\mathbb{Z}\}_{s=1}^k \quad (a_1, \ldots, a_k \in \mathbb{Z} \text{ and } n_1, \ldots, n_k \in \mathbb{Z}^+)$$

of residue classes forms an $m$-cover of $\mathbb{Z}$, then $\sum_{s=1}^k 1/n_s \geq m$, and the equality holds if and only if (2) is an exact $m$-cover of $\mathbb{Z}$. This becomes apparent if we calculate

$$\sum_{s=1}^{k} |\{0 \leq x < N : x \equiv a_s \pmod{n_s}\}| = \sum_{x=0}^{N-1} |\{1 \leq s \leq k : x \equiv a_s \pmod{n_s}\}|$$

where $N$ is the least common multiple of $n_1, \ldots, n_k$.

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In this paper we investigate properties of $m$-covers of $\mathbb{Z}$ in the form (1). In the next section we shall give three equivalent conditions for (1) to be an $m$-cover of $\mathbb{Z}$. One is that (1) covers $W$ consecutive integers at least $m$ times where

$$W=\left|\left\{\left\{\sum_{s\in I}\frac{1}{\beta_s}\right\}:I\subseteq\{1,\ldots,k\}\right\}\right|$$

($[x]$ and $\{x\}$ stand for the integral and fractional parts of a real $x$ respectively throughout the paper), the other two are finite systems of equalities (not inequalities) involving roots of unity. Our tools used to deduce them include Vandermonde determinants, Stirling numbers, a little analysis and linear algebra.

In Sections 3 and 4 we will derive a number of results including the following ones:

(I) Let (1) be an $m$-cover of $\mathbb{Z}$ and $J\subseteq\{1,\ldots,k\}$. Then

$$\left\{\sum_{s\in I}\frac{1}{\beta_s}\right\}=\left\{\sum_{s\in J}\frac{1}{\beta_s}\right\}\quad\text{for some } I\subseteq\{1,\ldots,k\} \text{ with } I\neq J,$$

provided $\sum_{s=1}^k 1/\beta_s=m$ (e.g. (1) is an exact $m$-cover of $\mathbb{Z}$ with $\alpha_s\in\mathbb{Z}$ and $\beta_s\in\mathbb{Z}^+$ for $s=1,\ldots,k$) we have $\sum_{s\in I}1/\beta_s=\sum_{s\in J}1/\beta_s$ for some $I\subseteq\{1,\ldots,k\}$ with $I\neq J$ if $\emptyset\neq J\subset\{1,\ldots,k\}$, when $\sum_{s\in I}1/\beta_s=\sum_{s\in J}1/\beta_s$ for no $I\subseteq\{1,\ldots,k\}$ with $I\neq J$ there are at least $m$ nonzero integers of the form $\sum_{s\in I}1/\beta_s-\sum_{s\in J}1/\beta_s$ where $I\subseteq\{1,\ldots,k\}$.

(II) Let $k\geq l\geq 0$ be integers. Then $2^{k-l}(l+1)$ is the smallest $n\in\mathbb{Z}^+$ such that any system of $k$ arithmetic sequences with at least $l$ equal common differences covers an arithmetic sequence at least $m$ times if it covers $n$ consecutive terms in the sequence at least $m$ times.

The last section contains some unsolved problems related to possible extensions.

**2. Characterizations of $m$-covers.** Let us provide several technical lemmas the first of which serves as the starting point of our new approach.

LEMMA 1. *Let $m\in\mathbb{Z}^+$ and $x\in\mathbb{R}$. Then* (1) *covers $x$ at least $m$ times if and only if*

$$\prod_{s=1}^k(1-r^{1/\beta_s}e^{2\pi i(\alpha_s-x)/\beta_s})=o((1-r)^{m-1})\quad(r\to 1).\tag{3}$$

Proof. Set $I=\{1\leq s\leq k: x\in\alpha_s+\beta_s\mathbb{Z}\}$ and $I'=\{1,\ldots,k\}\setminus I$.

Clearly,

$$\lim_{r\to 1}\frac{\prod_{s=1}^{k}(1-r^{1/\beta_s}e^{2\pi i(\alpha_s-x)/\beta_s})}{(1-r)^{|I|}}$$
$$=\lim_{r\to 1}\prod_{s\in I'}(1-r^{1/\beta_s}e^{2\pi i(\alpha_s-x)/\beta_s})\cdot\lim_{r\to 1}\prod_{s\in I}\frac{1-r^{1/\beta_s}}{1-r}$$
$$=\prod_{s\in I'}(1-e^{2\pi i(\alpha_s-x)/\beta_s})\cdot\prod_{s\in I}\frac{d}{dr}(r^{1/\beta_s})\Big|_{r=1}$$
$$=\prod_{s\in I'}(1-e^{2\pi i(\alpha_s-x)/\beta_s})\cdot\prod_{s\in I}\beta_s^{-1}\neq 0,$$

and hence

$$\lim_{r\to 1}\frac{\prod_{s=1}^{k}(1-r^{1/\beta_s}e^{2\pi i(\alpha_s-x)/\beta_s})}{(1-r)^{m-1}}$$
$$=\lim_{r\to 1}\frac{\prod_{s=1}^{k}(1-r^{1/\beta_s}e^{2\pi i(\alpha_s-x)/\beta_s})}{(1-r)^{|I|}}(1-r)^{|I|-m+1}$$
$$=\begin{cases}0 & \text{if } |I|>m-1,\\ \infty & \text{if } |I|<m-1.\end{cases}$$

Now it is apparent that $|I|\ge m$ if and only if (3) holds. We are done.

LEMMA 2. *Let $\theta_1,\ldots,\theta_n$ be real numbers with distinct fractional parts. For any $\varepsilon>0$ there exists a $\delta>0$ such that if*

$$\Big|\sum_{t=1}^{n}e^{2\pi i s\theta_t}x_t\Big|<\delta$$

*for every $s=1,\ldots,n$ then $|x_t|<\varepsilon$ for all $t=1,\ldots,n$.*

Proof. Let $A$ be the matrix $(e^{2\pi i s\theta_t})_{\substack{1\le s\le n\\ 1\le t\le n}}$. Then

$$\frac{|A|}{e^{2\pi i\theta_1}e^{2\pi i\theta_2}\ldots e^{2\pi i\theta_n}}=\begin{vmatrix}1 & 1 & \ldots & 1\\ e^{2\pi i\theta_1} & e^{2\pi i\theta_2} & \ldots & e^{2\pi i\theta_n}\\ (e^{2\pi i\theta_1})^2 & (e^{2\pi i\theta_2})^2 & \ldots & (e^{2\pi i\theta_n})^2\\ \ldots & \ldots & \ldots & \ldots\\ (e^{2\pi i\theta_1})^{n-1} & (e^{2\pi i\theta_2})^{n-1} & \ldots & (e^{2\pi i\theta_n})^{n-1}\end{vmatrix}$$

is a determinant of Vandermonde's type. As $|A|\neq 0$ the inverse matrix of $A$ exists; we denote it by $B=(b_{st})_{\substack{1\le s\le n\\ 1\le t\le n}}$.

Let $b=\max\{|b_{st}|:s,t=1,\ldots,n\}>0$ and $\delta=\varepsilon/(bn)$. Let $x_1,\ldots,x_n$

be any complex numbers, and set

$$y_s = \sum_{t=1}^{n} e^{2\pi i s\theta_t} x_t \quad \text{for } s = 1, \ldots, n.$$

Let

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \vec{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}.$$

Then $\vec{x} = BA\vec{x} = B\vec{y}$. If $|y_s| < \delta$ for every $s = 1, \ldots, n$, then

$$|x_s| = \Big|\sum_{t=1}^{n} b_{st} y_t\Big| \le \sum_{t=1}^{n} b|y_t| < bn\delta = \varepsilon \quad \text{for all } s = 1, \ldots, n.$$

This concludes the proof.

LEMMA 3. *Let* $m \in \mathbb{Z}^+$. *Then*

$$\sum_{n=0}^{m-1} a_n t^{n-m+1} = o(1) \quad (t \to 0) \tag{4}$$

*if and only if* $a_0 = \ldots = a_{m-1} = 0$.

Proof. The "if" direction is trivial. When $a_0, \ldots, a_{m-1}$ are not all zero, for the least $k$ such that $a_k \ne 0$ we have

$$\sum_{n=0}^{m-1} a_n (x^{-1})^{n-m+1} = \sum_{n=k}^{m-1} a_n x^{m-1-n} \sim a_k x^{m-1-k} \quad (x \to \infty),$$

which contradicts (4). This ends the proof.

LEMMA 4. *Let* $n \ge m > 0$ *be integers and* $a_1, \ldots, a_n$ *distinct numbers. Then the system*

$$\begin{cases} x_1 + \ldots + x_n = 0, \\ a_1 x_1 + \ldots + a_n x_n = 0, \\ a_1^2 x_1 + \ldots + a_n^2 x_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_1^{m-1} x_1 + \ldots + a_n^{m-1} x_n = 0, \end{cases} \tag{5}$$

*is equivalent to*

$$\begin{cases} a_{11} x_1 + \ldots + a_{1n} x_n = 0, \\ a_{21} x_1 + \ldots + a_{2n} x_n = 0, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1} x_1 + \ldots + a_{mn} x_n = 0, \end{cases} \tag{6}$$

*where*

$$a_{st} = \prod_{\substack{i=1\\ i\neq s}}^{m} \frac{a_i - a_t}{a_i - a_s} \quad \textit{for } s = 1, \ldots, m \textit{ and } t = 1, \ldots, n.$$

P r o o f. Rewrite (5) in the form

$$\begin{cases} x_1 + \ldots + x_m = -\sum\limits_{m<t\le n} x_t, \\ a_1x_1 + \ldots + a_mx_m = -\sum\limits_{m<t\le n} a_tx_t, \\ a_1^2x_1 + \ldots + a_m^2x_m = -\sum\limits_{m<t\le n} a_t^2x_t, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_1^{m-1}x_1 + \ldots + a_m^{m-1}x_m = -\sum\limits_{m<t\le n} a_t^{m-1}x_t. \end{cases}$$

By Cramer's rule, this says that

$$x_s = \begin{vmatrix} 1 & \ldots & 1 & -\sum_{m<t\le n} x_t & 1 & \ldots & 1 \\ a_1 & \ldots & a_{s-1} & -\sum_{m<t\le n} a_tx_t & a_{s+1} & \ldots & a_m \\ a_1^2 & \ldots & a_{s-1}^2 & -\sum_{m<t\le n} a_t^2x_t & a_{s+1}^2 & \ldots & a_m^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1^{m-1} & \ldots & a_{s-1}^{m-1} & -\sum_{m<t\le n} a_t^{m-1}x_t & a_{s+1}^{m-1} & \ldots & a_m^{m-1} \end{vmatrix}$$

$$\times \begin{vmatrix} 1 & \ldots & 1 \\ a_1 & \ldots & a_m \\ a_1^2 & \ldots & a_m^2 \\ \ldots & \ldots & \ldots \\ a_1^{m-1} & \ldots & a_m^{m-1} \end{vmatrix}^{-1}$$

$$= -\sum_{m<t\le n} x_t \begin{vmatrix} 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\ a_1 & \ldots & a_{s-1} & a_t & a_{s+1} & \ldots & a_m \\ a_1^2 & \ldots & a_{s-1}^2 & a_t^2 & a_{s+1}^2 & \ldots & a_m^2 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ a_1^{m-1} & \ldots & a_{s-1}^{m-1} & a_t^{m-1} & a_{s+1}^{m-1} & \ldots & a_m^{m-1} \end{vmatrix}$$

$$\times \begin{vmatrix} 1 & \ldots & 1 \\ a_1 & \ldots & a_m \\ a_1^2 & \ldots & a_m^2 \\ \ldots & \ldots & \ldots \\ a_1^{m-1} & \ldots & a_m^{m-1} \end{vmatrix}^{-1}$$

$$= -\sum_{m<t\le n} x_t \frac{\prod_{1\le i<s}(a_t-a_i)\cdot\prod_{s<i\le m}(a_i-a_t)\cdot\prod_{\substack{1\le i<j\le m\\ i,j\ne s}}(a_j-a_i)}{\prod_{1\le i<s}(a_s-a_i)\cdot\prod_{s<i\le m}(a_i-a_s)\cdot\prod_{\substack{1\le i<j\le m\\ i,j\ne s}}(a_j-a_i)}$$

$$= -\sum_{m<t\le n} a_{st}x_t \qquad \text{(Vandermonde)}$$

for every $s=1,\ldots,m$, i.e.

$$\sum_{t=1}^{m}\delta_{st}x_t+\sum_{m<t\le n}a_{st}x_t=0 \quad \text{for } s=1,\ldots,m$$

where $\delta_{st}$ is the Kronecker delta. Since $a_{st}=\delta_{st}$ for $s,t=1,\ldots,m$, we have finished the proof.

Now we are ready to present

Theorem 1. *Let $\mathcal{A}=\{\alpha_s+\beta_s\mathbb{Z}\}_{s=1}^{k}$, where $\alpha_1,\ldots,\alpha_k\in\mathbb{R}$ and $\beta_1,\ldots,\beta_k\in\mathbb{R}^+$. Let $m\in\mathbb{Z}^+$ and*

$$S=\left\{0\le\theta<1:\left\{\sum_{s\in I}\frac{1}{\beta_s}\right\}=\theta \text{ for some } I\subseteq\{1,\ldots,k\}\right\}.$$

*Let*

$$V(\theta)=\left\{\sum_{s\in I}\frac{1}{\beta_s}:I\subseteq\{1,\ldots,k\} \text{ and } \sum_{s\in I}\frac{1}{\beta_s}-\theta\in\mathbb{Z}\right\}$$

*and $U(\theta)$ be a set of $m$ distinct numbers comparable with $V(\theta)$ (i.e. $|U(\theta)|=m$, and either $U(\theta)\subseteq V(\theta)$ or $U(\theta)\supseteq V(\theta)$). Then the following statements are equivalent*:

(a) *$\mathcal{A}$ is an $m$-cover of $\mathbb{Z}$.*

(b) *$\mathcal{A}$ covers $|S|$ consecutive integers at least $m$ times.*

(c) *For each $\theta\in S$,*

$$\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}}(-1)^{|I|}\binom{[\sum_{s\in I}1/\beta_s]}{n}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}=0 \tag{7}$$

*holds for every $n=0,1,\ldots,m-1$. (As usual $\binom{x}{n}$ denotes $\frac{x(x-1)\ldots(x-n+1)}{1\cdot2\cdot\ldots\cdot(n-1)n}$.)*

(d) *For any $\theta\in S$,*

$$\sum_{v\in V(\theta)}a_{uv}f(v)=0 \quad \text{for all } u\in U(\theta), \tag{8}$$

*where*

$$a_{uv}=\prod_{\substack{x\in U(\theta)\\ x\neq u}}\frac{x-v}{x-u}\quad \textit{and}\quad f(v)=\sum_{\substack{I\subseteq\{1,\dots,k\}\\ \Sigma_{s\in I}1/\beta_s=v}}(-1)^{|I|}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}.$$

P r o o f. (a)⇒(b). This is obvious.

(b)⇒(c). Suppose that each of $x+1,\dots,x+|S|$ is covered by $\mathcal{A}$ at least $m$ times, where $x$ is an integer. By Lemma 1 for every $n=1,\dots,|S|$ we have

$$\begin{aligned}0&=\lim_{r\to1}\frac{\prod_{s=1}^{k}(1-r^{1/\beta_s}e^{2\pi i(\alpha_s-x-n)/\beta_s})}{(1-r)^{m-1}}\\&=\lim_{r\to1}\Big((1-r)^{1-m}\\&\quad\times\sum_{I\subseteq\{1,\dots,k\}}(-1)^{|I|}r^{\Sigma_{s\in I}1/\beta_s}e^{2\pi i\Sigma_{s\in I}(\alpha_s-x)/\beta_s}e^{-2\pi in\Sigma_{s\in I}1/\beta_s}\Big)\\&=\lim_{r\to1}\sum_{\theta\in S}F(r,\theta)e^{-2\pi in\theta},\end{aligned}$$

where

$$F(r,\theta)=\sum_{\substack{I\subseteq\{1,\dots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}}(-1)^{|I|}r^{\Sigma_{s\in I}1/\beta_s}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}e^{-2\pi ix\theta}/(1-r)^{m-1}.$$

Let $\varepsilon$ be an arbitrary positive number. By Lemma 2 there is an $\eta>0$ such that if

$$\Big|\sum_{\theta\in S}e^{-2\pi in\theta}x_\theta\Big|<\eta$$

for every $n=1,\dots,|S|$ then $|x_\theta|<\varepsilon$ for all $\theta\in S$. Since

$$\sum_{\theta\in S}F(r,\theta)e^{-2\pi in\theta}=o(1)\quad(r\to1)\qquad\text{for } n=1,\dots,|S|,$$

there exists a $\delta>0$ such that whenever $|r-1|<\delta$,

$$\Big|\sum_{\theta\in S}F(r,\theta)e^{-2\pi in\theta}\Big|<\eta\qquad\text{for all } n=1,\dots,|S|$$

and hence by the above $|F(r,\theta)|<\varepsilon$ for each $\theta\in S$. This shows that $\lim_{r\to1}F(r,\theta)=0$ for every $\theta\in S$.

For any $\theta\in S$ we have

$$\begin{aligned}0&=\lim_{r\to 1}\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}}(-1)^{|I|}r^{\Sigma_{s\in I}1/\beta_s}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}/(1-r)^{m-1}\\
&=\lim_{t\to 0}\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}}(-1)^{|I|}(1-t)^{[\Sigma_{s\in I}1/\beta_s]+\theta}t^{1-m}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}\\
&=\lim_{t\to 0}\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}}(-1)^{|I|}\sum_{n=0}^{[\Sigma_{s\in I}1/\beta_s]}\binom{[\sum_{s\in I}1/\beta_s]}{n}(-t)^n t^{1-m}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}\\
&=\lim_{t\to 0}\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}}\Bigg((-1)^{|I|}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}\\
&\qquad\times\sum_{\substack{n=0\\ n\le[\Sigma_{s\in I}1/\beta_s]}}^{m-1}\binom{[\sum_{s\in I}1/\beta_s]}{n}(-1)^n t^{n-m+1}\Bigg)\\
&=\lim_{t\to 0}\sum_{n=0}^{m-1}(-1)^n\Bigg(\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}}(-1)^{|I|}\binom{[\sum_{s\in I}1/\beta_s]}{n}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}\Bigg)t^{n-m+1}.\end{aligned}$$

In view of Lemma 3, (7) holds for every $n=0,1,\ldots,m-1$. Therefore part (c) follows.

(c)⇒(d). Fix $\theta\in S$. For each $n=0,1,\ldots,m-1$,

$$x^n=\sum_{j=0}^{n}S(n,j)x(x-1)\ldots(x-j+1)$$

where $S(n,j)$ $(0\le j\le n)$ are Stirling numbers of the second kind, so by (c) we have

$$\begin{aligned}&\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}}(-1)^{|I|}\Big[\sum_{s\in I}1/\beta_s\Big]^n e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}\\
&\qquad=\sum_{j=0}^{n}j!S(n,j)\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}}(-1)^{|I|}\binom{[\sum_{s\in I}1/\beta_s]}{j}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}=0,\end{aligned}$$

i.e.

$$\sum_{v\in V(\theta)}\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \Sigma_{s\in I}1/\beta_s=v}}(-1)^{|I|}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}[v]^n=0.$$

Case 1: $|V(\theta)| \le m$. In this case

$$\sum_{v\in V(\theta)} [v]^n f(v) = 0 \quad \text{for every } n = 0,1,\ldots,|V(\theta)|-1.$$

Hence (8) holds since $f(v) = 0$ for all $v \in V(\theta)$ (Vandermonde).

Case 2: $|V(\theta)| > m$. In this case, $U(\theta) \subset V(\theta)$ and

$$\sum_{v\in U(\theta)} [v]^n f(v) + \sum_{v\in V(\theta)\setminus U(\theta)} [v]^n f(v) = 0$$

for each $n = 0,1,\ldots,m-1$. According to Lemma 4,

$$\sum_{v\in V(\theta)} a_{uv} f(v) = \sum_{v\in V(\theta)} \Bigg( \prod_{\substack{x\in U(\theta)\\ x\ne u}} \frac{[x]-[v]}{[x]-[u]} \Bigg) f(v) = 0 \quad \text{for all } u \in U(\theta).$$

So in either case we have (8).

(d)⇒(a). Assume that (d) holds. Let $\theta \in S$. For $u, v \in U(\theta)$,

$$a_{uv} = \prod_{\substack{x\in U(\theta)\\ x\ne u}} \frac{x-v}{x-u} = \begin{cases} 1 & \text{if } u = v,\\ 0 & \text{if } u \ne v.\end{cases}$$

Case 1: $|V(\theta)| \le m$. In this case $V(\theta) \subseteq U(\theta)$. As

$$f(u) = \sum_{v\in V(\theta)} a_{uv} f(v) = 0 \quad \text{for each } u \in V(\theta),$$

we get

$$\sum_{v\in V(\theta)} f(v)[v]^n = 0 \quad \text{for all } n = 0,1,2,\ldots$$

Case 2: $|V(\theta)| > m$. In this case $U(\theta) \subset V(\theta)$, so for any $u \in U(\theta)$ and $v \in V(\theta)$ we have $\{u\} = \{v\} = \theta$ and hence $[u]-[v] = u - v$. Since

$$\sum_{v\in V(\theta)} \Bigg( \prod_{\substack{x\in U(\theta)\\ x\ne u}} \frac{[x]-[v]}{[x]-[u]} \Bigg) f(v) = \sum_{v\in V(\theta)} a_{uv} f(v) = 0$$

for every $u \in U(\theta)$, it follows from Lemma 4 that

$$\sum_{v\in V(\theta)} f(v)[v]^n = \sum_{v\in U(\theta)} [v]^n f(v) + \sum_{v\in V(\theta)\setminus U(\theta)} [v]^n f(v) = 0$$

for all $n = 0,1,\ldots,m-1$.

In both cases,

$$\sum_{v\in V(\theta)} f(v)[v]^n = 0 \quad \text{for } n = 0,1,\ldots,m-1.$$

Thus for each nonnegative integer $n < m$,

$$\sum_{v\in V(\theta)} f(v)\binom{[v]}{n} = \sum_{v\in V(\theta)} f(v)\sum_{j=0}^{n}(-1)^{n-j}s(n,j)[v]^j/n!$$
$$= \frac{1}{n!}\sum_{j=0}^{n}(-1)^{n-j}s(n,j)\sum_{v\in V(\theta)} f(v)[v]^j = 0,$$

where $s(n,j)$ $(0\le j\le n)$ are Stirling numbers of the first kind, i.e.

$$\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|}\binom{[\sum_{s\in I}1/\beta_s]}{n}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}$$
$$= \sum_{v\in V(\theta)}\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \Sigma_{s\in I}1/\beta_s=v}}\binom{[v]}{n}(-1)^{|I|}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s} = 0.$$

Therefore by the proof of (b)$\Rightarrow$(c),

$$\lim_{r\to 1}\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|}r^{\Sigma_{s\in I}1/\beta_s}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}/(1-r)^{m-1}$$
$$= \lim_{t\to 0}\sum_{n=0}^{m-1}(-1)^n\Bigg(\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}}(-1)^{|I|}\binom{[\sum_{s\in I}1/\beta_s]}{n}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}\Bigg)t^{n-m+1}$$
$$= 0.$$

Now for every integer $x$,

$$\prod_{s=1}^{k}(1-r^{1/\beta_s}e^{2\pi i(\alpha_s-x)/\beta_s})$$
$$= \sum_{I\subseteq\{1,\ldots,k\}}(-1)^{|I|}r^{\Sigma_{s\in I}1/\beta_s}e^{2\pi i\Sigma_{s\in I}(\alpha_s-x)/\beta_s}$$
$$= \sum_{\theta\in S}e^{-2\pi ix\theta}\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}}(-1)^{|I|}r^{\Sigma_{s\in I}1/\beta_s}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}$$
$$= \sum_{\theta\in S}e^{-2\pi ix\theta}o((1-r)^{m-1}) = o((1-r)^{m-1}) \quad (r\to 1).$$

Applying Lemma 1 we then obtain part (a).

The proof of Theorem 1 is now complete.

**3. Reciprocals of common differences.** In 1989 M. Z. Zhang [19] showed the following surprising result analytically: Provided that (2) is a cover of $\mathbb{Z}$, $\sum_{s\in I} 1/n_s \in \mathbb{Z}^+$ for some $I \subseteq \{1,\ldots,k\}$. Here we give

THEOREM 2. *Let* (1) *be a cover of* $\mathbb{Z}$. *Then for any* $J \subseteq \{1,\ldots,k\}$ *there is an* $I \subseteq \{1,\ldots,k\}$ *with* $I \neq J$ *such that*

$$\sum_{s\in I}\frac{1}{\beta_s} - \sum_{s\in J}\frac{1}{\beta_s} \in \mathbb{Z}. \tag{9}$$

Proof. Set $\theta = \{\sum_{s\in J} 1/\beta_s\}$. By Theorem 1,

$$\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|}\binom{[\sum_{s\in I}1/\beta_s]}{0} e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s} = 0,$$

that is,

$$\sum_{\substack{J\neq I\subseteq\{1,\ldots,k\}\\ \{\Sigma_{s\in I}1/\beta_s\}=\theta}} (-1)^{|I|} e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s} = -(-1)^{|J|}e^{2\pi i\Sigma_{s\in J}\alpha_s/\beta_s}.$$

Therefore

$$\left\{I\subseteq\{1,\ldots,k\} : I\neq J \text{ and } \left\{\sum_{s\in I}\frac{1}{\beta_s}\right\}=\theta\right\}\neq\emptyset.$$

We are done.

In the case $J=\emptyset$, Theorem 2 yields a generalization of Zhang's result ([19]).

Provided that (1) is an $m$-cover of $\mathbb{Z}$ with $m\in\mathbb{Z}^+$, Theorem 2 asserts that for any $J\subseteq\{1,\ldots,k\}$,

$$S(J) = \left\{I\subseteq\{1,\ldots,k\} : I\neq J \text{ and } \sum_{s\in I}\frac{1}{\beta_s}-\sum_{s\in J}\frac{1}{\beta_s}\in\mathbb{Z}\right\} \tag{10}$$

is nonempty. This becomes trivial if

$$\sum_{s\in I}\frac{1}{\beta_s} = \sum_{s\in J}\frac{1}{\beta_s} \quad \text{for some } I\subseteq\{1,\ldots,k\} \text{ with } I\neq J. \tag{11}$$

What can we say about

$$Z(J) = \left\{\sum_{s\in I}\frac{1}{\beta_s}-\sum_{s\in J}\frac{1}{\beta_s} : I\in S(J)\right\} \tag{12}$$

if it does not contain zero? The following theorem gives us more information.

THEOREM 3. *Assume that* (1) *is an* $m$-*cover of* $\mathbb{Z}$. *Let* $J$ *be a subset of* $\{1,\ldots,k\}$ *such that* (11) *fails*, *i.e.* $0\notin Z(J)$ *where* $S(J)$ *and* $Z(J)$ *are given by* (10) *and* (12). *Then*

(i) $|Z(J)| \ge m$ *and hence*

$$\sum_{s=1}^{k}\frac{1}{\beta_s} \ge md(J)+\Big\{\sum_{s\in J}\frac{1}{\beta_s}\Big\} \ge m, \tag{13}$$

*where* $d(J)$ *is the least positive integer that can be written as the difference of two* (*distinct*) *numbers of the form*

$$\sum_{s\in I}\frac{1}{\beta_s} \in \mathbb{Z}+\sum_{s\in J}\frac{1}{\beta_s} \quad \textit{where } I\subseteq\{1,\dots,k\}.$$

(ii) *When* $d(J) \ge [\sum_{s=1}^{k} 1/\beta_s]/m$, $d(J)$ *equals* $[\sum_{s=1}^{k} 1/\beta_s]/m$ *and divides* $[\sum_{s\in J} 1/\beta_s]$, *and for every* $n=0,1,\dots,m$ *there exist at least*

$$\binom{m}{n}\Big/\binom{m}{m[\sum_{s\in J}1/\beta_s]/[\sum_{s=1}^{k}1/\beta_s]}$$

*subsets* $I$ *of* $\{1,\dots,k\}$ *such that*

$$\sum_{s\in I}\frac{1}{\beta_s} = \frac{n}{m}\Big[\sum_{s=1}^{k}\frac{1}{\beta_s}\Big]+\Big\{\sum_{s\in J}\frac{1}{\beta_s}\Big\}, \tag{14}$$

*hence*

$$|S(J)| \ge 2^m\Big/\binom{m}{m[\sum_{s\in J}1/\beta_s]/[\sum_{s=1}^{k}1/\beta_s]}-1 \quad \textit{and} \quad |Z(J)|=m.$$

Proof. Let $\theta=\{\sum_{s\in J}1/\beta_s\}$, $V(\theta)$, $U(\theta)$ and $f(x)$ be as in Theorem 1. If $|V(\theta)|\le m$, then $V(\theta)\subseteq U(\theta)$, hence by Theorem 1 for all $u\in V(\theta)\subseteq U(\theta)$,

$$f(u)=\sum_{v\in V(\theta)}\Big(\prod_{\substack{x\in U(\theta)\\ x\ne u}}\frac{x-v}{x-u}\Big)f(v)=0,$$

which is impossible since $0\notin Z(J)$ and

$$f\Big(\sum_{s\in J}\frac{1}{\beta_s}\Big)=(-1)^{|J|}e^{2\pi i\Sigma_{s\in J}\alpha_s/\beta_s}\ne 0.$$

Thus $|V(\theta)|>m$.

(i) Let $v_0<v_1<\dots<v_m$ be the first $m+1$ elements of $V(\theta)$ in ascending order. Clearly

$$1+|Z(J)|=|Z(J)\cup\{0\}|=\Big|\Big\{v-\sum_{s\in J}\frac{1}{\beta_s}: v\in V(\theta)\Big\}\Big|=|V(\theta)|\ge m+1$$

and

$$\sum_{s=1}^{k}\frac{1}{\beta_s}\ge\max_{v\in V(\theta)}v\ge v_m=\sum_{i=0}^{m-1}(v_{i+1}-v_i)+v_0\ge md(J)+\theta.$$

(ii) If $|V(\theta)| > m+1$ then

$$\sum_{s=1}^{k} \frac{1}{\beta_s} \geq \max_{v \in V(\theta)} v \geq v_m + 1 \geq 1 + md(J) + \theta.$$

Now suppose that $d(J) \geq [\sum_{s=1}^{k} 1/\beta_s]/m$. Then we must have $|V(\theta)| = m+1$, thus $V(\theta) = \{v_0, v_1, \ldots, v_m\}$ and $|Z(J)| = |V(\theta)| - 1 = m$. As

$$md(J) \geq \left[\sum_{s=1}^{k} \frac{1}{\beta_s}\right] \geq [v_m] = v_0 - \theta + \sum_{i=0}^{m-1}(v_{i+1} - v_i) \geq [v_0] + md(J),$$

$$md(J) = \left[\sum_{s=1}^{k} \frac{1}{\beta_s}\right], \quad [v_0] = 0$$

and

$$[v_n] = v_0 - \theta + \sum_{i=0}^{n-1}(v_{i+1} - v_i) = 0 + \sum_{i=0}^{n-1} d(J) = nd(J)$$

for $n = 1, \ldots, m$.

Choose $0 \leq j \leq m$ such that $v_j = \sum_{s \in J} 1/\beta_s$. Then

$$j = \frac{[v_j]}{d(J)} = m\left[\sum_{s \in J} \frac{1}{\beta_s}\right] \Big/ \left[\sum_{s=1}^{k} \frac{1}{\beta_s}\right].$$

Set

$$U'(\theta) = \{v_i : 0 \leq i \leq m,\ i \neq j\}.$$

By Theorem 1, for any $n = 0, 1, \ldots, m$ with $n \neq j$,

$$\begin{aligned} 0 &= \sum_{v \in V(\theta)} \Bigg( \prod_{\substack{x \in U'(\theta) \\ x \neq v_n}} \frac{x - v}{x - v_n} \Bigg) f(v) = \sum_{t=0}^{m} \Bigg( \prod_{\substack{i=0 \\ i \neq j,n}}^{m} \frac{v_i - v_t}{v_i - v_n} \Bigg) f(v_t) \\ &= \sum_{t=0}^{m} \Bigg( \prod_{\substack{i=0 \\ i \neq j,n}}^{m} \frac{id(J) + \theta - (td(J) + \theta)}{id(J) + \theta - (nd(J) + \theta)} \Bigg) f(v_t) \\ &= \sum_{t=0}^{m} \Bigg( \prod_{\substack{i=0 \\ i \neq j,n}}^{m} \frac{i - t}{i - n} \Bigg) f(v_t) \\ &= f(v_n) + \Bigg( \prod_{\substack{i=0 \\ i \neq j,n}}^{m} \frac{i - j}{i - n} \Bigg) f(v_j). \end{aligned}$$

Since

$$\prod_{\substack{i=0\\ i\neq j,n}}^{m} \frac{i-j}{i-n} = \frac{\prod_{i=0,\, i\neq j}^{m}(i-j)}{n-j} \Big/ \frac{\prod_{i=0,\, i\neq n}^{m}(i-n)}{j-n}$$
$$= -\frac{\prod_{i=0}^{j-1}(i-j)\cdot\prod_{i=j+1}^{m}(i-j)}{\prod_{i=0}^{n-1}(i-n)\cdot\prod_{i=n+1}^{m}(i-n)}$$
$$= -\frac{(-1)^j j!(m-j)!}{(-1)^n n!(m-n)!} = (-1)^{j-n+1}\binom{m}{n}\Big/\binom{m}{j},$$

we have

$$\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \Sigma_{s\in I}1/\beta_s=nd(J)+\theta}} (-1)^{|I|}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}$$
$$= f(v_n) = -(-1)^{j-n+1}\binom{m}{n}\binom{m}{j}^{-1} f\Big(\sum_{s\in J}\frac{1}{\beta_s}\Big)$$
$$= (-1)^{j-n}\binom{m}{n}\binom{m}{j}^{-1}(-1)^{|J|}e^{2\pi i\Sigma_{s\in J}\alpha_s/\beta_s}$$

and hence

$$\Big|\Big\{I\subseteq\{1,\ldots,k\}: \sum_{s\in I}\frac{1}{\beta_s} = \frac{n}{m}\Big[\sum_{s=1}^{k}\frac{1}{\beta_s}\Big] + \Big\{\sum_{s\in J}\frac{1}{\beta_s}\Big\}\Big\}\Big|$$
$$= \sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \Sigma_{s\in I}1/\beta_s=nd(J)+\theta}} 1$$
$$\geq \Big|\sum_{\substack{I\subseteq\{1,\ldots,k\}\\ \Sigma_{s\in I}1/\beta_s=nd(J)+\theta}} (-1)^{|I|}e^{2\pi i\Sigma_{s\in I}\alpha_s/\beta_s}\Big| = \binom{m}{n}\Big/\binom{m}{j};$$

therefore

$$1+|S(J)| = \Big|\Big\{I\subseteq\{1,\ldots,k\}: \sum_{s\in I}\frac{1}{\beta_s}\in V(\theta)\Big\}\Big|$$
$$= \sum_{n=0}^{m}\Big|\Big\{I\subseteq\{1,\ldots,k\}: \sum_{s\in I}\frac{1}{\beta_s} = v_n = nd(J)+\theta\Big\}\Big|$$
$$\geq \sum_{n=0}^{m}\binom{m}{n}\Big/\binom{m}{j} = 2^m\Big/\binom{m}{j}.$$

This ends the proof.

Now let us apply Theorem 3 to those $m$-covers (1) with $\sum_{s=1}^{k}1/\beta_s = m$.

THEOREM 4. *Let* (1) *be an* $m$*-cover of* $\mathbb{Z}$ *with* $\sum_{s=1}^{k} 1/\beta_s = m \in \mathbb{Z}^+$, *which happens if* (1) *is an exact* $m$*-cover of* $\mathbb{Z}$ *by residue classes. Then*

(i) *For every* $l = 1, \ldots, k-1$ *we have*

$$\sum_{s=l+1}^{k} \frac{1}{\beta_s} \ge \frac{1}{\max\{\beta_1, \ldots, \beta_l\}}. \tag{15}$$

(ii) *For any* $\emptyset \ne J \subset \{1, \ldots, k\}$ *there exists an* $I \subseteq \{1, \ldots, k\}$ *with* $I \ne J$ *such that*

$$\sum_{s \in I} \frac{1}{\beta_s} = \sum_{s \in J} \frac{1}{\beta_s}, \tag{16}$$

*furthermore when* $\sum_{s \in J} 1/\beta_s \in \mathbb{Z}$ *there are at least*

$$\binom{m}{\sum_{s \in J} 1/\beta_s} \ge m > 1$$

*subsets* $I$ *of* $\{1, \ldots, k\}$ *satisfying* (16).

Proof. (i) For $l = 1, \ldots, k-1$ (15) follows from part (ii) in the case $J = \{l+1, \ldots, k\}$, so we proceed to the proof of part (ii).

(ii) If (11) fails then by part (i) of Theorem 3 and the equality $\sum_{s=1}^{k} 1/\beta_s = m$ we must have

$$\left\{ \sum_{s \in J} \frac{1}{\beta_s} \right\} = 0, \quad \text{i.e.} \quad \sum_{s \in J} \frac{1}{\beta_s} \in \mathbb{Z}.$$

Observe that

$$0 < \sum_{s \in J} \frac{1}{\beta_s} < \sum_{s=1}^{k} \frac{1}{\beta_s} = m.$$

If $\sum_{s \in J} 1/\beta_s \in \mathbb{Z}$, then $m > 1$ and $\sum_{s \in J} 1/\beta_s = n$ for some $n = 1, \ldots, m-1$, by part (ii) of Theorem 3 there are at least $\binom{m}{n}/\binom{m}{m} = \binom{m}{n} \ge m$ subsets $I$ of $\{1, \ldots, k\}$ such that

$$\sum_{s \in I} \frac{1}{\beta_s} = \frac{n}{m} \left[ \sum_{s=1}^{k} \frac{1}{\beta_s} \right] + \left\{ \sum_{1 \le s \le k} \frac{1}{\beta_s} \right\} = n = \sum_{s \in J} \frac{1}{\beta_s}.$$

We are done.

Remark. In 1992 Z. W. Sun ([17]) proved that if (2) is an exact $m$-cover of $\mathbb{Z}$ then for each $n=1, \ldots, m$ there exist at least $\binom{m}{n}$ subsets $I$ of $\{1, \ldots, k\}$ such that $\sum_{s \in I} 1/n_s$ equals $n$. The lower bounds $\binom{m}{n}$ $(1 \le n \le m)$ are best possible, and the Riemann zeta function was used in the proof.

From Theorem 3 we can also deduce the following theorem which extends Zhang's result ([19]) and the theorem of Sun [17] even in the case $l = k$.

THEOREM 5. *Let* (1) *be an $m$-cover of $\mathbb{Z}$ and $l$ a positive integer not exceeding $k$ such that*

$$\min\left\{1, \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_l}\right\} > \sum_{l<t\le k} \frac{1}{\beta_t}, \tag{17}$$

*where $\sum_{l<t\le k} 1/\beta_t$ is considered to be zero for $l = k$. Then*

(i) *There are at least $m$ positive integers representable by*

$$\sum_{s\in I} \frac{1}{\beta_s} - \sum_{l<t\le k} \frac{1}{\beta_t}, \qquad \text{where } I \subseteq \{1, \ldots, k\}, \tag{18}$$

*thus we have*

$$\sum_{s=1}^{l} \frac{1}{\beta_s} = \sum_{s=1}^{k} \frac{1}{\beta_s} - \sum_{l<t\le k} \frac{1}{\beta_t} \ge m. \tag{19}$$

(ii) *Provided that any positive integer less than $[\sum_{s=1}^k 1/\beta_s]/m$ cannot be expressed as the difference of two integers of the form* (18), *$[\sum_{s=1}^k 1/\beta_s]$ is divisible by $m$ and for each $n = 0, 1, \ldots, m$ there are at least $\binom{m}{n}$ subsets $I$ of $\{1, \ldots, k\}$ such that*

$$\sum_{s\in I} \frac{1}{\beta_s} = \frac{n}{m}\left[\sum_{s=1}^{k} \frac{1}{\beta_s}\right] + \sum_{l<t\le k} \frac{1}{\beta_t}, \tag{20}$$

*hence there exist at least $2^m - 1$ subsets $I$ of $\{1, \ldots, k\}$ with*

$$\sum_{s\in I} \frac{1}{\beta_s} \in \mathbb{Z}^+ + \sum_{l<t\le k} \frac{1}{\beta_t}.$$

Proof. Let $J = \{1 \le t \le k : t > l\}$. By (17),

$$\left[\sum_{t\in J} \frac{1}{\beta_t}\right] = 0 \quad \text{and} \quad \left\{\sum_{t\in J} \frac{1}{\beta_t}\right\} = \sum_{l<t\le k} \frac{1}{\beta_t}.$$

For any $I \subseteq \{1, \ldots, k\}$, if $I \subset J$ then

$$0 < \sum_{t\in J} \frac{1}{\beta_t} - \sum_{s\in I} \frac{1}{\beta_s} < 1,$$

and if $I \not\subseteq J$ then

$$\sum_{s\in I} \frac{1}{\beta_s} - \sum_{t\in J} \frac{1}{\beta_t} \ge \min\left\{\frac{1}{\beta_s} : 1 \le s \le l\right\} - \sum_{l<t\le k} \frac{1}{\beta_t} > 0.$$

So (11) fails, moreover $Z(J)$ given by (12) contains only positive integers. Applying Theorem 3 we obtain the desired results.

Erdős conjectured (before 1950) that if (2) is a cover of $\mathbb{Z}$ with $1 < n_1 < n_2 < \ldots < n_k$ then $\sum_{s=1}^k 1/n_s > 1$. H. Davenport, L. Mirsky, D. Newman and R. Radó confirmed this conjecture (independently) by proving that if (2) is an exact cover of $\mathbb{Z}$ with $1 < n_1 \le \ldots \le n_{k-1} \le n_k$ then $n_{k-1} = n_k$. For further improvements see Znám [20], M. Newman [10], Porubský [11, 12], M. A. Berger, A. Felzenbaum and A. S. Fraenkel [1]. The best record in this direction is the following result due to the author which is partially announced in [15] and completely proved in [16]: Let $\lambda_1, \ldots, \lambda_k$ be complex numbers and $n_0 \in \mathbb{Z}^+$ a period of the function

$$\sigma(x) = \sum_{\substack{s=1 \\ x \equiv a_s \ (\mathrm{mod}\ n_s)}}^{k} \lambda_s.$$

If $d \in \mathbb{Z}^+$ does not divide $n_0$ and

$$\sum_{\substack{s=1 \\ d \mid n_s,\, a_s \equiv a \,(\mathrm{mod}\, d)}}^{k} \frac{\lambda_s}{n_s} \neq 0 \quad \text{for some integer } a,$$

then

$$|\{a_s \bmod d : 1 \le s \le k,\ d \mid n_s\}| \ge \min_{\substack{0 \le s \le k \\ d \nmid n_s}} \frac{d}{\gcd(d, n_s)} \ge p(d),$$

where $p(d)$ is the least prime divisor of $d$. Here we have

THEOREM 6. *Let* (1) *be an $m$-cover of $\mathbb{Z}$ with $\beta_1 \le \ldots \le \beta_{k-l} < \beta_{k-l+1} = \ldots = \beta_k$ where $1 \le l < k$. Then either*

$$l \ge \beta_k / \max\{1, \beta_{k-l}\} \quad (> 1 \text{ if } \beta_k > 1), \tag{21}$$

*or there are at least $m$ positive integers in the form*

$$\sum_{s \in I} \frac{1}{\beta_s} - \frac{l}{\beta_k}, \quad \text{where } I \subseteq \{1, \ldots, k\}, \tag{22}$$

*and hence*

$$\sum_{s=1}^{k} \frac{1}{\beta_s} > \sum_{s=1}^{k-l} \frac{1}{\beta_s} = \sum_{s=1}^{k} \frac{1}{\beta_s} - \frac{l}{\beta_k} \ge m. \tag{23}$$

(*Also*, $\sum_{s=1}^k 1/\beta_s > \sum_{s=1}^k 1/\beta_k \ge k \ge m$ *if* $\beta_k \le 1$.)

Proof. Clearly $l < \beta_k / \max\{1, \beta_{k-l}\}$ if and only if

$$\min\left\{1, \frac{1}{\beta_1}, \ldots, \frac{1}{\beta_{k-l}}\right\} > \sum_{k-l < t \le k} \frac{1}{\beta_t} \ (= l/\beta_k).$$

Therefore Theorem 6 follows from part (i) of Theorem 5.

Note that when $\beta_{k-l} \geq 1$ and $\beta_k/\beta_{k-l} \in \mathbb{Z}$

$$\beta_k/\max\{1, \beta_{k-l}\} = \beta_k/\beta_{k-l} \geq p(\beta_k/\beta_{k-l}) \quad (\geq p(\beta_k) \text{ if } \beta_{k-l}, \beta_k \in \mathbb{Z}).$$

**4. Some local-global results.** In 1958 S. K. Stein [14] conjectured that whenever the residue classes in (2) are pairwise disjoint and the moduli $n_1, \ldots, n_k > 1$ are distinct there exists an integer $x$ with $1 \leq x \leq 2^k$ such that $x$ is not covered by (2). Erdős [6] confirmed this conjecture with $k \cdot 2^k$ instead of $2^k$. Since the Davenport–Mirsky–Newman–Radó result indicates that an exact cover of $\mathbb{Z}$ by (finitely many) residue classes cannot have its moduli distinct and greater than one, Erdős proposed the stronger conjecture that any system of $k$ residue classes not covering all the integers omits a positive integer not exceeding $2^k$. Both conjectures have some local-global character. In 1969 R. B. Crittenden and C. L. Vanden Eynden [2] claimed their positive answer to the stronger conjecture. Later in [3] a long indirect and awkward proof was given for $k \geq 20$, the authors concluded the paper with the statements: "Of course it remains to show the conjecture is true for $k < 20$. This may be checked by more special arguments."

In 1970 Crittenden and Vanden Eynden [4] conjectured further that if all the moduli $n_s$ in (2) are greater than an integer $0 \leq l < k$ then (2) is a cover of $\mathbb{Z}$ if it covers all the integers in the interval $[1, 2^{k-l}(l+1)]$. In contrast with the Crittenden–Vanden Eynden conjecture we give

THEOREM 7. *For any $m \in \mathbb{Z}^+$, (1) is an $m$-cover of $\mathbb{Z}$ if it covers $2^{k-M}(M+1)$ consecutive integers at least $m$ times, where*

$$M = \max_{1\leq t\leq k} |\{1 \leq s \leq k : \beta_s = \beta_t\}|. \tag{24}$$

Proof. Let $\beta > 0$ be a number such that $J = \{1 \leq s \leq k : \beta_s = \beta\}$ has cardinality $M$. As

$$\begin{aligned}
&\left|\left\{\left\{\sum_{s\in I}\frac{1}{\beta_s}\right\} : I \subseteq \{1,\ldots,k\}\right\}\right| \\
&\quad \leq \left|\left\{\sum_{s\in I\cap J}\frac{1}{\beta_s} + \sum_{s\in I\setminus J}\frac{1}{\beta_s} : I \subseteq \{1,\ldots,k\}\right\}\right| \\
&\quad \leq \left|\left\{\sum_{s\in I}\frac{1}{\beta} : I \subseteq J\right\}\right| \cdot \left|\left\{\sum_{s\in I}\frac{1}{\beta_s} : I \subseteq \{1,\ldots,k\}\setminus J\right\}\right| \\
&\quad \leq \left|\left\{\frac{|I|}{\beta} : I \subseteq J\right\}\right| \cdot |\{I : I \subseteq \{1,\ldots,k\}\setminus J\}| \\
&\quad = (|J|+1)\cdot 2^{k-|J|} = 2^{k-M}(M+1),
\end{aligned}$$

Theorem 1 implies Theorem 7.

The following example noted by Crittenden and Vanden Eynden [4] shows that the number $g(k, M) = 2^{k-M}(M+1)$ in Theorem 7 is best possible.

EXAMPLE. Let $M = n - 1 \in \mathbb{Z}^+$. Consider the system $A$ consisting of the following $k \geq M$ residue classes:

$$\begin{gathered} 1 + n\mathbb{Z}, \quad 2 + n\mathbb{Z}, \quad \ldots, \quad M + n\mathbb{Z}, \\ n + 2n\mathbb{Z}, \quad 2n + 2^2 n\mathbb{Z}, \quad \ldots, \quad 2^{k-M-1} n + 2^{k-M} n\mathbb{Z}. \end{gathered}$$

Observe that $A$ together with $2^{k-M} n\mathbb{Z}$ forms an exact cover of $\mathbb{Z}$. So $A$ covers positive integers from 1 to $2^{k-M}(M+1) - 1$, but it does not cover all the integers.

Result (II) stated in Section 1 follows from Theorem 7 and Example, since (1) covers $\alpha + \beta x$ (where $\alpha \in \mathbb{R}$, $\beta \in \mathbb{R}^+$ and $x \in \mathbb{Z}$) at least $m$ times if and only if $\{\frac{\alpha_s - \alpha}{\beta} + \frac{\beta_s}{\beta}\mathbb{Z}\}_{s=1}^k$ covers $x$ at least $m$ times, and $2^{k-l}(l+1) \geq 2^{k-M}(M+1)$ if $k \geq M \geq l > 0$. (The case $l = 0$ can be reduced to the case $l = 1$.)

**5. Several open problems.** Theorem 1 tells us that (2) is a cover of $\mathbb{Z}$ if it covers integers from 1 to

$$\left|\left\{\left\{\sum_{s\in I} \frac{1}{\beta_s}\right\} : I \subseteq \{1, \ldots, k\}\right\}\right| \leq 2^k \leq 2^{n_1 + \ldots + n_k}.$$

This suggests

PROBLEM 1. Can we find a polynomial $P$ with integer coefficients such that a finite system (2) of residue classes forms a cover of $\mathbb{Z}$ whenever it covers all positive integers not exceeding $P(n_1 + \ldots + n_k)$?

In 1973 L. J. Stockmeyer and A. R. Meyer proved that the problem whether there exists an integer not covered by a given (2) is NP-complete. In 1991 S. P. Tung [18] extended this result to algebraic integer rings. If the required $P$ in Problem 1 exists, then there is a polynomial time algorithm to decide whether (2) covers all the integers or not. So a positive answer to Problem 1 would imply that NP = P.

By appearances Theorems 2–7 involve no roots of unity. Perhaps vast generalizations of them could be made.

PROBLEM 2. Let $A_1, \ldots, A_k$ be sets of natural numbers having positive densities $d(A_1), \ldots, d(A_k)$ respectively. If no $A_s$ contains $m_s \in \mathbb{Z}^+$ consecutive integers, does $\bigcup_{s=1}^k A_s$ have density 1 when it covers $m_1 \ldots m_k$ arbitrarily large consecutive integers? Suppose that $\{A_s\}_{s=1}^k$ covers all the natural numbers; does there exist, for any $J \subseteq \{1, \ldots, k\}$, an $I \subseteq \{1, \ldots, k\}$

with $I \neq J$ such that

$$\sum_{s\in I} d(A_s) - \sum_{s\in J} d(A_s) \in \mathbb{Z}?$$

PROBLEM 3. Let $K$ be an algebraic number field and $O_K$ the ring of algebraic integers in $K$. Let $a_1, \ldots, a_k \in O_K$ and $A_1, \ldots, A_k$ be ideals of $O_K$ with norms $N(A_1), \ldots, N(A_k)$ respectively. If $\{a_s + A_s\}_{s=1}^k$ forms an exact $m$-cover of $O_K$ for some $m \in \mathbb{Z}^+$, is it true that for any $\emptyset \neq J \subseteq \{1, \ldots, k\}$ there exists a subset $I$ of $\{1, \ldots, k\}$ with $I \neq J$ such that

$$\sum_{s\in I} \frac{1}{N(A_s)} = \sum_{s\in J} \frac{1}{N(A_s)}?$$

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