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ABSTRACT

Numerical integration plays an important role in satellite orbit determination. This paper presents the general philosophy of numerical integration, a description of the often used multistep numerical integration algorithms pertinent to orbit determination, and the derivation of the formulas and their various forms used in these multistep algorithms. The coefficients for different forms of these formulas are presented in rational form up to order fifteen in the appendix.

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INTRODUCTION: GENERAL PHILOSOPHY

Many problems involving ordinary differential equations cannot be solved explicitly or analytically. It is for this reason that numerical techniques for approximating solutions of such equations were developed. The advent of high speed computers which can handle the tedious arithmetic involved has made these techniques even more attractive and useful. Using a computer, it is possible to extend these numerical techniques to a degree of precision far higher than any hand calculation could ever achieve.

Of particular interest are the *discrete variable methods* which yield approximate solutions of the problem $y' = f(x, y)$ at a set of discrete points x, $x + h$, $x + 2h$, . . . where h is the *step size*. In general, the discrete variable methods applied to initial value problems can be classified as either *one-step* methods or *multistep* methods. The one-step methods require knowledge of the value of the function at only the previous point while the multistep methods require this knowledge at a certain number of preceding values. That is, to approximate the value of the function at $x + h$, a one-step method would need only knowledge of the value of the function at x while a multistep method would require this knowledge at the points x, $x - h$, $x - 2h$, $x - 3h$, . . ., $x - nh$.

At first, one might think that the one-step methods would be more advantageous in obtaining the approximations since they require only one previous value, one *backpoint*. However, the error committed in using the formulas of any one-step process over a given interval is generally larger than the error incurred in a multistep method. Also, to go one step forward with a one-step method requires more evaluations of the function, and, in the multistep method, increasing the *order* (the number of backpoints used) does not necessarily require a concomitant increase in evaluations. Furthermore, since large orders of a multistep method are easily attained, multistep methods are highly accurate with relatively large increments of the independent variable.

In the realm of orbital dynamics, the use of numerical techniques is virtually dictated. It is almost impossible to solve analytically (i.e., explicitly) those equations which represent the motion of a satellite. Analytical solutions such as Brouwer or Two Body Motion are sometimes employed, but at best they use only limited approximations of the real forces which affect a satellite's motion. With the numerical approach, the expressions of these forces do not have to be truncated after the first few terms: they can be expressed in their entirety.

Some of the computer programs which use numerical methods to compute the motion of artificial satellites are:

D.O.D.S. — Definitive Orbit Determination System
May 15, 1968
Space Systems Analysis and Computer Programming Services
Contract NAS 5-10022
Prepared by
Scientific Satellite Systems Department
Federal Systems Division
International Business Machines Corporation
Gaithersburg, Maryland

Noname — An Orbit and Geodetic Parameter Estimation System
Aug. 1968
Contract Number NAS-5-9756-71D
Prepared by
Wolf Research & Development Corporation
Applied Sciences Department
College Park, Maryland
Prepared for
Mission and Trajectory Analysis Division
National Aeronautics and Space Administration
G.S.F.C., Greenbelt, Maryland

Lungfish — Lunar Gravitational Field in Spherical Harmonics
Feb. 1966
Contract No. NAS1-4998
Prepared for the Space Mechanics Division of the Langley Research Center
Prepared by Computer Usage Company, Inc.

Trace — Trace-C Powered Flight Trajectory Determination Program
May 1965
Report No. TOR-469(5352)-1
Prepared by Aerospace Corp. —
C. S. Christensen, A. R. Jacobsen and R. J. Mercer

This paper describes how multistep numerical integration is started with a one-step process, exemplified by the Runge-Kutta method; how the multistep process is used in orbit determination, exemplified by Cowell type formulas; and derivation of predictor and corrector formulas for equations of the first and second orders. Also included, in the appendix, are the coefficients for the multistep methods discussed in the text.

In the discussion, y and f are 3-space vectors. The independent variable is x, while $|y| = (y_1^2 + y_2^2 + y_3^2)^{1/2}$.

DESCRIPTION OF INTEGRATION METHOD

I Starting the Multistep

The multistep numerical integration method of solving differential equations requires a knowledge of preceding values (backpoints). Consider the initial value problem

$$y' = f(x, y(x))$$

$$y(x_0) = y_0 .$$

We need to know the values $y(x_1) = y_1$, $y(x_2) = y_2, \ldots, y(x_{m-1}) = y_{m-1}, y(x_m) = y_m$ where $x_1 = x + h$, $x_2 = x + 2h, \ldots, x_{m-1} = x + (m-1)h$, $x_m = x + mh$, h being the step size. These values are needed to determine from evaluation of $y' = f(x, y(x))$ — more simply written $f(x, y)$ — the backpoints y'_m, $y'_{m-1}, \ldots, y'_2, y'_1, y'_0$ required by the multistep algorithms. (In physical terms, this may be considered as having for each x_i a position y_i and a velocity y'_i.)

To produce the initial backpoints used to start the multistep process, a one-step numerical integration method such as Euler's method, Taylor's expansion, Runge-Kutta, etc., is used. Each of these methods requires a knowledge of only one preceding value of $y(x)$. Thus the initial value $y(x_0) = y_0$ is sufficient to initiate the one-step "starter" for a multistep process.

A commonly used one-step method is the Runge-Kutta which computes $y_1, y_2, \ldots$ as follows: Given the initial value problem

$$y' = f(x,y)$$

$$y(x_0) = y_0 .$$

The formula used is

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 3k_3 + 4k_4)$$

$$n = 0, 1, 2, \ldots$$

where

$$k_1 = hf(x_n, y_n)$$

$$k_2 = hf\left(x_n + \frac{h}{2}, y_n - \frac{k_1}{2}\right)$$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_n + h, y_n + k_3)$$

As can be seen from the above equations, a fourth order Runge-Kutta process requires four evaluations of the derivative $y' = f(x, y)$ for each step forward.

By way of remark, the following should be considered. Applying this Runge-Kutta process to each of the three (usually complex) equations of motion of a satellite to produce position and velocity coordinates is inefficient. Furthermore, to achieve the required accuracy necessary in orbit determination analysis, the step size h must be very small. The error incurred by this fourth order Runge-Kutta is of the order h^5 while the corresponding local error for a multistep process is of the order h^{P+1} where P is the order of the multistep method which is usually higher than 4. Thus, the step size of the Runge-Kutta starter *must* be a fraction of the step size of the multistep process. This is an important consideration in programming the multistep algorithms.

There do exist multistep methods used as starters. These methods employ a time-consuming, iterative procedure to produce each backpoint and it is questionable whether they are more efficient than the one-step methods. In any event, the time required to set up the starting table of initial backpoints for the multistep process is usually a fraction of the total computation time. Any gains in efficiency accrued by these iterative schemes are, at most, marginal while the simplicity of the one-step methods make them desirable.

II. The Multistep Algorithm

Assuming now that for the initial value problem

$$y' = f(x, y)$$

$$y(x_0) = y_0$$

we have generated the backpoints $y'_m, y'_{m-1}, \ldots y'_2, y'_1, y'_0$ by some single step process. (We may write $y'_0 = f_0, y'_1 = f_1, \ldots, y'_m = f_m$ to mean $y'_i = f(x_0 + ih, y(x_0 + ih))$.) With this set of backpoints, $y_0, y_1, y_2, \ldots y_{m-1}, y_m$, the multistep process can be started. These values are used in an *extrapolator* or *predictor* to compute y_{m+1}. The predictor considered here is the Adams-Bashforth (Henrici) which has the form

$$y_{m+1} = y_m + h\left\{\alpha_0 \nabla^0 y'_m + \alpha_1 \nabla^1 y'_m + \alpha_2 \nabla^2 y'_m + \ldots + \alpha_n \nabla^n y'_m\right\}$$

where ∇^i represents a difference operator (discussed later) operating on y'_m and employing the backpoints $y'_m, y'_{m-1}, \ldots, y'_{m-n+2}, y'_{m-n+1}$.

The predicted value of y_{m+1} is used with x_{m+1} to evaluate

$$y' = f(x, y)$$

for y'_{m+1}. This value of y'_{m+1} is then employed in a *corrector* formula which yields a new value for y_{m+1}. The corrector discussed here is the Adams-Moulton (Henrici) which has the form

$$y_{m+1} = y_m + h\left\{a_0^* \nabla^0 y'_{m+1} + a_1^* \nabla^1 y'_{m+1} + a_2^* \nabla^2 y'_{m+1} + \ldots + a_n^* \nabla^n y'_{m+1}\right\}$$

We now have two values for y_{m+1}: a predicted value, say ${}^p y_{m+1}$, and a corrected value, say ${}^c y_{m+1}$. These two values are compared. If the absolute value of their difference, $|{}^c y_{m+1} - {}^p y_{m+1}|$, is not less than a given tolerance, the ${}^{c_1} y_{m+1}$ is used (i.e., substituted for ${}^p y_{m+1}$) with x_{m+1} to again evaluate $f(x, y)$ for a new value of y'_{m+1}. The corrector is then used again with this new value of y'_{m+1} to calculate a new y_{m+1}. This iteration process on the corrector is repeated until $|{}^{c_{i+1}} y_{m+1} - {}^{c_i} y_{m+1}|$, where ${}^{c_1} y_{m+1} = {}^p y_{m+1}$, meets the tolerance. A simple flow chart may describe this more clearly. See Figure 1.

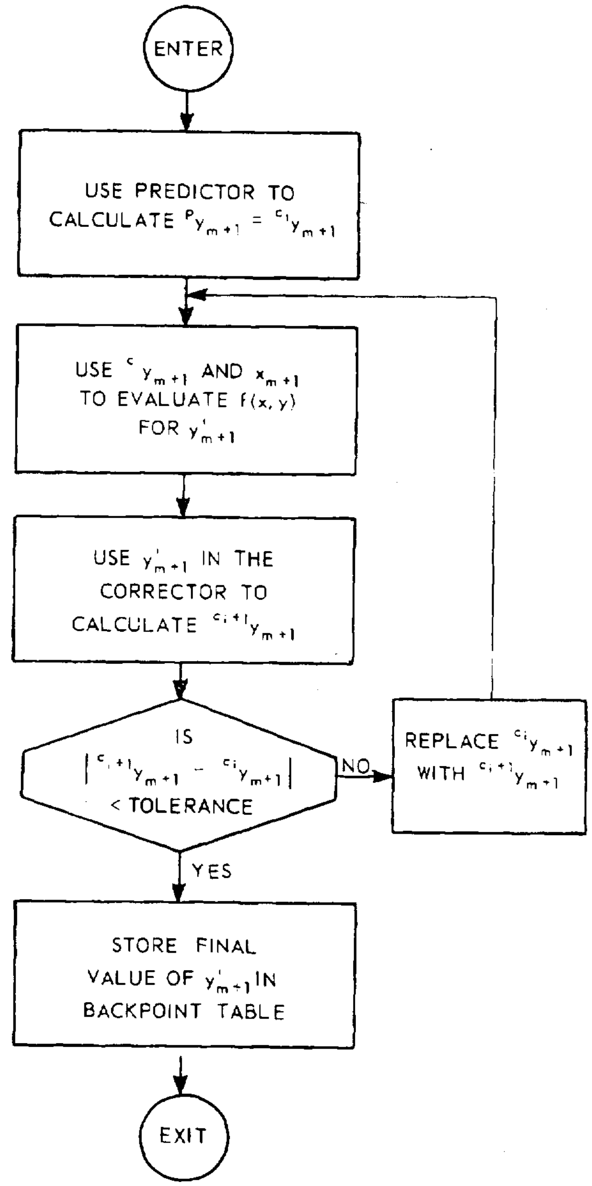

Figure 1—Predictor-corrector algorithm applied to the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$.

When the iteration process has converged (i.e., the criterion on $|{}^{c_{i+1}} y_{m+1} - {}^{c_i} y_{m+1}|$ has been satisfied), the final computed value for y'_{m+1} is entered in the backpoint table. Then, where the points $y'_0, y'_1, \ldots, y'_{m-1}, y'_m$ were used to determine y'_{m+1}, the points $y'_1, y'_2, \ldots, y'_m, y'_{m+1}$ are now used to determine y'_{m+2}. Etc.

Note that in the Adams-Bashforth predictor, no knowledge of the value y_{m+1} being derived is needed while such knowledge (namely a value for y'_{m+1}) is needed in the Adams-Moulton corrector.

Equations like Adams-Moulton corrector (closed form equations) have smaller truncation errors as well as desirable stabilizing characteristics. The predictor is used to obtain an estimated value for y_{m+1} good enough to keep the number of corrector iterations low. This predictor-corrector algorithm is well known and it has been shown by various authors that for a sufficiently small step size, h, the successive corrected values obtained converge to the unique solution of the closed form equation provided the function being numerically integrated is sufficiently smooth.

The above discussion considered numerical calculations for deriving values of y (and concomitantly y') at discrete points from the initial value problem

$$y' = f(x, y)$$

$$y(x_0) = y_0 .$$

The same technique could be used on any initial value problem of the form

$$y^{(n)} = f(x, y^{(n-1)})$$

$$y^{(n-1)}(x_0) = y_0^{(n-1)}$$

to solve for $y^{(n-1)}(x_i)$. In particular, we are interested in calculating y'_{m+1} from the backpoints y''_m, y''_{m-1}, ... since, in general, satellite orbit determination involves the initial value problem

$$y'' = f(x, y, y')$$

$$y'(x_0) = y'_0$$

$$y(x_0) = y_0 .$$

This could be approached by generating an initial set of backpoints for y'' and y'; then using y''_m, y''_{m-1}, ... to calculate y'_{m+1} and using y'_m, y'_{m-1}, ... to calculate y_{m+1} employing the same technique described above in both steps. However, certain advantages accrue if we use a mathematically equivalent technique which derives y_{m+1} directly from the backpoints y''_m, y''_{m-1}, ... For one, it is necessary to keep only one set of backpoints — the retention of y'_m, y'_{m-1}, ... is obviated. Secondly, we often must work with the problem

$$y'' = f(x, y)$$

$$y(x_0) = y_0$$

when only conservative forces are involved (i.e., no drag or other energy dissipating forces). In this situation, when y''_{m+1} has been satisfactorially determined, y'_{m+1} can be calculated by evaluating the corrector

$$y'_{m+1} = y'_m + h \left\{ \alpha_0^* \nabla^0 y''_{m+1} + \alpha_1^* \nabla^1 y''_{m+1} + \dots + \alpha_n^* \nabla^n y''_{m+1} \right\}$$

only once.

Consider now, working with the initial value problem

$$y'' = f(x, y)$$

$$y'(x_0) = y'_0$$

$$y(x_0) = y_0 .$$

Here, the predictor-corrector approach is the same. The difference exists in the polynomials: in particular, the coefficients are different. The formulas considered here are generally referred to as Cowell type formulas. They are:

Störmer Predictor

$$y_{m+1} = 2y_m - y_{m-1} + h^2 \left\{ \beta_0 \nabla^0 y''_m + \beta_1 \nabla^1 y''_m + \beta_2 \nabla^2 y''_m + \dots \beta_n \nabla^n y''_m \right\}$$

Cowell Corrector

$$y_{m+1} = 2y_m - y_{m-1} + h^2 \left\{ \beta_0^* \nabla^0 y''_{m+1} + \beta_1^* \nabla^1 y''_{m+1} + \beta_2^* \nabla^2 y''_{m+1} + \dots + \beta_n^* \nabla^n y''_{m+1} \right\} .$$

In the most general form of the initial value problem

$$y'' = f(x, y, y')$$

$$y'(x_0) = y'_0$$

$$y(x_0) = y_0 ,$$

y'_{m+1} is derived from the backpoints y''_m, y''_{m-1}, ... using the Adams formulas while y_{m+1} is obtained from the same backpoint set using the Cowell formulas. In testing for convergence of the corrector formulas, the sum $|{}^{c_{i+1}}y'_{m+1} - {}^{c_i}y'_{m+1}| + |{}^{c_{i+1}}y_{m+1} - {}^{c_i}y_{m+1}|$ is compared to the tolerance. A flow chart of the process is given in Figure 2.

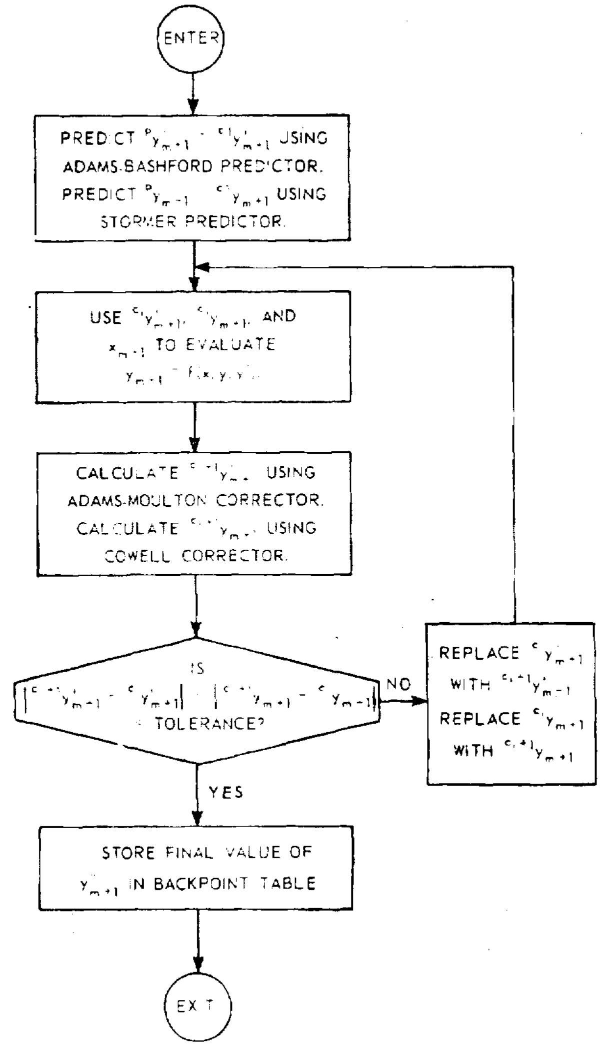

Figure 2—Predictor-corrector algorithm applied to the initial value problem $y'' = f(x,y,y')$, $y'(x_0) = y'_0$, $y(x_0) = y_0$.

III. Derivation of Multistep Formulas

These foregoing techniques are referred to as numerical integration. This appellation originates from the derivation of the methods. Consider again

$$y' = f(x, y)$$

$$y(x_0) = y_0 .$$

Integrating both sides between x_m and x_{m+1}, we have

$$y_{m+1} - y_m = \int_{x_m}^{x_{m+1}} y'(s)ds$$

or

$$y_{m+1} = y_m + \int_{x_m}^{x_{m+1}} f(s)ds$$

where $f(s)$ denotes $f(s, y(s))$.

By replacing $f(s)$ by a Newtonian type interpolating polynomial and integrating, it is possible to derive the Adams type polynomials which are used to approximate the expression

$$\int_{x_m}^{x_{m+1}} f(s)ds .$$

The error generated by replacing the function being integrated with a polynomial which is, effectively, integrated is usually obtained by integrating the local error associated with the interpolating polynomial. For example, it can be shown (Henrici) that the local error expression for formulas of the above type is of the form

$$R_n = C h^{p+1} y^{(p+1)} (\xi)$$

where p is the order of the method, h the step size, ξ is a value between the largest and smallest values of x on the interval (x_p, x_{p+1}), and C is a constant specific to the formula.

The Cowell type formulas can be derived by a double integration of $y'' = f(x,y)$ and again employing a Newtonian type interpolating polynomial (Henrici). These derivations are complex. A simpler approach using difference operators avoids much of the difficulty involved in integrating the interpolating polynomials. This is the derivation given here. Using this approach, the operator definitions lead naturally to the Adams-Moulton corrector. It is derived first. The other formulas follow easily from this derivation: first, the Adams-Bashforth predictor, then the Cowell corrector, and finally the Störmer predictor.

In the ensuing derivations, some confusion may arise between the subscripts m and $m+1$. The predictors are derived for y_{m+1}, the correctors for y_m. This is of no real importance since the same backpoints can be labelled either as y_m, y_{m-1}, y_{m-2}, ... or y_{m+1}, y_m, y_{m-1}, ...

A. Preliminary Definitions and Relationships

In order to derive the formulas for multistep numerical integration, it is useful to develop several tools. Consider the following *difference tables* (Figures 3 and 4). The first column is formed by defining the values $f(x+ih)$, $i = 0, 1, 2, \ldots$ for forward differences and $f(x-ih)$ for backward differences. The second columns are formed from differences of successive values of the first column. The third columns, from differences of the second. And so forth. (In both tables, the subtrahend is the value *above* the minuend in each column.)

$f(x)$			
	$f(x+h) - f(x)$		
$f(x+h)$		$f(x+2h) - 2f(x+h) + f(x)$	
	$f(x+2h) - f(x+h)$		$f(x+3h) - 3f(x+2h) + 3f(x+h) - f(x)$
$f(x+2h)$		$f(x+3h) - 2f(x+2h) + f(x+h)$	
	$f(x+3h) - f(x+2h)$		
$f(x+3h)$			

Figure 3—Forward difference table.

$f(x-3h)$			
	$f(x-2h) - f(x-3h)$		
$f(x-2h)$		$f(x-h) - 2f(x-2h) + f(x-3h)$	
	$f(x-h) - f(x-2h)$		$f(x) - 3f(x-h) + 3f(x-2h) - f(x-3h)$
$f(x-h)$		$f(x) - 2f(x-h) + f(x-2h)$	
	$f(x) - f(x-h)$		
$f(x)$			

Figure 4—Backward difference table.

From these tables, we derive the following operator definitions:

Forward Difference Operator (delta)

$$\Delta f(x) = f(x+h) - f(x) \tag{1a}$$

$$\Delta^2 f(x) = \Delta\big(\Delta f(x)\big) = f(x+2h) - 2f(x+h) + f(x)$$

$$\Delta^n f(x) = \Delta\big(\Delta^{n-1} f(x)\big) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f\big(x + (n-i)h\big) \tag{1b}$$

Backward Difference Operator (nabla)

$$\nabla f(x) = f(x) - f(x-h) \tag{2a}$$

$$\nabla^2 f(x) = \nabla\big(\nabla f(x)\big) = f(x) - 2f(x-h) + f(x-2h)$$

$$\nabla^n f(x) = \nabla\big(\nabla^{n-1} f(x)\big) = \sum_{i=0}^{n} (-1)^i \binom{n}{i} f(x - ih) \tag{2b}$$

These definitions simplify our difference tables. See Figures 5 and 6.

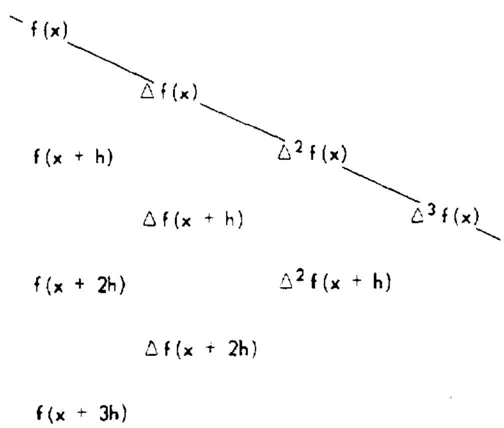

Figure 5—Forward difference table written in forward difference operator notation.

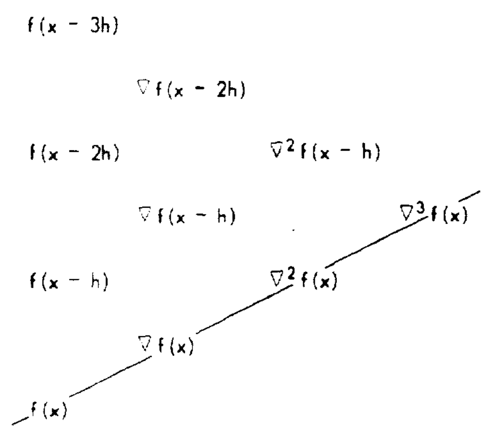

Figure 6—Backward difference table written in backward difference operator notation.

In addition to the difference operators, we define:

Identity Operator

$$I f(x) = f(x) \tag{3}$$

Shift Operator

$$E f(x) = f(x + h) \tag{4}$$

$$E^{\eta} f(x) = f(x + \eta h)$$

(η may be any real number)

Differential Operator

$$D f(x) = f'(x) \tag{5}$$

$$D^{n} f(x) = f^{(n)}(x).$$

On these operators, we define an algebra where, for any two operators L_1 and L_2, $L_1 \pm L_2$ means the results of L_2 operating on $f(x)$ are to be added to or subtracted from the results of L_1 operating on $f(x)$; while multiplication, L_1 times L_2, means L_1 operating on the results of L_2 operating on $f(x)$. For example,

$$I f(x) - E^{-1} f(x) = f(x) - f(x - h) = \nabla f(x).$$

$$\begin{aligned}
\Delta\nabla f(x) &= \Delta\left[f(x) - f(x - h)\right] \\
&= \Delta f(x) - \Delta f(x - h) \\
&= f(x + h) - f(x) - \left[f(x + h - h) - f(x - h)\right] \\
&= f(x + h) - 2 f(x) + f(x - h).
\end{aligned}$$

It can be shown (Hildebrand) that these operators follow the laws of commutivity, associability, and distribution.

With these definitions, we derive the relationships

$$\nabla = I - E^{-1} \tag{6}$$

$$E = (I - \nabla)^{-1} = \frac{I}{I - \nabla} \tag{7}$$

$$\Delta = E - I. \tag{8}$$

Then from Equations (7) and (8),

$$\Delta = E - I = \frac{I}{I - \nabla} - I = \frac{I - I^2 + I\nabla}{I - \nabla}.$$

But, $I^2 = I$ and $I\nabla = \nabla$. Hence

$$\Delta = \frac{\nabla}{I - \nabla} \tag{9}$$

In addition to the above operator definitions and relationships, we need the series representations for e^x, $\frac{x}{1-x}$, $\frac{1}{1-x}$, and $-\log(1-x)$, and formulas for series multiplication and series division:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots = \sum_{i=0}^{\infty} \frac{x^i}{i!} \tag{10}$$

$$\frac{x}{1-x} = x + x^2 + x^3 + \dots = \sum_{i=0}^{\infty} x^{i+1} \tag{11}$$

$$\frac{1}{1-x} = 1 + x + x^2 + \dots = \sum_{i=0}^{\infty} x^i \tag{12}$$

$$-\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} \dots = x \sum_{i=0}^{\infty} \frac{x^i}{i+1} \tag{13}$$

For series division and multiplication, let the series s_1 and s_2 be the arguments of the operation and s_3 the result. We define

$$s_1 = 1 + a_1 x + a_2 x^2 + \dots = \sum_{i=0}^{\infty} a_i x^i$$

where

$$a_0 = 1 .$$

$$s_2 = 1 + b_1 x + b_2 x^2 + \cdots = \sum_{i=0}^{\infty} b_i x^i$$

where

$$b_0 = 1 ,$$

and for the resultant series $s_3 = s_1 s_2$ or $s_3 = s_1/s_2$ we desire

$$s_3 = 1 + c_1 x + c_2 x^2 + \dots = \sum_{i=0}^{\infty} c_i x^i$$

where

$$c_0 = 1 .$$

Then,

Series multiplication is defined as

$$s_1 s_2 = s_3 = 1 + (b_1 + a_1)x + (b_2 + a_1 b_1 + a_2)x^2 + (b_3 + a_1 b_2 + a_2 b_1 + a_3)x^3 + \ldots$$

$$= \sum_{i=0}^{\infty} x^i \left(\sum_{j=0}^{i} b_{i-j} a_j \right) \quad (14)$$

where

$$a_0 = b_0 = 1$$

and,

Series division is defined as

$$s_1/s_2 = s_3 = 1 + (a_1 - b_1)x + \left[a_2 - (b_1 c_1 + b_2) \right] x^2 + \left[a_3 - (b_1 c_2 + b_2 c_1 + b_3) \right] x^3 + \ldots$$

$$= 1 + \sum_{i=1}^{\infty} x^i \left(a_i - \sum_{j=1}^{i} b_j c_{i-j} \right) . \quad (15)$$

where

$$c_0 = 1 .$$

Note that series division is a recursive definition requiring $c_0, c_1, c_2, \ldots, c_{n-1}$ to compute the n^{th} coefficient, c_n, of the n^{th} term of the s_3 series. Note also, where $s_1 = 1$, series division reduces to

$$1/s_2 = 1 + \sum_{i=1}^{\infty} x^i \ -\left(\sum_{j=1}^{i} b_j c_{i-j} \right) . \quad (16)$$

where

$$c_0 = 1$$

since $a_i = 0$ for $i > 0$.

B. Derivation of Formulas

Consider now the Taylor's expansion of an interpolating polynomial

$$p(x+h) = p(x) + \frac{h}{1!}p^{(1)}(x) + \frac{h^2}{2!}p^{(2)}(x) + \ldots + \frac{h^n}{n!}p^{(n)}(x).$$

Using the shift operator $Ep(x) = p(x+h)$, the differential operator $D^n p(x) = p^{(n)}(x)$, and the identity operator $Ip(x) = p(x)$, we have

$$Ep(x) = \left(I + \frac{h}{1!}D + \frac{h^2}{2!}D^2 + \ldots + \frac{h^n}{n!}D^n\right) p(x).$$

(Note that this is a finite expansion for any given n since $p(x)$ is a polynomial, hence has only n derivatives.)

Then, by Equation (10) the expansion of e^x,

$$E = e^{hD}$$

or, by relationship (7) is

$$(I - \nabla)^{-1} = e^{hD}.$$

Taking the log of both sides,

$$-\log(I - \nabla) = hD$$

or

$$I = \frac{hD}{-\log(I - \nabla)}.$$

Multiplying both sides by ∇,

$$\nabla = h\left[\frac{\nabla}{-\log(I - \nabla)}\right] D \tag{17}$$

and employing Equation (13), the expansion of $-\log(1 - x)$, we have

$$\nabla = h \left[\frac{\nabla}{\nabla \sum_{i=0}^{\infty} \frac{\nabla^i}{i+1}} \right] D = h \left[\frac{I}{\sum_{i=0}^{\infty} \frac{\nabla^i}{i+1}} \right] D$$

which by series division (16) is

$$\nabla = h \left[\sum_{i=0}^{n} \alpha_i^* \nabla^i \right] D \tag{18}$$

where n is the order of the interpolating polynomial and

$$\alpha_0^* = 1, \qquad \alpha_i^* = - \sum_{j=1}^{i} \frac{\alpha_{i-j}^*}{j+1} \tag{19}$$

This is the Adams-Moulton Corrector. Some of the coefficients, α_i^*, are given in Table 1. For $i = 0$ to $i = 15$, see Table 2 in the appendix.

Applying this to our initial value problem

$$y'' = f(x, y, y')$$

$$y'(x_0) = y_0'$$

$$y(x_0) = y_0$$

Table 1

Coefficients of Adams-Moulton Corrector.						
i	0	1	2	3	4	5
α_i^*	1	$-\frac{1}{2}$	$-\frac{1}{12}$	$-\frac{1}{24}$	$-\frac{19}{720}$	$-\frac{3}{160}$

to obtain a corrected value, ${}^c y_m'$, when y_m' and y_m have been predicted, an approximation of y_m'' calculated, and the other $n - 1$ backpoints $y_{m-1}'', y_{m-2}'', \ldots, y_{m-n+1}''$ determined, we have

$$\nabla y_m' = y_m' - y_{m-1}'$$

$$= h \left\{ I - \frac{1}{2}\nabla - \frac{1}{12}\nabla^2 - \frac{1}{24}\nabla^3 \ldots \right\} y_m''$$

or

$$y'_m = y'_{m-1} + h\Big\{y''_m - \frac{1}{2}\left[y''_m - y''_{m-1}\right]$$

$$- \frac{1}{12}\left[y''_m - 2y''_{m-1} + y''_{m-2}\right]$$

$$- \frac{1}{24}\left[y''_m - 3y''_{m-1} + 3y''_{m-2} + y''_{m-3}\right]$$

$$+ \alpha_n^* \left[y''_m - \binom{n}{1} y''_{m-1} + \binom{n}{2} y''_{m-2} - \binom{n}{3} y''_{m-3} + \ldots - (-1)^n y''_{m-n}\right]\Big\} . \qquad (20)$$

We now wish to develop the Adams-Bashforth predictor. Consider again Equation (17) and multiply both sides by relationship (7) noting that $\nabla E = \Delta$. Then

$$\nabla E = \Delta = h\left[\frac{(I - \nabla)^{-1}\nabla}{-\log(I - \nabla)}\right] D = h\left[\frac{\frac{\nabla}{I - \nabla}}{-\log(I - \nabla)}\right] D.$$

Now, employing Equations (11) and (13), the series representations respectively for $\frac{x}{1 - x}$ and $-\log(1 - x)$, we have

$$\Delta = h\left[\frac{\nabla \sum_{i=0}^{\infty} \nabla^i}{\nabla \sum_{i=0}^{\infty} \frac{\nabla^i}{i + 1}}\right] D = h\left[\frac{\sum_{i=0}^{\infty} \nabla^i}{\sum_{i=0}^{\infty} \frac{\nabla^i}{i + 1}}\right] D ,$$

which by Equation (15) series division is

$$\Delta = h\left[\sum_{i=0}^{n} \alpha_i \nabla^i\right] D \qquad (21)$$

where n is the number of backpoints (i.e., the order of the method) and

$$\alpha_0 = 1, \qquad \alpha_i = 1 - \sum_{j=1}^{i} \frac{\alpha_{i-j}}{j+1} . \tag{22}$$

Some of the α_i are given in Table 2. These coefficients are given rational form for $i = 0$ to $i = 15$ in Table 2 of the appendix.

Note that the derivation involved infinite series. However, since these operator relationships are valid for polynomials, the corresponding series are finite. Hence, there exists n such that $\alpha_i = 0$ for all $i > n$.

Table 2

Coefficients of Adams-Bashforth Predictor.

i	0	1	2	3	4	5
α	1	$\frac{1}{2}$	$\frac{5}{12}$	$\frac{3}{8}$	$\frac{251}{720}$	$\frac{95}{288}$

Thus,

$$\Delta y'_m = y'_{m+1} - y'_m$$

$$= h \left\{ I + \frac{1}{2} \nabla + \frac{5}{12} \nabla^2 + \frac{3}{8} \nabla^3 + \ldots \alpha_n \nabla^n \right\} y''_m$$

or

$$y'_{m+1} = y'_m + h \left\{ y''_m + \frac{1}{2} \left[y''_m - y''_{m-1} \right] \right.$$

$$+ \frac{5}{12} \left[y''_m - 2y''_{m-1} + y''_{m-2} \right]$$

$$+ \frac{3}{8}\left[y''_m - 3y''_{m-1} + 3y''_{m-2} - y''_{m-3}\right]$$

.

.

.

$$+ a_n \left[y''_m - \binom{n}{1} y''_{m-1} + \binom{n}{2} y''_{m-2} + \binom{n}{3} y''_{m-3} + \dots + (-1)^n y''_{m-n}\right]\Bigg\} \quad (23)$$

As previously noted, we have the problem of calculating y_m from the backpoints y''_{m-1}, y''_{m-2}, ... To achieve this, consider once again Equation (17). By squaring both sides we immediately have a formula involving $D^2 y = y''$.

$$\nabla^2 = h^2 \left[\frac{\nabla}{-\log(I - \nabla)}\right]^2 D^2 . \quad (24)$$

It is possible to obtain an expression for $\left[\frac{\nabla}{-\log(I - \nabla)}\right]^2$ merely by squaring the series representation for $\left[\frac{\nabla}{-\log(I - \nabla)}\right]$. However, a more suitable formulation can be derived as follows:

Consider

$$\left[-\log(I - \nabla)\right]^2 = D^{-1} D \left[-\log(I - \nabla)\right]^2$$

where D^{-1} is the *informal* integration operator (Hildebrand), the inverse of the differential operator. Then

$$D^{-1} D \left[-\log(I - \nabla)\right]^2 = D^{-1}\, 2\, \frac{\left[-\log(I - \nabla)\right]}{I - \nabla}$$

$$= D^{-1}\, 2 \left[\left(\frac{\nabla}{I - \nabla}\right)\left(\sum_{j=0}^{\infty} \frac{\nabla^j}{j+1}\right)\right] \quad \text{from (13)}$$

$$= D^{-1}\, 2 \left[\left(\nabla \sum_{j=0}^{\infty} \nabla^j\right)\left(\sum_{j=0}^{\infty} \frac{\nabla^j}{j+1}\right)\right] \quad \text{from (11)}$$

$$= D^{-1}\, 2 \left[1 + \left(1 + \frac{1}{2}\right)\nabla + \left(1 + \frac{1}{2} + \frac{1}{3}\right)\nabla^2 + \ldots + \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{i+1}\right)\nabla^j + \ldots\right]$$

$$= D^{-1}\left[\sum_{j=0}^{\infty} \frac{2H_{j+1}\,\nabla^{j+1}}{j+1}\right]$$

where

$$H_m = \sum_{k=0}^{m} \frac{1}{k+1} \qquad m = 0, 1, 2, \ldots$$

Then, by integrating (i.e. using the operator D^{-1}),

$$\left[-\log(I - \nabla)\right]^2 = \sum_{j=0}^{\infty} \frac{2H_{j+1}\,\nabla^{j+2}}{j+2}$$

$$= 2\nabla^2 \sum_{j=0}^{\infty} \frac{H_{j+1}\,\nabla^j}{j+2}\,. \qquad (25)$$

Using this expression in Equation (24),

$$\nabla^2 = h^2 \left[\frac{\nabla^2}{2\nabla^2 \sum_{j=0}^{\infty} \frac{H_{j+1}\,\nabla^j}{j+2}}\right] D^2$$

which by series division (15) is

$$\nabla^2 = h^2 \left[\sum_{i=0}^{n} \beta_i^{*}\, \nabla^i\right] D^2 \qquad (26)$$

where

$$\beta_0^{*} = 1$$

and

$$\beta_i^* = -\sum_{j=1}^{i} \frac{2H_{j+1}}{j+2} \beta_{i-j}^* , \tag{27}$$

$$H_m = \sum_{k=1}^{m} \frac{1}{k}$$

and n is the order of the method. This is the Cowell corrector. Some of the coefficients, β_i^*, are given in Table 3. For β_i^*, $i = 0$ to $i = 15$, in rational form see Table 4 of the appendix.

Table 3

Coefficients of Cowell Corrector						
i	0	1	2	3	4	5
β_i^*	1	-1	$\frac{1}{12}$	0	$-\frac{1}{240}$	$-\frac{1}{240}$

Thus,

$$\nabla^2 y_m = y_m - 2y_{m-1} - y_{m-2}$$

$$= h \left\{ I - \frac{1}{2} \nabla + \frac{1}{12} \nabla^2 + 0 \nabla^3 - \frac{1}{240} \nabla^4 + \dots \right\} y_m''$$

or

$$y_m = 2y_{m-1} + y_{m-2} + h \left\{ y_m'' - \frac{1}{2} \left[y_m'' - y_{m-1}'' \right] \right.$$

$$+ \frac{1}{12} \left[y_m'' - 2y_{m-1}'' + y_{m-2}'' \right] + 0$$

$$- \frac{1}{240} \left[y_m'' - 4y_{m-1}'' + 6y_{m-2}'' - 4y_{m-3}'' + y_{m-4}'' \right]$$

$$\vdots$$

$$\left. + \beta_n^* \left[y_m'' - \binom{n}{1} y_{m-1}'' + \binom{n}{2} y_{m-2}'' - \binom{n}{3} y_{m-3}'' + \dots + (-1)^n y_{m-n}'' \right] \right\} . \tag{28}$$

As in the case of Equation (19) we need an extrapolator or predictor. This can be derived in the same manner as Equation (21), only this time, multiplying Equation (24) by relationship (7),

$$\nabla^2 E = h^2 \left[\frac{\nabla}{-\log(I - \nabla)}\right] \left(\frac{I}{I - \nabla}\right) D^2 .$$

Using Equations (25) and (12)

$$\nabla^2 E = h^2 \left(\sum_{j=0}^{\infty} \frac{2H_{j+1} \nabla^j}{j + 2}\right) \left(\sum_{j=0}^{\infty} \nabla^j\right) D^2$$

which by series multiplication (14) is

$$\nabla^2 E = h^2 \sum_{i=0}^{n} \beta_i \nabla^i \qquad (29)$$

where

$$\beta_0 = 1$$

and

$$\beta_i = 1 - \sum_{j=1}^{i} \frac{2H_{j+1}}{j + 2} \beta_{i-j} . \qquad (30)$$

This is the Störmer predictor. Several of the coefficients, β_i, are given in Table 4. For β_i in rational form for $i = 0$ to $i = 15$, see Table 3 of the appendix.

Table 4

Coefficients of Störmer Predictor						
i	0	1	2	3	4	5
β_i	1	0	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{19}{240}$	$\frac{3}{40}$

Thus,

$$\nabla^2 E\, y_m = y_{m+1} - 2y_m + y_{m-1}$$

$$= h^2 \left\{ I + 0\nabla + \frac{1}{12}\nabla^2 + \frac{1}{12}\nabla^3 + \dots \right\} y_m''$$

or

$$y_{m+1} = 2y_m - y_{m-1} + h^2 \Big\{ y''_m + 0$$

$$+ \frac{1}{12} \left[y''_m - 2y''_{m-1} + y''_{m-2} \right]$$

$$+ \frac{1}{12} \left[y''_m - 3y''_{m-1} + 3y''_{m-2} - y''_{m-3} \right] +$$

$$\vdots$$

$$+ \beta_n \left[y''_m - \binom{n}{1} y''_{m-1} + \binom{n}{2} y''_{m-2} - \binom{n}{3} y''_{m-3} + \cdots + (-1)^n y''_{m-n} \right] \Big\} . \qquad (31)$$

In recapitulation, we have derived the following formulas for numerically solving at discrete points the initial value problem

$$y'' = f(x, y, y')$$

$$y'(x_0) = y'_0$$

$$y(x_0) = y_0 .$$

The Adams-Bashforth predictor

$$\nabla y'_{m+1} = \Delta y'_m = y'_{m+1} - y' = h \sum_{i=0}^{n} \alpha_i \nabla^i y''_m$$

where

$$\alpha_0 = 1$$

and

$$\alpha_i = 1 - \sum_{j=1}^{i} \frac{\alpha_{i-j}}{j+1}$$

which is used to produce a first approximation of y'_{m+1} for iteration in the Adams-Moulton corrector

$$\nabla y'_m = y'_m - y'_{m-1} = h \sum_{i=0}^{n} \alpha_i^* \nabla^i y''_m$$

where

$$\alpha_i^* = 1$$

and

$$\alpha_i^* = - \sum_{i=0}^{i} \frac{\alpha_{i-j}^*}{j+1} \, ;$$

and the Störmer predictor

$$\nabla^2 y_{m+1} = \nabla^2 E y_m = y_{m+1} - 2y_m + y_{m-1} = h^2 \sum_{i=0}^{n} \beta_i \nabla^i y''_m$$

where

$$\beta_0 = 1$$

and

$$\beta_i = 1 - \sum_{j=1}^{i} \frac{2H_{j+1}}{j+2} \beta_{i-j}$$

$$H_m = \sum_{k=1}^{m} \frac{1}{k}$$

which produces a first approximation of y_{m+1} for iteration in the Cowell corrector

$$\nabla^2 y_m = y_m - 2y_{m-1} + y_{m-2} = h^2 \sum_{i=0}^{n} \beta_i^* \nabla^i y''_m$$

where

$$\beta_0^* = 1$$

and

$$\beta_i^* = -\sum_{j=1}^{i} \frac{2H_{j+1}}{j+2} \beta_{i-j}^*$$

$$H_m = \sum_{k=1}^{m} \frac{1}{k}.$$

C. The Summed Form

It has been established (Henrici) that algebraic equivalents known as the *summed* forms of the foregoing equations considerably reduce the propagation of round-off error. These summed forms can be derived by defining the operators ∇^{-1} and ∇^{-2} as the inverses of ∇^1 and ∇^2

$$\nabla^{-1}\nabla = I, \qquad \nabla^{-2}\nabla^2 = I$$

and defining

$$\nabla^{-1} y_m'' = {}^{I}S_m \tag{32}$$

$$\nabla^{-2} y_m'' = \nabla^{-1}({}^{I}S_m) = {}^{II}S_m. \tag{33}$$

Then, applying ∇ to ${}^{I}S_{m+1} = \nabla^{-1} y_{m+1}''$ we have

$$\nabla\nabla^{-1} y_{m+1}'' = \nabla({}^{I}S_{m+1})$$

$$y_{m+1}'' = {}^{I}S_{m+1} - {}^{I}S_m$$

or

$${}^{I}S_{m+1} = {}^{I}S_m + y_{m+1}''. \tag{34}$$

Also, applying ∇ to ${}^{II}S_{m+1} = \nabla^{-1}({}^{I}S_{m+1})$, we have

$$\nabla\nabla^{-1}({}^{I}S_{m+1}) = \nabla({}^{II}S_{m+1})$$

$${}^{I}S_{m+1} = {}^{II}S_{m+1} - {}^{II}S_m$$

which, by using Equation (34), becomes

$$^{II}S_{m+1} = {}^{II}S_m + {}^{I}S_m + y''_{m+1} . \tag{35}$$

Then, multiplying both sides of the Adams-Bashforth predictor and the Adams-Moulton corrector by ∇^{-1}, and similarly multiplying both sides of the Störmer predictor and Cowell corrector by ∇^{-2} and using identities (34) and (35) we derive the following summed forms:

Adams-Bashforth Predictor Summed Form

$$\nabla^{-1}\nabla y'_{m+1} = y'_{m+1} = h\left\{\alpha_0\,{}^{I}S_m + \alpha_1 y''_m + \sum_{i=2}^{n} \alpha_i \nabla^{i-1} y'_m\right\} \tag{36}$$

where

$$\alpha_0 = 1$$

and

$$\alpha_i = 1 - \sum_{j=0}^{i} \frac{\alpha_{i-j}}{j+1}$$

Adams-Moulton Corrector Summed Form

$$\nabla^{-1}\nabla y'_m = y'_m = h\left\{\alpha_0^*\,{}^{I}S_m + (\alpha_0^* + \alpha_1^*)\, y''_m + \sum_{i=2}^{n} \alpha_i^* \nabla^{i-1} y'_m\right\} \tag{37}$$

where

$$\alpha_0^* = 1$$

and

$$\alpha_i^* = -\sum_{j=0}^{i} \frac{\alpha_{i-j}^*}{j+1}$$

Störmer Predictor Summed Form

$$\nabla^{-2}\nabla^{2} y_{m+1} = y_{m+1} = h^{2}\left\{\beta_0 \, {}^{II}S_m + \beta_1 \, {}^{I}S_m + \beta_2 \, y''_m + \sum_{i=3}^{n} \beta_i \nabla^{i-2} y''_m \right\} \tag{38}$$

where

$$\beta_0 = 1$$

and

$$\beta_i = 1 - \sum_{j=0}^{i} \frac{2H_{j+1}}{j+2} \beta_{i-j}$$

Cowell Corrector Summed Form

$$\nabla^{-2}\nabla^{2} y_m = y_m = h^{2}\left\{\beta_0^* \, {}^{II}S_m + (\beta_0^* + \beta_1^*) \, {}^{I}S_m + (\beta_0^* + \beta_1^* + \beta_2^*) \, y''_m + \sum_{i=3}^{n} \beta_i^* \nabla^{i-2} y''_m \right\} \tag{39}$$

where

$$\beta_0^* = 1$$

and

$$\beta_i^* = - \sum_{j=0}^{i} \frac{2H_{j+1}}{j+2} \beta_{i-j}^*$$

The meaning of ${}^{I}S_m$ and ${}^{II}S_{m+1}$ can best be seen from their positions in an extended difference table (Figure 7). Examination of this table shows that the sums can be maintained by relationships (34) and (35)

$${}^{I}S_{m+1} = {}^{I}S_m + y''_{m+1}$$

$${}^{II}S_{m+1} = {}^{II}S_m + {}^{I}S_m + y''_{m+1}$$

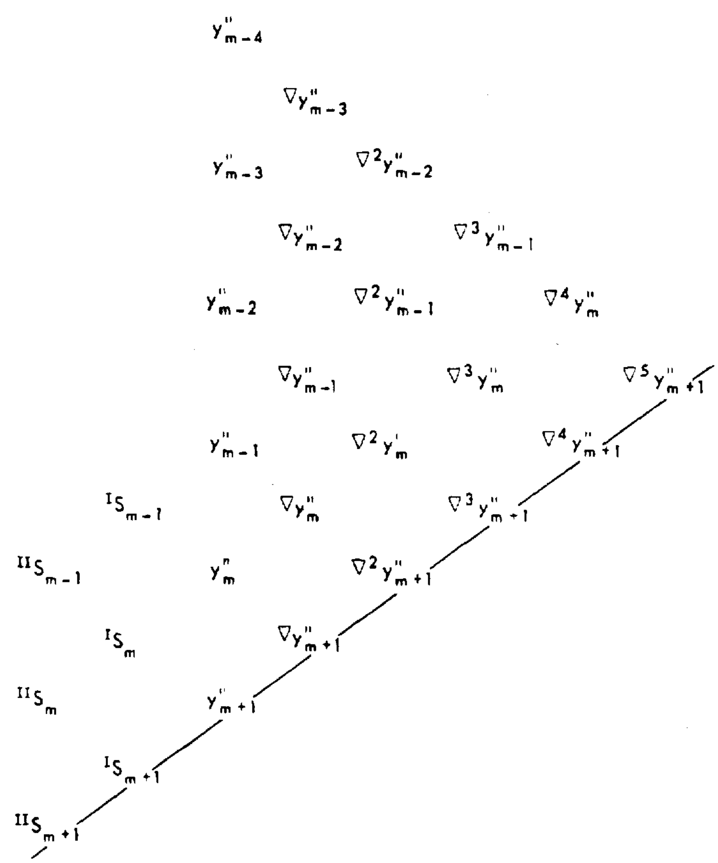

Figure 7—Extended difference table showing ${}^{I}S_m$ and ${}^{II}S_m$.

but that initial values for some ${}^{I}S_m$ and ${}^{II}S_m$ must be supplied. These initial values can be determined by inverting the corrector formulas (${}^{I}S_m$ is eliminated from the Cowell corrector since its coefficient, $\beta_0^* + \beta_1^*$, is zero) and solving respectively for ${}^{I}S_{m-1}$ and ${}^{II}S_{m-1}$

$$ {}^{I}S_{m-1} = \frac{y''_{m-1}}{h} - \left[\frac{1}{2} y''_{m-1} + \alpha_2^* \nabla y''_{m-1} + \alpha_3^* \nabla^2 y''_{m-1} + \dots\right] \tag{40} $$

$$ {}^{II}S_{m-1} = \frac{y''_{m-1}}{h} - \left[\frac{1}{12} y''_{m-1} + \beta_3^* \nabla y''_{m-1} + \beta_4^* \nabla^2 y''_{m-1} + \dots\right]. \tag{41} $$

D. Ordinate Forms

All of the foregoing formulas involved difference operators. They are thus known as the *difference* forms and *summed difference* forms. Another useful form of these formulas which can be used under certain circumstances is the *ordinate* forms.

When using the difference forms, the order of the method can be dynamically changed as the problem dictates. That is, on the basis of the number of corrector iterations, the order of the method (the number of backpoints) could be increased (or perhaps decreased) to improve accuracy (or lower computation time). However, in satellite orbit determination, the functions are usually smooth enough so that the order of the method can be fixed. This permits us to take advantage of the ordinate forms of the Cowell and Adams type formulas.

In using the difference forms, it is necessary to maintain a table of backpoints and tables of differences. The ordinate forms enable us to rely solely on the table of backpoints thus obviating the computation and maintenance of the difference tables. This simplifies the integration process and enhances calculation speed.

Consider the Adams-Bashforth predictor (21) substituting definition (2b) for ∇^i:

$$y'_{m+1} = y'_m + h \left\{ \sum_{i=0}^{n} \alpha_i \left(\sum_{j=0}^{i} (-1)^j \binom{i}{j} y''_{m-j} \right) \right\}$$

Expanding the expression in brackets and denoting y''_{m-j} by Z_j, we have

$$\alpha_0 (-1)^0 \binom{0}{0} Z_0 +$$

$$\alpha_1 (-1)^0 \binom{1}{0} Z_0 + \alpha_1 (-1)^1 \binom{1}{1} Z_1$$

$$\alpha_2 (-1)^0 \binom{2}{0} Z_0 + \alpha_2 (-1)^1 \binom{2}{1} Z_1 + \alpha_2 (-1)^2 \binom{2}{2} Z_2 +$$

$$\alpha_3 (-1)^0 \binom{3}{0} Z_0 + \alpha_3 (-1)^1 \binom{3}{1} Z_1 + \alpha_3 (-1)^2 \binom{3}{2} Z_3 + \alpha_3 (-1)^3 \binom{3}{3} Z_3 +$$

$$\vdots$$

$$\alpha_n (-1)^0 \binom{n}{0} Z_0 + \alpha_n (-1)^1 \binom{n}{1} Z_1 + \alpha_n (-1)^2 \binom{n}{2} Z_2 + \alpha_n (-1)^3 \binom{n}{3} Z_3 + \dots + \alpha_n (-1)^n \binom{n}{n} Z_n .$$

Then collecting the coefficients of like ordinates, the expression becomes

$$Z_0(-1)^0\left[\alpha_0\binom{0}{0}+\alpha_1\binom{1}{0}+\alpha_2\binom{2}{0}+\alpha_3\binom{3}{0}+\alpha_4\binom{4}{0}+\ldots+\alpha_n\binom{n}{0}\right]$$

$$+Z_1(-1)^1\left[\alpha_1\binom{1}{1}+\alpha_2\binom{2}{1}+\alpha_3\binom{3}{1}+\alpha_4\binom{4}{1}+\ldots+\alpha_n\binom{n}{1}\right]$$

$$+Z_2(-1)^2\left[\alpha_2\binom{2}{2}+\alpha_3\binom{3}{2}+\alpha_4\binom{4}{2}+\ldots+\alpha_n\binom{n}{2}\right]$$

$$+Z_3(-1)^3\left[\alpha_3\binom{3}{3}+\alpha_4\binom{4}{3}+\ldots+\alpha_n\binom{n}{3}\right]$$

$$\vdots$$

$$+Z_{n-1}(-1)^{n-1}\left[\alpha_{n-1}\binom{n-1}{n-1}+\alpha_n\binom{n}{n-1}\right]$$

$$+Z_n(-1)^n\left[\alpha_n\binom{n}{n}\right]$$

or

$$y'_{m+1} = y'_m + y''_m\sum_{i=0}^{n}\alpha_i\binom{i}{0} - y''_{m-1}\sum_{i=1}^{n}\alpha_i\binom{i}{1} + y''_{m-2}\sum_{i=2}^{n}\alpha_i\binom{i}{2}$$

$$+ y''_{m-3}\sum_{i=3}^{n}\alpha_i\binom{i}{3} - \ldots + y''_{m-n+1}\sum_{i=n-1}^{n}\alpha_i\binom{i}{n-1} + y''_{m-n}\sum_{i=n}^{n}\alpha_i\binom{i}{n}$$

which can be represented as

$$y''_{m+1} = y'_m + \sum_{j=0}^{n}\sigma_j\, y''_{m-j}$$

where

$$\sigma_j = (-1)^j \sum_{i=j}^{n} a_i \binom{i}{j}$$

Sample calculations of the coefficients σ_j for a fifth order Adams-Bashforth predictor are given in Table 5. In like manner, the ordinate forms for any order of the summed and non-summed Cowell and Adams type formulas can be developed.

Table 5

Coefficients for Fixed, Fifth-Order, Ordinate Form Adams-Bashforth Predictor.

$$\sigma_0 = (-1)^0 \left[\binom{0}{0} 1 + \binom{1}{0} \frac{1}{2} + \binom{2}{0} \frac{5}{12} + \binom{3}{0} \frac{3}{8} + \binom{4}{0} \frac{251}{720} \right] = \frac{1901}{720}$$

$$\sigma_1 = (-1)^1 \left[\binom{1}{1} \frac{1}{2} + \binom{2}{1} \frac{5}{12} + \binom{3}{1} \frac{3}{8} + \binom{4}{1} \frac{251}{720} \right] = \frac{-1387}{360}$$

$$\sigma_2 = (-1)^2 \left[\binom{2}{2} \frac{5}{12} + \binom{3}{2} \frac{3}{8} + \binom{4}{2} \frac{251}{720} \right] = \frac{109}{30}$$

$$\sigma_3 = (-1)^3 \left[\binom{3}{3} \frac{3}{8} + \binom{4}{3} \frac{251}{720} \right] = \frac{-637}{360}$$

$$\sigma_4 = (-1)^4 \left[\binom{4}{4} \frac{251}{720} \right] = \frac{251}{720}$$

Thus, the ordinate forms for the non-summed integration formulas are

Adams-Bashforth Predictor Ordinate Form

$$y'_{m+1} = y'_m + h \sum_{j=0}^{n} \sigma_j \, y''(x_{m-j}, h)$$

where

$$\sigma_j = (-1)^j \sum_{i=j}^{n} a_i \binom{i}{j} \tag{40}$$

Adams-Moulton Corrector Ordinate Form

$$y'_m = y'_{m-1} + h \sum_{j=0}^{n} \sigma_j^* \, y''_{m-j}$$

where

$$\sigma_j^* = (-1)^j \sum_{i=j}^{n} \alpha_i^* \binom{i}{j} \tag{41}$$

Störmer Predictor Ordinate Form

$$y_{m+1} = 2y_m - y_{m-1} + h \sum_{j=0}^{n} \lambda_j \, y''_{m-j}$$

where

$$\lambda_j = (-1)^j \sum_{i=j}^{n} \beta_i \binom{i}{j} \tag{42}$$

Cowell Corrector Ordinate Form

$$y_m = 2y_{m-1} - y_{m-2} + h \sum_{j=0}^{n} \lambda_j^* \, y''_{m-j}$$

where

$$\lambda_j^* = (-1)^j \sum_{i=j}^{n} \beta_i^* \binom{i}{j} \tag{43}$$

The coefficients σ_j, σ_j^*, λ_j, λ_j^* are given in rational form in the appendix in Tables 5 through 8. Within each table, subtables are presented on the basis of $n = 0$ to $n = 15$.

The summed ordinate forms are

Adams-Bashforth Predictor Summed Ordinate Form

$$y_{m+1} = h \left\{ \alpha_0 \, {}^{I}S_m + \sum_{j=0}^{n} \sigma'_j \, y''_{m-j} \right\}.$$

$$\sigma'_j = (-1)^j \sum_{i=j}^{n} \alpha'_i \binom{i}{j}$$

where

$$\alpha'_i = \alpha_{i+1} \tag{44}$$

Adams-Moulton Corrector Summed Ordinate Form

$$y'_m = h\left\{\alpha_0\,{}^{I}S_m + \sum_{j=0}^{n}\sigma_j^{*\prime}\,y''_{m-j}\right\}$$

$$\sigma_j^{*\prime} = (-1)^j \sum_{i=j}^{n}\alpha_i^{*\prime}\binom{i}{j}$$

where

$$\alpha_0^{*\prime} = (\alpha_0^{*} + \alpha_1^{*})$$

and

$$\alpha_i^{*\prime} = \alpha_{i+1}^{*} \quad \text{for} \quad i > 0 \tag{45}$$

Störmer Predictor Summed Ordinate Form

$$y_{m+1} = h\left\{\beta_0\,{}^{II}S_m + \beta_1\,{}^{I}S_m + \sum_{j=0}^{n}\lambda'_j\,y''_{m-j}\right\}.$$

$$\lambda'_j = (-1)^j \sum_{i=j}^{n}\beta'_i\binom{i}{j}$$

where

$$\beta'_i = \beta_{i+2} \tag{46}$$

Cowell Corrector Summed Ordinate Form

$$y_m = h\left\{\beta_0\,{}^{II}S_m + (\beta_0 + \beta_1)\,{}^{I}S_m + \sum_{j=0}^{n}\lambda_j^{*\prime}\,y''_{m-j}\right\}$$

$$\lambda_j^{*'} = (-1)^j \sum_{i=j}^{n} \beta_i^{*'} \binom{i}{j}$$

where

$$\beta_0^{*'} = (\beta_0^* + \beta_1^* + \beta_2^*)$$

and

$$\beta_i^{*'} = \beta_{i-2}^* \tag{47}$$

The coefficients σ_j', $\sigma_j^{*'}$, λ_j', and $\lambda_j^{*'}$ are given in rational form in the appendix in Tables 9 through 12. Within each table, subtables are presented on the basis of $n = 0$ to $n = 15$.

REMARKS

In determining the orbits of artificial satellites, in which the equations that describe the satellite's motion are extremely complex, numerical integration methods are very fruitful. Predictor-corrector methods for numerically integrating ordinary differential equations are used because they are efficient and lead to accurate results. In general, predictor-corrector methods have the following advantages:

1. Generally only one or perhaps two evaluations of the function need be computed at each step of the integration whereas one-step methods require at least four or more evaluations of the function.

2. The difference between predicted and corrected values provides a measure of the error being made at each step of the integration. Thus this error, which is better known as the local error, can be used to control the stepsize employed in the integration .

Some disadvantages in using predictor-corrector methods are:

1. The process is not self-starting.

2. The process is highly complex to program.

The main sources of trouble that arise when using any type of numerical method for integrating ordinary differential equations are (Henrici):

1. Truncation error due to finite approximations for the derivatives.

2. Propagation errors (instability).

3. Round-off errors due to a finite number of decimal figures used to express the coefficients in the formulas.

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REFERENCES

Henrici, Peter, "Discrete Variable Methods in Ordinary Differential Equations," John Wiley & Sons, Inc., New York, 1962.

Hildebrand, F. B., "Introduction to Numerical Analysis," McGraw-Hill Book Company, Inc., 1956.

APPENDIX

The formulas for the coefficients presented in the following tables were programmed in fortran using a rational arithmetic package to eliminate the deterioration which would have been incurred using floating point arithmetic. This rational package consisted of the following subroutines:

(1) GCD - A function which uses Euclid's algorithm to compute the Greatest Common Divisor of two numbers.

$$[A_1, A_2] = \text{GCD} > 0$$

where GCD = 1 if $A_1 = 0$ or $A_2 = 0$ or if A_1 or A_2 is not integral.

(2) ADD - A subroutine which performs rational addition defined by

$$\frac{N_1}{D_1} + \frac{N_2}{D_2} = \frac{N_1\left(\frac{D_2}{[D_1, D_2]}\right) + N_2\left(\frac{D_1}{[D_1, D_2]}\right)}{D_2\left(\frac{D_1}{[D_1, D_2]}\right)} = \frac{\frac{N_3}{[D_3, N_3]}}{\frac{D_3}{[D_3, N_3]}} = \frac{N_4}{D_4} .$$

(3) SUB - A subroutine which performs rational subtraction defined by

$$\frac{N_1}{D_1} - \frac{N_2}{D_2} = \frac{N_1}{D_1} + \frac{(-N_2)}{D_2} = \frac{N_3}{D_3} .$$

(4) MPY - A subroutine which performs rational multiplication defined by

$$\frac{N_1}{D_1} \cdot \frac{N_2}{D_2} = \frac{\frac{N_1}{[N_1, D_2]} \cdot \frac{N_2}{[N_2, D_1]}}{\frac{D_1}{[N_2, D_1]} \cdot \frac{D_2}{[N_1, D_2]}} = \frac{N_3}{D_3} .$$

(5) GRBC - A subroutine which calculates the Generalized Rational Binomial Coefficient defined by

$$\binom{-S}{m} = \prod_{i=1}^{m} \frac{S - (i - 1)}{i}$$

where

$$m = 0, 1, 2, \ldots, \qquad s = \ldots -2, -1, 0, 1, 2, \ldots$$

and

$$\binom{0}{m} = 0 \text{ for } m > 0, \; \binom{-S}{0} = 1 .$$

(6) HS – A subroutine which rationally computes the coefficients of the Harmonic Series defined by

$$H_k = \sum_{i=1}^{k} \frac{1}{i} .$$

These subroutines were so constructed that the numerator and denominator of any result were relatively prime (i.e. $(N, D) = 1$). Also, the sign of any term was carried by the numerator while the denominator was kept positive. A zero denominator was used to indicate loss of integral significance in the computation of a term.

These subroutines were used by a main routine to calculate the coefficients of the difference forms of the Cowell type formulas. A subroutine was used to calculate the coefficients for the ordinate forms. A final machine language subroutine was used to format and print the coefficients in rational form.

Tables 1-4 give the coefficients of the difference formulas. The coefficients for the summed difference formulas are not presented since they can easily be taken from the non-summed coefficient tables. Tables 5-8 present the coefficients for the non-summed ordinate forms of the formulas. Tables 9-12 give the coefficients for the summed ordinate forms. Although the lower order ordinate forms are essentially meaningless, they are included in the tables to provide completeness.

Table 1

Adams-Bashforth Predictor,
Non-Summed Difference Form

α_0	1
α_1	1/2
α_2	5/12
α_3	3/8
α_4	251/720
α_5	95/288
α_6	19087/60480
α_7	5257/17280
α_8	1070017/3628800
α_9	25713/89600
α_{10}	26842253/95800320
α_{11}	4777223/17418240
α_{12}	703604254357/2615348736000
α_{13}	106364763817/402361344000
α_{14}	1166309819657/4483454976000
α_{15}	25221445/98402304

Table 2

Adams-Moulton Corrector,
Non-Summed Difference Form

α_0^*	1
α_1^*	-1/2
α_2^*	-1/12
α_3^*	-1/24
α_4^*	-19/720
α_5^*	-3/160
α_6^*	-863/60480
α_7^*	-275/24192
α_8^*	-33953/3628800
α_9^*	-8183/1036800
α_{10}^*	-3250433/479001600
α_{11}^*	-4671/788480
α_{12}^*	-13695779093/2615348736000
α_{13}^*	-2224234463/475517952000
α_{14}^*	-132282840127/31384184832000
α_{15}^*	-2639651053/689762304000

Table 3

Störmer Predictor,
Non-Summed Difference Form

β_0	1
β_1	0
β_2	1/12
β_3	1/12
β_4	19/240
β_5	3/40
β_6	863/12096
β_7	275/4032
β_8	33953/518400
β_9	8183/129600
β_{10}	3250433/53222400
β_{11}	4671/78848
β_{12}	13695779093/237758976000
β_{13}	2224234463/39626496000
β_{14}	132282840127/2414168064000
β_{15}	2639651053/49268736000

Table 4

Cowell Corrector,
Non-Summed Difference Form

β_0^*	1
β_1^*	-1/1
β_2^*	1/12
β_3^*	0
β_4^*	-1/240
β_5^*	-1/240
β_6^*	-221/60480
β_7^*	-19/6048
β_8^*	-9829/3628800
β_9^*	-407/172800
β_{10}^*	-330157/159667200
β_{11}^*	-24377/13305600
β_{12}^*	-4281164477/2615348736000
β_{13}^*	-70074463/47551795200
β_{14}^*	-1197622087/896690995200
β_{15}^*	-97997951/80472268800

Table 5

Adams-Bashforth Predictor, Non-Summed Ordinate Form

Order = 1	σ_0	1
Order = 2	σ_0	3/2
	σ_1	-1/2
Order = 3	σ_0	23/12
	σ_1	-4/3
	σ_2	5/12
Order = 4	σ_0	55/24
	σ_1	-59/24
	σ_2	37/24
	σ_3	-3/8
Order = 5	σ_0	1901/720
	σ_1	-1387/360
	σ_2	109/30
	σ_3	-637/360
	σ_4	251/720
Order = 6	σ_0	4277/1440
	σ_1	-2641/480
	σ_2	4991/720
	σ_3	-3649/720
	σ_4	959/480
	σ_5	-95/288
Order = 7	σ_0	198721/60480
	σ_1	-18637/2520
	σ_2	235183/20160
	σ_3	-10754/945
	σ_4	135713/20160
	σ_5	-5603/2520
	σ_6	19087/60480
Order = 8	σ_0	16083/4480
	σ_1	-1152169/120960
	σ_2	242653/13440
	σ_3	-296053/13440
	σ_4	2102243/120960
	σ_5	-115747/13440
	σ_6	32863/13440
	σ_7	-5257/17280

Order = 9

σ_0	14097247/3628800
σ_1	-21562603/1814400
σ_2	47738393/1814400
σ_3	-69927631/1814400
σ_4	862303/22680
σ_5	-45586321/1814400
σ_6	19416743/1814400
σ_7	-4832053/1814400
σ_8	1070017/3628800

Order = 10

σ_0	4325321/1036800
σ_1	-104995189/7257600
σ_2	6648317/181440
σ_3	-28416361/453600
σ_4	569181919/3628800
σ_5	-522386081/3628800
σ_6	15788639/453600
σ_7	-2357683/181440
σ_8	20884811/7257600
σ_9	-25713/89600

Order = 11

σ_0	2132509567/479001600
σ_1	-2067948781/119750400
σ_2	1572737587/31933440
σ_3	-1621376209/19958400
σ_4	3539798831/26611200
σ_5	-82260679/623700
σ_6	2492064913/26611200
σ_7	-186080291/3991680
σ_8	2472634817/159667200
σ_9	-52841941/17107200
σ_{10}	26842253/95800320

Order = 12

σ_0	4527766399/958003200
σ_1	-6477936721/319334400
σ_2	12726645437/191600640
σ_3	-15064372973/106444800
σ_4	35689892561/159667200
σ_5	-41290273229/159667200
σ_6	35183920883/159667200
σ_7	-625551749/4561920
σ_8	923636629/15206400
σ_9	-17410248271/958003200
σ_{10}	30082309/9123840
σ_{11}	-4777223/17418240

Order = 13

σ_0	13064406523627/2615348736000
σ_1	-931781102989/39625496000
σ_2	5963794194517/72648576000
σ_3	-10498491598103/52306974720
σ_4	20731767690131/58118860800
σ_5	-34266767915049/72648576000
σ_6	2291133014533/4864860000
σ_7	-2826800577631/8072064000
σ_8	2253957178793/11623772100
σ_9	-20237791373837/261534873600
σ_{10}	4508814555201/217945728000
σ_{11}	-1694398344921/48432384000
σ_{12}	703604254357/2615348736000

Order = 14

σ_0	905730205/172204032
σ_1	-1409707506779621/5230697472000
σ_2	89541175419277/871782912000
σ_3	-34412222659093/124540416000
σ_4	570885914358161/10461394944000
σ_5	-31457535950413/38745907200
σ_6	134046425652457/145297152000
σ_7	-350379127127677/435891456000
σ_8	310429955675453/581188608000
σ_9	-10320787460413/38745907200
σ_{10}	72227659157949/74724249600
σ_{11}	-21029162113651/871782912000
σ_{12}	6460951197929/1743565824000
σ_{13}	-104364763817/402361344000

Order = 15

σ_0	13325653736373/2414168064000
σ_1	-60007679150257/1961511552000
σ_2	3960421470215481/31384184832000
σ_3	-25991762345039/70053984000
σ_4	25298910137081429/31384184832000
σ_5	-2614079370781733/1961511552000
σ_6	17023675553313503/10461394944000
σ_7	-2166615342637/1277025750
σ_8	13760072112094753/10461394944000
σ_9	-1544031478475483/1961511552000
σ_{10}	1600635679073597/4483454976000
σ_{11}	-58242613384023/490377888000
σ_{12}	859236476684231/31384184832000
σ_{13}	-696561442637/178319232000
σ_{14}	1166309819657/4483454976000

Order = 16

σ_0	362555126427073/62768369664000
σ_1	-2161567671245849/62768369664000
σ_2	7401613009731949/4828336128000
σ_3	-4372481080074367/8966909952000
σ_4	72558117072259733/62768369664000
σ_5	-131963191940828581/62768369664000
σ_6	62487713370967031/20922789888000
σ_7	-70006862970773983/20922789888000
σ_8	62029181421199481/20922789888000
σ_9	-129930094104237331/62768369664000
σ_{10}	10103478797549069/8966909952000
σ_{11}	-2674355637386529/5706215424000
σ_{12}	9038571752734087/62768369664000
σ_{13}	-1934443196892599/62768369664000
σ_{14}	30607182273669/8966909952000
σ_{15}	-25221445/98402304

Table 6

Adams-Moulton Corrector, Non-Summed Ordinate Form

Order = 1	σ_0^*	1
Order = 2	σ_0^*	1/2
	σ_1^*	1/2
Order = 3	σ_0^*	5/12
	σ_1^*	2/3
	σ_2^*	-1/12
Order = 4	σ_0^*	3/8
	σ_1^*	19/24
	σ_2^*	-5/24
	σ_3^*	1/24
Order = 5	σ_0^*	251/720
	σ_1^*	323/360
	σ_2^*	-11/30
	σ_3^*	53/360
	σ_4^*	-19/720
Order = 6	σ_0^*	95/288
	σ_1^*	1427/1440
	σ_2^*	-133/240
	σ_3^*	241/720
	σ_4^*	-173/1440
	σ_5^*	3/160
Order = 7	σ_0^*	19087/60480
	σ_1^*	2713/2520
	σ_2^*	-15487/20160
	σ_3^*	586/945
	σ_4^*	-6737/20160
	σ_5^*	263/2520
	σ_6^*	-863/60480
Order = 8	σ_0^*	5257/17280
	σ_1^*	139849/120960
	σ_2^*	-4511/4480
	σ_3^*	123133/120960
	σ_4^*	-88547/120960
	σ_5^*	1537/4480
	σ_6^*	-11351/120960
	σ_7^*	275/24192

Order = 9

σ_0^*	1070017/3628800
σ_1^*	2233547/1814400
σ_2^*	-2302297/1814400
σ_3^*	2797679/1814400
σ_4^*	-31457/22680
σ_5^*	1573169/1814400
σ_6^*	-645607/1814400
σ_7^*	156437/1814400
σ_8^*	-33953/3628800

Order = 10

σ_0^*	25713/89600
σ_1^*	9449717/7257600
σ_2^*	-1408913/907200
σ_3^*	200029/90720
σ_4^*	-8641823/3628800
σ_5^*	6755041/3628800
σ_6^*	-462127/453600
σ_7^*	335983/907200
σ_8^*	-116687/1451520
σ_9^*	8183/1036800

Order = 11

σ_0^*	26842253/95800320
σ_1^*	164046413/119750400
σ_2^*	-296725183/159667200
σ_3^*	12051709/3991680
σ_4^*	-33765029/8870400
σ_5^*	2227571/623700
σ_6^*	-21677723/8870400
σ_7^*	23643791/19958400
σ_8^*	-12318413/31933440
σ_9^*	9071219/119750400
σ_{10}^*	-3250433/479001600

Order = 12

σ_0^*	4777223/17418240
σ_1^*	1374799219/958003200
σ_2^*	-99642413/45619200
σ_3^*	36465037/9123840
σ_4^*	-102212233/17740800
σ_5^*	1007253581/159667200
σ_6^*	-91910491/17740800
σ_7^*	501289903/159667200
σ_8^*	-87064741/63866880
σ_9^*	384709327/958003200
σ_{10}^*	-68928781/958003200
σ_{11}^*	4671/788480

Order = 13

σ_0^*	703604254357/2615348736000
σ_1^*	6595204069/4402944000
σ_2^*	-551368413119/217945728000
σ_3^*	1346577425651/261534873600
σ_4^*	-485500845331/58118860800
σ_5^*	84400835489/8072064000
σ_6^*	-4874320027/486486000
σ_7^*	529394045911/72648576000
σ_8^*	-229882484333/58118860800
σ_9^*	406332786317/261534873600
σ_{10}^*	-30336027563/72648576000
σ_{11}^*	2724891251/39626496000
σ_{12}^*	-13695779093/2615348736000

Order = 14

σ_0^*	106364763817/402361344000
σ_1^*	741197087471/475517952000
σ_2^*	-168235945379/58118860800
σ_3^*	16964495066809/2615348736000
σ_4^*	-1748248003751/149448499200
σ_5^*	9575580965507/581188608000
σ_6^*	-7866111554491/435891456000
σ_7^*	1335017017153/87178291200
σ_8^*	-5797545653629/581188608000
σ_9^*	5124051955567/1046139494400
σ_{10}^*	-4590817802567/2615348736000
σ_{11}^*	1836420501/3773952000
σ_{12}^*	-69091417279/1046139494400
σ_{13}^*	2224234463/475517952000

Order = 15

σ_0^*	1164309819657/4483454976000
σ_1^*	3173185470929/1961511552000
σ_2^*	-102885148956217/31384184832000
σ_3^*	3933201478249/490377888000
σ_4^*	-71363086250691/4483454976000
σ_5^*	48849476129477/1961511552000
σ_6^*	-321201400274911/10461394944000
σ_7^*	38029005269/1277025750
σ_8^*	-236770944732449/10461394944000
σ_9^*	26159487787579/1961511552000
σ_{10}^*	-187504936597931/31384184832000
σ_{11}^*	137955863153/70053984000
σ_{12}^*	-14110480969927/31384184832000
σ_{13}^*	124922452271/1961511552000
σ_{14}^*	-132782940127/31384184832000

Order = 16

σ_0^*	25221445/98402304
σ_1^*	105145058757073/62768369664000
σ_2^*	-20997287611259/5706215424000
σ_3^*	612744541065337/62768369664000
σ_4^*	-189568380436667/8966909952000
σ_5^*	2285169598347733/62768369664000
σ_6^*	-3127453071993581/62768369664000
σ_7^*	1139313909617631/20922789888000
σ_8^*	-998783576755233/20922789888000
σ_9^*	677781959848881/20922789888000
σ_{10}^*	-1096355735402331/62768369664000
σ_{11}^*	64486158419969/8966909952000
σ_{12}^*	-137515713787319/62768369664000
σ_{13}^*	27219384284087/62768369664000
σ_{14}^*	-3867489367599/62768369664000
σ_{15}^*	2639851053/689762304000

Table 7

Störmer Predictor, Non-Summed Ordinate Form

Order = 1	λ_0	1
Order = 2	λ_0	1
	λ_1	0
Order = 3	λ_0	13/12
	λ_1	-1/6
	λ_2	1/12
Order = 4	λ_0	7/6
	λ_1	-5/12
	λ_2	1/3
	λ_3	-1/12
Order = 5	λ_0	299/240
	λ_1	-11/15
	λ_2	97/120
	λ_3	-2/5
	λ_4	19/240
Order = 6	λ_0	317/240
	λ_1	-133/120
	λ_2	187/120
	λ_3	-23/20
	λ_4	109/240
	λ_5	-3/40
Order = 7	λ_0	84199/60480
	λ_1	-15487/10080
	λ_2	52991/20160
	λ_3	-38963/15120
	λ_4	30731/20160
	λ_5	-5071/10080
	λ_6	863/12096
Order = 8	λ_0	22081/15120
	λ_1	-4511/2240
	λ_2	40933/10080
	λ_3	-300227/60480
	λ_4	9857/2520
	λ_5	-39017/20160
	λ_6	3319/6048
	λ_7	-275/4032

Order		
Order = 9	λ_0	5537111/3628800
	λ_1	-2302297/907200
	λ_2	5347667/907200
	λ_3	-7830749/907200
	λ_4	615621/72576
	λ_5	-5083159/907200
	λ_6	2161547/907200
	λ_7	-537217/907200
	λ_8	33953/518400
Order = 10	λ_0	1153247/725760
	λ_1	-1408913/453600
	λ_2	7409783/907200
	λ_3	-12642403/907200
	λ_4	29850337/1814400
	λ_5	-2460113/181440
	λ_6	6773151/907200
	λ_7	-2597333/907200
	λ_8	328541/518400
	λ_9	-3183/129600
Order = 11	λ_0	263465639/159667200
	λ_1	-296725183/79833600
	λ_2	1742930263/159667200
	λ_3	-424402351/19958400
	λ_4	2337301223/79833600
	λ_5	-1155556697/39916800
	λ_6	1637523663/79833600
	λ_7	-29064973/2851200
	λ_8	539497083/159667200
	λ_9	-53797223/79833600
	λ_{10}	3250433/53222400
Order = 12	λ_0	19494601/11404800
	λ_1	-99642413/22809600
	λ_2	40313623/2851200
	λ_3	-4955910683/159667200
	λ_4	278428507/5702400
	λ_5	-4496090419/79833600
	λ_6	955625177/19958400
	λ_7	-2374517119/79833600
	λ_8	1050348479/79833600
	λ_9	-627827071/159667200
	λ_{10}	84671/118800
	λ_{11}	-4671/78848
Order = 13	λ_0	4621155471343/2615348736000
	λ_1	-551368413119/108972864000
	λ_2	7835623954493/435891456000
	λ_3	-571608503363/13076743680
	λ_4	1499310871199/19372953600
	λ_5	-1851455205449/18162144000
	λ_6	3147964546373/31135104000
	λ_7	-1368797279699/18162144000
	λ_8	161456197531/3874590720
	λ_9	-1095689820701/65383718400
	λ_{10}	1967857329773/435891456000
	λ_{11}	-81782398949/108972364000
	λ_{12}	13695779093/237758976000

Order		
Order = 14	λ_0	681136420843/373621248000
	λ_1	-148235945379/29059430400
	λ_2	9744017123747/435891456000
	λ_3	-37076687599047/653837184000
	λ_4	20482239461389/174356582400
	λ_5	-25307804074469/145297152000
	λ_6	9605615337281/43589145600
	λ_7	-295967398557/1729728000
	λ_8	11133046558873/96864768000
	λ_9	-14839921713701/261534873600
	λ_{10}	117692703091/5331376000
	λ_{11}	-44710455301/8717829120
	λ_{12}	187186067207/237758976000
	λ_{13}	-2724234453/39625496000
Order = 15	λ_0	5357739561133/2853107712000
	λ_1	-102485146956217/15692092416000
	λ_2	34322393311201/1255367393280
	λ_3	-125041930211761/1569209241600
	λ_4	5408122701622671/31384184832000
	λ_5	-4454639434617463/15692092416000
	λ_6	3786744279520091/10461394944000
	λ_7	-94084230621037/261534873600
	λ_8	582610405386187/2092278988800
	λ_9	-2611731901394711/15692092416000
	λ_{10}	2366898122997363/31384184832000
	λ_{11}	-196730009641141/7846046208000
	λ_{12}	36239832148313/6276836966400
	λ_{13}	-2583707059781/3138418483200
	λ_{14}	1372282440127/24141680640000
Order = 16	λ_0	7577074249153/3723023104000
	λ_1	-209972287611259/2853107712000
	λ_2	103461289345973/3138418483200
	λ_3	-14518674965251/139485265920
	λ_4	1925847372615359/7846046208000
	λ_5	-13958696412680209/31384184832000
	λ_6	9887964365484539/15692092416000
	λ_7	-294803841434953/418455797760
	λ_8	81497235474541/130767436800
	λ_9	-13639159695198227/31384184832000
	λ_{10}	3708157829222323/15692092416000
	λ_{11}	-3082109827403329/31384184832000
	λ_{12}	15771040394797/523069747200
	λ_{13}	-40478026255543/6276836966400
	λ_{14}	1036213182041/12070840320000
	λ_{15}	-2639651053/49268736000

Table 8

Cowell Corrector, Non-Summed Ordinate Form

Order = 1	λ_0^*	1
Order = 2	λ_0^*	0
	λ_1^*	1
Order = 3	λ_0^*	1/12
	λ_1^*	5/6
	λ_2^*	1/12
Order = 4	λ_0^*	1/12
	λ_1^*	5/6
	λ_2^*	1/12
	λ_3^*	0
Order = 5	λ_0^*	19/240
	λ_1^*	17/20
	λ_2^*	7/120
	λ_3^*	1/60
	λ_4^*	-1/240
Order = 6	λ_0^*	3/40
	λ_1^*	209/240
	λ_2^*	1/60
	λ_3^*	7/120
	λ_4^*	-1/40
	λ_5^*	1/240
Order = 7	λ_0^*	863/12096
	λ_1^*	8999/10080
	λ_2^*	-769/20160
	λ_3^*	1987/15120
	λ_4^*	-1609/20160
	λ_5^*	263/10080
	λ_6^*	-221/60480
Order = 8	λ_0^*	275/4032
	λ_1^*	13831/15120
	λ_2^*	-2099/20160
	λ_3^*	811/3360
	λ_4^*	-11477/60480
	λ_5^*	29/315
	λ_6^*	-517/20160
	λ_7^*	19/6048

Order = 9

λ_0^*	33953/518400
λ_1^*	424759/453600
λ_2^*	-81629/453600
λ_3^*	11143/28350
λ_4^*	-27533/72576
λ_5^*	110563/453600
λ_6^*	-23017/226800
λ_7^*	5627/226800
λ_8^*	-7829/3628800

Order = 10

λ_0^*	3183/129600
λ_1^*	694499/725760
λ_2^*	-240181/907200
λ_3^*	536063/907200
λ_4^*	-613393/907200
λ_5^*	990713/1814400
λ_6^*	-54311/181440
λ_7^*	99431/907200
λ_8^*	-2711/113400
λ_9^*	407/172800

Order = 11

λ_0^*	3250433/53222400
λ_1^*	3124027/3193344
λ_2^*	-57128721/159667200
λ_3^*	16745741/19958400
λ_4^*	-88645069/79833600
λ_5^*	42375577/39916800
λ_6^*	-2342333/3193344
λ_7^*	7137837/19958400
λ_8^*	-18674153/159667200
λ_9^*	1833819/79833600
λ_{10}^*	-330157/159667200

Order = 12

λ_0^*	4671/78848
λ_1^*	79709557/79833600
λ_2^*	-73217741/159667200
λ_3^*	45558097/39916800
λ_4^*	-136911529/79833600
λ_5^*	76162079/39916800
λ_6^*	-126133369/79833600
λ_7^*	4401613/4989600
λ_8^*	-66740613/159667200
λ_9^*	9683229/79833600
λ_{10}^*	-3547721/159667200
λ_{11}^*	24377/13305600

Order = 13

λ_0^*	13895777093/237758976000
λ_1^*	221883255087/217945728000
λ_2^*	-246977242177/435891456000
λ_3^*	196341135019/130767436800
λ_4^*	-48921467453/19372953600
λ_5^*	116400317337/36324288000
λ_6^*	-96285993157/31135104000
λ_7^*	82048551887/36324288000
λ_8^*	-23619730793/19372953600
λ_9^*	63281534349/130767436800
λ_{10}^*	-56778633577/435891456000
λ_{11}^*	46804597737/217945728000
λ_{12}^*	-4281164477/2615348736000

Order		
Order = 14	λ_0^*	2224231463/39626496000
	λ_1^*	55362495961/53374464000
	λ_2^*	-143540241611/217945728000
	λ_3^*	157636465779/87178291200
	λ_4^*	-935007636363/261534873600
	λ_5^*	494112735397/96864768000
	λ_6^*	-204187600599/36324288000
	λ_7^*	1043426962817/217945728000
	λ_8^*	-12977594477/4151347200
	λ_9^*	264087268297/174356582400
	λ_{10}^*	-360735776113/653837184000
	λ_{11}^*	59464160519/435891456000
	λ_{12}^*	-9064067567/435891456000
	λ_{13}^*	70074463/47551795200
Order = 15	λ_0^*	132282640127/2414168064000
	λ_1^*	3341630986201/320246704000
	λ_2^*	-25204221139079/31384184832000
	λ_3^*	3741341671441/1569209241600
	λ_4^*	-30855021236321/6276836966400
	λ_5^*	122074052952359/15692092416000
	λ_6^*	-100765294790557/10461394944000
	λ_7^*	12254660322337/1307674368000
	λ_8^*	-2133206511431/293896998400
	λ_9^*	2643461754591/627683696640
	λ_{10}^*	-59274007071469/31384184832000
	λ_{11}^*	4434781236437/7846046208000
	λ_{12}^*	-4467039213359/31384184832000
	λ_{13}^*	63300396921/3138418483200
	λ_{14}^*	-1197622087/695690995200
Order = 16	λ_0^*	2439651053/49268736000
	λ_1^*	4214158807631/3923023104000
	λ_2^*	-29717237232529/31384184832000
	λ_3^*	9299656983377/3138418483200
	λ_4^*	-41239763079291/6276836966400
	λ_5^*	89695541544347/7846046208000
	λ_6^*	-54842553869569/3487131648000
	λ_7^*	90008734243873/5230697472000
	λ_8^*	-31328482761427/2092278988800
	λ_9^*	13108130887/980755776
	λ_{10}^*	-174045267344139/31384184832000
	λ_{11}^*	35654167080299/15692092416000
	λ_{12}^*	-21456775614309/31384184832000
	λ_{13}^*	232305003033/1569209241600
	λ_{14}^*	-123040957279/6276836966400
	λ_{15}^*	97997251/80472268800

Table 9

Adams-Bashforth Predictor, Summed Ordinate Form

Order = 1	α_0'	1/2
Order = 2	α_0'	11/12
	α_1'	-5/12
Order = 3	α_0'	31/24
	α_1'	-7/6
	α_2'	3/8
Order = 4	α_0'	1181/720
	α_1'	-177/80
	α_2'	341/240
	α_3'	-251/720
Order = 5	α_0'	2837/1440
	α_1'	-2543/720
	α_2'	17/5
	α_3'	-1201/720
	α_4'	95/288
Order = 6	α_0'	138241/60480
	α_1'	-309047/60480
	α_2'	198251/30240
	α_3'	-145477/30240
	α_4'	23077/12096
	α_5'	-19087/60480
Order = 7	α_0'	11603/4480
	α_1'	-104861/15120
	α_2'	1344989/120960
	α_3'	-20617/1890
	α_4'	156551/24192
	α_5'	-32371/15120
	α_6'	5257/17280
Order = 8	α_0'	10468447/3628800
	α_1'	-32656759/3628800
	α_2'	6980003/403200
	α_3'	-15407047/725760
	α_4'	12186649/725760
	α_5'	-3359933/403200
	α_6'	1227727/518400
	α_7'	-1070017/3628800

Order	Coefficient	Value
Order = 9	α_0'	3288521/1036800
	α_1'	-40987771/3628800
	α_2'	10219841/403200
	α_3'	-135352319/3628800
	α_4'	167287/4536
	α_5'	-9839609/403200
	α_6'	5393233/518400
	α_7'	-9401029/3628800
	α_8'	25713/89600

Order	Coefficient	Value
Order = 10	α_0'	1453507967/479001600
	α_1'	-2206095719/159667200
	α_2'	235733009/6652800
	α_3'	-407088591/9979200
	α_4'	1152537553/15966720
	α_5'	-1688873049/26611200
	α_6'	5376023/158400
	α_7'	-253022557/19958400
	α_8'	149484787/53222400
	α_9'	-26842253/95800320

Order	Coefficient	Value
Order = 11	α_0'	1669763199/958003200
	α_1'	-3966011741/239500800
	α_2'	1495154823/35481600
	α_3'	-1247363563/13305600
	α_4'	4144305961/31933440
	α_5'	-495967/3850
	α_6'	2087683637/22809600
	α_7'	-86656259/1900800
	α_8'	76795519/5068800
	α_9'	-144794759/47900160
	α_{10}'	4777223/17418240

Order	Coefficient	Value
Order = 12	α_0'	10449057787627/26153487360000
	α_1'	-51048495609647/26153487360000
	α_2'	32772419198593/523069747200
	α_3'	-24085098927479/174356582400
	α_4'	19055402071457/87178291200
	α_5'	-15761456733287/62270208000
	α_6'	13439569126937/62270208000
	α_7'	-11714849460703/87178291200
	α_8'	10381259604489/174356582400
	α_9'	-9320605566207/523069747200
	α_{10}'	748613348307/23775897600
	α_{11}'	-703404254357/26153487360000

Order	Coefficient	Value
Order = 13	α_0'	233526173/172204032
	α_1'	-59344946587373/26153487360000
	α_2'	104639289835229/1307674368000
	α_3'	-102675419234099/523069747200
	α_4'	121844691963321/348713164800
	α_5'	-40318232897599/87178291200
	α_6'	31975145463/69498000
	α_7'	-149631214658501/435891456000
	α_8'	66393001798471/348713164800
	α_9'	-13247642672623/174356582400
	α_{10}'	491703913717/23775897600
	α_{11}'	-9000055932083/26153487360000
	α_{12}'	106364763817/4023613440000

Order = 14

a_0'	10911485674373/2414168064000
a_1'	-818273652637263/31364184832000
a_2'	524691352929703/5230697472000
a_3'	-4247744706499627/15692092416000
a_4'	672136836963287/1255367393280
a_5'	-2780206445380617/3487131648000
a_6'	338680686327559/373621248000
a_7'	-7066463417427663/2615348736000
a_8'	1831406147461367/3487131648000
a_9'	-328673933138217/1255367393280
a_{10}'	135636420411007/1426553856000
a_{11}'	-124134305252953/5230697472000
a_{12}'	8802357320501/2414168064000
a_{13}'	-1166309619657/4483454976000

Order = 15

a_0'	299736756763073/627683696664000
a_1'	-116361307155361/3923023104000
a_2'	2586771098343187/20922789688000
a_3'	-356985279148297/980755776000
a_4'	1988442366270749/25107347865600
a_5'	-571195366208749/435891456000
a_6'	5010847670421097/2988969984000
a_7'	-2132356395131/1277025750
a_8'	9030884747790859/6974263296000
a_9'	-121630328435299/156920924160
a_{10}'	2006565473520353/5706215424000
a_{11}'	-3478073249303/29719872000
a_{12}'	130221619246627/4928336128000
a_{13}'	-2156804881129/560431872000
a_{14}'	25221445/98402304

Table 10

Adams-Moulton Corrector, Summed Ordinate Form

Order	Coefficient	Value
Order = 1	$\alpha_0^{*\prime}$	1/2
Order = 2	$\alpha_0^{*\prime}$	5/12
	$\alpha_1^{*\prime}$	1/12
Order = 3	$\alpha_0^{*\prime}$	3/8
	$\alpha_1^{*\prime}$	1/6
	$\alpha_2^{*\prime}$	-1/24
Order = 4	$\alpha_0^{*\prime}$	251/720
	$\alpha_1^{*\prime}$	59/240
	$\alpha_2^{*\prime}$	-29/240
	$\alpha_3^{*\prime}$	19/720
Order = 5	$\alpha_0^{*\prime}$	95/288
	$\alpha_1^{*\prime}$	77/240
	$\alpha_2^{*\prime}$	-7/30
	$\alpha_3^{*\prime}$	73/720
	$\alpha_4^{*\prime}$	-3/160
Order = 6	$\alpha_0^{*\prime}$	19087/60480
	$\alpha_1^{*\prime}$	23719/60480
	$\alpha_2^{*\prime}$	-11371/30240
	$\alpha_3^{*\prime}$	7331/30240
	$\alpha_4^{*\prime}$	-5449/60480
	$\alpha_5^{*\prime}$	863/60480
Order = 7	$\alpha_0^{*\prime}$	5257/17280
	$\alpha_1^{*\prime}$	6961/15120
	$\alpha_2^{*\prime}$	-66109/120960
	$\alpha_3^{*\prime}$	33/70
	$\alpha_4^{*\prime}$	-31523/120960
	$\alpha_5^{*\prime}$	1247/15120
	$\alpha_6^{*\prime}$	-275/24192
Order = 8	$\alpha_0^{*\prime}$	1070017/3628800
	$\alpha_1^{*\prime}$	1908311/3628800
	$\alpha_2^{*\prime}$	-299587/403200
	$\alpha_3^{*\prime}$	115963/145152
	$\alpha_4^{*\prime}$	-426809/725760
	$\alpha_5^{*\prime}$	112477/403200
	$\alpha_6^{*\prime}$	-273921/3628800
	$\alpha_7^{*\prime}$	33953/3628800

Order = 9

α_0^*	25713/89600
α_1^*	4277447/725760
α_2^*	-3498217/3628800
α_3^*	500327/403200
α_4^*	-6467/5670
α_5^*	2616161/3628800
α_6^*	-24019/80640
α_7^*	263077/3628800
α_8^*	-8183/1036800

Order = 10

α_0^*	26842253/95800320
α_1^*	103798439/159667200
α_2^*	-24115943/19958400
α_3^*	18071351/9979200
α_4^*	-159314453/79833600
α_5^*	25152927/15966720
α_6^*	-8660609/9979200
α_7^*	6322573/19958400
α_8^*	-11011481/159667200
α_9^*	3250433/479001600

Order = 11

α_0^*	4777223/17418240
α_1^*	8099901/11404800
α_2^*	-67283209/45619200
α_3^*	14380247/5702400
α_4^*	-517263181/159667200
α_5^*	76561/24948
α_6^*	-137204019/159667200
α_7^*	41021471/39916800
α_8^*	-107151937/319334400
α_9^*	15813379/239500800
α_{10}^*	-4671/788480

Order = 12

α_0^*	703604254357/2615348736000
α_1^*	2005806735343/2615348736000
α_2^*	-9227122844417/5230697472000
α_3^*	1180688004459/348713164800
α_4^*	-433079246049/87178291200
α_5^*	341749824023/62270208000
α_6^*	-282163137433/62270208000
α_7^*	240744462687/87178291200
α_8^*	-8366341105/6974263296
α_9^*	185189984759/523069747200
α_{10}^*	-166147043473/2615348736000
α_{11}^*	13695774073/2615348736000

Order = 13

α_0^*	106364763817/402361344000
α_1^*	307515172843/373621248000
α_2^*	-2709005666077/1307674368000
α_3^*	2309796746931/523069747200
α_4^*	-507942835493/69742632960
α_5^*	4007043002299/435891456000
α_6^*	-2215533/250250
α_7^*	2815716533573/435891456000
α_8^*	-175102023617/49816166400
α_9^*	144690945961/104613949440
α_{10}^*	-486772076771/1307674368000
α_{11}^*	160495253651/2615348736000
α_{12}^*	-2224234463/475517952000

Order = 14

$\alpha_0^{*'}$	1166309819657/4483454976000
$\alpha_1^{*'}$	2504631949133/2853107712000
$\alpha_2^{*'}$	-125553699585959/5230697472000
$\alpha_3^{*'}$	83195348546091/15692092416000
$\alpha_4^{*'}$	-64631301332531/6276836966400
$\alpha_5^{*'}$	50972790156553/3487131648000
$\alpha_6^{*'}$	-42070857451313/2615348736000
$\alpha_7^{*'}$	5116077905657/373621248000
$\alpha_8^{*'}$	-31173567791351/3487131648000
$\alpha_9^{*'}$	275977902895821/6276836966400
$\alpha_{10}^{*'}$	-24757711059413/15692092416000
$\alpha_{11}^{*'}$	2040667428953/5230697472000
$\alpha_{12}^{*'}$	-1366476396209/31384184832000
$\alpha_{13}^{*'}$	1327828840127/31384184832000

Order = 15

$\alpha_0^{*'}$	25221445/98402304
$\alpha_1^{*'}$	3654051145153/3923023104000
$\alpha_2^{*'}$	-172527345401401/62768369664000
$\alpha_3^{*'}$	2297797894083/326918592000
$\alpha_4^{*'}$	-886761467394133/62768369664000
$\alpha_5^{*'}$	3496017827389/156920924160
$\alpha_6^{*'}$	-192339437673109/6974263296000
$\alpha_7^{*'}$	349531097/13030875
$\alpha_8^{*'}$	-427489980816779/20922789888000
$\alpha_9^{*'}$	5256082896499/435891456000
$\alpha_{11}^{*'}$	-13579171932259/2510734786560
$\alpha_{12}^{*'}$	1748609541047/980755776000
$\alpha_{13}^{*'}$	-8530634387437/20922789888000
$\alpha_{14}^{*'}$	226717571111/3923023104000
$\alpha_{15}^{*'}$	-2639451053/689762304000

Table 11

Störmer Predictor, Summed Ordinate Form

Order		Value
Order = 1	λ_0'	1/12
Order = 2	λ_0'	1/6
	λ_1'	-1/12
Order = 3	λ_0'	59/240
	λ_1'	-29/120
	λ_2'	19/240
Order = 4	λ_0'	77/240
	λ_1'	-7/15
	λ_2'	73/240
	λ_3'	-3/40
Order = 5	λ_0'	23719/60480
	λ_2'	-11371/15120
	λ_3'	7381/10080
	λ_4'	-5449/15120
	λ_5'	863/12096
Order = 6	λ_0'	6961/15120
	λ_1'	-66109/60480
	λ_2'	79/70
	λ_3'	-31523/30240
	λ_4'	1247/3024
	λ_5'	-275/4032
Order = 7	λ_0'	1909311/3628800
	λ_1'	-2999587/201600
	λ_2'	115963/48384
	λ_3'	-425809/181440
	λ_4'	112477/60640
	λ_5'	-278921/604800
	λ_6'	33953/518400
Order = 8	λ_0'	427487/725760
	λ_1'	-3498217/1814400
	λ_2'	500327/134400
	λ_3'	-12934/2835
	λ_4'	2616161/725760
	λ_5'	-24019/13440
	λ_6'	263077/518400
	λ_7'	-8183/129600

Order = 9	λ_0'	103793439/159667200
	λ_1'	-24115443/9979200
	λ_2'	18071351/3326400
	λ_3'	-157314453/19958400
	λ_4'	25162927/3193344
	λ_5'	-8660609/1663200
	λ_6'	6322573/2851200
	λ_7'	-11011481/19958400
	λ_8'	3250433/53222400

Order = 10	λ_0'	8049801/11404800
	λ_1'	-67283209/22809600
	λ_2'	14330247/1700800
	λ_3'	-517263181/39916800
	λ_4'	382805/24948
	λ_5'	-137204019/26611200
	λ_6'	41021471/5702400
	λ_7'	-107151937/39916800
	λ_8'	15613379/26611200
	λ_9'	-4671/78848
	λ_{10}'	

Order = 11	λ_0'	2005806735343/2615348736000
	λ_1'	-922122844417/261534873600
	λ_2'	118068800459/11623772160
	λ_3'	-433079266849/21794572800
	λ_4'	341749826023/12454041600
	λ_5'	-247163137433/10378368000
	λ_6'	270244462687/12454041600
	λ_7'	-8366341105/371782912
	λ_8'	135189987759/58118860800
	λ_9'	-166147043473/261534873600
	λ_{10}'	13695779093/23775897600

Order = 12	λ_0'	307515172843/373621248000
	λ_1'	-2709005666077/653837184000
	λ_2'	2309796746931/174356582400
	λ_3'	-507942835493/17435658240
	λ_4'	4007043002299/87178291200
	λ_5'	-6646599/125125
	λ_6'	2816014533573/62270208000
	λ_7'	-175102023617/6227020800
	λ_8'	144690945961/11623772160
	λ_9'	-486772076771/1307674368000
	λ_{10}'	160495253651/237758976000
	λ_{11}'	-2224234443/39626496000
	λ_{12}'	

Order = 13

2504431949133/28531077712000
-12555499585959/2615348736000
33195148546091/5230697472000
-64631301332531/15692092416000
50772790156543/697426329600
-42070857451313/435891456000
5116077905657/53374464000
-31173587791351/435891456000
27597902395821/697426329600
-24757711059413/1569209241600
2041667423953/475517952000
-1866476396209/2615348736000
132282640127/2414168054000

Order = 14

λ'_0	3654051145153/3923023104000
λ'_1	-172527345401401/31384184832000
λ'_2	2272777894083/108972864000
λ'_3	-896761467394133/15692092416000
λ'_4	3476017827339/31384184832
λ'_5	-172334437693109/1162377216000
λ'_6	2447067679/13030875
λ'_7	-427439980316929/2615348736000
λ'_8	5256082896499/49432384000
λ'_9	-13579171932259/251073478656
λ'_{10}	1748809541047/89159616000
λ'_{11}	-8530634387437/1743565624000
λ'_{12}	226717570111/301771008000
λ'_{13}	-2639651053/49268736000

Table 12

Cowell Corrector, Summed Ordinate Form

Order = 1	$\lambda_0^{*\prime}$	1/12
Order = 2	$\lambda_0^{*\prime}$	1/12
	$\lambda_1^{*\prime}$	0
Order = 3	$\lambda_0^{*\prime}$	19/240
	$\lambda_1^{*\prime}$	1/120
	$\lambda_0^{*\prime}$	-1/240
Order = 4	$\lambda_0^{*\prime}$	3/40
	$\lambda_1^{*\prime}$	1/48
	$\lambda_2^{*\prime}$	-1/60
	$\lambda_3^{*\prime}$	1/240
Order = 5	$\lambda_0^{*\prime}$	863/12096
	$\lambda_1^{*\prime}$	67/1390
	$\lambda_2^{*\prime}$	-389/10080
	$\lambda_3^{*\prime}$	71/3780
	$\lambda_4^{*\prime}$	-221/60430
Order = 6	$\lambda_0^{*\prime}$	275/4032
	$\lambda_1^{*\prime}$	221/4320
	$\lambda_2^{*\prime}$	-2117/30240
	$\lambda_3^{*\prime}$	253/5040
	$\lambda_4^{*\prime}$	-1171/60480
	$\lambda_5^{*\prime}$	19/6048
Order = 7	$\lambda_0^{*\prime}$	33953/518400
	$\lambda_1^{*\prime}$	40769/604800
	$\lambda_2^{*\prime}$	-5353/48384
	$\lambda_3^{*\prime}$	13937/181440
	$\lambda_4^{*\prime}$	-14513/241920
	$\lambda_5^{*\prime}$	11729/604800
	$\lambda_6^{*\prime}$	-9829/3628800
Order = 8	$\lambda_0^{*\prime}$	8183/129600
	$\lambda_1^{*\prime}$	33827/403200
	$\lambda_2^{*\prime}$	-96827/604800
	$\lambda_3^{*\prime}$	135577/725760
	$\lambda_4^{*\prime}$	-4307/30240
	$\lambda_5^{*\prime}$	83287/1209600
	$\lambda_6^{*\prime}$	-34929/1814400
	$\lambda_7^{*\prime}$	407/172800

Order = 9

$\lambda_0^{*'}$	1250433/53222400
$\lambda_1^{*'}$	572741/5702400
$\lambda_2^{*'}$	-8701681/39916800
$\lambda_3^{*'}$	4025311/13305600
$\lambda_4^{*'}$	-917039/3193344
$\lambda_5^{*'}$	7370669/39916800
$\lambda_6^{*'}$	-1025779/13305600
$\lambda_7^{*'}$	754331/39916800
$\lambda_8^{*'}$	-330157/159667200

Order = 10

$\lambda_0^{*'}$	4671/78848
$\lambda_1^{*'}$	212153/1814400
$\lambda_2^{*'}$	-11334397/39916800
$\lambda_3^{*'}$	6073979/13305600
$\lambda_4^{*'}$	-41354987/79833600
$\lambda_5^{*'}$	663407/1596672
$\lambda_6^{*'}$	-3073447/13305600
$\lambda_7^{*'}$	3337047/39916800
$\lambda_8^{*'}$	-2962873/159667200
$\lambda_9^{*'}$	24377/13305600

Order = 11

$\lambda_0^{*'}$	13695779093/237758976000
$\lambda_1^{*'}$	34861746509/261534873600
$\lambda_2^{*'}$	-2309338501/6457651200
$\lambda_3^{*'}$	14230342079/21794572800
$\lambda_4^{*'}$	-19732542949/12454041600
$\lambda_5^{*'}$	409205237/494208000
$\lambda_6^{*'}$	-7157910969/12454041600
$\lambda_7^{*'}$	6130492139/21794572800
$\lambda_8^{*'}$	-1786550083/19372953600
$\lambda_9^{*'}$	4760318789/261534873600
$\lambda_{10}^{*'}$	-4281164477/2615348736000

Order = 12

$\lambda_0^{*'}$	2224234463/39626496000
$\lambda_1^{*'}$	6015577157/40236134400
$\lambda_2^{*'}$	-23681433587/65383718400
$\lambda_3^{*'}$	2479964869/2767564800
$\lambda_4^{*'}$	-58761423629/43589145600
$\lambda_5^{*'}$	91954909977/62270208000
$\lambda_6^{*'}$	-1103076741/1037836800
$\lambda_7^{*'}$	66917018671/87178291200
$\lambda_8^{*'}$	-27237000431/87178291200
$\lambda_9^{*'}$	17305222231/174356582400
$\lambda_{10}^{*'}$	-14683997/833976000
$\lambda_{11}^{*'}$	70074463/47551795200

Order = 13

$\lambda_0^{*'}$	132292640127/24141680064000
$\lambda_1^{*'}$	172913017/1162377216
$\lambda_2^{*'}$	-183706612697/348713164800
$\lambda_3^{*'}$	133373184587/112086374400
$\lambda_4^{*'}$	-467089093853/232475443200
$\lambda_5^{*'}$	26437359127/10378368000
$\lambda_6^{*'}$	-186038426051/74724249600
$\lambda_7^{*'}$	5304463979/2905943040
$\lambda_8^{*'}$	-77220056327/77491814400
$\lambda_9^{*'}$	308415783287/784504620800
$\lambda_{10}^{*'}$	-3800585233/83026944000
$\lambda_{11}^{*'}$	1525695617/87178291200
$\lambda_{12}^{*'}$	-1197622087/396690995200

Order = 14	$\lambda_0^{*'}$	2639651053/49268736000
	$\lambda_1^{*'}$	189733369019/1046139494400
	$\lambda_2^{*'}$	-14455328009/23247544320
	$\lambda_3^{*'}$	482751631389/313841848320
	$\lambda_4^{*'}$	-6025583753767/20922789888000
	$\lambda_5^{*'}$	2432562785657/581188608000
	$\lambda_6^{*'}$	-2395338127411/523069747200
	$\lambda_7^{*'}$	6965553271/1779148800
	$\lambda_8^{*'}$	-596016550799/232475443200
	$\lambda_9^{*'}$	3966335995783/3138418483200
	$\lambda_{10}^{*'}$	-2376718841769/5230697472000
	$\lambda_{11}^{*'}$	2179727217/19372953600
	$\lambda_{12}^{*'}$	-107753276923/6276836966400
	$\lambda_{13}^{*'}$	97997951/80472268600