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CPT violation and the standard model

Don Colladay and V. Alan Kostelecký
Department of Physics, Indiana University, Bloomington, Indiana 47405
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Spontaneous *CPT* breaking arising in string theory has been suggested as a possible observable experimental signature in neutral-meson systems. We provide a theoretical framework for the treatment of low-energy effects of spontaneous *CPT* violation and the attendant partial Lorentz breaking. The analysis is within the context of conventional relativistic quantum mechanics and quantum field theory in four dimensions. We use the framework to develop a *CPT*-violating extension to the minimal standard model that could serve as a basis for establishing quantitative *CPT* bounds.
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I. INTRODUCTION

Among the symmetries of the minimal standard model is invariance under *CPT*. Indeed, *CPT* invariance holds under mild technical assumptions for any local relativistic point-particle field theory [1–5]. Numerous experiments have confirmed this result [6], including in particular high-precision tests using neutral-kaon interferometry [7,8]. The simultaneous existence of a general theoretical proof of *CPT* invariance in particle physics and accurate experimental tests makes *CPT* violation an attractive candidate signature for nonparticle physics such as string theory [9,10].

The assumptions needed to prove the *CPT* theorem are invalid for strings, which are extended objects. Moreover, since the critical string dimensionality is larger than four, it is plausible that higher-dimensional Lorentz breaking would be incorporated in a realistic model. In fact, a mechanism is known in string theory that can cause spontaneous *CPT* violation [9] with accompanying partial Lorentz-symmetry breaking [11]. The effect can be traced to string interactions that are absent in conventional four-dimensional renormalizable gauge theory. Under suitable circumstances, these interactions can cause instabilities in Lorentz-tensor potentials, thereby inducing spontaneous *CPT* and Lorentz breaking. If in a realistic theory the spontaneous *CPT* and partial Lorentz violation extend to the four-dimensional spacetime, detectable effects might occur in interferometric experiments with neutral kaons [9,10], neutral B_d or B_s mesons [10,12], or neutral *D* mesons [10,13]. For example, the quantities parametrizing indirect *CPT* violation in these systems could be nonzero. There may also be implications for baryogenesis [14].

In the present paper, our goal is to develop within an effective-theory approach a plausible *CPT*-violating extension of the minimal standard model that provides a theoretical basis for establishing quantitative bounds on *CPT* invariance. The idea is to incorporate notions of spontaneous *CPT* and Lorentz breaking while maintaining the usual gauge structure and properties like renormalizability. To achieve this, we first establish a conceptual framework and a procedure for treating spontaneous *CPT* and Lorentz violation in the context of conventional quantum theory. We seek a general methodology that is compatible with desirable features like microscopic causality while being sufficiently detailed to permit explicit calculations.

We suppose that underlying the effective four-dimensional action is a complete fundamental theory that is based on conventional quantum physics [15] and is dynamically *CPT* and Poincaré invariant. The fundamental theory is assumed to undergo spontaneous *CPT* and Lorentz breaking. In a Poincaré-observer frame in the low-energy effective action, this process is taken to fix the form of any *CPT*- and Lorentz-violating terms.

Since interferometric tests of *CPT* violation are so sensitive, we focus specifically on *CPT* violation and the associated Lorentz-breaking issues in a low-energy effective theory without gravity [21]. For the most part, effects from derivative couplings and possible *CPT*-preserving but Lorentz-breaking terms in the action are disregarded, and any *CPT*-violating terms are taken to be small enough to avoid issues with standard experimental tests of Lorentz symmetry. A partial justification for the latter assumption is that the absence of signals for *CPT* violation in the neutral-kaon system provides one of the best bounds on Lorentz invariance.

Our focus on the low-energy effective model bypasses various important theoretical issues regarding the structure of the underlying fundamental theory and its behavior at scales above electroweak unification, including the origin and (renormalization-group) stability of the suppression of *CPT* breaking and the issue of mode fluctuations around Lorentz-tensor expectation values. Since these topics involve the Lorentz structure of the fundamental theory, they are likely to be related to the difficult hierarchy problems associated with compactification and the cosmological constant.

The ideas underlying our theoretical framework are described in Sec. II. A simple model is used to illustrate concepts associated with *CPT* and Lorentz breaking, including the possibility of eliminating some *CPT*-violating effects through field redefinitions. The associated relativistic quantum mechanics is discussed in Sec. III. Section IV contains a treatment of some issues in quantum field theory. A *CPT*-violating extension of the minimal standard model is provided in Sec. V, and the physically observable subset of

CPT-breaking terms is established. We summarize in Sec. VI. Some of the more technical results are presented in the appendixes.

II. BASICS

A. Effective model for spontaneous CPT violation

We begin our considerations with a simple model within which many of the basic features of spontaneous CPT violation can be examined. The model involves a single massive Dirac field $\psi(x)$ in four dimensions with Lagrangian density

$$\mathcal{L}=\mathcal{L}_0-\mathcal{L}', \tag{1}$$

where $\mathcal{L}_0$ is the usual free-field Dirac Lagrangian for a fermion ψ of mass m, and where $\mathcal{L}'$ contains extra CPT-violating terms to be described below. For the present discussion, we follow an approach in which the C, P, T and Lorentz properties of ψ are assumed to be conventionally determined by the free-field theory $\mathcal{L}_0$ and are used to establish the corresponding properties of $\mathcal{L}'$ [22]. This method is intrinsically perturbative, which is particularly appropriate here since any CPT-violating effects must be small. In Sec. II C, we consider the possibility of alternative definitions of C, P, T and Lorentz properties that could encompass the full structure of $\mathcal{L}$.

We are interested in possible forms of $\mathcal{L}'$ that could arise as effective contributions from spontaneous CPT violation in a more complete theory. To our knowledge, string theory forms the only class of (gauge) theories in four or more dimensions that are quantum consistent, dynamically Poincaré invariant, and known to admit an explicit mechanism [9] for spontaneous CPT violation triggered by interactions in the Lagrangian. However, to keep the treatment as general as possible we assume only that the spontaneous CPT violation arises from nonzero expectation values acquired by one or more Lorentz tensors T, so $\mathcal{L}'$ is taken to be an effective four-dimensional Lagrangian obtained from an underlying theory involving Poincaré-invariant interactions of ψ with T. The discussion that follows is independent of any specifics of string theory and should therefore be relevant to a nonstring model with spontaneous CPT violation, if such a model is eventually formulated.

Even applying the stringent requirement of dynamical Poincaré invariance, an unbroken realistic theory can in principle include terms with derivatives, powers of tensor fields, and powers of various terms quadratic in fermion fields. However, any CPT-breaking term that is to be part of a four-dimensional effective theory must have mass dimension four. In the effective Lagrangian, each combination of fields and derivatives of dimension greater than four therefore must have a corresponding weighting factor of a negative power $-k$ of at least one mass scale M that is large compared to the scale m of the effective theory. In a realistic theory with the string scenario, M might be the Planck mass or perhaps a smaller mass scale associated with compactification and unification. Moreover, since the expectations $\langle T\rangle$ of the tensors T are assumed to be Lorentz and possibly CPT violating, any terms that survive in $\mathcal{L}'$ after the spontaneous symmetry breaking must on physical grounds be suppressed, presumably by at least one power of m/M relative to the scale of the effective theory.

A hierarchy of possible terms in $\mathcal{L}'$ thus emerges, labeled by $k=0,1,2,\ldots$. Omitting Lorentz indices for simplicity, the leading terms with $k\leqslant 2$ have the schematic form

$$\mathcal{L}'\supset\frac{\lambda}{M^k}\langle T\rangle\cdot\overline{\psi}\Gamma(i\partial)^k\psi+\text{H.c.} \tag{2}$$

In this expression, the parameter λ is a dimensionless coupling constant, $(i\partial)^k$ represents k four-derivatives acting in some combination on the fermion fields, and Γ represents some gamma-matrix structure. Terms with $k\geqslant 3$ and with more quadratic fermion factors also appear, but these are further suppressed. Note that contributions of the form (2) arise in string theory [10]. Note also that naive power counting indicates the dominant terms with $k\leqslant 1$ are renormalizable.

For $k=0$, the above considerations indicate that the dominant terms of the form (2) must have expectations $\langle T\rangle\sim m^2/M$. In the present work, we focus primarily on this relatively simple case. Most of the general features arising from CPT and Lorentz violation together with some of our more specific results remain valid when terms with other values of k are considered, but it remains an open issue to investigate the detailed properties of terms with $k=1$ and expectations $\langle T\rangle\sim m$ or those with $k=2$ and expectations $\langle T\rangle\sim M$. Both these could in principle contribute leading effects in the low-energy effective action.

Each contribution to $\mathcal{L}'$ from an expression of the form (2) is a fermion bilinear involving a 4×4 spinor matrix Γ. Regardless of the complexity and number of the tensors T inducing the breaking, Γ can be decomposed as a linear combination of the usual 16 basis elements of the gamma-matrix algebra. Only the subset of these that produce CPT-violating bilinears are of interest for our present purposes, and they permit us to provide explicit and relatively simple expressions for the possible CPT-violating contributions to $\mathcal{L}'$.

For the case $k=0$ of interest here, we find two possible types of CPT-violating term:

$$\mathcal{L}'_a\equiv a_\mu\overline{\psi}\gamma^\mu\psi,\quad \mathcal{L}'_b\equiv b_\mu\overline{\psi}\gamma_5\gamma^\mu\psi. \tag{3}$$

For completeness, we provide here also the terms appearing for the case $k=1$, where we find three types of relevant contribution:

$$\mathcal{L}'_c\equiv\tfrac{1}{2}\,ic^\alpha\overline{\psi}\overleftrightarrow{\partial}_\alpha\psi,\quad \mathcal{L}'_d\equiv\tfrac{1}{2}\,d^\alpha\overline{\psi}\gamma_5\overleftrightarrow{\partial}_\alpha\psi,$$
$$\mathcal{L}'_e\equiv\tfrac{1}{2}\,ie^\alpha_{\mu\nu}\overline{\psi}\sigma^{\mu\nu}\overleftrightarrow{\partial}_\alpha\psi, \tag{4}$$

where $A\overleftrightarrow{\partial}_\mu B\equiv A\partial_\mu B-(\partial_\mu A)B$. In all these expressions, the quantities a_μ, b_μ, c^α, d^α, and $e^\alpha_{\mu\nu}$ must be real as consequences of their origins in spontaneous symmetry breaking and of the presumed hermiticity of the underlying theory. They are combinations of coupling constants, tensor expectations, mass parameters, and coefficients arising from the decomposition of Γ.

In keeping with their interpretation as effective coupling constants arising from a scenario with spontaneous symmetry breaking, a_μ, b_μ, c^α, d^α, and $e^\alpha_{\mu\nu}$ are invariant under CPT transformations. Together with the standard

CPT-transformation properties ascribed to ψ, this invariance causes the terms in Eqs. (3) and (4) to break CPT [23]. As discussed above, in the remainder of this work we restrict ourselves largely to the expressions in Eq. (3).

Allowing both kinds of term in Eq. (3) to appear in $\mathcal{L}'$ produces a model Lagrangian of the form

$$\mathcal{L}=\frac{1}{2}\, i\overline{\psi}\gamma^{\mu}\overleftrightarrow{\partial_{\mu}}\psi - a_{\mu}\overline{\psi}\gamma^{\mu}\psi - b_{\mu}\overline{\psi}\gamma_{5}\gamma^{\mu}\psi - m\overline{\psi}\psi. \quad (5)$$

The variational procedure generates a modified Dirac equation:

$$(i\gamma^{\mu}\partial_{\mu} - a_{\mu}\gamma^{\mu} - b_{\mu}\gamma_{5}\gamma^{\mu} - m)\psi = 0. \quad (6)$$

Associated with this Dirac-type equation is a modified Klein-Gordon equation. Proceeding with the usual squaring procedure, in which the Dirac-equation operator with opposite mass sign is applied to the Dirac equation from the left, leads to the Klein-Gordon-type expression

$$[(i\partial - a)^{2} - b^{2} - m^{2} + 2i\gamma_{5}\sigma^{\mu\nu}b_{\mu}(i\partial_{\nu} - a_{\nu})]\psi(x) = 0. \quad (7)$$

This equation is second order in derivatives, but unlike the usual Klein-Gordon case it contains off-diagonal terms in the spinor space. These may be eliminated by repeating the squaring procedure, this time applying the operator in Eq. (7) with opposite sign for the off-diagonal piece. The result is a fourth-order equation satisfied by each spinor component of any solution to the modified Dirac equation:

$$\{[(i\partial - a)^{2} - b^{2} - m^{2}]^{2} + 4b^{2}(i\partial - a)^{2} - 4[b^{\mu}(i\partial_{\mu} - a_{\mu})]^{2}\}\psi(x) = 0. \quad (8)$$

B. Continuous symmetries

Consider next the continuous symmetries of the model with Lagrangian (5). For definiteness, we begin with an analysis in a given oriented inertial frame in which values of the quantities a_{μ} and b_{μ} are assumed to have been specified. The effects of rotations and boosts are considered later.

The CPT-violating terms in Eq. (5) leave unaffected the usual global U(1) gauge invariance, which has conserved current $j^{\mu} = \overline{\psi}\gamma^{\mu}\psi$. Charge is therefore conserved in the model. These terms also leave unaffected the usual breaking of the chiral U(1) current $j_{5}^{\mu} = \overline{\psi}\gamma_{5}\gamma^{\mu}\psi$ due to the mass term. In what follows, we denote the volume integrals of the current densities j^{μ} and j_{5}^{μ} by J^{μ} and J_{5}^{μ}, respectively.

The model is also invariant under translations provided the tensor expectations are assumed constant, i.e., provided the possibility of CPT-breaking soliton-type solutions in the underlying theory is disregarded. This leads to a conserved canonical energy-momentum tensor $\Theta^{\mu\nu}$ given by

$$\Theta^{\mu\nu} = \frac{1}{2}\, i\overline{\psi}\gamma^{\mu}\overleftrightarrow{\partial^{\nu}}\psi, \quad \partial_{\mu}\Theta^{\mu\nu} = 0, \quad (9)$$

and a corresponding conserved four-momentum P^{μ}. These expressions have the same form as in the free theory. Note, however, that constancy of the energy and momentum does not necessarily imply conventional behavior under boosts or rotations. Note also that the presence of the CPT-violating terms in the Dirac equation destroys the usual symmetrizability property of $\Theta^{\mu\nu}$. The antisymmetric part $\Theta^{[\mu\nu]}$ is

$$\Theta^{[\mu\nu]} \equiv \Theta^{\mu\nu} - \Theta^{\nu\mu} = -\frac{1}{4}\,\partial_{\alpha}[\overline{\psi}\{\gamma^{\alpha},\sigma^{\mu\nu}\}\psi] - a^{[\mu}j^{\nu]} - b^{[\mu}j_{5}^{\nu]}, \quad (10)$$

which is no longer a total divergence. The conventional construction of a symmetric energy-momentum tensor, involving a subtraction of this antisymmetric part from the canonical energy-momentum tensor, would affect the conserved energy and momentum and is therefore presumably inapplicable in the present case. The implications of this for a more complete low-energy effective theory that includes gravity remain to be explored.

Next, consider the effect of Lorentz transformations, i.e., rotations and boosts. Conventional Lorentz transformations in special relativity relate observations made in two inertial frames with differing orientations and velocities. These transformations can be implemented as coordinate changes, and we call them observer Lorentz transformations. It is also possible to consider transformations that relate the properties of two particles with differing spin orientation or momentum within a specific oriented inertial frame. We call these particle Lorentz transformations. For free particles under usual circumstances, the two kinds of transformation are (inversely) related. However, this equivalence fails for particles under the action of a background field.

The reader is warned to avoid confusing observer Lorentz transformations (which involve coordinate changes) or particle Lorentz transformations (which involve boosts on particles or localized fields but *not* on background fields) with a third type of Lorentz transformation that within a specified inertial frame boosts all particles and fields simultaneously, including background ones. The latter are sometimes called (inverse) active Lorentz transformations. For the case of conventional free particles, they coincide with particle Lorentz transformations. We have chosen to avoid applying the terms active and passive here because they are insufficient to distinguish the three kinds of transformation and because in any case their interpretation varies in the literature.

The distinction between observer and particle transformations is relevant for the present model, where the CPT-violating terms can be regarded as arising from constant background fields a_{μ} and b_{μ}. The point is that these eight quantities transform as two four-vectors under observer Lorentz transformations and as eight scalars under particle Lorentz transformations, whereas they are coupled to currents that transform as four-vectors under both types of transformation. This means that observer Lorentz symmetry is still an invariance of the model, but the particle Lorentz group is (partly) broken.

Physical situations with features like this can readily be identified. For example, an electron with momentum perpendicular to a uniform background magnetic field moves in a circle. Suppose in the same observer frame we instantaneously increase the magnitude of the electron momentum without changing its direction, causing the electron to move in a circle of larger radius. This (instantaneous) particle boost leaves the background field unaffected. However, if instead an observer boost perpendicular to the magnetic field is applied, the electron no longer moves in a circle. This is viewed in the new inertial frame as an $E\times B$ drift caused by the presence of an electric field. In this example, the background magnetic field transforms into a different electromagnetic field under observer boosts but (by definition) is unchanged by particle boosts, in analogy to the transformation of a_μ and b_μ in the CPT-violating model.

From the viewpoint of this example, the unconventional aspect of the CPT-violating model is merely that the constant fields a_μ and b_μ are a global feature of the model. They cannot be regarded as arising from localized experimental conditions, which would cause them to transform under particle Lorentz transformations as four-vectors rather than as scalars. The behavior of a_μ and b_μ as background fields and hence as scalars under particle Lorentz transformations is a consequence of their origin as nonzero expectation values of Lorentz tensors in the underlying theory. These Lorentz-tensor expectations break those parts of the particle Lorentz group that cannot be implemented as unitary transformations on the vacuum. This is in parallel with other situations involving spontaneous symmetry breaking, such as ones commonly encountered in the treatment of internal symmetries.

The preservation of observer Lorentz symmetry is an important feature of the model. It is a consequence of observer Lorentz invariance of the underlying fundamental theory. This symmetry is unaffected by the appearance of tensor expectation values by virtue of its implementation via coordinate transformations. As an illustration of its use in the effective model, we show that it permits a further classification of types of CPT-violating term according to the observer Lorentz properties of a_μ and b_μ. Thus, for example, if b_μ is future timelike in one inertial frame, it must be future timelike in all frames. This implies that a class of inertial frames can be found in which $b_\mu=b(1,0,0,0)$, where calculations are potentially simplified. A similar argument for the lightlike or spacelike cases shows that the CPT-violating physics of the four components of b_μ can in each case be reduced to knowledge of its Lorentz type and a single number specifying its magnitude. Inertial frames within this ideal class are determined by the little group of b_μ, which can in turn be used to simplify (partially) the form of a_μ.

The reader is cautioned that the class of inertial frames selected in this way may be distinct from experimentally relevant inertial frames such as, for example, those defined using the microwave background radiation and interpreting the dipole component in terms of the motion of the Earth. The point is that, given an inertial frame, the process of spontaneous Lorentz violation in the underlying theory is assumed to produce some values of a_μ and b_μ. In this specific inertial frame, there is no reason *a priori* why these values should take the ideal form described above. One is merely assured of the existence of some frame in which the ideal form can be attained.

The current $J^{\lambda\mu\nu}$ for particle Lorentz transformations takes the usual form when expressed in terms of the energy-momentum tensor:

$$J^{\lambda\mu\nu}=x^{[\mu}\Theta^{\lambda\nu]}+\tfrac{1}{4}\overline{\psi}\{\gamma^\lambda,\sigma^{\mu\nu}\}\psi. \qquad (11)$$

This current is conserved at the level of the underlying theory with spontaneous symmetry breaking, but in the effective low-energy theory where the spontaneous breaking appears as an explicit symmetry violation the conservation property is destroyed. In the latter case, the corresponding Lorentz charges $M^{\mu\nu}$ obey

$$\frac{dM^{\mu\nu}}{dt}=-a^{[\mu}J^{\nu]}-b^{[\mu}J_5^{\nu]}. \qquad (12)$$

Given explicit values of a_μ and b_μ in some inertial frame, Eq. (12) can be used directly to determine which Lorentz symmetries are violated. Note that if either a_μ or b_μ vanishes, the Lorentz group is broken to the little group of the nonzero four-vector. This means that the *largest* Lorentz-symmetry subgroup that can remain as an invariance of the model Lagrangian (5) is SO(3), E(2), or SO(2,1). Since a_μ and b_μ represent two four-vectors in four-dimensional spacetime, they can define a two-dimensional plane. Transformations involving the two orthogonal dimensions have no effect on this plane. This means that the *smallest* Lorentz-symmetry subgroup that can remain is a compact or noncompact U(1).

In a realistic low-energy effective theory, CPT-violating terms would break the particle Lorentz group in a manner related to the breaking given by Eq. (12). Since no zeroth-order CPT violation has been observed in experiments, CPT-violating effects in the string scenario are expected to be suppressed by at least one power of the Planck mass relative to the scale of the effective theory. However, the interesting and involved issue of exactly how small the magnitudes of a_μ and b_μ (or their equivalents in a realistic model) must be to satisfy current experimental constraints lies beyond the scope of the present work. We confine our remarks here to noting that the partial breaking of particle Lorentz invariance discussed above generates an effective boost dependence in the CPT-breaking parameters. This could provide a definite experimental signature for our framework if CPT violation were detected at some future date.

C. Field redefinitions

For the discussions in the previous subsections, we adopted a practical approach to the definition of CPT and Lorentz transformations. It involves treating C, P, T and Lorentz properties of ψ as being defined via the free-field theory $\mathcal{L}_0$ and subsequently using them to establish the symmetry properties of $\mathcal{L}'$. This approach requires caution, however, because in principle alternative definitions of the symmetry transformations could exist that would leave the full theory $\mathcal{L}$ invariant.

Consider first an apparently CPT- and Lorentz-violating model formed with a_μ only, defined in a given inertial frame by the Lagrangian

$$\mathcal{L}[\psi]=\mathcal{L}_0[\psi]-\mathcal{L}'_a[\psi]. \qquad (13)$$

Introducing in this frame a field redefinition of ψ by a spacetime-dependent phase,

$$\chi=\exp(ia\cdot x)\psi, \qquad (14)$$

the Lagrangian expressed in terms of the new field is $\mathcal{L}[\psi=\exp(-ia\cdot x)\chi]\equiv\mathcal{L}_0[\chi]$. This shows that the model is equivalent to a conventional free Dirac theory, in which there is no CPT or Lorentz breaking, and thereby provides an example of redefining symmetry transformations to maintain invariance [24].

The connection between the Poincaré generators in the two forms of the theory can be found explicitly by substituting $\psi=\psi[\chi]$ in the Poincaré generators for $\mathcal{L}[\psi]$ and extracting the combinations needed to reproduce the usual Poincaré generators for $\mathcal{L}_0[\chi]$. We find that the charge and chiral currents j^μ and j_5^μ take the same functional forms in both theories but that the form of the canonical energy-momentum tensor changes,

$$\Theta^{\mu\nu}=\frac{1}{2}\,i\overline{\chi}\gamma^\mu\overleftrightarrow{\partial^\nu}\chi+a^\nu\overline{\chi}\gamma^\mu\chi, \tag{15}$$

producing a corresponding change in the Lorentz current $J^{\lambda\mu\nu}$. This means that in the original theory $\mathcal{L}[\psi]$ we could introduce modified Poincaré currents $\widetilde{\Theta}^{\mu\nu}$ and $\widetilde{J}^{\lambda\mu\nu}$ that have corresponding conserved charges generating an unbroken Poincaré algebra. These currents are given as functionals of ψ by

$$\widetilde{\Theta}^{\mu\nu}=\Theta^{\mu\nu}-a^\nu j^\mu,\quad \widetilde{J}^{\lambda\mu\nu}=J^{\lambda\mu\nu}-x^{[\mu}a^{\nu]}j^\lambda. \tag{16}$$

The existence of this connection between the two theories depends critically on the existence of the conserved current j^μ. In the model (5) with both a_μ and b_μ terms, the component $\mathcal{L}_a'$ can be eliminated by a field redefinition as before but there is no similar transformation removing $\mathcal{L}_b'$ because conservation of the chiral current j_5^μ is violated by the mass. In the massless limit of this model the chiral current is conserved, and we can eliminate both a_μ and b_μ via the field redefinition

$$\chi=\exp(ia\cdot x-ib\cdot x\gamma_5)\psi. \tag{17}$$

For the situation with $m\neq 0$, however, this redefinition would introduce spacetime-dependent mass parameters.

The term $\mathcal{L}_a'$ in Eq. (3) is reminiscent of a local U(1) coupling, although there is no local U(1) invariance in the theory (5). It is natural and relevant to our later considerations of the standard model to ask how the above discussion of field redefinitions is affected if the U(1) invariance of the original theory is gauged. Then, the term $\mathcal{L}_a'$ has the same form as a coupling to a constant background electromagnetic potential. At the classical level, this would be expected to have no effect since it is pure gauge. However, a conventional quantum-field gauge transformation involving both ψ and the electromagnetic potential A_μ cannot eliminate a_μ, since the theory is invariant under such transformations. Instead, the electromagnetic field can be taken as the sum of a classical c-number background field $\mathcal{A}_\mu$ and a quantum field A_μ, whereupon a_μ can be regarded as contributing to an effective $\mathcal{A}_\mu$. Conventional classical gauge transformations can be performed on the c-number potential $\mathcal{A}_\mu$, while leaving the quantum fields ψ and A_μ unaffected. This changes the Lagrangian but should not change the physics. In fact, the resulting gauge-transformed Lagrangian is unitarily equivalent to the original one under a field redefinition on ψ of the form discussed above for the ungauged model.

To summarize, in the gauged theory the CPT-breaking term $\mathcal{L}_a'$ can be interpreted as a background gauge choice and eliminated via a field redefinition as in the ungauged case. We note in passing that related issues arise for certain nonlinear gauge choices [25] and in the context of efforts to interpret the photon as a Nambu-Goldstone boson arising from (unphysical) spontaneous Lorentz breaking [26–32]. In typical models of the latter type, a four-vector bilinear condensate $\langle\overline{\psi}\gamma_\mu\psi\rangle$ plays a role having some similarities to that of a_μ.

The model (5) involves only a single fermion field. All CPT-violating effects can also be removed from certain theories describing more than one fermion field in which each fermion has a term of the form $\mathcal{L}_a'$. For example, this is possible if there is no fermion mixing and each such CPT-violating term involves the same value of a_μ, or if the fermions have no interactions or mixings that acquire spacetime-dependence upon performing the field redefinitions. However, in generic multifermion theories with CPT violation involving fermion-bilinear terms, it is impossible to eliminate all CPT-breaking effects through field redefinitions. Nonetheless, since Lagrangian terms that spontaneously break CPT necessarily involve paired fermion fields, at least one of the quantities a_μ can be removed. This means that only differences between values of a_μ are observable. Examples appear in the context of the CPT-violating extension of the standard model discussed in Sec. V.

III. RELATIVISTIC QUANTUM MECHANICS

In this section, we discuss some aspects of relativistic quantum mechanics based on Eq. (6), with ψ regarded as a four-component wave function. The results obtained provide further insight into the nature of the CPT-violating terms and are precursors to the quantum field theory. The analogous treatment in the context of the standard model involves several fermion fields, for which CPT-violating terms of the form $\mathcal{L}_a'$ cannot be altogether eliminated. We therefore explicitly include the quantity a_μ in the following analysis, even though it could be eliminated by a field redefinition for the simple one-fermion case. In fact, the reinterpretation of negative-energy solutions causes the explicit effects of a_μ to be more involved than might otherwise be expected.

The modified Dirac equation (6) can be solved by assuming the usual plane-wave dependence,

$$\psi(x)=e^{-i\lambda_\mu x^\mu}w(\vec{\lambda}). \tag{18}$$

In this equation, $w(\vec{\lambda})$ is a four-component spinor satisfying

$$(\lambda_\mu\gamma^\mu-a_\mu\gamma^\mu-b_\mu\gamma_5\gamma^\mu-m)w(\vec{\lambda})=0. \tag{19}$$

For a nontrivial solution to exist, the determinant of the matrix acting on $w(\vec{\lambda})$ in this equation must vanish. This means that $\lambda^\mu\equiv(\lambda^0,\vec{\lambda})$, where [33] $\lambda^0=\lambda^0(\vec{\lambda})$, must satisfy the requirement

$$[(\lambda-a)^2-b^2-m^2]^2+4b^2(\lambda-a)^2-4[b^\mu(\lambda_\mu-a_\mu)]^2=0. \tag{20}$$

This condition can also be obtained directly from Eq. (8) and the assumption (18).

The dispersion relation (20) is a quartic equation for $\lambda^0(\vec{\lambda})$. Although the Euler reducing cubic has a relatively elegant form, in part because Eq. (20) contains no term cubic in λ^0, the algebraic solutions to this equation are not particularly transparent. Even without examining the analytical results, however, certain features of the solutions can be established. One is that all four roots must be real, due to Hermiticity of the quantum-mechanical Hamiltonian

$$H\psi \equiv i\frac{\partial \psi}{\partial t} = (-i\gamma^0\vec{\gamma}\cdot\vec{\nabla} + a_\mu\gamma^0\gamma^\mu - b_\mu\gamma_5\gamma^0\gamma^\mu + m\gamma^0)\psi. \tag{21}$$

Another stems from the invariance of the quartic under the interchange $(\lambda_\mu - a_\mu)\to -(\lambda_\mu - a_\mu)$, which implies that to each solution $\lambda^0_+(\vec{\lambda})$ there corresponds a second solution $\lambda^0_-(\vec{\lambda})$ given by

$$\lambda^0_-(\vec{\lambda}) = -\lambda^0_+(-\vec{\lambda}+2\vec{a}) + 2a^0. \tag{22}$$

This equation and the invariance of the quartic under the interchange $b_\mu \to -b_\mu$ show that, unlike the conventional Dirac case, the magnitudes of the eigenenergies of the four roots all differ generically as a direct consequence of the CPT-violating terms [34].

Another qualitatively different feature of the present model is that under certain conditions the roots $\lambda^0(\vec{\lambda})$ of the dispersion relation can display cusps. For a conventional dispersion relation, the energy is a smooth function of each three-momentum component for both timelike and spacelike four-momenta, while there is a cusp at the origin for the lightlike case. By examining discontinuities in the derivatives of the roots $\lambda^0(\vec{\lambda})$ with respect to the components of $\vec{\lambda}$, we have demonstrated that the criterion for cusps to appear in the present model with $m^2>0$ is that b_μ be timelike. The derivation is most straightforward using observer Lorentz invariance to select one of the canonical frames listed in Appendix B, for which exact solutions to the dispersion relations can be found. The presence of cusps appears to have no directly observable consequences, in part because their size is governed by the magnitude of b_μ, which is highly suppressed in a realistic situation.

The assumption that the CPT-violating quantities a_μ and b_μ are small relative to the scale m of the low-energy theory implies the dispersion relation (20) must have two positive-valued roots $\lambda^0_{+(\alpha)}(\vec{\lambda})$ and two negative-valued roots $\lambda^0_{-(\alpha)}(\vec{\lambda})$, where $\alpha=1,2$. Since these roots are eigenvalues of the time-translation operator, the corresponding wave functions can be termed positive- and negative-energy states, respectively. Useful approximate solutions for $\lambda^0_{\pm(\alpha)}(\vec{\lambda})$ that are valid to second order for arbitrary small a_μ and b_μ are given in Eq. (62) of Appendix A. Some exact solutions valid for various important special cases are provided in Appendix B.

Within conventional relativistic quantum mechanics, negative-energy states are deemed to be filled, forming the Dirac sea. When a negative-energy state is excited to a positive-energy one, it leaves a hole appearing to be a particle with opposite energy, momentum, spin, and charge to that of the negative-energy state. In the present model, however, when a negative-energy state moving in a CPT-violating background with parameters a_μ and b_μ is excited to a positive-energy one, it leaves a hole appearing to be a particle with opposite values as before but moving in a CPT-violating background with parameters $-a_\mu$ and b_μ instead. This is because the term $\mathcal{L}'_a$ is odd under charge conjugation. The same effect can be seen explicitly by constructing the charge-conjugate Dirac equation for the model. We find

$$(i\gamma^\mu\partial_\mu + a_\mu\gamma^\mu - b_\mu\gamma_5\gamma^\mu - m)\psi^c = 0, \tag{23}$$

where as usual $\psi^c \equiv C\overline{\psi^T}$ and C is the charge-conjugation matrix.

The eigenfunctions corresponding to the two negative eigenvalues $\lambda^0_{-(\alpha)}(\vec{\lambda})$ can be reinterpreted as positive-energy, reversed-momentum wave functions in the usual way. We introduce momentum-space spinors $u^{(\alpha)}(\vec{p})$, $v^{(\alpha)}(\vec{p})$ via the definitions

$$\psi^{(\alpha)}(x) = \exp(-ip_u^{(\alpha)}\cdot x)u^{(\alpha)}(\vec{p}),$$
$$\psi^{(\alpha)}(x) = \exp(+ip_v^{(\alpha)}\cdot x)v^{(\alpha)}(\vec{p}), \tag{24}$$

where the four-momenta are given by

$$p_u^{(\alpha)} \equiv (E_u^{(\alpha)},\vec{p}), \quad E_u^{(\alpha)}(\vec{p}) = \lambda^0_{+(\alpha)}(\vec{p}),$$
$$p_v^{(\alpha)} \equiv (E_v^{(\alpha)},\vec{p}), \quad E_v^{(\alpha)}(\vec{p}) = -\lambda^0_{-(\alpha)}(-\vec{p}). \tag{25}$$

The general forms of $u^{(\alpha)}(\vec{p})$ and $v^{(\alpha)}(\vec{p})$ are given in Appendix A. The relation between the spinors u and v is determined by the charge-conjugation matrix and the charge-conjugate Dirac equation (23). For example, $u^{(2)}(\vec{p},a_\mu,b_\mu) \propto v^{(1)c}(\vec{p},-a_\mu,b_\mu)$, where the dependence on a_μ and b_μ has been explicitly restored for clarity. The symmetry (22) of the dispersion relation then connects the two sets of energies by

$$E_v^{(2,1)}(\vec{p}) = E_u^{(1,2)}(\vec{p}+2\vec{a}) - 2a^0. \tag{26}$$

The exact eigenenergies for the various canonical cases are provided in Appendix B, while Appendix C contains explicit solutions for the eigenspinors in the special case $\vec{b}=0$.

The four spinors $u^{(\alpha)}(\vec{p})$, $v^{(\alpha)}(\vec{p})$ are orthogonal. Their normalization can be freely chosen, although imposing the condition $(\psi^c)^c=\psi$ provides a partial constraint. Our choice leads to orthonormality conditions given by

$$u^{(\alpha)\dagger}(\vec{p})u^{(\alpha')}(\vec{p}) = \delta^{\alpha\alpha'}\frac{E_u^{(\alpha)}}{m},$$
$$v^{(\alpha)\dagger}(\vec{p})v^{(\alpha')}(\vec{p}) = \delta^{\alpha\alpha'}\frac{E_v^{(\alpha)}}{m},$$
$$u^{(\alpha)\dagger}(\vec{p})v^{(\alpha')}(-\vec{p}) = 0, \quad v^{(\alpha)\dagger}(-\vec{p})u^{(\alpha')}(\vec{p}) = 0. \tag{27}$$

Note, however, that the Lorentz breaking precludes a simple generalization of the orthonormality relations involving the Dirac-conjugate spinors $\overline{u^{(\alpha)}}(\vec{p})$ and $\overline{v^{(\alpha)}}(\vec{p})$ instead of the Hermitian-conjugate spinors $u^{(\alpha)\dagger}(\vec{p})$ and $v^{(\alpha)\dagger}(\vec{p})$. Equation (27) produces the completeness relation.

$$\sum_{\alpha=1}^{2}\left[\frac{m}{E_u^{(\alpha)}(\vec{p})}\,u^{(\alpha)}(\vec{p})\otimes u^{(\alpha)\dagger}(\vec{p}) + \frac{m}{E_v^{(\alpha)}(-\vec{p})}\,v^{(\alpha)}(-\vec{p})\otimes v^{(\alpha)\dagger}(-\vec{p})\right]=I. \quad (28)$$

We remark in passing that another useful result is the modified Gordon identity

$$\overline{u^{(\alpha')}}(\vec{p}\,')\gamma^\mu u^{(\alpha)}(\vec{p}) = \frac{1}{2m}\,\overline{u^{\alpha}}'(\vec{p}\,')[p_u'^{(\alpha')\mu}+p_u^{(\alpha)\mu}-2a^\mu + i\sigma^{\mu\nu}(p_{u\nu}'^{(\alpha')}-p_{u\nu}^{(\alpha)}-2\gamma_5 b_\nu)]u^{(\alpha)}(\vec{p}). \quad (29)$$

The general solution to the modified Dirac equation (6) can be written as a superposition of the four spinors $u^{(\alpha)}$, $v^{(\alpha)}$:

$$\psi(x)=\int \frac{d^3p}{(2\pi)^3}\sum_{\alpha=1}^{2}\left[\frac{m}{E_u^{(\alpha)}}\,b_{(\alpha)}(\vec{p})e^{-ip_u^{(\alpha)}\cdot x}u^{(\alpha)}(\vec{p}) + \frac{m}{E_v^{(\alpha)}}\,d^*_{(\alpha)}(\vec{p})e^{ip_v^{(\alpha)}\cdot x}v^{(\alpha)}(\vec{p})\right], \quad (30)$$

where $b_{(\alpha)}(\vec{p})$, $d^*_{(\alpha)}(\vec{p})$ are the usual complex weights for the momentum expansion. We remind the reader that in this expression the a_μ and b_μ dependence of the energies and the spinors is understood.

In the above expressions, the four-momenta are eigenvalues of the translation operators and hence are conserved quantities. They therefore represent canonical energy and momentum rather than kinetic energy and momentum. A distinction of this type occurs in many physical systems, such as a charged particle moving in an electromagnetic field. In the present case this means, for example, that the canonical four-momenta are *not* related to velocity as are the usual kinetic four-momenta in special relativity. The actual relationship can be explored by using the velocity operator, given in relativistic quantum mechanics by $\vec{v}\equiv d\vec{x}/dt=i[H,\vec{x}]=\gamma^0\vec{\gamma}$, where H is the Hamiltonian (21). The expectation value of this operator for a given wave packet is the vector-current integral and gives the (group) velocity of the packet.

As an explicit example, consider the special case $\vec{b}=0$, for which the eigenenergies and eigenspinors are provided in Appendixes B and C, respectively. Suppose a wave packet of energy E and momentum $\vec{p}$ is constructed as a superposition of positive-energy spin-up solutions. A short calculation produces

$$\langle\vec{v}\rangle=\left\langle\frac{(|\vec{p}-\vec{a}|-b^0)}{(E-a^0)}\,\frac{(\vec{p}-\vec{a})}{|\vec{p}-\vec{a}|}\right\rangle. \quad (31)$$

It follows that the velocity is related to the energy and momentum by

$$\gamma m=E-a^0, \quad \gamma m\vec{v}=\vec{p}-\vec{a}-b^0\,\frac{(\vec{p}-\vec{a})}{|\vec{p}-\vec{a}|}, \quad (32)$$

where $\gamma=1/\sqrt{1-v^2}$ as usual. These are just the usual special-relativistic results shifted by (small) amounts controlled by the CPT-violating terms. Note that four-momentum conservation shows that the wave-packet velocity is constant, as usual.

Even in conventional Dirac quantum mechanics, the above notion of velocity involves subtleties associated with the presence of negative-energy solutions. For example, the velocity operator does *not* commute with the usual Dirac Hamiltonian. In the present CPT-violating model, additional subtleties arise. For example, it follows from the properties of the roots of the dispersion relation (20) or from the above discussion that for the special case of timelike b_μ the velocity near the origin is not in one-to-one correspondence with the conserved momentum. Perhaps of more interest is that in the general case the velocity operator in the energy basis has additional off-diagonal components, even in the positive-energy sector. For the example above, these oscillate transverse to $(\vec{p}-\vec{a})$ with relatively large period of order b_0^{-1}. They provide time-independent corrections to the velocity eigenvalues, but only at order b_0^2. The implications of these features for possible bounds on b_μ are therefore likely to be limited but remain to be explored.

A related approach to the notion of velocity is to take the derivative of the energy with respect to the momentum. For the special case $\vec{b}=0$, this definition produces the same result as above and moreover can be obtained without the explicit wave function. It therefore provides a relatively simple method of investigating velocity-related issues. For example, causality of the model is related to the restriction of the group velocity to below the velocity of light. If causality is satisfied, the criterion

$$|v^j|\equiv\left|\frac{\partial E}{\partial p^j}\right|<1 \quad (33)$$

should be obeyed for each $j=1,2,3$. Observer Lorentz invariance makes it sufficient to examine the various canonical cases. Calculating with the expressions in Appendix B, we find that the criterion (33) is satisfied for all values of a_μ and b_μ. This supports the notion that causality is maintained. Although the Lorentz breaking does affect quantum wave propagation, it apparently is mild enough to avoid superluminal signals.

Our treatment in this section of the relativistic quantum mechanics of a single fermion in the presence of CPT-violating terms could be further developed to allow for interactions with conventional applied fields, along the lines of the usual Dirac theory. In detail, this lies outside the scope of the present work. We remark, however, that standard Green-function methods should be applicable. In particular, we can introduce a generalized Feynman propagator $S_F(x-x')$ satisfying

$$(i\gamma^\mu\partial_\mu - a_\mu\gamma^\mu - b_\mu\gamma_5\gamma^\mu - m)S_F(x-x') = \delta^4(x-x') \tag{34}$$

and obeying the usual Feynman boundary conditions. It has integral representation

$$S_F(x-x') = \int_{C_F} \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} \times \frac{1}{p_\mu\gamma^\mu - a_\mu\gamma^\mu - b_\mu\gamma_5\gamma^\mu - m}, \tag{35}$$

where C_F is the direct analogue of the usual Feynman contour in p_0 space, passing below the two negative-energy poles and above the positive-energy ones. Appendix D contains some remarks about this propagator, including a closed-form integration for the case $\vec{b}=0$.

IV. QUANTUM FIELD THEORY

In this section, we discuss a few aspects of the quantum field theory associated with the model Lagrangian (5). As in the usual Dirac case, direct canonical quantization is unsatisfactory, and the quantization condition is found instead by imposing positivity of the conserved energy.

Promoting the Fourier coefficients in the expansion (30) to operators on a Hilbert space, we can obtain from Eq. (9) an expression for the normal-ordered conserved energy $P_0 = \int d^3x{:}\Theta^0{}_0{:}$. This expression is positive definite for $a^0 < m$, provided the following nonvanishing anticommutation relations are imposed:

$$\{b_{(\alpha)}(\vec{p}), b^\dagger_{(\alpha')}(\vec{p}\,')\} = (2\pi)^3 \frac{E_u^{(\alpha)}}{m} \delta_{\alpha\alpha'}\delta^3(\vec{p}-\vec{p}\,'),$$

$$\{d_{(\alpha)}(\vec{p}), d^\dagger_{(\alpha')}(\vec{p}\,')\} = (2\pi)^3 \frac{E_v^{(\alpha)}}{m} \delta_{\alpha\alpha'}\delta^3(\vec{p}-\vec{p}\,'). \tag{36}$$

For simplicity, the dependence on a_μ and b_μ is suppressed in these and subsequent equations. The corresponding equal-time field anticommutators are given by

$$\{\psi_j(t,\vec{x}), \psi_k^\dagger(t,\vec{x}\,')\} = \delta_{jk}\delta^3(\vec{x}-\vec{x}\,'),$$

$$\{\psi_j(t,\vec{x}), \psi_k(t,\vec{x}\,')\} = \{\psi_j^\dagger(t,\vec{x}), \psi_k^\dagger(t,\vec{x}\,')\} = 0, \tag{37}$$

where the spinor indices j,k are explicitly shown.

Using these expressions, the normal-ordered conserved charge becomes

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_{\alpha=1}^{2} \left[\frac{m}{E_u^{(\alpha)}} b^\dagger_{(\alpha)}(\vec{p}) b_{(\alpha)}(\vec{p}) - \frac{m}{E_v^{(\alpha)}} d^\dagger_{(\alpha)}(\vec{p}) d_{(\alpha)}(\vec{p}) \right]. \tag{38}$$

Similarly, the normal-ordered conserved four-momentum is

$$P_\mu = \int \frac{d^3p}{(2\pi)^3} \sum_{\alpha=1}^{2} \left[\frac{m}{E_u^{(\alpha)}} p^{(\alpha)}_{u\mu} b^\dagger_{(\alpha)}(\vec{p}) b_{(\alpha)}(\vec{p}) + \frac{m}{E_v^{(\alpha)}} p^{(\alpha)}_{v\mu} d^\dagger_{(\alpha)}(\vec{p}) d_{(\alpha)}(\vec{p}) \right]. \tag{39}$$

The reader can verify that the operator P_μ generates spacetime translations by determining the commutation relation with the field ψ: $i[P_\mu, \psi(x)] = \partial_\mu\psi(x)$.

The creation and annihilation operators can be written in terms of the fields as

$$b_{(\alpha)}(\vec{p}) = \int d^3x\, e^{ip_u^{(\alpha)}\cdot x}\, \overline{u^{(\alpha)}}(\vec{p})\gamma^0\psi(x),$$

$$d^\dagger_{(\alpha)}(\vec{p}) = \int d^3x\, e^{-ip_v^{(\alpha)}\cdot x}\, \overline{v^{(\alpha)}}(\vec{p})\gamma^0\psi(x). \tag{40}$$

The vacuum state $|0\rangle$ of the Hilbert space obeys

$$b_{(\alpha)}(\vec{p})|0\rangle = 0, \quad d_{(\alpha)}(\vec{p})|0\rangle = 0. \tag{41}$$

Acting on $|0\rangle$, the creation operators $b^\dagger_{(\alpha)}(\vec{p})$ and $d^\dagger_{(\alpha)}(\vec{p})$ produce particles and antiparticles with four-momenta $p_u^{(\alpha)\mu}$ and $p_v^{(\alpha)\mu}$, respectively. The reinterpretation (25), which is based on the usual heuristic arguments in relativistic quantum mechanics, therefore makes sense in the field-theoretic framework. As expected, the usual fourfold degeneracy of the eigenstates of the Hamiltonian for a given three-momentum is broken by the CPT-violating terms.

The above expressions can be used to establish various results for the field theory with nonzero a_μ and b_μ. For example, the (time-dependent) commutation relation between the conserved charges P_μ and the quantum operators $M^{\mu\nu} = \int d^3x{:}J^{0\mu\nu}{:}$, obtained from the operator form of the currents (11), is found to be

$$i[P^\lambda, M^{\mu\nu}] = -g^{\lambda[\mu}P^{\nu]} - g^{\lambda 0}(a^{[\mu}J^{\nu]} + b^{[\mu}J_5^{\nu]}), \tag{42}$$

where $J^\mu = \int d^3x{:}j^\mu{:}$ and $J_5^\mu = \int d^3x{:}j_5^\mu{:}$ are integrals of the charge and chiral currents. The $\lambda=0$ component of this equation is the quantum-field analogue of Eq. (12).

The generalizations of the equal-time anticommutation relations (37) to unequal times must be solutions to the modified Dirac equation in each variable and must reduce to the usual results in the limit where a_μ and b_μ vanish. The correct expressions can in principle be derived by evolving forward in time one of the two fields in each anticommutator of Eq. (37). We write

$$\{\psi(x), \overline{\psi}(x')\} = iS(x-x'),$$

$$\{\psi(x), \psi(x')\} = \{\overline{\psi}(x), \overline{\psi}(x')\} = 0, \tag{43}$$

where

$$S(x-x') = \int_C \frac{d^4p}{(2\pi)^4} e^{-ip\cdot(x-x')} \times \frac{1}{p_\mu\gamma^\mu - a_\mu\gamma^\mu - b_\mu\gamma_5\gamma^\mu - m}. \tag{44}$$

In this expression, C is the analogue of the usual closed contour in p_0 space encircling all the poles in the anticlockwise direction. Some comments about this anticommutator function are given in Appendix E, along with a closed-form integration for $\vec{b}=0$. For this case, we have checked explicitly that the anticommutators (43) are detemined by the integral (44).

The anticommutators (43) are relevant to the causal structure of the quantum theory. In particular, for the case $\vec{b}=0$ the results in Appendix E can be used to show that the anticommutator of two fields separated by a spacelike interval vanishes:

$$\{\psi_\alpha(x),\overline{\psi_\beta}(x')\}=0, \quad (x-x')^2<0. \tag{45}$$

Observables constructed out of bilinear products of field operators and separated by a spacelike interval therefore commute. This means that the quantum field theory with timelike b_μ preserves microscopic causality in all associated observer frames. The breaking of Lorentz invariance and the distortions relative to conventional propagation are apparently mild enough to exclude superluminal signals, in agreement with the result from relativistic quantum mechanics. The result might be anticipated since observer Lorentz invariance holds and the particle Lorentz breaking involves only local terms in the Lagrangian. A direct analytical proof of microscopic causality in the quantum field theory for the cases of lightlike or spacelike b_μ would be of interest but is hampered by the complexity of the integral (44).

We next turn to issues associated with interacting field theory. For the most part, since the CPT-violating terms in the model Lagrangian can be treated exactly, any added conventional interactions can be handled with standard methods. In what follows, we suppose that the Dirac fermion in the model has interactions with one or more other fields that are of a type acceptable within a conventional approach, and we discuss effects from CPT-violating terms.

Essentially all the standard assumptions underlying treatments of conventional interacting field theories can reasonably be made in the present context. Thus, for example, the property of observer Lorentz invariance ensures that consistent quantization can be established in all observer frames once it is established in a given frame. Much of the usual analysis is performed in a given observer frame, which in the present context means that the values of a_μ and b_μ are fixed. Distinct effects are to be expected only in calculations for which particle Lorentz covariance plays an essential role. For the most part, matters proceed in a straightforward manner at the level of the general framework of interacting field theory. One exception we have found is the explicit derivation of the Källén-Lehmann spectral representation for the vacuum expectation value of the field anticommutator, which normally takes advantage of both CPT and particle Lorentz covariance [35–38]. The spectral representation could be used to investigate microscopic causality of the interacting theory, although the conventional local interactions we consider seem unlikely to introduce difficulties in this regard. In any case, it remains an open issue to obtain the spectral representation in the present case, where additional four-vectors appear in the theory.

The construction of the in and out fields and the definition of the S matrix can be implemented in the normal manner. The Lehmann-Symanzik-Zimmermann (LSZ) reduction procedure for S-matrix elements generates expressions involving vacuum expectation values of time-ordered products of the interacting fields, as usual, but with external-leg factors for fermions involving the modified Dirac operator $(i\gamma^\mu\partial_\mu - a_\mu\gamma^\mu - b_\mu\gamma_5\gamma^\mu - m)$ or its conjugate $(i\gamma^\mu\overleftarrow{\partial}_\mu + a_\mu\gamma^\mu + b_\mu\gamma_5\gamma^\mu + m)$.

The canonical Dyson formalism for the perturbation series of time-ordered products of interacting fields in terms of in fields can be applied without encountering difficulties. Standard expressions emerge, including the vacuum bubbles. Wick's theorem holds, reducing the time-ordered products into normal-ordered products and pair contractions of in fields. For fermion in fields, the vacuum expectation value of a pairwise contraction can be shown explicitly to be the generalized Feynman propagator introduced in the previous section: $\langle 0|T\psi(x)\overline{\psi}(x')|0\rangle = iS_F(x-x')$. In a momentum-space Feynman diagram, the corresponding propagator is

$$S_F(p)=\frac{1}{p_\mu\gamma^\mu - a_\mu\gamma^\mu - b_\mu\gamma_5\gamma^\mu - m}. \tag{46}$$

The momentum-space Feynman rules require this modified propagator for internal fermion lines and modified spinors on external fermion lines, but are otherwise unchanged from those of a conventional theory. For example, translational invariance insures that energy and momentum are conserved in any process and so the standard four-momentum δ functions emerge.

Since the CPT-violating terms are renormalizable by naive power counting, we anticipate no difficulties with the usual renormalization program. Details of loop calculations lie beyond the scope of the present work and remain an interesting open issue. We remark in passing that the form (D3) of the propagator given in Appendix D shows that a_μ cancels around all closed fermion loops in analogy with Furry's theorem.

We also expect the unitarity of the S matrix to be unaffected by CPT-violating terms. Since a complete Hilbert-space solution exists in the pure fermion case, there are no hidden or inaccessible states that could generate nonunitarity of the type appearing in the first stage of Gupta-Bleuler quantization of quantum electrodynamics, for example. Moreover, the interaction Hamiltonian is Hermitian. In any event, any nonunitarity appearing in a realistic model based on a unitary fundamental theory would presumably be a signal that the domain of validity of the effective low-energy theory is being breached.

In determining physically relevant quantities such as cross sections or transition rates, kinematic factors appear. For the most part, these are straightforward to obtain. A subtlety arises in the calculation of a physical cross section because the standard definition involves the notion of incident flux defined in terms of incoming particle velocity. Since the velocity-momentum relation has corrections involving a_μ and b_μ [cf. Eq. (32)], there are corresponding small modifications to the kinematic factors in standard cross-section formulas expressed in terms of conserved momenta. In a realistic case, these are unobservable because they are suppressed. This should be contrasted with the CPT-violating corrections arising in an amplitude from the modified fermion propagator, which are also suppressed but might be detected in interferometric experiments using neutral-meson oscillations [10].

V. STANDARD-MODEL EXTENSION

In this section, we consider the possibility of generalizing the minimal standard model by adding CPT-violating terms within a self-consistent framework of the type described in the previous sections. Since CPT violation has not been observed in nature, any CPT-violating constants appearing in an extension of the minimal standard model must be small. In what follows, we assume that these constants are singlets under the unbroken gauge group, but as before behave under particle Lorentz transformations as tensors with an odd number of Lorentz indices. Our primary goal is to obtain an explicit and realistic model for CPT-violating interactions that could serve as a basis for establishing quantitative CPT bounds.

The discussion in the previous sections is limited to CPT-breaking fermion bilinears. However, other types of terms violating CPT in the Lagrangian could in principle originate from spontaneous symmetry breaking. We adopt here a general approach, investigating possible CPT-violating extensions to the minimal standard model such that the SU(3)×SU(2)×U(1) gauge structure is maintained. To preserve naive power-counting renormalizability at the level of the unbroken gauge group, we restrict attention to terms involving field operators of mass dimension four or less. The simultaneous requirements of gauge invariance, suitable mass dimensionality, and CPT violation allow relatively few new terms in the action [39].

Any Lagrangian term must be formed from combinations of covariant derivatives and fields for leptons, quarks, gauge bosons, and Higgs bosons. We consider first allowed CPT-violating Lagrangian extensions involving fermions. Inspection shows that the only possibilities satisfying the above criteria are pure fermion-bilinear terms without derivatives [40]. In the present context involving many fermions, the analysis given in the previous sections of such terms requires some generalization. Since SU(3) invariance precludes quark-lepton couplings, we can treat the lepton and quark sectors separately as usual.

Consider first the lepton sector. Denote the left- and right-handed lepton multiplets by

$$L_A=\begin{pmatrix}\nu_A\\ l_A\end{pmatrix}_L, \quad R_A=(l_A)_R, \tag{47}$$

where

$$\psi_R\equiv\frac{1}{2}(1+\gamma_5)\psi, \quad \psi_L\equiv\frac{1}{2}(1-\gamma_5)\psi, \tag{48}$$

as usual, and where $A=1,2,3$ labels the lepton flavor: $l_A \equiv(e,\mu,\tau)$, $\nu_A\equiv(\nu_e,\nu_\mu,\nu_\tau)$. Then, the most general set of CPT-violating lepton bilinears consistent with gauge invariance is

$$\mathcal{L}^{CPT}_{\text{lepton}}=-(a_L)_{\mu AB}\overline{L_A}\gamma^\mu L_B-(a_R)_{\mu AB}\overline{R_A}\gamma^\mu R_B. \tag{49}$$

The constant flavor-space matrices $a_{L,R}$ are Hermitian. The presence of the γ^μ factor allows fields of only one handedness to appear in a given term while maintaining gauge invariance. This contrasts with conventional Yukawa couplings, in which fields of both handedness appear and invariance is ensured by the presence of the Higgs doublet.

After spontaneous symmetry breaking, the mass eigenstates (denoted with carets) are constructed with standard unitary transformations

$$\nu_{LA}=(U^\nu_L)_{AB}\hat{\nu}_{LB}, \quad l_{LA}=(U^l_L)_{AB}\hat{l}_{LB}, \quad l_{RA}=(U^l_R)_{AB}\hat{l}_{RB}. \tag{50}$$

The CPT-violating term in Eq. (49) becomes

$$\mathcal{L}^{CPT}_{\text{lepton}}=-(\hat{a}_{\nu L})_{\mu AB}\overline{\hat{\nu}_{LA}}\gamma^\mu\hat{\nu}_{LB}-(\hat{a}_{lL})_{\mu AB}\overline{\hat{l}}_{LA}\gamma^\mu\hat{l}_{LB} -(\hat{a}_{lR})_{\mu AB}\overline{\hat{l}}_{RA}\gamma^\mu\hat{l}_{RB}, \tag{51}$$

where each matrix of constants $\hat{a}_\mu$ is obtained from the corresponding a_μ via unitary rotation with the corresponding matrix U: $\hat{a}_\mu=U^\dagger a_\mu U$.

Not all the couplings $\hat{a}_\mu$ are observable. The freedom to redefine fields allows some couplings to be eliminated, in analogy to the discussion of Sec. II C for the model Lagrangian. Consider, for example, general field redefinitions of the form

$$\widetilde{\nu}_{LA}=(V^\nu_L)_{AB}\hat{\nu}_{LB}, \quad \widetilde{l}_{LA}=(V^l_L)_{AB}\hat{l}_{LB}, \quad \widetilde{l}_{RA}=(V^l_R)_{AB}\hat{l}_{RB}, \tag{52}$$

where the matrices $V(x^\mu)$ are unitary in generation space and have the form $V=\exp(iH_\mu x^\mu)$ with H_μ Hermitian. Then, in each kinetic term of the generic form $i\overline{\hat{\psi}}\gamma^\mu\partial_\mu\hat{\psi}$, the redefinition (52) generates an apparent CPT-violating term of the form $-\overline{\widetilde{\psi}}\gamma^\mu\exp(iH_\lambda x^\lambda)H_\mu\exp(-iH_\nu x^\nu)\widetilde{\psi}$. A suitable choice of H_μ might therefore remove a CPT-violating term involving $\hat{a}_\mu$ from Eq. (51). The matrices V must be chosen to leave unaffected the Yukawa couplings and the conventional nonderivative couplings defining the neutrino fields as weak eigenstates. It can be shown that V^l_L must be a diagonal matrix of phases, with $V^l_R=V^l_L$ and $V^\nu_L=(U^l_L)^\dagger U^\nu_L V^l_L$. The freedom therefore exists to redefine the fields so as to eliminate, say, the three diagonal elements of the CPT-violating matrix in the neutrino sector. Note that the existence of this choice obviates possible theoretical issues arising from the combination of massless fields at zero temperature and small negative energy shifts induced by CPT-breaking terms [41].

Thus, omitting tildes and carets, the general CPT-violating extension of the lepton sector of the minimal standard model has the form

$$\mathcal{L}^{CPT}_{\text{lepton}}=-(a_\nu)_{\mu AB}\overline{\nu_A}\frac{1}{2}(1+\gamma_5)\gamma^\mu\nu_B-(a_l)_{\mu AB}\overline{l_A}\gamma^\mu l_B -(b_l)_{\mu AB}\overline{l_A}\gamma_5\gamma^\mu l_B, \tag{53}$$

where we have used Eq. (48) to replace left- and right-handed couplings with vector and axial vector couplings, and where $(a_\nu)_{\mu AA}=0$. Note that Eq. (53) includes terms breaking individual lepton numbers, although total lepton number remains conserved. Flavor-changing transitions therefore exist in principle but are unobservable if the CPT-violating couplings are sufficiently suppressed. For example, a fractional suppression of order 10^{-17} or smaller might occur in the string scenario [10].

Consider next the quark sector. The SU(3) symmetry ensures that all three quark colors of any given flavor must have the same CPT-violating current coupling. We can therefore disregard the color space in what follows, and a construction analogous to that for the lepton sector can be applied. Denote the left- and right-handed components of the quark fields by

$$Q_A=\begin{pmatrix} u_A \\ d_A \end{pmatrix}_L, \quad U_A=(u_A)_R, \quad D_A=(d_A)_R, \tag{54}$$

where $A=1,2,3$ labels the quark flavors: $u_A\equiv(u,c,t)$, $d_A\equiv(d,s,b)$. Then, at the unbroken-symmetry level, the most general CPT-violating coupling is

$$\mathcal{L}^{CPT}_{\text{quark}}=-(a_Q)_{\mu AB}\overline{Q_A}\gamma^\mu Q_B-(a_U)_{\mu AB}\overline{U_A}\gamma^\mu U_B -(a_D)_{\mu AB}\overline{D_A}\gamma^\mu D_B. \tag{55}$$

As before, the constant flavor-space matrices $a_{Q,U,D}$ are Hermitian.

After spontaneous symmetry breaking the mass eigenstates are obtained with the standard unitary transformations

$$u_{LA}=(U^u_L)_{AB}\hat{u}_{LB}, \quad u_{RA}=(U^u_R)_{AB}\hat{u}_{RB},$$
$$d_{LA}=(U^d_L)_{AB}\hat{d}_{LB}, \quad d_{RA}=(U^d_R)_{AB}\hat{d}_{RB}. \tag{56}$$

The CPT-violating expression (55) becomes

$$\mathcal{L}^{CPT}_{\text{quark}}=-(\hat{a}_{uL})_{\mu AB}\overline{\hat{u}}_{LA}\gamma^\mu\hat{u}_{LB}-(\hat{a}_{dL})_{\mu AB}\overline{\hat{d}}_{LA}\gamma^\mu\hat{d}_{BL} -(\hat{a}_{uR})_{\mu AB}\overline{\hat{u}}_{RA}\gamma^\mu\hat{u}_{RB}-(\hat{a}_{dR})_{\mu AB}\overline{\hat{d}}_{RA}\gamma^\mu\hat{d}_{RB}. \tag{57}$$

As in the lepton sector, each constant matrix $\hat{a}_\mu$ is obtained from the corresponding a_μ via the appropriate unitary rotation.

Again, field redefinitions can be used to eliminate some CPT violation. Consider field redefinitions of the form

$$\tilde{u}_{LA}=(V^u_L)_{AB}\hat{u}_{LB}, \quad \tilde{u}_{RA}=(V^u_R)_{AB}\hat{u}_{RB},$$
$$\tilde{d}_{LA}=(V^d_L)_{AB}\hat{d}_{LB}, \quad \tilde{d}_{RA}=(V^d_R)_{AB}\hat{d}_{RB}, \tag{58}$$

where as before the matrices $V=\exp(iH_\mu x^\mu)$ are unitary in generation space. In this case, invariance of the Yukawa and nonderivative couplings, including the Cabbibo-Kobayashi-Maskawa mixings, requires the effect of the matrices V to reduce to multiplication by a single phase. For example, we can choose this phase so that the condition $(a_{uL})_{\mu 11}=(a_{uR})_{\mu 11}$ holds. This removes the CPT-breaking vector coupling from the u-quark sector.

Omitting tildes and carets, the general CPT-violating extension of the quark sector of the minimal standard model is therefore

$$\mathcal{L}^{CPT}_{\text{quark}}=-(a_u)_{\mu AB}\overline{u_A}\gamma^\mu u_B-(b_u)_{\mu AB}\overline{u_A}\gamma_5\gamma^\mu u_B -(a_d)_{\mu AB}\overline{d_A}\gamma^\mu d_B-(b_d)_{\mu AB}\overline{d_A}\gamma_5\gamma^\mu d_B, \tag{59}$$

where $(a_u)_{\mu 11}\equiv 0$. Again, these terms include small flavor-changing effects that are unobservable if the suppression is sufficiently small, such as that of fractional order 10^{-17} or smaller possible in the string scenario. In contrast, the diagonal contributions might be detected in interferometric experiments that measure the phenomenological parameters δ_P for indirect CPT violation in oscillations of neutral-P mesons, where P is one of K, D, B_d, or B_s. Each quantity δ_P is proportional to the difference between the diagonal elements of the effective Hamiltonian governing the time evolution of the corresponding P-$\overline{P}$ system. Explicit expressions for δ_P in terms of quantities closely related to those in Eq. (59) have been given in Ref. [10].

The two equations (53) and (59) represent allowed CPT-violating extensions of the fermion sector of the minimal standard model. Next, we briefly consider other CPT-violating terms without fermions.

The only CPT-violating term involving the Higgs field and satisfying our criteria is a derivative coupling of the form

$$\mathcal{L}^{CPT}_{\text{Higgs}}=ik^\mu\phi^\dagger D_\mu\phi+\text{H.c.}, \tag{60}$$

where k^μ is a CPT-violating constant, D_μ is the covariant derivative, and ϕ is the usual SU(2)-doublet Higgs field. Let us proceed under the assumption that no self-consistency issues arise for a scalar field that breaks CPT and Lorentz invariance, so that standard methods apply. Then, Eq. (60) represents a contribution to the Higgs-Z^0_μ sector of the model. Disregarding possible CPT-preserving but Lorentz breaking contributions to the static potential, it can be shown that the term (60) produces a (stable) modification of the standard symmetry-breaking pattern to include an expectation value for the Z^0_μ field with magnitude proportional to k_μ. Several kinds of effect ensue but if, as expected, the quantities k^μ are sufficiently small then it can be shown that the results are either unobservable or produce additional contributions to the fermion-bilinear terms already considered.

It is also possible to find CPT-violating terms satisfying our criteria and involving only the gauge fields. They are of the form

$$\mathcal{L}^{CPT}_{\text{gauge}}=k_{3\kappa}\epsilon^{\kappa\lambda\mu\nu}\text{Tr}(G_\lambda G_{\mu\nu}+\tfrac{2}{3}G_\lambda G_\mu G_\nu) +k_{2\kappa}\epsilon^{\kappa\lambda\mu\nu}\text{Tr}(W_\lambda W_{\mu\nu}+\tfrac{2}{3}W_\lambda W_\mu W_\nu) +k_{1\kappa}\epsilon^{\kappa\lambda\mu\nu}B_\lambda B_{\mu\nu}+k_{0\kappa}B^\kappa, \tag{61}$$

where $k_{3\kappa}$, $k_{2\kappa}$, $k_{1\kappa}$, and $k_{0\kappa}$ are CPT-violating constants. Here, G_μ, W_μ, B_μ are the (matrix-valued) SU(3), SU(2), U(1) gauge bosons, respectively, and $G_{\mu\nu}$, $W_{\mu\nu}$, $B_{\mu\nu}$ are the corresponding field strengths. The first three of these terms can be shown to leave unaffected the symmetry-breaking pattern, and we expect only unobservable effects for sufficiently small CPT-violating constants [42]. The field entering the term with coupling $k_{0\kappa}$ is of dimension one. It appears to produce a linear instability in the theory because it involves the photon, in which case it cannot emerge from a fundamental theory with a stable ground state.

VI. SUMMARY

In this paper, we have developed a framework for treating spontaneous CPT and Lorentz breaking in the context of conventional effective field theory. The underlying action is assumed to be consistent and fully CPT and Poincaré invariant, with solutions exhibiting spontaneous CPT and Lorentz breaking. The effective low-energy field theory then remains translationally invariant and covariant under changes of observer inertial frame, but violates CPT and partially breaks covariance under particle boosts.

Our focus has primarily been on Lagrangian terms that involve CPT-violating fermion bilinears, which are relevant for experiments bounding CPT in meson interferometry. In principle, these terms can be treated exactly because they are quadratic. We have investigated the relativistic quantum mechanics and the quantum field theory of a model for a Dirac fermion involving CPT violation. The analysis suggests that effective field theories with spontaneous CPT breaking have desirable properties like microscopic causality and renormalizability. The existence of consistent theories of this type is reasonable since they are analogous to conventional field theories in a nonvanishing background. Additional interactions appear minimally affected by the CPT violation, and the effects are largely restricted to modifications on fermion lines.

Within the framework developed, we have constructed a CPT-violating generalization of the minimal standard model that could be used in establishing quantitative CPT bounds. The criteria of gauge invariance and power-counting renormalizability constrain the extension to a relatively simple form, involving the extra terms given in Eqs. (53), (59), (60), and (61). It has been previously been suggested [9,10] that the properties of neutral-meson systems $P\bar{P}$, where P is one of K, D, B_d, or B_s, are well suited to interferometric tests of spontaneous CPT violation, with the experimentally measurable parameters for (indirect) CPT violation being explicitly related to certain diagonal elements of the quark-sector CPT-breaking matrices given in Eq. (59). Investigating the current experimental constraints on the other CPT-violating parameters introduced here is an interesting open topic and could lead to additional signals for CPT violation.

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APPENDIX A: EIGENSPINORS OF THE DIRAC EQUATION

Treating the CPT-violating parameters a_μ and b_μ as small relative to m, the four roots $\lambda^0_{\pm(\alpha)}(\vec{\lambda})$, $\alpha=1,2$, of the dispersion relation (20) are given to second order by

$$\lambda^0_{\pm(\alpha)}(\vec{\lambda}) = \pm\left\{ m^2+(\vec{\lambda}-\vec{a})^2+2(-1)^\alpha\left[b_0^2(\vec{\lambda}-\vec{a})^2+\vec{b}^2 m^2+(\vec{b}\cdot(\vec{\lambda}-\vec{a}))^2 \mp 2b_0\vec{b}\cdot(\vec{\lambda}-\vec{a})\sqrt{m^2+(\vec{\lambda}-\vec{a})^2}\right]^{1/2} + b_0^2+\vec{b}^2 \mp \frac{2b_0\vec{b}\cdot(\vec{\lambda}-\vec{a})}{\sqrt{m^2+(\vec{\lambda}-\vec{a})^2}} \right\}^{1/2} + a_0. \quad \text{(A1)}$$

This equation produces exact solutions to the dispersion relation in any of the special cases for which $b_0\vec{b}\cdot(\vec{\lambda}-\vec{a})=0$. The eigenenergies of the four spinors $u^{(\alpha)}(\vec{p})$, $v^{(\alpha)}(\vec{p})$ defined in Eq. (24) can be obtained by combining Eq. (A1) with Eq. (25).

In the general case, the four spinor eigensolutions can be written in the Pauli-Dirac representation as

$$u^{(\alpha)}(\vec{p}) = N_u^{(\alpha)}\begin{pmatrix}\phi^{(\alpha)}\\ X_u^{(\alpha)}\phi^{(\alpha)}\end{pmatrix},$$

$$v^{(\alpha)}(\vec{p}) = N_v^{(\alpha)}\begin{pmatrix}X_v^{(\alpha)}\chi^{(\alpha)}\\ \chi^{(\alpha)}\end{pmatrix}. \quad \text{(A2)}$$

In the first of these equations, $N_u^{(\alpha)}$ is an arbitrary spinor normalization factor and $X_u^{(\alpha)}$ is a spinor matrix defined by

$$X_u^{(\alpha)} = \frac{(E_u^{(\alpha)}-a_0+m+\vec{b}\cdot\vec{\sigma})[(\vec{p}-\vec{a})\cdot\vec{\sigma}-b_0]}{(E_u^{(\alpha)}-a_0+m)^2-\vec{b}^2}. \quad \text{(A3)}$$

The analogous quantity $X_v^{(\alpha)}$ for the second equation in (A2) can be found by replacing all subscripts u by v in Eq. (A3) and implementing the substitutions $a_\mu \to -a_\mu$, $b_\mu \to -b_\mu$ wherever these quantities explicitly appear. The quantities $\phi^{(\alpha)}$ and $\chi^{(\alpha)}$ are two-component spinors satisfying the eigenvalue equations

$$\vec{\kappa}_u^{(\alpha)}\cdot\vec{\sigma}\phi^{(\alpha)} = \eta_u^{(\alpha)}\phi^{(\alpha)}, \quad \vec{\kappa}_v^{(\alpha)}\cdot\vec{\sigma}\chi^{(\alpha)} = \eta_v^{(\alpha)}\chi^{(\alpha)}, \quad \text{(A4)}$$

with $[\vec{\kappa}_u^{(\alpha)}]^2 = [\eta_u^{(\alpha)}]^2$. Here, the vector $\vec{\kappa}_u^{(\alpha)}$ and the scalar $\eta_u^{(\alpha)}$ are given by

$$\vec{\kappa}_u^{(\alpha)}=2[(p_{u\mu}-a_\mu)b^\mu+mb_0](\vec{p}-\vec{a})$$
$$-[(p_u-a)^2+b^2+m^2+2m(E_u^{(\alpha)}-a_0)]\vec{b},$$
$$\eta_u^{(\alpha)}=2(E_u^{(\alpha)}-a_0)\vec{b}^2-2b_0\vec{b}\cdot(\vec{p}-\vec{a})$$
$$-(E_u^{(\alpha)}-a_0+m)[(p_u-a)^2-b^2-m^2]. \tag{A5}$$

The analogous quantities with subscripts v are given by the same substitutions as before.

APPENDIX B: EXACT EIGENENERGIES FOR CANONICAL CASES

For the case where b_μ is timelike, observer Lorentz invariance can be used to select a canonical frame in which $\vec{b}=0$. In this frame, we find the exact eigenenergies after reinterpretation are

$$E_u^{(\alpha)}=[m^2+(|\vec{p}-\vec{a}|+(-1)^\alpha b_0)^2]^{1/2}+a_0,$$
$$E_v^{(\alpha)}=[m^2+(|\vec{p}+\vec{a}|-(-1)^\alpha b_0)^2]^{1/2}-a_0, \tag{B1}$$

where $\alpha=1,2$ as usual.

For the case of spacelike b_μ, an observer frame can be chosen in which $b^0=0$. After reinterpretation the exact eigenenergies become

$$E_u^{(\alpha)}=[m^2+(\vec{p}-\vec{a})^2$$
$$+(-1)^\alpha 2\sqrt{m^2\vec{b}^2+(\vec{b}\cdot(\vec{p}-\vec{a}))^2}+\vec{b}^2]^{1/2}+a_0,$$
$$E_v^{(\alpha)}=[m^2+(\vec{p}+\vec{a})^2$$
$$-(-1)^\alpha 2\sqrt{m^2\vec{b}^2+(\vec{b}\cdot(\vec{p}+\vec{a}))^2}+\vec{b}^2]^{1/2}-a_0. \tag{B2}$$

Finally, for the lightlike case $b_\mu b^\mu=0$ the exact eigenvalues of the dispersion relation after reinterpretation are

$$E_u^{(\alpha)}=[m^2+(\vec{p}-\vec{a}-(-1)^\alpha\vec{b})^2]^{1/2}+a_0+(-1)^\alpha b_0,$$
$$E_v^{(\alpha)}=[m^2+(\vec{p}+\vec{a}+(-1)^\alpha\vec{b})^2]^{1/2}-a_0-(-1)^\alpha b_0. \tag{B3}$$

These last expressions hold in all observer frames.

APPENDIX C: EXPLICIT SOLUTION FOR $\vec{b}=0$

For the special case $\vec{b}=0$, the eigenenergies are given by Eq. (B1) and the eigenspinors can be written in a relatively simple form. Introducing momentum-space spinors via Eq. (24), we find in the Pauli-Dirac basis the expressions

$$u^{(\alpha)}(\vec{p})=\left(\frac{E_u^{(\alpha)}(E_u^{(\alpha)}-a_0+m)}{2m(E_u^{(\alpha)}-a_0)}\right)^{1/2}\begin{pmatrix}\phi^{(\alpha)}(\vec{p}-\vec{a})\\ \dfrac{-(-1)^\alpha|\vec{p}-\vec{a}|-b_0}{E_u^{(\alpha)}-a_0+m}\,\phi^{(\alpha)}(\vec{p}-\vec{a})\end{pmatrix},$$
$$v^{(\alpha)}(\vec{p})=\left(\frac{E_v^{(\alpha)}(E_v^{(\alpha)}+a_0+m)}{2m(E_v^{(\alpha)}+a_0)}\right)^{1/2}\begin{pmatrix}\dfrac{-(-1)^\alpha|\vec{p}+\vec{a}|+b_0}{E_v^{(\alpha)}+a_0+m}\,\phi^{(\alpha)}(\vec{p}+\vec{a})\\ \phi^{(\alpha)}(\vec{p}+\vec{a})\end{pmatrix}, \tag{C1}$$

where $\alpha=1,2$ as usual and where we have chosen the normalization of the spinors so that Eq. (27) is satisfied. In Eq. (C1), the two two-component spinors $\phi^{(\alpha)}(\vec{\lambda})$ are the eigenvectors of $\vec{\sigma}\cdot\hat{\lambda}$ with eigenvalues $-(-1)^\alpha$. If the spherical-polar angles that $\vec{\lambda}$ subtends are specified as (θ,ϕ), then the spinors $\phi^{(\alpha)}(\vec{\lambda})$ are given explicitly by

$$\phi^{(1)}(\vec{\lambda})=\begin{pmatrix}\cos\dfrac{\theta}{2}\\ \sin\dfrac{\theta}{2}\,e^{i\phi}\end{pmatrix},\quad \phi^{(2)}(\vec{\lambda})=\begin{pmatrix}-\sin\dfrac{\theta}{2}\,e^{-i\phi}\\ \cos\dfrac{\theta}{2}\end{pmatrix}. \tag{C2}$$

Note that the structure of the CPT-violating terms forces the spinors (C1) and their generalizations in Appendix A to involve helicity-type states. In the limit of vanishing CPT violation, the solutions (C1) reduce to standard Dirac spinors in the helicity basis.

APPENDIX D: PROPAGATOR FUNCTIONS

It can be shown that the generalized Feynman propagator determined by Eqs. (34) and (35) has the form

$$S_F(x-x')=(i\gamma^\lambda\partial_\lambda-a_\lambda\gamma^\lambda-b_\lambda\gamma_5\gamma^\lambda+m)(i\gamma^\mu\partial_\mu-a_\mu\gamma^\mu+b_\mu\gamma_5\gamma^\mu+m)(i\gamma^\nu\partial_\nu-a_\nu\gamma^\nu+b_\nu\gamma_5\gamma^\nu-m)\Delta_F(x-x'), \tag{D1}$$

where

$$\Delta_F(y)=\int_{C_F}\frac{d^4p}{(2\pi)^4}\,e^{-ip\cdot y}\,\frac{1}{[(p-a)^2-b^2-m^2]^2+4b^2(p-a)^2-4[b^\mu(p_\mu-a_\mu)]^2},\tag{D2}$$

with C_F the same contour in the p_0 plane as that in Eq. (35).

Direct integration for the special case $\vec{b}=0$ gives

$$S_F(x-x')=e^{-ia\cdot(x-x')}(i\gamma^\mu\partial_\mu+b_0\gamma_5\gamma^0+m)(-\partial^2-m^2-b_0^2+2ib_0\gamma_5\gamma^0\gamma^j\partial_j)\Delta_F(x-x'),\tag{D3}$$

where

$$\Delta_F(x-x')=\frac{1}{16\pi^2}\,\frac{\sin b_0r}{b_0r}\begin{cases}2iK_0(m\sqrt{r^2-t^2}), & r^2>t^2,\\ \pi H_0^{(2)}(m\sqrt{t^2-r^2}), & r^2<t^2,\end{cases}\tag{D4}$$

where r is the radial spherical-polar coordinate. In this expression, K_0 is a modified Bessel function and $H_0^{(2)}$ is a Hankel function of the second kind. The result (D3) reduces to the standard one in the limit $a_\mu=b_0=0$. Note that the propagator is singular on the light cone, as usual.

APPENDIX E: ANTICOMMUTATOR FUNCTIONS

The anticommutator function $S(x-x')$ defined in Eq. (44) can be shown to be given by

$$S(x-x')=(i\gamma^\lambda\partial_\lambda-a_\lambda\gamma^\lambda-b_\lambda\gamma_5\gamma^\lambda+m)(i\gamma^\mu\partial_\mu-a_\mu\gamma^\mu+b_\mu\gamma_5\gamma^\mu+m)(i\gamma^\nu\partial_\nu-a_\nu\gamma^\nu+b_\nu\gamma_5\gamma^\nu-m)\Delta(x-x'),\tag{E1}$$

where

$$i\Delta(y)=\int_{C}\frac{d^4p}{(2\pi)^4}\,e^{-ip\cdot y}\,\frac{1}{[(p-a)^2-b^2-m^2]^2+4b^2(p-a)^2-4[b^\mu(p_\mu-a_\mu)]^2},\tag{E2}$$

with C being the contour of Eq. (44) in the p_0 plane.

For the special case $\vec{b}=0$, direct integration gives

$$S(x-x')=e^{-ia\cdot(x-x')}(i\gamma^\mu\partial_\mu+b_0\gamma_5\gamma^0+m)(-\partial^2-m^2-b_0^2+2ib_0\gamma_5\gamma^0\gamma^j\partial_j)\Delta(x-x'),\tag{E3}$$

where

$$i\Delta(x-x')=-\frac{1}{8\pi}\,\frac{\sin b_0r}{b_0r}\begin{cases}J_0(m\sqrt{t^2-r^2}), & t>r,\\ 0, & -r<t<r,\\ -J_0(m\sqrt{t^2-r^2}), & t<-r,\end{cases}\tag{E4}$$

where r is the radial spherical-polar coordinate and J_0 is a Bessel function. The expression (E3) reduces to the standard result in the limit $a_\mu=b_0=0$.

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[41] Neutrino-mass terms in a nonminimal standard model would exclude some of the redefinitions but would also avoid any zero-mass issues.

[42] The term involving $k_{1\kappa}$ modifies the photon propagator, so $k_{1\kappa}$ can be bounded by terrestrial, astrophysical, and cosmological experiments. See Colladay and Kostelelcký [39].